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# From the Dirac Operator to Wess-Zumino Models on Spatial Lattices

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**Abstract:** We investigate two-dimensional Wess-Zumino models in the continuum and on spatial lattices in detail. We show that a non-antisymmetric lattice derivative not only excludes chiral fermions but in addition introduces supersymmetry breaking lattice artifacts. We study the nonlocal and antisymmetric SLAC derivative which allows for chiral fermions without doublers and minimizes those artifacts. The supercharges of the lattice Wess-Zumino models are obtained by dimensional reduction of Dirac operators in high-dimensional spaces. The normalizable zero modes of the models with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry are counted and constructed in the weak- and strong-coupling limits. Together with known methods from operator theory this gives us complete control of the zero mode sector of these theories for arbitrary coupling.

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# 1 Introduction

Ever since its invention supersymmetry has been an important subject in high-energy physics beyond the standard model. It is considered to be a necessary ingredient to bridge the gap between the scale of electroweak symmetry breaking and the much larger unification scale. Nowadays, supersymmetric theories cover the whole range from supersymmetric classical mechanics [1], quantum mechanics [2, 3], scalar and gauge field theories [4] to string- and $M$-theory [5]. They allow for the construction of low-energy effective actions, as for the $\mathcal{N}=2$ Seiberg-Witten model [6] or the formulation of certain duality relations, like in the original Maldacena conjecture for gauge theories with $\mathcal{N}=4$ extended supersymmetry [7].

The non-perturbative effects in supersymmetric theories, and in particular, the dynamical breaking of supersymmetry are a subject of intensive studies. At present time the lattice formulation is the only tool for *systematic* investigations of such effects, and lattice simulations provide the means of doing reliable calculations in the strong-coupling regime or near a phase transition point. After the pioneering work of Dondi and Nicolai [8] there has been an ongoing effort into formulating, understanding and simulating supersymmetric theories on the lattice [9, 10, 11, 12]. Recent lattice results, e.g. on the breaking of supersymmetry, have been obtained in [13, 14, 15].

A commonly accepted guiding principle in any good lattice calculation is to build in as many of the symmetries of the continuum model as possible, such that the lattice results respect these symmetries identically. However, often these are conflicting requirements and not all symmetries can be incorporated on the lattice. This in turn introduces subtle lattice artifacts into the formalism, which one may not get rid of in the continuum limit. For example, lattice regularizations of supersymmetric theories generically *break large parts of supersymmetry*, and it is a nontrivial problem to recover supersymmetry in the continuum limit. However, there are discretizations with highly nonlocal derivative operators, for which supersymmetry is manifestly realized [8, 11]. Alternatively, for two-dimensional models one can discretize only space (time remains continuous) such that a subalgebra of the $\mathcal{N}=1$ supersymmetry algebra,

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu$$

remains intact [9, 10, 16]. That subalgebra then determines the spectral properties of the super-Hamiltonian $H$. The *fermion doubling* for naive lattice derivatives [17, 18] is another apparently unrelated notorious example of such lattice artifacts. For bosons there is no such problem. However, if we try to preserve part of supersymmetry then the fermionic mirror states lead to doublers in the bosonic sector as well.

In this paper we study continuum and lattice versions of two-dimensional Wess-Zumino (WZ) models. Similar to the original four-dimensional theory [19], these models contain scalar and fermion fields coupled by a Yukawa term. A particular version possesses $\mathcal{N}=2$ supersymmetry and has been the subject of analytic [20, 21] and numerical [22] studies.

In section 2 we consider the off-shell formulation for a general class of continuum models and derive the supersymmetry transformations and Noether currents. Particular emphasis is put on the form of the central charges [23].

In section 3 we turn to the lattice version of the models. We show that for real and *antisymmetric* lattice derivatives the $\mathcal{N}=1$ algebra can be represented on free fields. The local left- and right-derivatives are not antisymmetric and the anticommutator of the corresponding supercharges does not yield the discretized Hamiltonian for the free model. If we insist that supersymmetry is realised on free fields without fermion and boson doubling then we must allow for nonlocal derivatives on the lattice. One particular such derivative, the SLAC operator, is introduced in this section. The numerical results for this operator concerning supersymmetry in lower-dimensional systems are in excellent agreement with continuum results. In section 4 we show how to derive the models with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry on a spatial lattice by a suitable *dimensional reduction* of a high-dimensional Euclidean Dirac operator. In the process of reduction the Dirac matrices and coordinates turn into Majorana spinors and scalar fields on the lattice. We count and *construct* the normalisable eigenstates of $H$ with zero energy both in the weak and strong-coupling limits. In particular we find that the $\mathcal{N}=2$ models with $\phi^{2q}$ interaction admit $q^N$ such states if $N$ is the number of spatial lattice sites.

In section 4 we bridge the gap between strong- and weak-coupling regimes for models with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry with the help of powerful methods from operator theory. Using a theorem by Kato we prove that the zero modes in the strong-coupling limit survive for intermediate couplings as long as the coupling constant of the leading term in the potential does not vanish. We comment on what we expect to happen in the continuum limit of the $\mathcal{N}=2$ models, where only $q$ of the $q^N$ zero modes survive [24]. We also comment on recent lattice simulations of the two dimensional Wess-Zumino model by Beccaria et al. [25]. Some technical details concerning the nonlocal SLAC operator and the proof that the transition from strong to intermediate couplings is governed by a relative compact perturbation are relegated to the appendix.

## 2 Wess-Zumino Models in $1+1$ Dimensions

In the off-shell formulation two-dimensional parity invariant Wess-Zumino models contain a set of, say $d$, triples, each containing a real scalar $\phi$, Majorana spinor $\psi$ and auxiliary field $F$. In a Majorana representation for the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu},\quad\text{with}\quad \gamma^0\gamma^{\mu\dagger}\gamma^0=\gamma^\mu,\quad \eta=\mathrm{diag}(1,-1),\tag{1}$$

the Majorana spinors are real.

The supersymmetry algebra is spanned by $\mathcal{N}$ Hermitian spinorial supercharges $Q^{(I)}$, $I=1,\dots,\mathcal{N}$, by the Hermitian two-momentum $P_\mu$ and by the (anti-)symmetric matrix of Hermitian central charges $Z_{\rm S}^{IJ}$ $(Z_{\rm A}^{IJ})$ and has the form

$$\{Q_\alpha^{(I)},\bar Q_\beta^{(J)}\}=2\left(\delta^{IJ}\not{P}_{\alpha\beta}+\mathrm{i}\delta_{\alpha\beta}\mathcal{Z}_{\rm A}^{IJ}+\mathrm{i}\gamma_{*\alpha\beta}\mathcal{Z}_{\rm S}^{IJ}\right),\quad \gamma_*=\gamma^0\gamma^1, \tag{2}$$

with spinor index $\alpha=1,2$.

In component fields the Lagrangian of the models with $\mathcal{N}=1$ supersymmetry reads [26]

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi^a\partial^\mu\phi_a-F^aW_{,a}+\tfrac{1}{2}F^aF_a+\tfrac{\mathrm{i}}{2}\bar\psi^a\not{\partial}\psi_a-\tfrac{1}{2}W_{,ab}\,\bar\psi^a\psi^b, \tag{3}$$

where the *superpotential* $W$ depends on the dimensionless scalar fields $\phi^1,\dots,\phi^d$. We denoted the derivative of $W$ with respect to $\phi^a$ by $W_{,a}$ and employed the Einstein summation convention. For Wess-Zumino models the target spaces are $\mathbb{R}^d$ with Euclidean metric $\delta_{ab}$.

Now we consider the most general *linear* off-shell supersymmetry transformation of the fields. Since $(\phi^a,\psi^a,F^a)$ have mass dimensions $(0,\frac{1}{2},1)$ respectively, such transformations have the form [26]

$$\begin{aligned}\delta_\epsilon\phi^a&=\bar\epsilon(A\psi)^a,\\ \delta_\epsilon\psi^a&=\mathrm{i}\not{\partial}(B\phi)^a\epsilon+(CF)^a\epsilon,\\ \delta_\epsilon F^a&=\mathrm{i}\bar\epsilon\not{\partial}(D\psi)^a,\end{aligned} \tag{4}$$

where, for example, $(A\psi)^a=A^a{}_b\psi^b$. The constant matrices $A,B,C,D$ must be real for the supersymmetry variations to be Hermitian fields. The requirement that $\mathcal{L}$ transforms into a divergence implies the following algebraic relations for these matrices and the real symmetric matrix $W''=(W_{,ab})$,

$$A+B^T=0,\quad D+C^T=0, \tag{5}$$

$$A^TW''+W''C=0,\quad W''A^T+CW''=0. \tag{6}$$

It follows that

$$\delta\mathcal{L}=\bar\epsilon\,\partial_\mu V^\mu+\Delta\quad\text{with}\quad\Delta=-\tfrac{1}{2}W_{,abc}\left(\bar\epsilon A^a{}_d\psi^d\right)\left(\bar\psi^b\psi^c\right).$$

Free models have quadratic superpotentials and $\Delta$ is identically zero. For interacting models we may exploit the Fierz identity

$$(\bar\psi^a\psi^b)(\psi^c\psi^d)+(\bar\psi^a\psi^d)(\psi^b\psi^c)+(\bar\psi^a\psi^c)(\psi^d\psi^b)=0$$

to prove that $\Delta$ vanishes, provided

$$W''A=A^TW'' \tag{7}$$

holds true. Then the action is left invariant by the transformations (4) and the corresponding conserved Noether current reads

$$J^\mu=(\partial^\mu\phi-\gamma_*\epsilon^{\mu\nu}\partial_\nu\phi)^a\,(A\psi)^a-\mathrm{i}(CW')^a\gamma^\mu\psi^a,\quad W'=(\partial W/\partial\phi^a). \tag{8}$$

In what follows, employing (5), we express the matrices $B$ and $D$ in terms of $A$ and $C$. We consider $\mathcal{N}$ supersymmetries (4) with matrices $(A_I, C_I)$ and denote the corresponding supersymmetry transformations by $\delta_\epsilon^{(I)}$. For all pairs $(A_I, C_I)$ the conditions (6) and (7) must hold for the Lagrangian to be invariant. These conditions severely restrict the form of the superpotential $W$. We also demand that two supersymmetry transformations close on translations (later we shall comment on the possibility of central charges)

$$\left[\delta_{\epsilon_1}^{(I)}, \delta_{\epsilon_2}^{(J)}\right]\Phi = 2\mathrm{i}\delta^{IJ}(\bar\epsilon_2\gamma^\mu\epsilon_1)\partial_\mu\Phi, \tag{9}$$

and this puts further restrictions on the matrices. For the scalar and the auxiliary field the condition (9) read

$$A_I A_J^T + A_J A_I^T = C_I^T C_J + C_J^T C_I = 2\delta_{IJ}\mathbb{1} \quad \text{and} \quad A_I C_J - A_J C_I = 0. \tag{10}$$

In particular all matrices are orthogonal, such that the two conditions in (6) coincide. Actually, the last relation implies that the algebra (9) is realized on the Majorana fields as well.

The transformation $\delta_\epsilon^{(I)}$ is generated by the Noether charge corresponding to $J_I^\mu$ in (8),

$$Q^{(I)} = \int \mathrm{d}x \left((\pi - \phi'\gamma_*)_a (A_I\psi)^a - \mathrm{i}(C_I W')_a \gamma^0\psi^a\right), \quad \pi^a = \dot\phi^a, \tag{11}$$

where we have set $(\mathrm{d}\phi^a/\mathrm{d}x) = \phi'$.

**Canonical structure:** The canonical structure is more transparent in the on-shell formulation. This is obtained from the off-shell one by replacing $F_a$ by $W_{,a}$. The nontrivial equal time (anti)commutators between the scalar fields, their conjugated momentum fields $\pi^a = \dot\phi^a$ and the Majorana fields read

$$\{\psi_\alpha^a(x), \psi_\beta^b(y)\} = \delta_{\alpha\beta}\delta^{ab}\delta(x-y) \quad \text{and} \quad [\phi^a(x), \pi^b(y)] = \mathrm{i}\delta^{ab}\delta(x-y). \tag{12}$$

The Hamiltonian is the Legendre transform of the Lagrangian,

$$H = \int \mathrm{d}x\, \mathcal{H}, \qquad \mathcal{H} = \tfrac{1}{2}\pi\cdot\pi + \tfrac{1}{2}\phi'\cdot\phi' + \tfrac{1}{2}W'\cdot W' + \tfrac{1}{2}\psi^\dagger h_{\mathrm{F}}\psi, \tag{13}$$

where, for example, $\pi\cdot\pi = \pi_a\pi^a$. We have introduced the *Hermitian* Dirac-Hamiltonian

$$(h_{\mathrm{F}})_{ab} = -\mathrm{i}\gamma_*\partial_x\delta_{ab} + \gamma^0 W_{,ab} \equiv (h_{\mathrm{F}}^0)_{ab} + \gamma^0 W_{,ab}\,. \tag{14}$$

The action is invariant under spacetime translations generated by Noether charges

$$P_0 = \mathcal{H} \quad \text{and} \quad P_1 = \int \mathrm{d}x\, \left(\pi\cdot\phi' + \tfrac{\mathrm{i}}{2}\bar\psi\gamma^0\psi'\right), \tag{15}$$

and under supersymmetry transformations (4) generated by the above supercharges $Q^{(I)}$. By using the relations (6,10) one proves that the $Q^{(I)}$ satisfy the super-algebra (2) with central charges

$$\mathcal{Z}_A^{IJ} = 0 \quad \text{and} \quad \mathcal{Z}_S^{IJ} = -\int \phi'\cdot\big(A_I C_J\big)W', \tag{16}$$

where we have neglected ambiguous surface terms containing the Majorana fields only. Note, that the integrand is a total derivative, since the *integrability conditions* for the existence of a potential $U(\phi(x))$ with

$$\phi' \cdot (A_I C_J) W' = \frac{\mathrm{d}U}{\mathrm{d}x} = U' \cdot \phi'$$

is that $A_I C_J W''$ is a symmetric matrix. But this follows from the condition (6).

In most explicit calculations we choose the Majorana representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = \mathrm{i}\sigma_3 \quad \text{and} \quad \gamma_* = \gamma^0\gamma^1 = -\sigma_1 \tag{17}$$

such that the superalgebra takes the simple form

$$\begin{aligned} \{Q_1^{(I)}, Q_1^{(J)}\} &= 2\left(H\delta^{IJ} + \mathcal{Z}_S^{IJ}\right), \\ \{Q_2^{(I)}, Q_2^{(J)}\} &= 2\left(H\delta^{IJ} - \mathcal{Z}_S^{IJ}\right), \\ \{Q_1^{(I)}, Q_2^{(J)}\} &= 2\left(P_1\delta^{IJ} + \mathcal{Z}_A^{IJ}\right). \end{aligned} \tag{18}$$

$\mathcal{N} = 1$ **supersymmetry:** There is always at least one solution to the constraints (5,6,7) and (10) for an arbitrary superpotential $W$, namely

$$A_1 = -B_1 = -C_1 = D_1 = \mathbb{1}. \tag{19}$$

Solving for the auxiliary field, $F_a = W_{,a}$, the on-shell transformations take the form

$$\delta_\epsilon^{(1)}\phi = \bar{\epsilon}\psi, \quad \delta_\epsilon^{(1)}\psi = \left(-\mathrm{i}\partial\!\!\!/\phi - W'\right)\epsilon, \tag{20}$$

and the corresponding supercharge reads

$$Q^{(1)} = \int \mathrm{d}x \, \left(\pi - \phi'\gamma_* + \mathrm{i}W'\gamma^0\right) \cdot \psi \, . \tag{21}$$

For vanishing spinors the only non-trivial central charge is

$$\mathcal{Z}_S = \int \mathrm{d}x \, \frac{\mathrm{d}W}{\mathrm{d}x}. \tag{22}$$

$\mathcal{N} = 2$ **extended supersymmetry:** We assume that the model (3) admits a second supersymmetry besides the solution (19). The conditions (10) imply

$$A_2 = -C_2 = I, \quad I = -I^T, \quad I^2 = -\mathbb{1}. \tag{23}$$

The matrix $I$ defines a complex structure and exists for all target spaces $\mathbb{R}^d$ with even dimension $d$. The conditions in (6) and (7) on the superpotential both reduce to

$$IW'' + W''I = 0, \tag{24}$$

which means that the superpotential is a harmonic function of the scalar fields, in agreement with the general analysis in [26]. On-shell, the second supersymmetry has the form

$$\delta_\epsilon^{(2)}\phi = \bar\epsilon I\psi, \quad \delta_\epsilon^{(2)}\psi = \left(\mathrm{i}\partial\!\!\!/ I\phi - IW'\right)\epsilon, \tag{25}$$

and is generated by the Noether-supercharge

$$Q^{(2)} = \int \mathrm{d}x\ \left(\pi - \phi'\gamma_* - \mathrm{i}W'\gamma^0\right)\cdot(I\psi). \tag{26}$$

For vanishing spinor fields the central charges read

$$\mathcal{Z}_A^{IJ} = 0 \quad \text{and} \quad \left(\mathcal{Z}_S^{IJ}\right) = \sigma_3 \int \mathrm{d}x\ \frac{\mathrm{d}W}{\mathrm{d}x} - \sigma_1 \int \mathrm{d}x\ \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{27}$$

where $U$ is the imaginary part of the analytic function $F(\phi^1 + \mathrm{i}\phi^2) = W + \mathrm{i}U$ with real part $W$.

For the models with $\mathcal{N} = 2$ supersymmetry there exists a concise formulation in which two real scalars are combined to a complex scalar, and two Majorana spinors are combined to a Dirac spinor. For example, for the target space $\mathbb{R}^2$ we set

$$\phi = \frac{1}{\sqrt{2}}\left(\phi^1 + \mathrm{i}\phi^2\right), \quad \psi = \frac{1}{\sqrt{2}}\left(\psi^1 + \mathrm{i}\gamma_*\psi^2\right). \tag{28}$$

The harmonic superpotential is the real part of a holomorphic function,

$$W(\phi, \bar\phi) = F(\phi) + \bar F(\bar\phi), \tag{29}$$

and the on-shell Lagrangian takes the form

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^\dagger + \mathrm{i}\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}|F'|^2 - F''\bar\psi P_+\psi - \bar F''\bar\psi P_-\psi, \tag{30}$$

where $F'$ is the derivative of $F$ with respect to the *complex* field $\phi$ and we have introduced the chiral projectors

$$P_\pm = \tfrac{1}{2}(\mathbb{1} + \gamma_*). \tag{31}$$

Along with the real scalar fields one combines the corresponding conjugate momentum fields to a complex momentum, $\pi = (\pi^1 - \mathrm{i}\pi^2)/\sqrt{2}$, such that

$$[\phi(x), \pi(y)] = \mathrm{i}\delta(x-y) \quad \text{and} \quad \{\psi_\alpha, \psi_\beta^\dagger\} = \delta_{\alpha\beta}. \tag{32}$$

The complex supercharge takes the form

$$Q = \tfrac{1}{2}\left(Q^{(1)} + \mathrm{i}\gamma_* Q^{(2)}\right) = \left(\pi - \bar\phi' + \mathrm{i}F'\gamma^0\right)P_+\psi + \left(\bar\pi + \phi' + \mathrm{i}\bar F'\gamma^0\right)P_-\psi. \tag{33}$$

and satisfies the anticommutation relations

$$\{Q, Q\} = 0 \quad \text{and} \quad \{Q, \bar Q\} = P\!\!\!/ + \gamma_*\mathcal{Z}_S^{11} - \mathcal{Z}_S^{12}. \tag{34}$$

**Higher supersymmetries:** Next we show that with the absence of central charges there is no third linear off-shell supersymmetry besides (20) and (25). To be compatible with the first transformation in (20), the orthogonal matrices $A_3$ and $C_3$ must be antisymmetric and of opposite sign. The conditions (10) between the second and third supersymmetry imply

$$[I, A_3] = \{I, A_3\} = 0,$$

which is impossible for orthogonal matrices $I$ and $A_3$. We conclude that the models (3) admit at most two linear off-shell supersymmetries.

Let us mention that, if we allow for central charges in the superalgebra, there exist further supersymmetries. But the corresponding models are massive free models. They can be derived by a dimensional reduction of the free $\mathcal{N} = 2$ model in 4 dimensions.

## 3 Lattice Formulations of Wess-Zumino Models

As ultraviolet-cutoff we discretize space, introduce a spatial lattice with $N$ equidistant sites and choose periodic boundary conditions. The time is kept continuous such that time translations remain symmetries generated by the Hamiltonian. Following [9] we try to preserve at least that subalgebra of (2) which involves $H$.

The fields of the supersymmetric model in the Hamiltonian formulation are discretized as follows,

$$(\phi^a(x), \pi^a(x), \psi^a(x)) \longrightarrow (\phi^a(n), \pi^a(n), \psi^a(n)), \quad n = 1, \dots, N, \tag{35}$$

where the lattice spacing has been set to one. On a space-lattice the derivative becomes a difference operator the particular choice of which is left open for the moment being. We *define* the lattice Hamiltonian as square of the discretized supercharge $Q_1$. For interacting theories it consists of the discretized Hamiltonian of the continuum theory plus a lattice counterpart of the central charge.

On-shell the $\mathcal{N} = 1$ model contains $d \in \{1, 2, \dots\}$ Hermitian scalar fields $\phi^a(n)$ and $d$ Majorana spinors $\psi^a(n)$ on $N$ lattice sites ($n = 1, \dots, N$). The fields obey the non-trivial canonical (anti-)commutation relations

$$[\phi^a(n), \pi^b(n')] = \mathrm{i}\delta^{ab}\delta(n, n') \quad \text{and} \quad \{\psi^a_\alpha(n), \psi^b_\beta(n')\} = \delta^{ab}\delta_{\alpha\beta}\delta(n, n'). \tag{36}$$

We choose a Majorana representation such that the $\psi^a$ are Hermitian two component spinors.

When we put the supercharge on a space-lattice, we must choose the lattice derivative in the term

$$-\int \phi'\gamma_*\psi = \int \phi\gamma_*\psi' = \mathrm{i}\int \phi h^0_{\mathrm{F}}\psi$$

in (11). Since we do not want to specify $\partial$ at this point we make the general ansatz for the Hermitian Dirac-Hamiltonian

$$h^0_{\rm F}={\rm i}\delta_{ab}\begin{pmatrix}0&\partial\\-\partial^\dagger&0\end{pmatrix}\ ,\quad\text{with}\quad \partial\partial^\dagger=\partial^\dagger\partial\equiv-\triangle \tag{37}$$

and a real $\partial$ with correct continuum limit. $\partial$ must be real, since it should map Majorana spinors into Majorana spinors. Let us define its symmetric and antisymmetric parts

$$\partial_{\rm S}=\tfrac{1}{2}(\partial+\partial^\dagger),\quad \partial_{\rm A}=\tfrac{1}{2}(\partial-\partial^\dagger)\quad\text{with}\quad [\partial_{\rm A},\partial_{\rm S}]=0,\quad \partial_{\rm A}^2-\partial_{\rm S}^2=\triangle. \tag{38}$$

The last two properties follow from our assumption $[\partial,\partial^\dagger]=0$ in (37). Since

$$h^0_{\rm F}\stackrel{(17)}{=}-{\rm i}\gamma_*\partial_{\rm A}-\gamma^0\partial_{\rm S}, \tag{39}$$

*chirality is preserved* for massless fermions if $\partial=\partial_{\rm A}$ is antisymmetric, in which case $h^0_{\rm F}=-{\rm i}\gamma_*\partial_{\rm A}$. Thus, if $\partial$ is antisymmetric and local then, according to some long-standing no-go theorems there is fermion doubling. There are many such theorems, and we mention only two, one due to Nielsen and Ninomiya [17] and a later elaboration due to Friedan [18]. No-go theorems are notorious in that people find a way around them, and following Friedans work, Lüscher [27] and others did so. Below we circumvent the no-go theorems by using a nonlocal and antisymmetric derivative.

However, most lattice derivative are not antisymmetric in which case $h^0_{\rm F}$ contains a momentum dependent mass term $-\gamma^0\partial_{\rm S}$. Such a chirality violating term has been introduced by Wilson [28] to raise the masses of the unwanted doublers to values of order of the cutoff, thereby decoupling them from continuum physics.

As discretized supercharge (21) we take

$$Q^{(1)}=(\pi,\psi)+{\rm i}(\phi,h^0_{\rm F}\psi)+{\rm i}(W',\gamma^0\psi). \tag{40}$$

A careful calculation yields the following anticommutation relations,

$$\tfrac{1}{2}\{Q^{(1)}_\alpha,Q^{(1)}_\beta\}=(\not{P}\gamma^0)_{\alpha\beta}-{\rm i}(\gamma^1)_{\alpha\beta}(W',\partial_{\rm A}\phi)-\delta_{\alpha\beta}(W',\partial_{\rm S}\phi) \tag{41}$$

with energy and momentum

$$\begin{aligned}2P_0&=&(\pi,\pi)-(\phi,\triangle\phi)+\big(W',W'\big)+(\psi,h_{\rm F}\psi)\,,\\ 2P_1&=&2\big(\partial_{\rm A}\phi,\pi\big)-\big(\psi,\gamma_*h^0_{\rm F}\psi\big)\,,\quad h_{\rm F}=h^0_{\rm F}+\gamma^0W''.\end{aligned} \tag{42}$$

To arrive at these results one uses the identity

$$(\pi,\partial\phi)+{\rm i}(\partial^\dagger\psi_1,\psi_1)=(\partial\phi,\pi)-{\rm i}(\psi_1,\partial^\dagger\psi_1),$$

which holds for any real difference operator $\partial$. The superalgebra can be rewritten as

$$\tfrac{1}{2}\{Q^{(1)},\bar{Q}^{(1)}\}=\not{P}+{\rm i}\gamma_*(W',\partial_{\rm A}\phi)-\gamma^0(W',\partial_{\rm S}\phi). \tag{43}$$

The last term is absent in the superalgebra (2) and breaks Lorentz covariance explicitly. This lattice artifact originates in the Wilson term $-\gamma^0\partial_\mathrm{S}$ in (39). This term must vanish in the continuum limit. One may wonder whether there exist other improvement terms we could add to a local $-\mathrm{i}\gamma_*\partial_\mathrm{A}$ in order to avoid the fermion doubling. However, since for Majorana fermions the terms

$$(\psi,\partial_\mathrm{S}\psi),\quad (\psi,\gamma^1\partial_\mathrm{S}\psi),\quad (\psi,\gamma_*\partial_\mathrm{S}\psi)$$

are constant or zero, all terms but $\gamma^0\partial_\mathrm{S}$ do not show up in the right hand side of (42) and we obtain the same result as if we had chosen $h_\mathrm{F}=-\mathrm{i}\gamma_*\partial_\mathrm{A}$. Hence only the Wilson term $\sim\gamma^0\partial_\mathrm{S}$ can be used to avoid the fermion doubling. This argument does not apply to theories with several Majorana fermions and in particular to models with extended supersymmetry.

Models with $\mathcal{N}=2$ supersymmetry contain the second supercharge in (26), the lattice version of which reads

$$Q^{(2)}=(\pi,I\psi)+\mathrm{i}(\phi,h_\mathrm{F}^0I\psi)-\mathrm{i}(W',\gamma^0I\psi),\tag{44}$$

and satisfies the same anticommutation relations as $Q^{(1)}$, up to a sign change of the last two terms in (43). The anticommutator of two lattice charges reads

$$\tfrac{1}{2}\{Q^{(I)},\bar{Q}^{(J)}\}=\delta^{IJ}\not{P}+\mathrm{i}\gamma_*\mathcal{Z}_S^{IJ}+\mathcal{Z}_L^{IJ},\tag{45}$$

where the 'would-be' central charges

$$\mathcal{Z}_S^{IJ}=\sigma_3^{IJ}(W',\partial_\mathrm{A}\phi)-(\sigma_1)^{IJ}(U',\partial_\mathrm{A}\phi)\tag{46}$$

approach the central charges (27) of the continuum model. To arrive at (45) one needs the harmonicity of the superpotential which in turn implies the existence of a function $U(\phi)$ with $IW'=U'$, and this function enters the central charges. However, since the *Leibniz rule* never holds on the lattice, the integrands $W'\cdot\partial_\mathrm{A}\phi$ and $U'\cdot\partial_\mathrm{A}\phi$ in (46) are not just total derivatives as in the continuum and as a consequence the terms $\mathcal{Z}_S^{IJ}$ are not central to the algebra. The annoying terms

$$\mathcal{Z}_L^{IJ}=-(\sigma_3)^{IJ}\gamma^0(W',\partial_\mathrm{S}\phi)+(\sigma_1)^{IJ}\left(\gamma^0(U',\partial_\mathrm{S}\phi)-\mathrm{i}(\pi,I\partial_\mathrm{S}\phi)-\tfrac{\mathrm{i}}{2}(\psi,I\partial_\mathrm{S}\psi)\right)\tag{47}$$

in (45) are pure lattice artifacts and vanish for antisymmetric lattice derivatives.

**Free Wess-Zumino model ($\mathcal{N}=1$):** For simplicity we consider the free model with scalars of equal mass. The superpotential reads $W=\frac{1}{2}m\phi_a\phi^a$ and with $W'=m\phi$ the 'would-be' central charge vanishes,

$$(W',\partial_\mathrm{A}\phi)=m(\phi,\partial_\mathrm{A}\phi)=0.\tag{48}$$

As Hamiltonian we *choose* the square of the supercharges,

$$\begin{aligned}H&=\tfrac{1}{2}\{Q_1,Q_1\}=\tfrac{1}{2}\{Q_2,Q_2\}=P_0-m(\phi,\partial_\mathrm{S}\phi),\\ 2P_0&=(\pi,\pi)+\big(\phi,(-\triangle+m^2)\phi\big)+(\psi,h_\mathrm{F}\psi)\,,\end{aligned}\tag{49}$$

where the Dirac-Hamiltonian for the non-interacting model is just

$$h_{\mathrm{F}} = -\mathrm{i}\gamma_* \partial_{\mathrm{A}} + \gamma^0(m - \partial_{\mathrm{S}}) \quad \text{with} \quad h_{\mathrm{F}}^2 = (-\triangle + m^2 - 2m\partial_{\mathrm{S}})\mathbb{1}_2 \tag{50}$$

and $-\triangle = \partial\partial^\dagger$. For antisymmetric derivatives the pure lattice artifacts containing $\partial_{\mathrm{S}}$ vanish and with $2P_1 = \{Q_1, Q_2\}$ we obtain the familiar algebra

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu, \quad [Q_\alpha, P_\mu] = 0, \quad [P_0, P_1] = 0. \tag{51}$$

We conclude that the $\mathcal{N} = 1$ superalgebra in $1+1$ dimensions can be represented as a *free Wess-Zumino model* on a space lattice.

## 3.1 Lattice Derivatives

At this point some words about lattice derivatives are in order. At first instance one may think that the local right- and left derivatives

$$(\partial_{\mathrm{R}}f)(n) = f(n+1) - f(n) \quad \text{and} \quad (\partial_{\mathrm{L}}f)(n) = f(n) - f(n-1) \tag{52}$$

are ideal candidates for a lattice derivative. With respect to the $\ell_2$-scalar product of two lattice functions,

$$(f, g) = \sum_{n=1}^{N} f(n)g(n), \tag{53}$$

the adjoint of the left-derivative is minus the right-derivative, $\partial_{\mathrm{L}}^\dagger = -\partial_{\mathrm{R}}$. Both derivatives share the property that $(1, \partial_{\mathrm{R}}f) = (1, \partial_{\mathrm{L}}f) = 0$. But the corresponding momenta $\hat{p}_{\mathrm{L}} = -\mathrm{i}\partial_{\mathrm{L}}$ and $\hat{p}_{\mathrm{R}} = -\mathrm{i}\partial_{\mathrm{R}}$ are not Hermitian and possess complex eigenvalues,

$$\lambda_k(\hat{p}_{\mathrm{R}}) = \bar{\lambda}_k(\hat{p}_{\mathrm{L}}) = 2\mathrm{e}^{\mathrm{i}p_k/2}\sin\frac{p_k}{2}, \qquad \text{with} \quad p_k = 2\pi k/N, \qquad \text{and} \quad k = 1, \dots, N.$$

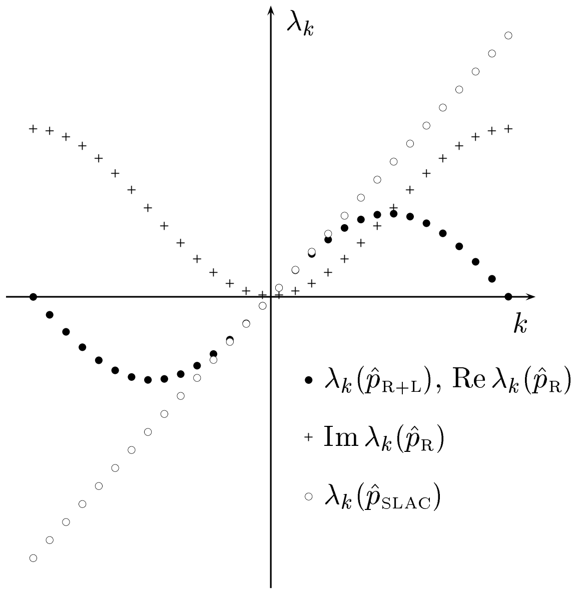
If we insist on a Hermitian momentum we could choose the *antisymmetric* derivative operator

$$\partial_{\mathrm{R+L}} = \tfrac{1}{2}(\partial_{\mathrm{R}} + \partial_{\mathrm{L}}) = -\partial_{\mathrm{R+L}}^T \tag{54}$$

which is used in many lattice calculations. The $N$ real eigenvalues of $\hat{p}_{\mathrm{R+L}}$ read

$$\lambda_k(\hat{p}_{\mathrm{R+L}}) = \sin p_k = \mathrm{Re}\,\left(\lambda_k(\hat{p}_{\mathrm{R}})\right),$$

and waves with the shortest wavelength, that is with $p_k$ at the boundary of the first Brillouin zone, are zero modes of $\partial_{\mathrm{R+L}}$. Hence, by trying to preserve the hermiticity of $\hat{p}$ in this naive way immediately introduces spurious zero modes that are responsible for the fermion doubling problem.




A third alternative for the lattice momentum is the Hermitian and nonlocal SLAC operator $\hat{p}_{\mathrm{SLAC}} = -\mathrm{i}\partial_{\mathrm{SLAC}}$ introduced by Drell, Weinstein and Yankielowicz [29] with real eigenvalues $p_k$. This operator has no spurious mirror states in the first Brillouin zone. The eigenvalues of $\hat{p}_{\mathrm{SLAC}}$ are eigenvalues of the momentum operator in the continuum and the difference between the lattice and continuum results are minimized. In the figure on the left we have plotted the eigenvalues of the various lattice operators. The real parts of the eigenvalues of $\hat{p}_{\mathrm{R}}$ and $\hat{p}_{\mathrm{L}}$ are just the eigenvalues of $\hat{p}_{\mathrm{R+L}}$. The eigenvalues of $\hat{p}_{\mathrm{R+L}}$ are twofold degenerate. The SLAC operator has the same dispersion relation as the momentum in the continuum.

Besides $\partial_{\mathrm{R}}, \partial_{\mathrm{L}}, \partial_{\mathrm{R+L}}$ and $\partial_{\mathrm{SLAC}}$ there are many other local and nonlocal candidates for lattice derivatives with the correct naive continuum limit. However, it is easy to see that no linear difference operator will obey the Leibniz rule. Many problems in supersymmetric lattice theories are exactly due to this fact, see [8].

In order to better understand the dependency of the spectrum and doubling phenomenon on the lattice derivative we consider the following one-parameter interpolating family of *ultra-local* difference operators

$$\partial_\alpha = \tfrac{1}{2}(1+\alpha)\partial_{\mathrm{R}} + \tfrac{1}{2}(1-\alpha)\partial_{\mathrm{L}} = \partial_{\mathrm{S}} + \partial_{\mathrm{A}}, \tag{55}$$

with symmetric and antisymmetric parts

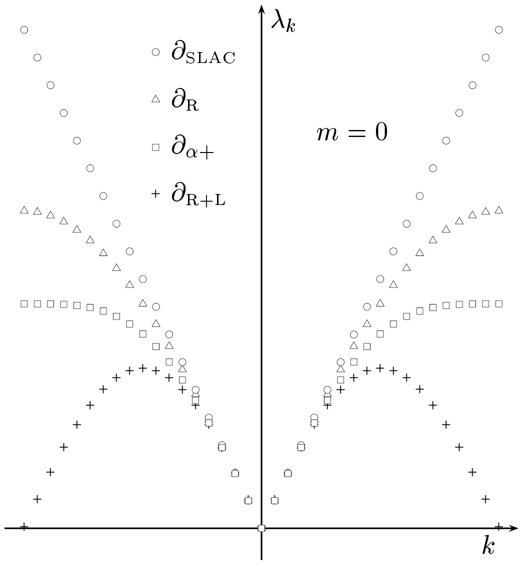
$$\partial_{\mathrm{S}} = \tfrac{1}{2}\alpha(\partial_{\mathrm{R}} - \partial_{\mathrm{L}}) = \tfrac{1}{2}\alpha\partial_{\mathrm{R}}\partial_{\mathrm{L}} \quad \text{and} \quad \partial_{\mathrm{A}} = \tfrac{1}{2}(\partial_{\mathrm{R}} + \partial_{\mathrm{L}}) = \partial_{\mathrm{R+L}}. \tag{56}$$

When the parameter $\alpha$ varies from 1 to $-1$, then $\partial_\alpha$ interpolates between $\partial_{\mathrm{R}}$ and $\partial_{\mathrm{L}}$. For $\alpha = 0$ we obtain the antisymmetric operator $\partial_{\mathrm{A}}$ in (52).

The $2N$ eigenvalues of the Hermitian *Dirac-Hamiltonian* (50) depend on the deformation parameter as follows,

$$\lambda_k(\alpha) = \lambda_{N-k}(\alpha) = \pm\sqrt{m^2 + 4\alpha(\alpha+m)\sin^2(\tfrac{1}{2}p_k) + (1-\alpha^2)\sin^2(p_k)}, \tag{57}$$

where $p_k = 2\pi k/N$ and $k$ runs from 0 to $N-1$. For the extreme cases $\alpha = 0, 1$ we obtain




the eigenvalues

$$\lambda_k(0) = \pm\sqrt{m^2+\sin^2(p_k)}$$

with multiplicity 4 and

$$\lambda_k(1) = \pm\sqrt{m^2+4(1+m)\sin^2(\tfrac{1}{2}p_k)}$$

with multiplicity 2. This should be compared with the eigenvalues on the *continuous* interval of 'length' $N$,

$$\lambda_k = \pm\sqrt{m^2+p_k^2} \tag{58}$$

with multiplicity 2. One can show that for $\alpha$ greater then $\alpha_+$ or less then $\alpha_-$, where

$$4\alpha_\pm = \pm\left(\sqrt{m^2+8}\mp m\right),$$

*all* eigenvalues have the same multiplicity as in the continuum. In particular, for massless fermions there are no doublers for $\alpha^2 > 1/2$. However, for $\alpha \in [\alpha_-, \alpha_+]$ some eigenvalues have multiplicity four. In the above figure we have plotted the *positive* eigenvalues of $h_{\rm F}$ for $\alpha = 0, 1, \alpha_+$. For comparison we have depicted the positive eigenvalues of $h_{\rm F}$ for the nonlocal SLAC derivative

$$(\partial_{\rm SLAC})_{n\neq n'} = (-)^{n-n'}\frac{\pi/N}{\sin\left(\pi(n-n')/N\right)} \quad \text{and} \quad (\partial_{\rm SLAC})_{nn} = 0. \tag{59}$$

Despite being nonlocal the SLAC derivative has many advantages as compared to the local operators $\partial_{\rm R}, \partial_{\rm L}$ or $\partial_{\rm R+L}$: it is *antisymmetric* such that for massless fermions chiral symmetry is preserved. By construction the $2N$ real eigenvalues of $h_{\rm F} = -i\gamma_*\partial_{\rm SLAC} + \gamma^0 m$ are identical to the $2N$ lowest eigenvalues of the continuum operator on the interval of 'length' $N$, (58). For this reason $\partial_{\rm SLAC}$ has been called *ideal lattice operator* in the literature. We do not expect that unwanted nonlocal counterterms [30] are required for the two-dimensional supersymmetric Wess-Zumino models. This is certainly the case for the finite models with extended supersymmetry. For the model with $\mathcal{N}=1$ supersymmetry the same should be true since it does not contain gauge fields which couple to high momentum modes at the edge of the Brillouin zone. Indeed, in [31] is has been claimed that $\partial_{\rm SLAC}$ approaches an ultra-local operator when $N$ tends to infinity, except for a border matrix. In the appendix we give a detailed analysis of this interesting operator.

### 3.2 On the Quality of Lattice Derivatives in Supersymmetric QM

It is enlightening to retreat to quantum-mechanical systems and study the supercharges

$$Q = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad \text{with} \quad A = \partial + W, \quad A^\dagger = \partial^\dagger + W, \tag{60}$$

and in particular the quality of lattice approximations for different lattice derivatives $\partial$ in $A$. The supercharge squares to

$$Q^2 = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}, \tag{61}$$

with isospectral discretized Schrödinger operators

$$\begin{aligned} AA^\dagger &= \partial\partial^\dagger + \partial W + W\partial^\dagger + W^2 \\ A^\dagger A &= \partial^\dagger\partial + \partial^\dagger W + W\partial + W^2. \end{aligned} \tag{62}$$

They have identical spectra, up to possible zero modes. If the Leibniz rule held on the lattice, if $\partial$ was antisymmetric and if we could replace $\partial W$ by $W' + W\partial$, then we would find the super-Hamiltonian of supersymmetric quantum mechanics in the continuum,

$$H = \begin{pmatrix} \partial\partial^\dagger + W' + W^2 & 0 \\ 0 & \partial^\dagger\partial - W' + W^2 \end{pmatrix}. \tag{63}$$

The difference between $Q^2$ and $H$ is the analog of the last two terms in (43) and the difference in their spectra is a good measure for the suitability of the chosen lattice derivative as regards supersymmetry and the speed with which the continuum limit is approached. In the following figure we have plotted the eigenvalues of $Q^2$ and $H$ for $\partial = \partial_{\text{SLAC}}$, denoted by $Q^2_{\text{SLAC}}$ and $H_{\text{SLAC}}$ and for $\partial = \partial_{\text{R}}$, denoted by $Q^2_{\text{naiv}}$ and $H_{\text{naiv}}$. We took the superpotential $W = \lambda x^2$ which gives rise to the supersymmetric anharmonic oscillator.

The lowest 57 eigenvalues of $Q^2$ and $H$ are almost identical for the SLAC derivatives and the lowest 90 eigenvalues of $H_{\text{SLAC}}$ agree with the exact values (calculated on a much finer grid). These results clearly demonstrate the high precision of the SLAC derivative in low-dimensional supersymmetric systems. It does not matter whether we discretise the supercharge or the super-Hamiltonian as long as we choose the SLAC derivative. After this detour to quantum mechanics we now return to supersymmetric field theories.

## 4 From the Dirac Operator to the Lattice $\mathcal{N} = 1$ WZ Model

In this section we relate the supercharges and Hamiltonians of two-dimensional Wess-Zumino models on a spatial lattice to suitable Dirac operators. We shall use the results in [32] on the (extended) supersymmetries of $\mathrm{i}\not{D}$ in arbitrary dimensions, specialized to

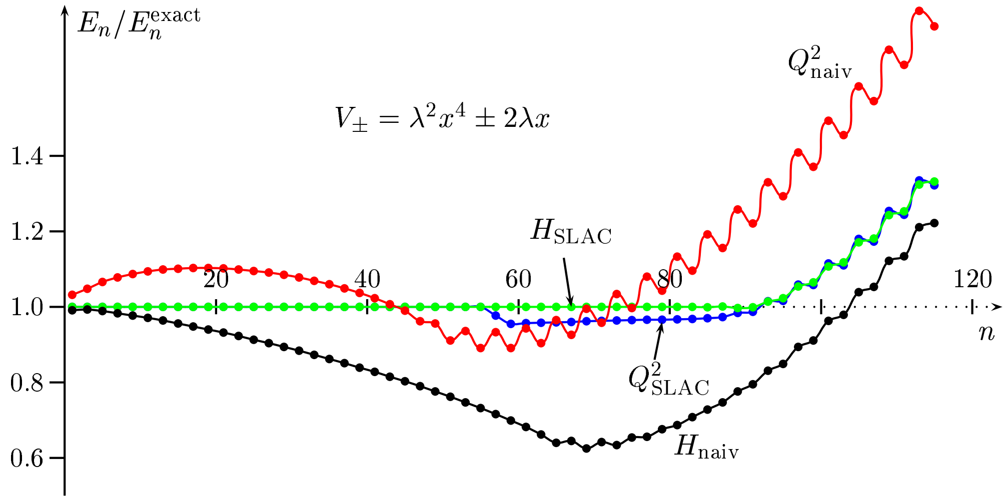



Figure 1: *Eigenvalues of $Q^2$ and $H$ for the SLAC derivative and the right-derivative for $N = 180$ lattice points, length $L = 30$ and $\lambda = 1$.*

flat space and perform a dimensional reduction such that the supercharges of the lattice models can be related to the reduced $\mathrm{i}\not{D}$.

To see how Dirac operators relate to multi-dimensional supersymmetric matrix-Schrödinger operators we generalize the reduction of the two-dimensional Dirac equation to the Nicolai-Witten operator to higher dimensions. For that purpose we dimensionally reduce the Euclidean operator

$$\mathrm{i}\not{D} = \mathrm{i}\Gamma^\mu D_\mu\,, \qquad D_\mu = \partial_\mu + \mathrm{i}A_\mu\,, \quad \Gamma^{\mu\dagger} = \Gamma^\mu, \quad \mu = 1, \dots, 2N, \tag{64}$$

from the products of cylinders,

$$M = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{N\text{-times}} \times \underbrace{S^1 \times \dots \times S^1}_{N\text{-times}} = \mathbb{R}^N \times T^N \tag{65}$$

to the factor $\mathbb{R}^N$. We take $x^1, \dots, x^N$ as coordinates on $\mathbb{R}^N$ and $\theta^1, \dots, \theta^N$ as coordinates on the torus $T^N$, respectively. We dimensionally reduce by assuming that the Abelian gauge potential is independent of the angles $\theta^n$. Then the Dirac operator commutes with the (angular)momenta $-\mathrm{i}\partial_{\theta^n}$ and we may set $\partial_{\theta^n} = 0$.

### 4.1 Reduction to Models with $\mathcal{N} = 1$

If we further set $A_1(x) = \dots = A_N(x) = 0$ and define $\bar{n} = N + n$, then the square of the reduced Dirac operator takes the form

$$-\not{D}^2 \equiv 2H = p_n p_n + A_{\bar{n}} A_{\bar{n}} - \mathrm{i}\Gamma^n \Gamma^{\bar{m}} \partial_n A_{\bar{m}}, \quad \text{where} \quad p_n = \frac{1}{\mathrm{i}} \frac{\partial}{\partial x^n}. \tag{66}$$

Note that the reduced operator $H$ contains no first-derivative terms. It can be identified with the Hamiltonian of a two-dimensional $\mathcal{N}=1$ WZ-model on a spatial lattice, if we set

$$x_n=\phi(n),\quad p_n=\pi(n)\quad\text{and}\quad \begin{pmatrix}\Gamma^n\\ \Gamma^{\bar n}\end{pmatrix}=\sqrt{2}\psi(n). \tag{67}$$

It follows with (17) that $(\psi,\gamma^0\psi)=-\mathrm{i}\Gamma^n\Gamma^{\bar n}$ holds true. If we further assume that the non-vanishing components of $A_\mu$ have the form

$$A_{\bar n}=-\big(\partial^\dagger\phi\big)(n)+W'(\phi(n))\quad\text{with}\quad \partial^\dagger=\partial_{\mathrm{S}}-\partial_{\mathrm{A}}, \tag{68}$$

then we find

$$-\frac{1}{\sqrt{2}}\,\mathrm{i}\not{D}=\underbrace{(\pi,\psi_1)+(W',\psi_2)}_{A_0}\underbrace{-(\phi,\partial\psi_2)}_{A_1}=Q_1^{(1)} \tag{69}$$

with $Q^{(1)}$ in (40). We conclude that the Hamiltonian reads

$$H=P_0+(W',\partial_{\mathrm{A}}\phi)-(W',\partial_{\mathrm{S}}\phi), \tag{70}$$

with $P_0$ from (42). Thus we have proved that the super-Hamiltonian of the $\mathcal{N}=1$ Wess-Zumino model on a lattice with $N$ sites is just the square of the Dirac operator in $2N$ dimensions, dimensionally reduced from $\mathbb{R}^N\times T^N$ to $\mathbb{R}^N$.

## 4.2 Ground State of the Free Model

For the massive non-interacting model we have $2W=m\phi^2$. The corresponding Hamiltonian is the sum of two commuting operators, of the bosonic part

$$H_{\mathrm{B}}=\tfrac{1}{2}(\pi,\pi)+\tfrac{1}{2}(\phi,A^2\phi),\qquad A^2=-\triangle+m\partial_{\mathrm{S}}+m^2, \tag{71}$$

and the fermionic one

$$H_{\mathrm{F}}=\tfrac{1}{2}(\psi,h_{\mathrm{F}}\psi),\quad h_{\mathrm{F}}=-\mathrm{i}\gamma_*\partial_{\mathrm{A}}+\gamma^0(m-\partial_{\mathrm{S}}). \tag{72}$$

We assume that the parameters are such that $A^2$ is positive. Near the continuum limit this is always the case if the physical mass is positive. The ground state wave function(al) of the supersymmetric Hamiltonian factorizes,

$$\Psi_0=\Psi_{\mathrm{B}}\Psi_{\mathrm{F}}\quad\text{with}\quad H_{\mathrm{B}}\Psi_{\mathrm{B}}=E_{\mathrm{B}}\Psi_{\mathrm{B}}\quad\text{and}\quad H_{\mathrm{F}}\Psi_{\mathrm{F}}=E_{\mathrm{F}}\Psi_{\mathrm{F}}.$$

We choose the field representation for the scalar field, such that

$$\pi(n)=\frac{1}{\mathrm{i}}\frac{\partial}{\partial\phi(n)}\quad\text{and}\quad\Psi_{\mathrm{B}}=\Psi_{\mathrm{B}}(\phi). \tag{73}$$

The bosonic factor $\Psi_{\rm B}$ is Gaussian

$$\Psi_{\rm B} = {\rm c}\cdot\exp\left(-\tfrac{1}{2}(\phi, A\phi)\right) \quad \text{and} \quad E_{\rm B} = \tfrac{1}{2}{\rm tr}A. \tag{74}$$

Here $A$ is the positive root of the positive and Hermitian $A^2$ in (71). For the family of operators in (55) the trace of $A$ is just half the sum of the positive eigenvalues in (57).

To find $\Psi_{\rm F}$ we introduce the (two-component) eigenfunctions $v_k$ of $h_{\rm F}$ with positive eigenvalues. Since the Hermitian matrix $h_{\rm F}$ is imaginary the $v_k$ cannot be real and we have

$$h_{\rm F}v_k = \lambda_k v_k \Longleftrightarrow h_{\rm F}\bar{v}_k = -\lambda_k\bar{v}_k \quad (\lambda_k > 0). \tag{75}$$

The eigenvectors are orthogonal with respect to the *Hermitian* scalar product,

$$(v_k, v_{k'}) = \sum_{n,\alpha=1,2} \bar{v}_{k\alpha}(n)v_{k'\alpha}(n) = \delta_{kk'} \quad \text{and} \quad (\bar{v}_k, v_{k'}) = 0. \tag{76}$$

Now we expand the Majorana spinors in terms of this orthonormal basis,

$$\psi(n) = \sum_{k=1}^{N}\left(\chi_k v_k(n) + \chi_k^\dagger \bar{v}_k(n)\right), \quad \text{where} \quad \chi_k = (v_k, \psi), \;\; \chi_k^\dagger = (\bar{v}_k, \psi) \tag{77}$$

are one-component complex objects with anticommutation relations

$$\{\chi_k, \chi_{k'}\} = 0 \quad \text{and} \quad \{\chi_k, \chi_{k'}^\dagger\} = \delta_{kk'}. \tag{78}$$

Inserting the expansion (77) into $H_{\rm F}$ yields

$$H_{\rm F} = \tfrac{1}{2}\sum_{k:\lambda_k>0}\lambda_k\left(\chi_k^\dagger\chi_k - \chi_k\chi_k^\dagger\right). \tag{79}$$

It follows that the ground state of $H_{\rm F}$ is the Fock vacuum which is annihilated by all annihilation operators $\chi_1, \dots, \chi_N$ and has energy

$$E_{\rm F} = -\tfrac{1}{2}\sum_{k:\lambda_k>0}\lambda_k. \tag{80}$$

Since $h_{\rm F}^2 = \mathbb{1}_2 \otimes A^2$ we conclude, that the positive eigenvalues of $h_{\rm F}$ are identical to the eigenvalues of $A$ such that

$$E = E_{\rm B} + E_{\rm F} = 0.$$

Since $\Psi_0$ is normalizable for $A > 0$ we see that the Hamiltonian admits a supersymmetric ground state for *all choices of the lattice derivative* $\partial$, provided $A$ is positive.

### 4.3 Ground State for Strong Coupling

In the extreme case of very strong self-coupling of the scalar field we may neglect the derivative term in the supercharge (69) [9]. Then $Q$ and $H$ are the sum of $N$ identical and commuting quantum mechanical operators, each defined on a given lattice site. Hence, the ground state is a product state, $\Psi_0(\phi) = \otimes_n \psi_0(\phi_n)$. The operators on a *fixed lattice site* read

$$Q_1^{(1)} = \frac{1}{\mathrm{i}}\psi_1 \frac{\partial}{\partial\phi} + \psi_2 W'(\phi) \quad \text{and} \quad H = -\frac{\partial^2}{\partial\phi^2} + W'^2 - \mathrm{i}\psi_1\psi_2\, W''. \tag{81}$$

A normalizable zero-energy state is annihilated by $Q_1^{(1)}$,

$$\psi_0(\phi) = \mathrm{e}^{-\mathrm{i}\psi_1\psi_2 W(\phi)}\omega_0, \tag{82}$$

where $\omega_0$ is a constant two-component spinor. It is well-known [33] and follows at once from (82) that supersymmetry is unbroken if $p$ in

$$W = \frac{1}{p}\lambda\phi^p + O(\phi^{p-1}), \quad \lambda \neq 0, \tag{83}$$

is even and it is broken if $p$ is odd. Note that $-\mathrm{i}\psi_1\psi_2$ is Hermitian and has eigenvalues $\pm 1$ and that for even $p$ the state $\psi_0$ is normalisable if

$$\mathrm{i}\,\mathrm{sign}\,(\lambda)\psi_1\psi_2\omega_0 = \omega_0 \quad \text{or} \quad (\psi_1 - \mathrm{i}\,\mathrm{sign}\,(\lambda)\psi_2)\,\omega_0 = 0.$$

To summarize, for even $p$ the $\mathcal{N} = 1$ Wess-Zumino model on the spatial lattice has always exactly one normalizable zero mode in the strong-coupling limit. For $\lambda > 0$ this product state has the form

$$\Psi_0(\phi) = \exp\left(-\sum_{n=1}^{N} W(\phi_n)\right)\Omega_0, \qquad \big(\psi_1(n) - \mathrm{i}\psi_2(n)\big)\Omega_0 = 0, \;\; \forall\, n. \tag{84}$$

In particular, for $\phi^4$-models supersymmetry is broken in the strong-coupling limit whereas it is unbroken for $\phi^6$-models.

## 5 From the Dirac Operator to the Lattice $\mathcal{N} = 2$ WZ Model

It is known, that on flat spacetime the Euclidean Dirac operator admits two supersymmetries if the field strength commutes with an antisymmetric and orthogonal matrix $I$, which defines a complex structure [32]. The two real supercharges

$$Q_1 = \frac{1}{\sqrt{2}}\,\mathrm{i}\,\Gamma^\mu D_\mu \quad \text{and} \quad Q_2 = \frac{1}{\sqrt{2}}\,\mathrm{i}\,I^\mu{}_\nu\Gamma^\nu D_\nu \tag{85}$$

form the superalgebra

$$\tfrac{1}{2}\{Q_i, Q_j\} = \delta_{ij} H. \tag{86}$$

They can be combined to a *nilpotent* complex charge

$$Q = \frac{1}{\sqrt{2}}\left(Q_1 + \mathrm{i} Q_2\right) \tag{87}$$

and its adjoint $Q^\dagger$, in terms of which the supersymmetry algebra takes the form

$$H = \tfrac{1}{2}\{Q, Q^\dagger\}, \quad Q^2 = Q^{\dagger 2} = 0 \quad \text{and} \quad [Q, H] = 0. \tag{88}$$

To obtain $\mathcal{N} = 2$ lattice models on $N$ sites we consider the Dirac operator on

$$M = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{2N\text{-times}} \times \underbrace{S^1 \times \ldots \times S^1}_{2N\text{-times}} = \mathbb{R}^{2N} \times T^{2N} \tag{89}$$

in contrast to the $2N$-dimensional space in (65). Since the field strength commutes with the complex structure $I$ it is very convenient to introduce the corresponding complex coordinates on $M$,

$$z^n = x^n + \mathrm{i} x^{\bar n} = x^n + \mathrm{i}\theta^n, \quad \bar n = 2N + n, \quad n, \bar n \in \{1, \ldots, 2N\}, \tag{90}$$

and fermionic annihilation and creation operators,

$$\psi^n = \tfrac{1}{2}\left(\Gamma^n + \mathrm{i}\Gamma^{\bar n}\right), \quad \psi^{\dagger n} = \tfrac{1}{2}\left(\Gamma^n - \mathrm{i}\Gamma^{\bar n}\right) \quad \text{with} \quad \{\psi^n, \psi^{\dagger m}\} = \delta^{mn}. \tag{91}$$

The condition that $F_{\mu\nu}$ commutes with the complex structure $I$ implies the existence of a real superpotential $\chi(z, \bar z)$, such that [32]

$$Q = 2\mathrm{i} \mathrm{e}^{-\chi} \left( \sum_{n=1}^{2N} \psi^n \frac{\partial}{\partial z^n} \right) \mathrm{e}^{\chi}.$$

## 5.1 Reduction to Models with $\mathcal{N} = 2$

Again we perform a dimensional reduction by assuming that $\chi$ does not depend on the compact variables $\theta^n$,

$$\chi = \chi\left(x^1, \ldots, x^{2N}\right) \tag{92}$$

and that the angular momenta $\partial_{\theta^n}$ vanish. In the sector with vanishing angular momenta the complex charge simplify to

$$Q = \mathrm{e}^{-\chi} Q_0 \mathrm{e}^{\chi}, \quad \text{where} \quad Q_0 = \mathrm{i} \sum_{n=1}^{2N} \psi^n \frac{\partial}{\partial x^n}, \tag{93}$$

since the complex $z^n$-derivative becomes the real $x^n$-derivative in this sector. This dimensional reduced supercharge and its adjoint generate the superalgebra (88) with supersymmetric matrix-Schrödinger operator

$$H=\frac{1}{2}\{Q,Q^\dagger\}=\underbrace{-\frac{1}{2}\Delta+\frac{1}{2}(\nabla\chi,\nabla\chi)+\frac{1}{2}\Delta\chi}_{H_{\rm B}}-\underbrace{\sum\psi^{n\dagger}\chi_{,nm}\psi^m}_{H_{\rm F}}. \tag{94}$$

For example, for $\chi=-\lambda r$ this is just the Hamiltonian of the supersymmetrized Hydrogen atom which has been introduced and solved in [34]. It is evident from the representations (93) and (94) that $Q$ decreases and $Q^\dagger$ increases the eigenvalue of the number operator

$$N=\sum\psi^{n\dagger}\psi^n \tag{95}$$

by one and $H$ commutes with $N$. The eigenvalues of $N$ are $0,1,\dots,2N$.

As before, we interpret the $2N$ coordinates $x^n$ and annihilation operators $\psi^n$ as values of two scalar and one Dirac field on a one-dimensional lattice with $N$ lattice sites. More precisely, we make the following identifications for $n=1,2,\dots,N$,

$$\begin{aligned}\phi(n)&=\begin{pmatrix}x^{2n-1}\\x^{2n}\end{pmatrix},\qquad \pi(n)=\begin{pmatrix}p_{2n-1}\\p_{2n}\end{pmatrix},\\ \psi(n)&=\begin{pmatrix}\psi^{2n-1}\\\psi^{2n}\end{pmatrix},\qquad \psi^\dagger(n)=\begin{pmatrix}\psi^{\dagger\,2n-1}\\\psi^{\dagger 2n}\end{pmatrix}.\end{aligned} \tag{96}$$

The free supercharge (93) takes the form

$$Q_0=\mathrm{i}\sum_{n=1}^N\psi(n)\frac{\partial}{\partial\phi(n)}\quad,\quad Q_0^\dagger=\sum_{n=1}^N\psi^\dagger(n)\frac{\partial}{\partial\phi(n)}. \tag{97}$$

The remaining task is to find a superpotential $\chi$ giving rise to interacting lattice Wess-Zumino models. Since $\chi$ should be real we use a representation for the two-dimensional $\gamma$-matrices such that $\mathrm{i}\gamma_*$ and $\gamma^0$ are real,

$$\gamma^0=\sigma_3,\quad \gamma^1=\mathrm{i}\sigma_1,\quad \gamma_*=-\sigma_2, \tag{98}$$

in order to obtain a real Dirac-Hamiltonian,

$$h_{\rm F}^m=-\mathrm{i}\gamma_*\partial+m\gamma^0=\begin{pmatrix}m&\partial\\-\partial&-m\end{pmatrix}.$$

As explained above, $\partial$ need not be anti-Hermitian in which case we take

$$h_{\rm F}^m=\begin{pmatrix}m&\partial\\\partial^\dagger&-m\end{pmatrix}=-\mathrm{i}\gamma_*\partial_{\rm A}+m\gamma^0-\mathrm{i}\gamma^1\partial_{\rm S}\quad\text{with}\quad(h_{\rm F}^m)^2=(-\triangle+m^2)\mathbb{1}_2, \tag{99}$$

such that $h_{\rm F}^m$ is real and Hermitian. Note that the term containing $\partial_{\rm S}$ is not a momentum dependent mass term as in (39). We have been lead to a different type of Wilson term as compared to the $\mathcal{N}=1$ model since we have chosen a different representation for the $\gamma$-matrices. The earlier Majorana representation (17) is not useful in the present context, since it would lead to a complex $\chi$ in (93).

The term $H_{\rm F}$ in (94) must contain the free Dirac-Hamiltonian and this condition implies

$$-\frac{\partial^2\chi^m}{\partial\phi_\alpha(n)\partial\phi_\beta(n')}=(h^m_{\rm F})_{\alpha\beta,nn'},\quad \alpha,\beta=1,2,\quad n,n'=1,2,\dots,N. \tag{100}$$

Hence we expect that the real function

$$\chi^m=-\tfrac{1}{2}(\phi,h^m_{\rm F}\phi), \tag{101}$$

is the superpotential for a $\mathcal{N}=2$ supersymmetric model. For these models we use the following inner products

$$(\phi,\tilde\phi)=\sum_{\alpha,n}\phi_{\alpha,n}\tilde\phi_{\alpha,n}\quad\text{and}\quad(\psi,\tilde\psi)=\sum_{\alpha,n}\psi^\dagger_\alpha(n)\tilde\psi_\alpha(n), \tag{102}$$

for scalar doublets and Dirac spinors, respectively. The corresponding supercharge

$$Q=\mathrm{i}e^{-\chi^m}Q_0\,e^{\chi^m}=\mathrm{i}\sum_n\psi(n)\left(\frac{\partial}{\partial\phi}-h^m_{\rm F}\phi\right) \tag{103}$$

and its adjoint give rise to the following super Hamiltonian,

$$H_{\rm B}=-\tfrac{1}{2}(\pi,\pi)+\tfrac{1}{2}\big(\phi,(-\triangle+m^2)\phi\big),\quad H_{\rm F}=(\psi,h^m_{\rm F}\psi). \tag{104}$$

Note that the superpotential $\chi^m$ is a harmonic function, $\triangle\chi^m=0$, and thus there is no constant contribution to $H_{\rm B}$. The charges and Hamiltonian act on function(al)s in the Hilbert space of the $\mathcal{N}=2$ lattice models

$$\mathcal{H}=\underbrace{\mathfrak{h}\otimes\cdots\otimes\mathfrak{h}}_{N-\text{times}},\quad\text{where}\quad\mathfrak{h}=L_2(\mathbb{R}^2)\otimes\mathbb{C}^4 \tag{105}$$

is the Hilbert space for the degrees of freedom on one lattice site.

Now we turn to *interacting models* by replacing the mass term in

$$\chi^m=-\tfrac{1}{2}(\phi,h^0_{\rm F}\phi)+\sum_n f\big(\phi(n)\big),\qquad f(\phi)=\tfrac{1}{2}m(\phi_2^2-\phi_1^2),$$

given by the quadratic harmonic function $f$, by an arbitrary harmonic function $f(\phi)$ of the two variables $\phi_1$ and $\phi_2$,

$$\chi=-\tfrac{1}{2}(\phi,h^0_{\rm F}\phi)+\sum f\big(\phi(n)\big),\quad\text{where}\quad\Delta f=0. \tag{106}$$

The supercharge and its adjoint are calculated as

$$Q=e^{-\chi}Q_0e^{\chi}\quad\text{and}\quad Q^\dagger=e^{\chi}Q_0^\dagger e^{-\chi} \tag{107}$$

with $Q_0$ and $Q_0^\dagger$ from (97). After some algebra one finds for the bosonic part of $H = \frac{1}{2}\{Q, Q^\dagger\}$ the formula

$$H_{\rm B} = \tfrac{1}{2}(\pi,\pi) - \tfrac{1}{2}(\phi,\triangle\phi) + \tfrac{1}{2}\left(\frac{\partial f}{\partial\phi},\frac{\partial f}{\partial\phi}\right) + \left(\frac{\partial g}{\partial\phi_1},\partial^\dagger\phi_1\right) - \left(\frac{\partial g}{\partial\phi_2},\partial\phi_2\right) \tag{108}$$

where the harmonic functions $f$ and $g$ are the real and imaginary parts of an analytic function, such that

$$\frac{\partial f}{\partial\phi_1} = \frac{\partial g}{\partial\phi_2} \quad \text{and} \quad \frac{\partial f}{\partial\phi_2} = -\frac{\partial g}{\partial\phi_1}. \tag{109}$$

For the fermionic part of the Hamiltonian one obtains

$$H_{\rm F} = (\psi, h_{\rm F}^0\psi) - \big(\psi,\gamma^0\Gamma(\phi)\psi\big)\,, \qquad \Gamma(\phi) = f_{,11}(\phi) - {\rm i}\gamma_* f_{,12}(\phi). \tag{110}$$

The last term contains the Yukawa coupling between scalar and Dirac fields. Note that the last two terms in (108) can be rewritten as

$$\mathcal{Z} = -\left(\frac{\partial g}{\partial\phi},\partial_{\rm A}\phi\right) + \left(\frac{\partial g}{\partial\phi},\sigma_3\partial_{\rm S}\phi\right). \tag{111}$$

In the continuum limit the first term on the right becomes a surface term commuting with the supercharges and the second term, which is a lattice artifact, must vanish. Thus it is natural to set

$$H_{\rm B} + H_{\rm F} = P_0 + \mathcal{Z} \tag{112}$$

and interpret the first term

$$P_0 = \tfrac{1}{2}(\pi,\pi) - \tfrac{1}{2}(\phi,\Delta\phi) + (\psi,h_{\rm F}^0\psi) + \tfrac{1}{2}\left(\frac{\partial f}{\partial\phi},\frac{\partial f}{\partial\phi}\right) - \big(\psi,\gamma^0\Gamma(\phi)\psi\big) \tag{113}$$

as energy and the second term as 'would be' *central charge* $\mathcal{Z}$ in (111). This agrees with our interpretation for solitonic configurations saturating the BPS-bound. To see that more clearly we consider the energy of a purely bosonic static solution,

$$E = P_0 = -\tfrac{1}{2}(\phi,\Delta\phi) + \tfrac{1}{2}\left(\frac{\partial f}{\partial\phi},\frac{\partial f}{\partial\phi}\right). \tag{114}$$

From the very construction it is evident, that there is a BPS-bound. One just adds the non-negative operator $H_{\rm B}$ in (108) to the non-negative operator one gets when changing the signs of $f$ and $g$ and finds

$$E \geq |\mathcal{Z}|. \tag{115}$$

For example, a cubic superpotential $f + {\rm i}g = \lambda\phi^3/3$ leads to a $\phi^4$-models with

$$P_0 = \tfrac{1}{2}(\pi,\pi) - \tfrac{1}{2}(\phi,\Delta\phi) + (\psi,h_{\rm F}^0\psi) + \tfrac{1}{2}\lambda^2\,(\phi,\phi)^2 - \big(\psi,\gamma^0\Gamma(\phi)\psi\big)\,. \tag{116}$$

It contains a scalar and pseudoscalar Yukawa interaction with

$$\Gamma(\phi) = 2\lambda(\phi_1 + \mathrm{i}\gamma_* \phi_2).$$

The would-be central charge is cubic in the scalar fields and reads

$$\mathcal{Z} = 2\lambda\left(\phi_1\phi_2, \partial^\dagger\phi_1\right) - \lambda\left(\phi_1^2 - \phi_2^2, \partial\phi_2\right). \tag{117}$$

Before turning to the discussion of the ground state we note, that the conserved number operator

$$N = \sum_n \psi^\dagger(n)\psi(n) \tag{118}$$

leads to a decomposition of the Hilbert space (105) in orthogonal subspaces labelled by the *fermion number*,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{2N-1} \oplus \mathcal{H}_{2N}, \qquad N\big|_{\mathcal{H}_p} = p\mathbb{1}. \tag{119}$$

The nilpotent supercharge $Q$ decreases $N$ by one, $Q^\dagger$ increases it by one and the super-Hamiltonian commutes with $N$,

$$[N, Q] = -Q, \quad [N, Q^\dagger] = Q^\dagger \quad \text{and} \quad [N, H] = 0. \tag{120}$$

We call the subspace $\mathcal{H}_p$ $p$-particle sector. The states in the zero-particle sector are annihilated by $Q$ and those in the $2N$-particle sector by $Q^\dagger$.

## 5.2 Ground State of the Free Model

The Hermitian lattice Dirac-Hamiltonian $h_{\mathrm{F}}^m$ in (99) is real and can be diagonalized by an orthogonal matrix $S$,

$$h_{\mathrm{F}}^m = S^{-1}DS, \quad D = \mathrm{diag}(d_1, d_2, \ldots, d_{2N}). \tag{121}$$

We rotate the field-variables with $S$,

$$\xi = S\phi, \quad \eta = S\psi \quad \text{and} \quad \eta^\dagger = S\psi^\dagger.$$

The new fields still obey the standard anticommutation relations, e.g.

$$\{\eta_\alpha^\dagger(n), \eta_\beta(m)\} = \delta_{\alpha\beta}\delta_{nm}, \tag{122}$$

and the transformed supercharges read

$$Q = \mathrm{i}\eta\left(\frac{\partial}{\partial\xi} - D\xi\right) \quad \text{and} \quad Q^\dagger = \mathrm{i}\eta^\dagger\left(\frac{\partial}{\partial\xi} + D\xi\right) \tag{123}$$

and show, that the new degrees of freedom decouple. Hence the ground state must have the product form

$$\Psi_0 = \exp\left(-\tfrac{1}{2}\sum |d_a|\xi_a^2\right)|\Omega\rangle \tag{124}$$

and the supercharges act on this state as follows,

$$Q\Psi_0 = 2\mathrm{i}\sum_{a:\, d_a>0} d_a\eta_a\xi_a\Psi_0 \quad , \quad Q^\dagger\Psi_0 = -2\mathrm{i}\sum_{a:\, d_a<0} d_a\eta_a^\dagger\xi_a\Psi_0. \tag{125}$$

This way we arrive at the following conditions for this state to be invariant,

$$d_a > 0 \Longrightarrow \eta_a|\Omega\rangle = 0 \quad \text{and} \quad d_a < 0 \Longrightarrow \eta_a^\dagger|\Omega\rangle = 0. \tag{126}$$

This leads to the *unique* normalizable ground state (124) with

$$|\Omega\rangle = \prod_{d_a<0}\eta_a^\dagger|0\rangle. \tag{127}$$

which is annihilated by the supercharges and hence has vanishing energy. There are $N$ positive and $N$ negative eigenvalues of $h_{\mathrm{F}}^m$ such that the invariant vacuum state lies in the middle sector $\mathcal{H}_N$ in the decomposition (119) of the Hilbert space. All fermionic states with negative energies are filled. This is just the Dirac-sea filling prescription. Note that our result is the lattice version of the continuum result for the ground state,

$$\Psi_0 = \exp\left(-\tfrac{1}{2}\int\phi_\alpha\sqrt{-\triangle+m^2}\phi_\alpha\right)|\Omega\rangle.$$

## 5.3 Ground States for Strong Coupling

In the strong-coupling limit we may neglect the spatial derivatives such that the supercharges and the Hamiltonian becomes the sum of $N$ commuting operators, each defined on one lattice site [21]. The operators *on a given site* take the form

$$Q = \mathrm{i}\psi\left(\nabla + \nabla f\right), \quad H = -\tfrac{1}{2}\triangle + \tfrac{1}{2}(\nabla f, \nabla f) - \psi^\dagger f''\psi. \tag{128}$$

Now we explicitly construct the ground state for the harmonic superpotential

$$f(\phi) = \frac{\lambda}{p}\,\mathrm{Re}\,\phi^p, \qquad \phi = \phi_1 + \mathrm{i}\phi_2 = r\mathrm{e}^{\mathrm{i}\theta}, \tag{129}$$

which gives rise to a supersymmetric *anharmonic oscillator* on the Hilbert space $\mathfrak{h} = L_2(\mathbb{R}^2)\times\mathbb{C}^4$. The bosonic part of $H$ reads

$$H_{\mathrm{B}} = -\tfrac{1}{2}\triangle + V \quad \text{with} \quad V = \tfrac{1}{2}\lambda^2 r^{2p-2}, \tag{130}$$

and its fermionic part

$$H_{\mathrm{F}} = \lambda(p-1)\,\psi^\dagger \begin{pmatrix} -\mathrm{Re}\,\phi^{p-2} & \mathrm{Im}\,\phi^{p-2} \\ \mathrm{Im}\,\phi^{p-2} & \mathrm{Re}\,\phi^{p-2} \end{pmatrix} \psi \tag{131}$$

It is useful to note that $H$ commutes with the operator

$$J = L + S, \quad S = -s\psi^\dagger\sigma_2\psi, \quad s = \tfrac{1}{2}(p-2) \tag{132}$$

and that the ground state must reside in the two-dimensional sector with particle number $N = \psi^\dagger\psi = 1$, since the restriction of $H_{\mathrm{F}}$ to the zero- and two-particle sectors vanish and $H_{\mathrm{B}} > 0$. The one-particle sector is spanned by the following two eigenstates of $S$,

$$|\uparrow\rangle = \left(\psi_1^\dagger - \mathrm{i}\psi_2^\dagger\right)|0\rangle \quad \text{and} \quad |\downarrow\rangle = \left(\psi_1^\dagger + \mathrm{i}\psi_2^\dagger\right)|0\rangle \tag{133}$$

with eigenvalues 1 and $-1$, respectively. Here $|0\rangle$ denotes the Fock-vacuum which is annihilated by the annihilation operators $\psi_\alpha$. Diagonalising $J$ in this sector leads to the ansatz

$$\psi_{0j}(\phi) = R_{j+}(r)e^{i(j-s)\theta}\,|\uparrow\rangle + R_{j-}(r)e^{i(j+s)\theta}\,|\downarrow\rangle, \tag{134}$$

where the $J$-eigenvalue $j$ is integer for even $p$ and half-integer for odd $p$. Inserting into $Q\psi_j = Q^\dagger\psi_j = 0$ yields the following coupled system of first order differential equations for the radial functions

$$R'_{j\pm}(r) - \frac{s \mp j}{r}R_{j\pm}(r) + \lambda r^{p-1}R_{j\mp}(r) = 0.$$

The square integrable solutions are Bessel functions

$$R_{j\pm}(r) = c\,r^{p-1}\,\mathrm{K}_{\frac{1}{2}\pm\frac{j}{p}}\left(\tfrac{\lambda}{p}r^p\right) \quad \text{with} \quad j \in \{-s, -s+1, \dots, s-1, s\}. \tag{135}$$

The number of supersymmetric ground states of the models with $\phi^{2p-2}$ self-interaction is just $p-1$. The $(p-1)^N$ normalizable invariant eigenstates are

$$\Psi_{0,j_1,\dots,j_N} = \bigotimes_{n=1}^{N}\psi_{0j_n}(\phi_n) \in \mathfrak{h}_1 \times \dots \times \mathfrak{h}_N. \tag{136}$$

For example, for the $\mathcal{N}=2$ model with $\phi^4$ interaction there exist $2^N$ normalizable zero modes in the strong-coupling limit. This number diverges in the thermodynamic limit. On the other hand, there is exactly one normalisable zero mode when one switches off the self-interaction. This discrepancy between the number of supersymmetric ground states in the weak and strong-coupling regimes becomes even more puzzling when one takes into account certain rigorous theorems on the stability of such states under analytic perturbations discussed in the following section. The zero modes in (136) with radial functions (135) have been constructed previously in [21, 33] and [35].

# 6 From Strong to Weak Couplings

## 6.1 Perturbation Theory and Zero Modes

Let us recall a well known result for perturbation theory of zero modes in supersymmetric quantum mechanics [33]. We consider the $\mathcal{N} = 1$ case and denote the single Hermitian supercharge by $Q_0$,

$$Q_0^2 = H_0, \quad \{\Gamma, Q_0\} = 0, \quad \Gamma^\dagger = \Gamma, \quad \Gamma^2 = \mathbb{1}. \tag{137}$$

In addition, we define the projection operators

$$\mathcal{P}_\pm = \tfrac{1}{2}(\mathbb{1} \pm \Gamma) \tag{138}$$

which project on the $\pm 1$ eigenspaces of $\Gamma$. These eigenspaces are denoted by $\mathcal{H}_{\mathrm{B/F}}$. In the following we assume that there are no zero modes in $\mathcal{H}_{\mathrm{F}}$ and at least one zero mode $\Psi_0$ in the bosonic sector $\mathcal{H}_{\mathrm{B}}$. We perturb the operator $Q_0$ by an operator $\epsilon Q_1$ with real parameter $\epsilon$, $Q(\epsilon) = Q_0 + \epsilon Q_1$, where $\{Q_1, \Gamma\} = 0$. We want to solve the eigenvalue equation

$$Q(\epsilon)\Psi(\epsilon) = \lambda(\epsilon)\Psi(\epsilon),$$

with $\lambda(0) = 0$ and $\Psi(0) = \Psi_0$. We consider the following formal power series in $\epsilon$,

$$\Psi(\epsilon) = \Psi_0 + \sum_{k=1}^{\infty} \epsilon^k \Psi_k, \quad \lambda(\epsilon) = \sum_{k=1}^{\infty} \epsilon^k \lambda_k.$$

**Proposition:** Under the assumptions above one has $\lambda(\epsilon) = 0$ and $\Gamma\Psi(\epsilon) = \Psi(\epsilon)$ in the sense of formal power series.

*Proof by induction:* To order $\epsilon^0$ the proposition holds. Let us assume it holds up to order $\epsilon^{j-1}$. To order $\epsilon^j$ we obtain the equation

$$Q_0\Psi_j + Q_1\Psi_{j-1} = \lambda_j\Psi_0. \tag{139}$$

Taking the scalar product with $\Psi_0$ yields

$$\lambda_j = (\Psi_0, Q_1\Psi_{j-1}).$$

Since $\Gamma$ squares to $\mathbb{1}$ and anticommutes with the perturbation we find

$$\lambda_j = (\Gamma^2\Psi_0, Q_1\Psi_{j-1}) = -(\Gamma\Psi_0, Q_1\Gamma\Psi_{j-1}) = -(\Psi_0, Q_1\Psi_{j-1}) = -\lambda_j,$$

which proves that $\lambda_j = 0$. Furthermore, with

$$Q_0\Gamma\Psi_j = -\Gamma Q_0\Psi_j = \Gamma Q_1\Psi_{j-1} = -Q_1\Psi_{j-1} = Q_0\Psi_j,$$

we conclude

$$Q_0 \mathcal{P}_- \Psi_j = 0,$$

where we used the projection operator $\mathcal{P}_-$ introduced in (138). As $\mathcal{P}_- \Psi_j$ is a zero mode of $Q_0$ it follows by assumption that it resides in $\mathcal{H}_\mathrm{B}$. But as $\mathcal{P}_-$ projects onto $\mathcal{H}_\mathrm{F}$, we conclude $\mathcal{P}_- \Psi_j = 0$ or $\Psi_j \in \mathcal{H}_\mathrm{B}$. This then proves our statement. Note that the statement has been proved in the sense of formal power series only. In case $\lambda(\epsilon)$ is not analytic at $\epsilon = 0$ the result above maybe misleading.

## 6.2 The $\mathcal{N} = 1$ Case

In what follows, we compare the strong-coupling results with the usual perturbation theory around minima of the potential.

In the case $\deg(W) = p$ even, supersymmetry is never broken, neither in the strong-coupling limit nor in perturbation theory. For even $p$ there is at least one minimum of the potential $V = \frac{1}{2}(W')^2$ with $V = 0$. The quadratic approximation of the potential at the critical points yields for each minimum one normalizable zero mode similar to the ground state of the free model. In contrast to the strong-coupling limit there may be more than one perturbative zero mode, but they always come in an odd number. The difference of bosonic and fermionic zero modes is $\pm 1$ as in the strong-coupling limit.

In the case $\deg(W) = p$ odd, the difference between the strong-coupling limit and perturbation theory is more severe. Supersymmetry is broken in the strong-coupling limit but it may be unbroken in perturbation theory. Let us consider an explicit example,

$$W(\phi) = \frac{g_2}{2}\phi^3 + g_0 \phi. \tag{140}$$

Perturbation theory for $g_0 < 0$ predicts one bosonic and one fermionic zero mode (unbroken supersymmetry), and broken supersymmetry for $g_0 > 0$. The strong-coupling limit states that supersymmetry is broken for all $g_0$.

In Appendix B.1 we provide the rigorous proof that $\lambda A_1$ with $A_1$ given in (69) is an *analytic perturbation* of $A_0$ in (69). This implies that all eigenvalues are analytic functions of the parameter $\lambda$. Assume now that in a finite range of the parameter $\lambda$ there is a ground state with energy zero. As an analytic function which vanishes in some finite range is identically zero, the number of zero modes changes at most at isolated points of the parameter space of $\lambda$. Furthermore, in the strong-coupling limit, we have either bosonic or fermionic zero modes. In subsection 6.1 we have proved that under this assumption a zero mode always remains a zero mode. We conclude that, generically, the number of zero modes is given by the number of zero modes in the strong-coupling limit. Generically, since for certain discrete values of $\lambda$ the number of zero modes could be enhanced. Moreover, as the index also depends analytically on the parameter $\lambda$, we are able to calculate this index in the strong-coupling limit.

In the continuum and infinite-volume limit these arguments may break down, as the estimates necessary for proving analyticity (see Appendix B.1) may not be valid anymore. In the *unbroken case* we can definitely conclude that the theory is still unbroken in the continuum and infinite-volume limit. Suppose we know that for any finite lattice there is at least one ground state with zero energy. As the limit of zero is again zero this mode survives in the limit. In the *case of broken supersymmetry* a non-zero energy eigenstate may become a zero mode in the continuum and infinite-volume limit, and supersymmetry may get restored in this limit although it is broken for all finite lattices.

Indeed, for negative $g_0$ in our example above, the scalar field has a non-vanishing vacuum expectation value and therefore the fermionic field $\psi$ acquires a non-zero mass. As there is no massless Goldstone fermion, supersymmetry has to be unbroken in this case [3].

Let us summarize. On a finite lattice, the *strong-coupling limit gives the correct number of zero modes of the full problem.* There is only one zero mode in the case where $\deg(W) = p$ is even, and otherwise there is no zero mode. Variations of the parameters in the superpotential of power less than $p$ do neither change the number of zero modes nor the index. For example, in the model with superpotential given in (140), it is impossible to have two phases of broken and unbroken supersymmetry (depending on the value of the parameter $g_0$) on a finite lattice. The numerical simulations in [25] may be interpreted as hinting towards such a phase transition in the continuum theory.

### 6.3 The $\mathcal{N} = 2$ Case

Similar to the case $\mathcal{N} = 1$, we prove in Appendix B.2 that the index in the strong-coupling limit is the same as for the full problem. This implies that we have at least $(p-1)^N$ zero modes for the theory on finite lattices. For the continuum theory in a finite volume, it was shown using methods of constructive field theory that the $\mathcal{N} = 2$ Wess-Zumino model is ultraviolet finite and that the index is given by $p - 1$ [24]. This seems to be in contradiction with our result, as the $(p-1)^N$ zero modes exist for all finite lattices and, by the same arguments as for the $\mathcal{N} = 1$ model, remain zero modes in the continuum limit.

We suggest the following solution for this problem. Remember that our lattice Hamiltonian $H$ contains not only the discretized version of the continuum Hamiltonian $P_0$ but also the central charge $\mathcal{Z}$, i.e.

$$H = P_0 + \mathcal{Z}. \tag{141}$$

Furthermore, both $P_0$ and $\mathcal{Z}$ contain the lattice derivative which couples fields at different lattice sites. If we choose in the strong-coupling limit a zero mode that varies from lattice point to lattice point, both $P_0$ and $\mathcal{Z}$ may become very large but will, nevertheless, add up to zero. In the continuum limit the energy $P_0$ may be infinite in which case this rapidly varying zero mode is only a lattice artifact. On the other hand, if we choose the same zero mode for each lattice site, then $P_0$ as well as $\mathcal{Z}$ should be zero in the continuum limit. Thus, there are exactly $p-1$ such modes. We are planning to test this proposal in a perturbative calculation of $P_0$ and $\mathcal{Z}$. The results will be presented elsewhere.

# 7 Conclusions

In this paper we have related Dirac operators defining supersymmetric quantum mechanical systems in high-dimensional spaces [32] to Wess-Zumino models on a spatial lattice in $1+1$ dimensions. After a very particular dimensional reduction the square of $\mathrm{i}\not{D}$ can be identified with the super-Hamiltonians of latticized Wess-Zumino models. This way we discovered a natural connection between discretized supersymmetric field theories and supersymmetric quantum mechanics.

We have recalled the continuum formulation of Wess-Zumino models and discussed their lattice versions. For the case of simple ($\mathcal{N}=1$) and extended ($\mathcal{N}=2$) supersymmetry, we have derived the corresponding Dirac operators. Furthermore, all ground states for the free massive models and the interacting theories in the limit of strong coupling have been constructed.

Different realizations of lattice derivatives have been discussed and their properties – in particular from the point of view of supersymmetric quantum mechanics – have been analyzed. Our results on the number of zero modes do not depend on the particular lattice derivative, as long as some mild assumptions are fulfilled.

Employing powerful theorems from functional analysis we were able to relate the strong and weak coupling regions. For $\mathcal{N}=1$ it turns out that generically the number of zero modes is determined by the strong-coupling limit. There is a single (no) zero mode, if the degree of the superpotential is even (odd). For $\mathcal{N}=2$ we find at least $(p-1)^N$ zero modes, where $p$ is the degree of the superpotential and $N$ the number of lattice sites. This number is far off the correct continuum result, which predicts $p-1$ zero modes, a serious problem which has been observed earlier in [21].

We have explained this paradox as follows: the lattice Hamiltonian $H$ does not only contain the continuum Hamiltonian $P_0$ but also additional terms which (for antisymmetric lattice derivatives) are to be interpreted as a lattice version of the central charge $\mathcal{Z}$. On the lattice, $P_0$ and $\mathcal{Z}$ cancel pairwise for the huge number of zero modes under discussion, even though neither $P_0$ nor $\mathcal{Z}$ is zero in the continuum limit, except for exactly $p-1$ of the modes.

Our Dirac operators clearly deserve further studies. For instance, the application of (standard) index theorems to the case at hand should reveal new information about the field theories. We also plan to extend our results to Dirac operators on curved manifolds, which can be reinterpreted as nonlinear $\sigma$-models from the field theory point of view.

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## A The SLAC Operator

In this appendix we introduce and discuss the nonlocal SLAC derivative. It can be used to define chiral fermions without fermion-doubling.

First we consider *real valued* scalar fields on the spatial lattice. They maybe interpreted as wave functions of a quantum mechanical system with Hilbert space $\mathbb{R}^N$, equipped with the scalar product

$$(\phi,\chi)=\sum_{n=1}^{N}\bar{\phi}(n)\chi(n).$$

Although the fields are real it is useful to embed them in the space of complex valued lattice fields. For a normalised function we interpret $|\phi(n)|^2$ as probability for finding the 'particle' at the lattice site $n$. Expectation values of functions of the 'position' operator $\hat{n}$ are

$$\langle f(\hat{n})\rangle_\phi=\sum\bar{\phi}(n)f(n)\phi(n). \tag{142}$$

We want to derive a similar formula for expectation values of functions of the momentum operator. For that aim we Fourier transform the wave function as follows

$$\tilde{\phi}(p_k)=\frac{1}{\sqrt{N}}\sum_{n=-N'}^{N'}\mathrm{e}^{-\mathrm{i}p_k n}\phi(n),\quad \text{where}\quad N'=\tfrac{1}{2}(N-1),\quad p_k=\frac{2\pi}{N}k. \tag{143}$$

The inverse Fourier transformation reads

$$\phi(n)=\frac{1}{\sqrt{N}}\sum_{k=-N'}^{N'}\mathrm{e}^{\mathrm{i}p_k n}\tilde{\phi}(p_k),\quad n=-N',\dots,N'. \tag{144}$$

We have chosen the symmetric summation to end up with a real difference operator. For periodic fields $n$ must be integer and this is the case for space lattices with an *odd number of sites*. For a normalized $\phi$ the Fourier transform $\tilde{\phi}$ is normalized as well and we may interpret $|\tilde{\phi}(p_k)|^2$ as probability for finding the 'momentum' $p_k$. With this interpretation we obtain

$$\langle f(\hat{p})\rangle_\phi=\sum_{k=-N'}^{N'}f(p_k)\,|\tilde{\phi}(p_k)|^2=\sum_{nn'}\bar{\phi}_n f(\hat{p})_{nn'}\phi_{n'} \tag{145}$$

with matrix elements

$$f(\hat{p})_{nn'} = \frac{1}{N} \sum_{k=-N'}^{N'} \mathrm{e}^{\mathrm{i}p_k(n-n')} f(p_k).$$

With the help of the generating function

$$Z(x) = \sum_{k=-N'}^{N'} \mathrm{e}^{\mathrm{i}p_k x} = \frac{\sin(\pi x)}{\sin(\pi x/N)}, \tag{146}$$

we can calculate all matrix elements of $f(\hat{p})$. Now we are ready to define the real, nonlocal and antisymmetric lattice operator $\partial_{\mathrm{SLAC}} = \mathrm{i}\hat{p}$. The matrix elements are

$$f(\partial_{\mathrm{SLAC}})_{nn'} = \frac{1}{N} f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) Z(x)\Big|_{x=n-n'}. \tag{147}$$

As expected $\partial_{\mathrm{SLAC}}$ is a Toeplitz matrix with elements

$$(\partial_{\mathrm{SLAC}})_{nn'} = (-)^{n-n'} \frac{\pi/N}{\sin\big(\pi(n-n')/N\big)}, \qquad \text{for} \quad n \neq n', \tag{148}$$

and the elements on the diagonal vanish, $(\partial_{\mathrm{SLAC}})_{nn} = 0$.

# B Analyticity of Perturbation

In the following we consider operators on the Hilbert space

$$\mathcal{H} = \mathrm{L}_2(\mathbb{R}^d, \mathrm{d}^d x) \otimes \mathbb{C}^D \tag{149}$$

for $D \in \mathbb{N}$ with norm

$$\|f\|^2 = \sum_{i=1}^{D} \|f_i\|^2_{\mathrm{L}_2}, \quad f = (f_1, \ldots, f_D) \in \mathcal{H}. \tag{150}$$

Here, $\|\cdot\|_{\mathrm{L}_2}$ denotes the familiar $\mathrm{L}_2$-norm.

## B.1 The $\mathcal{N} = 1$ Case

For the Wess-Zumino model on the lattice with $\mathcal{N} = 1$ supersymmetry we take $D = 2^N$, $d = N$ ($N$ = number of lattice points) and consider the (unperturbed) operator (69)

$$A_0 = \sum_{n=1}^{N} \big(-\mathrm{i}\psi_1(n)\partial_n + \psi_2(n) W'(x_n)\big). \tag{151}$$

We recall that $W$ is a polynomial of degree $\deg(W)=p>1$ and $\psi_\alpha(n)$ are Hermitian $D\times D-$matrices obeying the Clifford algebra

$$\{\psi_\alpha(n),\psi_\beta(n')\}=2\delta_{\alpha\beta}\delta(n,n'),\quad \alpha,\beta=1,2,\quad n,n'=1,2,\ldots,N. \tag{152}$$

The operator $A_0$ with domain of definition

$$\mathcal{D}(A_0)=C_{\rm c}^\infty(\mathbb{R}^N)\otimes\mathbb{C}^D$$

is essentially self-adjoint, where we write $C_{\rm c}^\infty(\mathbb{R}^N)$ for the $C^\infty$-functions with compact support in $\mathbb{R}^N$. A simple calculation using (152) shows

$$(A_0)^2=\sum_n\left(-\partial_n\partial_n+W'(x_n)W'(x_n)-\mathrm{i}\psi_1(n)\psi_2(n)W''(x_n)\right).$$

**Closure of the Operator $A_0$**

To determine the closure $\bar{A}_0$ of the operator $A_0$ we have to find the closure of its domain $\mathcal{D}(A_0)$ with respect to the norm

$$\|f\|^2_{A_0,a}=a\|f\|^2+\|A_0f\|^2,\quad a>0. \tag{153}$$

Note that these norms are equivalent for all $a>0$. Using the abbreviation

$$\rho_p=1+|x|^{p-1}, \tag{154}$$

we can prove the following

**Lemma:** There exist constants $a,b_1,b_2>0$ such that

$$\|f'\|^2+b_1\|\rho_pf\|^2\le\|f\|^2_{A_0,a}\le\|f'\|^2+b_2\|\rho_pf\|^2 \tag{155}$$

holds for all $f\in\mathcal{D}(A_0)$.

In the Lemma we used the short hand notation $\|f'\|^2=\sum_m\|\partial_mf\|^2$.

*Proof:* First, we show that only the degree $\deg(W)=p$ is important for terms like $\sum_n\|W'(x_n)f\|^2$ with $f\in\mathcal{D}(A_0)$. Indeed, we find

$$\sum_n\|W'(x_n)f\|^2\le Na_1\|\rho_pf\|^2,\quad a_1=\left\|\frac{W'(x_n)}{\rho_p}\right\|^2_\infty, \tag{156}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. The factor $N$ arises from the sum over $n$, as $a_1$ does not depend on $n$. Similar we obtain

$$\begin{aligned}\|\rho_pf\|^2 &= \left\|\frac{\rho_p}{\sqrt{1+\sum_nW'^2(x_n)}}\sqrt{1+\sum W'^2(x_n)}\,f\right\|^2\\ &\le a_2\left(\|f\|^2+\sum\|W'(x_n)f\|^2\right),\quad a_2=\left\|\frac{\rho_p}{\sqrt{1+\sum W'^2(x_n)}}\right\|^2_\infty.\end{aligned} \tag{157}$$

Now, it is easy to prove the second inequality in (155),

$$
\begin{aligned}
a\|f\|^2+\|A_0 f\|^2 &\overset{(156)}{\leq} \|f'\|^2+a\|f\|^2+Na_1\|\rho_p f\|^2+\sum_n \|f\|\,\|W''(x_n)f\| \\
&\leq \|f'\|^2+\underbrace{(a+Na_1+Na_3)}_{b_2}\|\rho_p f\|^2,
\end{aligned} \tag{158}
$$

with $a_3=\left\|\frac{W''(x_n)}{\rho_p}\right\|_\infty$. We used that the matrix-norm of $\psi_\alpha(n)$ is one, since its eigenvalues are $\pm 1$. In the last inequality we made use of $\|f\|\leq\|\,\rho_p f\|$ which holds for all $f\in\mathcal{D}(A_0)$.

The other inequality in (155) is more difficult to prove. With (157) we get

$$
a\|f\|^2+\|A_0 f\|^2 \geq \|f'\|^2+\frac{1}{a_2}\|\rho_p f\|^2+(a-1)\|f\|^2-\sum\|f\|\,\|W''(x_n)f\|. \tag{159}
$$

In order to obtain a positive constant $b_1$ in our lemma we must be rather careful with our estimates for the last term in (159). We introduce a ball of radius $R$ and split $f$ into two parts, $f=f_<+f_>$, where $f_<$ has its support inside the ball $f_>$ outside the ball. We obtain

$$
\sum_n \|f\|\,\|W''(x_n)f\|=\sum\left(\|f_<\|\,\|W''(x_n)f_<\|+\|f_>\|\,\|W''(x_n)f_>\|\right), \tag{160}
$$

where the terms containing both $f_<$ and $f_>$ vanishes. Let us now consider the two terms separately. First, we obtain

$$
\sum\|f_>\|\,\|W''(x_n)f_>\|\leq Na_4(R)\|\rho_p f\|^2, \quad a_4(R)=\left\|\frac{W''(x_n)}{\rho_p}\right\|_{\infty,>}, \tag{161}
$$

where we have introduced the supremum norm $\|\cdot\|_{\infty,>}=\sup_{|x|>R}\{|\cdot|\}$. For large $R$ we have $a_4(R)\sim 1/R$ such that $a_4$ gets arbitrarily small for big balls. Second, we obtain

$$
\sum\|f_<\|\,\|W''(x_n)f_<\|\leq Na_5(R)\|f\|^2, \quad a_5(R)=\|W''(x_n)\|_{\infty,<}, \tag{162}
$$

with $\|\cdot\|_{\infty,<}=\sup_{|x|<R}\{|\cdot|\}$. For $R\to\infty$ we have $a_5(R)\to\infty$. Altogether, we find

$$
a\|f\|^2+\|A_0 f\|^2\geq\|f'\|^2+(a-1-Na_5(R))\,\|f\|^2+\underbrace{(1/a_2-Na_4(R))}_{b_1}\|\rho_p f\|^2. \tag{163}
$$

In a first step we choose $R$ large enough such that $b_1$ is positive. In a second step we choose $a$ large such that the constant in front of $\|f\|^2$ is positive as well. This finishes the proof of our lemma.

Since all norms

$$
\|f\|_b^2\equiv\|f'\|^2+b\,\|\rho_p f\|^2 \tag{164}
$$

are equivalent for $b > 0$, the Lemma implies that these norms are equivalent to the norms $\|f\|^2_{A_0,a}$ in (153). Therefore, the closure of $\mathcal{D}(A_0)$ with respect to the norm (153) coincides with the closure with respect to $\|\cdot\|_b$. This closure is given by

$$\mathcal{D}(\bar{A}_0) = \left\{ f \in W_2^1(\mathbb{R}^N) \otimes \mathbb{C}^D : \|\rho_p f\| < \infty \right\} \equiv W_2^1(\mathbb{R}^N, \rho_p^2) \otimes \mathbb{C}^D. \tag{165}$$

Here $W_2^1(\mathbb{R}^N)$ is the Sobolev space with first weak-derivative in $\mathrm{L}_2$.

**Perturbation**

Let us perturb the operator $A_0$ by the operator $A_1$ in (69),

$$A_1 = -\sum_{m,n=1}^{N} x_m \, (\partial)_{mn} \, \psi_2(n). \tag{166}$$

The operator $A_1$ is self-adjoint with $\mathcal{D}(A_1) = \mathrm{L}_2(\mathbb{R}^N, \tilde{\rho}) \otimes \mathbb{C}^D \supset \mathcal{D}(\bar{A}_0)$, with $\tilde{\rho}$-weighted Lebesgue measure, where $\tilde{\rho}(x) = (1 + |x|)^2$. From the following Lemma we will derive useful information about the nature of the perturbation.

**Lemma:** For all $\lambda \in \mathbb{R}$ and *arbitrarily small* $\epsilon > 0$ there exists a $C_\epsilon > 0$ such that

$$\|\lambda A_1 f\| \leq \epsilon \|A_0 f\| + C_\epsilon \|f\|, \quad \forall f \in \mathcal{D}(\bar{A}_0). \tag{167}$$

*Proof:* We prove the inequality for all $f \in \mathcal{D}(A_0)$. Then it holds for all elements in the closure as well. As before we split $f = f_< + f_>$. First, we note

$$\|\lambda A_1 f_<\| \leq |\lambda| N^2 a(R) \, \|f\|, \quad a(R) = \|x_n\|_{\infty,<} \cdot \max\{|\partial_{mn}| : m,n = 1,\ldots,N\}. \tag{168}$$

For $R \to \infty$ the constant $a(R)$ tends to infinity. Next, we have

$$\begin{aligned} \|\lambda A_1 f_>\| &\overset{(155)}{\leq} |\lambda| N^2 b(R) \, (c\,\|f\| + \|A_0 f\|)\,, \\ b(R) &= \left\| \frac{x_n}{\rho_p(x)} \right\|_{\infty,>} \cdot \max\{|\partial_{mn}| : m,n = 1,\ldots,N\} \end{aligned} \tag{169}$$

for some positive constant $c$. For big $R$ the constant $b(R)$ tends to zero. We choose the ball big enough such that $|\lambda| N^2 b(R) = \epsilon$ and set $C_\epsilon = \epsilon c + |\lambda| N^2 a(R)$. Note that the latter constant may become huge.

**Self-adjointness**

We did prove that $\bar{A}_0$ is a self-adjoint operator. Clearly $\lambda A_1$ is symmetric on $\mathcal{D}(\bar{A}_0)$. Furthermore, (167) shows that $\lambda A_1$ is $\bar{A}_0$-bounded with relative bound less than one. The famous Kato-Rellich Theorem, see Theorem X.12 in [36], states that under these conditions the operator

$$Q_1(\lambda) = A_0 + \lambda A_1 \tag{170}$$

is self-adjoint with domain $\mathcal{D}(\bar{A}_0)$. We conclude that $Q_1(\lambda)$ is a family of self-adjoint operators with common domain of definition $\mathcal{D}(\bar{A}_0)$.

**Analyticity of Eigenvalues**

In the following we prove that $Q_1(\lambda)$ is an *analytic family* in the sense of Kato for all real $\lambda$. We have seen that $Q_1(\lambda)$ is self-adjoint for real $\lambda$. For a self-adjoint and analytic family it is known that the eigenvalues depend analytically on the parameter $\lambda$, see for example Theorem XII.13 in [36].

For an arbitrary real $\lambda_0$ the perturbation $\lambda_0 A_1$ is $\bar{A}_0$-bounded with arbitrary small relative bound (167). Then, it is easy to see that $A_1$ is $Q_1(\lambda_0)$-bounded. It follows that for small $\epsilon$ the operators $Q_1(\lambda_0+\epsilon)$ form an analytic family of type (A) [36] and therefore also an analytic family in the sense of Kato. But as $\lambda_0 \in \mathbb{R}$ is arbitrary, we haven proved that $Q_1(\lambda)$ is analytic for all real $\lambda$.

Actually, the cited Theorem XII.13 [36] above is only valid for isolated eigenvalues with finite degeneracy or equivalently for eigenvalue in the discrete spectrum. In the following we prove that the spectrum of $Q_1(\lambda)$ is discrete by proving this statement for its square, $H(\lambda) = Q_1(\lambda)^2$. $H(\lambda)$ is self-adjoint with domain of definition given by

$$\begin{aligned} \mathcal{D}\left(H(\lambda)\right) &\equiv \left\{f \in \mathcal{D}(\bar{A}_0) : Q_1(\lambda) f \in \mathcal{D}(\bar{A}_0)\right\} \\ &= W_2^2(\mathbb{R}^N, \rho') \otimes \mathbb{C}^D, \qquad \rho'(x) = \left(1 + |x|^{2p-2}\right)^2 \end{aligned} \tag{171}$$

and it is semibounded

$$H(\lambda) \geq 0. \tag{172}$$

Such operators possess entirely discrete spectra if and only if its resolvent is a compact operator, see Theorem XIII.64 in [36]. In the following we prove that $H(\lambda)$ has compact resolvent for all $\lambda \in \mathbb{R}$.

We must prove that the image of a bounded subset of the Hilbert space, say

$$\{f \in \mathcal{H} : \|f\| < 1\}, \tag{173}$$

is mapped to a precompact set under the map $(H-z)^{-1}$ for some $z$ in the resolvent of $H$. The image is given by

$$\{g \in \mathcal{D}(H) : \|(H-z)g\| < 1\}. \tag{174}$$

As earlier we split $g$ into $g_>$ and $g_<$ and obtain for large enough radii $R$ the inequality $\|g\| \leq \|g_<\| + \epsilon$. For a compact ball $\mathcal{B} = \{x \in \mathbb{R}^N : |x| \leq R\}$ we have Sobolev's embedding theorem and there is an $\epsilon$-net $g_j \in W_2^2(K, \rho')$, $j = 1, \ldots, N_\epsilon$ with $\|g_< - g_j\| < \epsilon$ for one $j \in \{1, \ldots, N_\epsilon\}$. We extend the $g_j$ by zero to the region outside the ball and obtain

$$\|g - g_j\| \leq 2\epsilon \tag{175}$$

for any $g$ in the image of the unit ball under $(H-z)^{-1}$ and a specific $j \in \{1, \ldots, N_\epsilon\}$. We conclude that there is a $2\epsilon$-net of the image and therefore the image is precompact. This completes our proof.

**Stability of the Index**
We have shown that the eigenvalues are analytic functions of the parameter $\lambda$ on the whole real axis. It follows at onces that the index – the difference of bosonic zero modes and fermionic zero modes – is also an analytic function and, as the index only takes on integer values, is constant.

An alternative and elegant proof of this statement can be given with the help of the theorem that a *relatively compact perturbation* does not change the index [37]. Indeed, inequality (167) implies that our perturbation is relatively compact[1].

## B.2 The $\mathcal{N}=2$ Case

After the detailed investigation of the $\mathcal{N}=1$ case we shorten our discussion for $\mathcal{N}=2$. In what follows we consider the real part of the complex supercharge in (107)

$$B_0=\sum_{n=1}^{N}\left(-\mathrm{i}\psi_1^1(n)\partial_{x_n}-\mathrm{i}\psi_1^2(n)\partial_{y_n}+\sum_{a=1}^{2}\psi_2^a(n)W_{,a}(x_n,y_n)\right) \tag{176}$$

in the strong-coupling limit. For $\mathcal{N}=2$ supersymmetry the $\psi_\alpha^a$ are $D$-dimensional Hermitian matrices obeying the Clifford algebra, with $D=2^{2N}$. The function $W(x,y)$ is harmonic and and thus is the real part of an analytic function $F(x+\mathrm{i}y)$. As in chapter 2 we use the notation $W_{,1}(x,y)=\partial_x W(x,y)$ and $W_{,2}(x,y)=\partial_y W(x,y)$. As domain of definition we take

$$\mathcal{D}(A_0)=C_c^\infty(\mathbb{R}^{2N})\otimes\mathbb{C}^D,\qquad D=2^{2N}, \tag{177}$$

such that $B_0$ is essentially self-adjoint. We introduce the potential

$$K(x,y)=\sum_a W_{,a}^2(x,y). \tag{178}$$

For large radii $r$ only the leading power of $W$ is relevant. Therefore, we may consider the particular case

$$W(x,y)=\frac{\kappa}{p}\mathrm{Re}\,z^p \tag{179}$$

for which we find $K(x,y)=\kappa r^{p-1}\to\infty$ in all directions for $r\to\infty$.

The perturbation contains the lattice derivative,

$$B_1=\sum_{m,n=1}^{2N}\left(x_m(\partial)_{mn}\psi_2^2(n)+y_m(\partial^\dagger)_{mn}\psi_2^1(n)\right).$$

Replacing in the estimates for the case with $\mathcal{N}=1$ supersymmetry the potential $W'(x_n)$ by $K(x_n,y_n)$ leads to analogous results in the $\mathcal{N}=2$ case. Again all eigenvalues are analytic functions of the parameter $\lambda$ and in particular the index does not depend on this parameter.

[1] We thank H. Triebel for the proof of this statement.

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