ON A MODEL OF A CLASSICAL RELATIVISTIC PARTICLE OF CONSTANT AND UNIVERSAL MASS AND SPIN

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Abstract. The deformation of the classical action for a point-like particle recently suggested by A. Staruszkiewicz gives rise to a spin structure which constrains the values of the invariant mass and the invariant spin to be the same for *any solution* of the equations of motion. Both these Casimir invariants, the square of the four-momentum vector and the square of the Pauli-Lubański vector, are shown to preserve the same fixed values also in the presence of an arbitrary external electromagnetic field. In the "free" case, in the centre-of-mass reference frame, the particle moves along a circle of fixed radius with arbitrary varying frequency. In a homogeneous magnetic field, a number of rotational "states" is possible with frequencies slightly different from the cyclotron frequency, and "phase-like" transitions with spin flops occure at some critical value of the particle's three-momentum.

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1 Introduction. The Staruszkiewicz model

From the times of Frenkel [1] and Mathisson [2], or from later works [3, 4], a lot of attempts have been undertaken to unambiguously formulate the dynamics of a classical spinning particle [5]-[10]. Most of them deal with generalizations of the classical point-mass Lagrangian ($-mc\sqrt{\dot{x}\dot{x}}$) through the introduction of terms with higher derivatives or "inner" variables, and then try to restrict the undesirable freedom by making use of some geometrical [11, 12] or symmetry [13, 14] considerations. A thorough analysis of extra variables responsible for the spin structure has been carried out by Hanson and Regge [15]. Rivas [16] proposed to make use of a complete set of parameters of the kinematical symmetry group and obtained considerable restrictions on the spin dynamics (so-called "atomic hypothesis").

However, a lot of problems arising within the classical description of spin still have to be solved. Corresponding Lagrangians, especially in their interaction part, are rather ambiguous, cumbersome and have no correspondence with generally accepted gauge theories and with the structure of the Lorentz force in the spinless limit in particular [17]. The Schrödinger *zitterbewegung* motion, being a common feature of different spin models, results in the problem of radiation of electromagnetic waves, and the invariant spin is not bound to preserve its value in the presence of external fields (as one expects for an innate characteristic of elementary particle). Thus, the "zitterbewegung" radius and the magnitude of spin are, as a rule, to be fixed "by hands". Generally, it is not quite clear which properties of the spin could arise in the framework of a successively classical relativistic model.

Meanwhile, in a short note [18] Staruszkiewicz has offered a fairly simple relativistic model for a classical particle with spin [1]. Specifically, instead of the consideration of a general local reference frame related to the orientation of the spin vector, he restricted additional degrees of freedom by a *single null vector* $k_\mu(\tau), (k \cdot k) = 0$ attached to each point of a trajectory of the classical particle. We note that Lorentz indices hereafter are labelled by Greek letters $\mu, \nu, ... = 0, 1, 2, 3$, whereas $(a \cdot c) = a_0 c_0 - \vec{a}\vec{c}$ denotes the Lorentz scalar product. The derivative $d/d\tau$ with respect to an invariant time parameter τ (identified later with the particle's proper time) is denoted by a dot.

[1] Recently we became informed that nearly the same model has been suggested and elaborated in 1994 in the paper [22]. We present and discuss it in the last section (Afterword) below.

Remarkably, if l is a new universal constant, a *fundamental length*, corresponding to this additionally introduced variable, one essentially has the only possibility to construct a dimensionless, relativistic and translation invariant quantity

$$\zeta = l^2 \frac{(\dot{k} \cdot \dot{k})}{(k \cdot \dot{x})^2}, \tag{1}$$

which is moreover reparametrization $\tau \mapsto g(\tau)$ and projective $k_\mu \mapsto h(\tau) k_\mu$ invariant, g and h being arbitrary smooth and monotonic (as to g) functions of the invariant time parameter τ.

Now one can write down the most general deformation of the well-known relativistic invariant point-mass action described by a position four-vector $x_\mu(\tau)$ and a null direction four-vector $k_\mu(\tau)$ in a unique and simple form [18]

$$S = \int L d\tau = -mc \int d\tau \sqrt{\dot{x} \cdot \dot{x}} f(\zeta). \tag{2}$$

In order to fix a particular form of the unknown function $f(\zeta)$ [2], Staruszkiewicz, inspired by the idea of "irreducibility" from the famous paper by E. Wigner [19], requires then for the values of both Casimir invariants of the Poincaré group to take one and the same value on any solution to the model (2). He asserts that such a requirement leads to two differential equations for the unknown $f(\zeta)$. According to [18], these equations have only one common solution that uniquely defines the form of the function $f(\zeta)$ to be

$$f = \sqrt{1 + \sqrt{-\zeta}}. \tag{3}$$

The two Casimir invariants associated with the action (2) are the four-momentum vector square

$$I_1 = (P \cdot P) \tag{4}$$

and the square

$$I_2 = (W \cdot W) \tag{5}$$

of the Pauli-Lubański pseudovector

$$W_\mu = -\frac{1}{2mc} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma, \tag{6}$$

responsible for the invariant (i.e. independent on the origin) and conserved spin structure. In equation (6) $M^{\nu\rho}$ is the (conserved) total angular momentum tensor of the Lagrangian (2). According to statements made in [18], for any solution of the equations of motion with the choice (3) for the function $f(\zeta)$, the invariants I_1 and I_2 take the only possible values

$$I_1 = (mc)^2 \quad \text{and} \quad I_2 = -\frac{1}{4}(mcl)^2. \tag{7}$$

Thus, in the framework of (generalized) classical mechanics of the "free" particle one is able to elegantly describe its spin structure and, moreover, to ensure the property of spin (and mass) "quantization". For some preliminary chosen "ethalon" units $\{m, c, l\}$, the Staruszkiewicz model *corresponds to a point relativistic particle of constant and universal mass and spin.* In particular, one can equate the fundamental length to the Compton wavelength $l = \hbar/mc$, so that the value of spin is then one half of the Planck constant,

[2] For the energy to be positive definite one requires $f(\zeta) > 0$ provided $m > 0$

$$S = \sqrt{-W \cdot W} = \frac{1}{2} mcl = \frac{1}{2} \hbar.$$

In view of these intriguing properties, our main goal in this article is to prove the statements of Staruszkiewicz in [18] and to generalize these to the case when the spinning particle interacts with an external electromagnetic field. We shall also study some unusual properties of exact solutions to the equations of motion for a "free" particle and for a particle in a homogeneous magnetic field.

2 Conservation laws and fixed invariants of the model

Let us first note that due to the above mentioned local symmetry $k_\mu \to h(\tau) k_\mu$ the quantity ζ in (1) depends only on the *ratio* of the components of the null vector k_μ, namely, on the components $n^a = k^a/k^0$, $a = 1, 2, 3$ of the unit three-vector $\vec{n} = \{n^a\}$, $\vec{n}^2 = 1$. Specifically, one obtains

$$-\zeta = l^2 \frac{\dot{\vec{n}} \dot{\vec{n}}}{(\dot{x}_0 - \vec{n} \dot{\vec{x}})^2}. \tag{8}$$

Thus, the individual components of the null vector k_μ are not determined, and only the *direction vector* $\vec{n} = \vec{k}/k^0$ will finally enter the equations of motion. However, to explicitly preserve relativistic invariance in the variational principle an additional term $\lambda(k \cdot k)$ with Lagrangian multiplier λ has to be added to the action (2).

We shall also from the very beginning switch on the electromagnetic interaction by introducing the usual and (modulo total time derivative) gauge invariant term $-(e/c)\dot{x}^\mu A_\mu$ with e being the *electric charge* of the particle and A_μ the *four-potential* of the external electromagnetic field. Thus, we shall deal with the action

$$S = \int L d\tau, \quad L = -mc\sqrt{(\dot{x} \cdot \dot{x})} f(\zeta) - \frac{e}{c}(\dot{x} \cdot A) + \lambda(k \cdot k). \tag{9}$$

Consider now the integrals of motion related to translational and rotational symmetries of the model (9). For the canonical four-momentum $\pi_\mu = -\partial L/\partial \dot{x}^\mu$ one obtains

$$\pi_\mu = P_\mu + \frac{e}{c} A_\mu, \quad P_\mu := \frac{mc}{\epsilon} f \dot{x}_\mu + B k_\mu, \tag{10}$$

where $\epsilon = \sqrt{\dot{x} \cdot \dot{x}}$, and

$$B = -2mc \frac{\epsilon}{(k \cdot \dot{x})} \zeta f'(\zeta), \tag{11}$$

with $'$ denoting differentiation with respect to ζ. Calculating the square of the *kinematical* four-momentum P_μ, one obtains for the first Casimir invariant

$$I_1 = (P \cdot P) = (mc)^2 (f^2 - 4\zeta f f'). \tag{12}$$

Now one is ready to write down the first set of the Euler-Lagrange equations of motion

$$\dot{P}_\mu = mc \frac{d}{d\tau} \left(\frac{f}{\epsilon} \dot{x}_\mu - \frac{2\epsilon\zeta f'}{(k \cdot \dot{x})} k_\mu \right) = F_\mu, \tag{13}$$

where F_μ stands for the four-vector of *Lorentz force*,

$$F_\mu = \frac{e}{c} F_{\mu\nu} \dot{x}^\nu, \tag{14}$$

with $F_{\mu\nu}$ being the electromagnetic field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial_\mu =: \frac{\partial}{\partial x^\mu}.$$

The "internal four-momentum" Q_μ is a direct analogue of the ordinary one with respect to additional degrees of freedom related to the null four-vector k_μ,

$$Q_\mu = -\frac{\partial L}{\partial \dot{k}^\mu} = G\dot{k}_\mu, \quad G = 2mc\epsilon l^2 \frac{f'}{(k\cdot\dot{x})^2}, \tag{15}$$

so that the second set of equations looks as follows:

$$\dot{Q}_\mu = -\frac{\partial L}{\partial k^\mu}, \;\rightarrow\; \frac{d}{d\tau}(G\dot{k}_\mu) = B\dot{x}_\mu - 2\lambda k_\mu, \tag{16}$$

where B turns out to be the same as in (11). We are now ready to introduce the *angular momentum* skew symmetric tensor

$$M_{\mu\nu} = x_{[\mu}P_{\nu]} + k_{[\mu}Q_{\nu]} = \frac{mc}{\epsilon} f x_{[\mu}\dot{x}_{\nu]} + Bx_{[\mu}k_{\nu]} + Gk_{[\mu}\dot{k}_{\nu]}, \tag{17}$$

with the notation $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$ to be used hereafter.

The angular momentum is conserved in the absence of an external field. Indeed, calculating with the help of (10,13,15,16) its invariant time derivative, one obtains

$$\dot{M}_{\mu\nu} = x_{[\mu}F_{\nu]}. \tag{18}$$

Let us now specify the form of the Pauli-Lubański pseudovector (6) with the expression for kinematical part P_μ from (10) to be used instead of the canonical four-momentum π_μ. Making also use of (17), one obtains

$$W_\mu = -\frac{Gf(\zeta)}{\epsilon}\varepsilon_{\mu\nu\rho\sigma}k^\nu \dot{k}^\rho \dot{x}^\sigma, \tag{19}$$

so that the expression for the second Casimir invariant (5) looks as follows:

$$I_2 = (W\cdot W) = \frac{G^2 f^2}{\epsilon^2}(k\cdot\dot{x})^2(\dot{k}\cdot\dot{k}) = 4(mcl)^2\zeta f^2 (f')^2. \tag{20}$$

Now, following Staruszkiewicz and equating the two Casimir invariants (12) and (20) to their required values (7), one arrives at two differential equations for the same unknown $f(\zeta)$, namely,

$$f^2 - 4\zeta f f' = 1 \tag{21}$$

and

$$-\zeta f^2 (f')^2 = \frac{1}{16}. \tag{22}$$

The first equation can easily be integrated leading (on account of (8), ζ is negative on solutions) to the following general solution:

$$f(\zeta) = \sqrt{1 + C\sqrt{-\zeta}}, \tag{23}$$

with C being an arbitrary constant. For $C = 0$, the model reduces to the canonical one for a free spinless particle. For nonzero $C \neq 0$, it is easy to verify that the function (23) satisfies also the second equation (22) provided $|C|=1$. Thus, one is finally left with *two different solutions* common to equations (21) and(22)

$$f(\zeta) := f_\pm = \sqrt{1 \pm \sqrt{-\zeta}}, \tag{24}$$

from which only the first one $f = f_+$ corresponds to that presented in [18]. The second case $f = f_-$ must be studied on equal footing. Below we shall see that a distinction between these two cases are essential and interesting from a physical point of view.

We see therefore that for the action (9) with fixed dimensional constants m, l and e and the function $f(\xi)$ taking one of the two obtained forms (24), expression (12) is identically a constant number; the same is true for expression (20). Thus, we have proved that both Casimir invariants take the same fixed values for any solution of the Staruszkiewicz model *in the presence of an arbitrary external electromagnetic field* provided the interaction is taken to be *minimal.* This is just the remarkable property one observes for a real quantum particle, whereas here it holds on a purely classical level. We are not aware of whether such a property holds in any other model of classical spin dynamics.

Making now use of the expressions (13) and (18) for the variation of the linear and angular momentum, one obtains for the derivative of the Pauli-Lubański pseudovector with respect to invariant time parameter

$$\dot{W}_\mu = -\frac{G}{2mc}\epsilon_{\mu\nu\rho\sigma}k^\nu \dot{k}^\rho F^\sigma. \tag{25}$$

On account of the constant (and universal) values of the Casimir invariants (12) and (20), the relations $(P\cdot\dot{P})=0$ and $(W\cdot\dot{W})=0$ must be fulfilled identically. On the other hand, one can explicitly obtain these products using (10) and (17) together with the equations of motions (13) and (18) respectively. Taking also into account the obvious identity

$$(\dot{x}\cdot F) = F_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \equiv 0,$$

one arrives at the following two relations:

$$0 \equiv (P\cdot\dot{P}) = -\frac{2mc^2\zeta f'}{(k\cdot\dot{x})}(k\cdot F), \quad 0 \equiv (W\cdot\dot{W}) = -\frac{mc^2 l^2}{4(k\cdot\dot{x})}(k\cdot F), \tag{26}$$

which should be fulfilled “on shell”. Thus, for any solution to the equations of motion *the null four-vector k_μ is necessarily orthogonal to the four-vector F_μ of the Lorentz force*,

$$(k\cdot F) = F_{\mu\nu}k^\mu\dot{x}^\nu \equiv 0 \tag{27}$$

Of course, the property (27) can be derived explicitly from the equations of motion themselves, without any direct account of the identities (12) and (20); however, this requires rather lengthy computations. From the condition (27) it also follows that the orientation of vector $\vec{n}$ is instantaneously in correspondence with the direction of electromagnetic field and, even at the initial instant, cannot be set in an arbitrary manner. Below (section 4) we shall consider this property in the particular case of a homogeneous magnetic field.

Let us now come back to the whole system of equations of motion (13),(16) and identify hereafter the parameter τ with proper time along the particle's trajectory so that one has

$$\epsilon = c\sqrt{(u\cdot u)} = c,$$

with $u_\mu = \dot{x}_\mu/c$ being the four-velocity unit vector. Eliminating also λ and passing thus to the unit three-vector $n^a = k^a/k^0$, one obtains from (16):

$$\frac{l^2}{c^2}\frac{d}{d\tau}\left(\frac{f^2 f'\dot{\vec{n}}}{Z^2}\right) = \frac{\zeta f f'}{Z}(u_0\vec{n} - \vec{u}), \tag{28}$$

where the abbreviation

$$Z := f(u_0 - \vec{n}\vec{u}) \equiv f(k \cdot u)/k_0 \tag{29}$$

was introduced. Separating now the spatial part in (13), we close the system of equations of motion as follows:

$$\frac{d}{d\tau}\left(f\vec{u} - \frac{2\zeta f f'}{Z}\vec{n}\right) = \frac{e}{mc}(u_0\vec{E} + \vec{u} \times \vec{H}), \tag{30}$$

$\vec{E}, \vec{H}$ being the electric and magnetic field strength vectors respectively. In addition, the remaining temporal part of (13) reads

$$\frac{d}{d\tau}\left(f u_0 - \frac{2\zeta f f'}{Z}\right) = \frac{e}{mc}\vec{E}\vec{u}, \tag{31}$$

and is not independent but follows from (30): it just represents the theorem for the change of energy.

In general, the equations for the four-momentum (30), (31) differ in structure from the canonical ones but reduce to the latter in the limit $l \to 0$. Later on we shall see if such reduction is in agreement with the additional equations (28) for the unit three-vector $\vec{n}$.

3 "Centre-of-mass frame" and dynamics of "free" spinning particle

Now we switch off the external electromagnetic field by setting $F_\mu = 0$ in order to see if some internal motion ("zitterbewegung") of the "charge-position vector" relative to the centre-of-mass can be accomplished by a "free" spinning particle, in analogy with many other classical spin models (see, e.g., the review in [13]). To start with, we can take into account the conservation of momentum (10) and angular momentum (17) in absence of a field. This allows one to pass, through a Lorentz boost and a three-translation, to a distinguished *centre-of-mass* reference frame defined by the following conditions:

$$P_a = 0 \quad (P_0 = mc), \quad M_{0a} = 0, \quad a = 1, 2, 3. \tag{32}$$

Equations (30),(31) now simplify to

$$\vec{u} - \frac{2\zeta f'}{Z}\vec{n} = 0,$$

and

$$u_0 - \frac{2\zeta f'}{Z} = \frac{1}{f},$$

where Z is defined by (29). Multiplying the first of these equations by $\vec{n}$ and taking in account the second one, one gets $Z = 1$. Now, after substitution of the particular form (24) of $f(\zeta)$, the first set of equations of motion takes the simple form

$$\vec{u} = \pm u\vec{n}, \quad u = |\vec{u}| = \frac{\sqrt{-\zeta}}{2f} = \frac{\omega_0 l}{2c}, \tag{33}$$

and

$$u_0 = \frac{1}{f} \pm u, . \tag{34}$$

where the characteristic "frequency" ω_0, depending on ζ, comes into play,

$$\omega_0 := \frac{c\sqrt{-\zeta}}{l f(\zeta)}, \tag{35}$$

and the different signs correspond to those of the functions $f = f_\pm$ in (24). Note that the direction three-vector $\vec{n}$ is, respectively, parallel or antiparallel to the three-velocity, and that equation (34) is a direct consequence of (33) as long as the function $f(\zeta)$ satisfies the defining relation (21).

The second set of equations (28) simplifies now to

$$\frac{d}{d\tau}\left(\frac{\dot{\vec{n}}}{\omega_0}\right) = -\omega_0\vec{n}, \tag{36}$$

and together with (33) forms a closed system defining the position vector $\vec{x}$ and the direction unit vector $\vec{n}$. Taking vector product of the above equations with $\vec{n}$, one finds that for any solution there exists a *conserved* pseudovector [3]

$$\vec{\nu} = \frac{1}{\omega_0}\vec{n}\times\dot{\vec{n}}, \quad \dot{\vec{\nu}} = 0. \tag{37}$$

Since this is *constant* and orthogonal to both $\dot{\vec{n}}$ and $\vec{n}$, the latter at all times lies in a fixed plane. From the parallelism (33) of $\vec{n}$ and the three-velocity vector $\vec{u}$ one concludes that the spatial motion of the particle is also in a *plane*.

As result, for the unit vector $\vec{n}$ the most general (up to a 3D rotation and time shift) form is as follows:

$$\vec{n} = \{\cos\kappa, \sin\kappa, 0\}, \tag{38}$$

with $\kappa = \kappa(\tau)$ being some function of the proper time parameter. Now equations (36) immediately lead to the relation

$$\dot{\kappa} = \pm\omega_0, \tag{39}$$

whence the dependence of ω_0 on time remains undetermined and different signs correspond to opposite directions of rotation (in definitely oriented centre-of-mass reference frame).

For both, the relative orientations of the three-vectors $\vec{n}$ and $\vec{u}$, in accord with (33), may be different. However, the direction of the spin three-vector $\vec{S} = \{W^a\}$ is the same for both orientations and depends only on the direction of rotation. Indeed, from the general expression (19) one has

$$\vec{S} = \{W^a\} = 2ml^2 f' f^2 \vec{n}\times\dot{\vec{n}} = \pm 2mcl(ff'\sqrt{-\zeta})\vec{e}_z = \pm\frac{1}{2}mcl\vec{e}_z, \tag{40}$$

where $\vec{e}_z$ is a unit vector along the z-axis, and the expression in parentheses is taken to be $1/4$ in accordance with the second defining equation (22) for the generating function $f(\zeta)$. Thus, the spin vector is orthogonal to the plane of rotation, and its sign depends only on the direction of rotation. As a whole, four different *degenerate* "configurations" are possible (Fig.1) [4]. As for the temporal component W_0 of the Pauli-Lubański four-vector, it is proportional to $(\dot{\vec{n}}\cdot\vec{n}\times\vec{u})$ and, therefore, zero. For the position vector one obtains then by integrating the equations (33)

$$\vec{x} = \frac{l}{2}\{\sin\kappa, -\cos\kappa, 0\}, \tag{41}$$

where the constant of integration was set to zero since the origin is placed at the centre-of-mass point (32), $M_{0a} = 0$. We see that *the "free" particle always moves along a circle with fixed radius*

$$R = \frac{l}{2}, \tag{42}$$

[3] This is in fact equal to the spin vector $\vec{S}$, see equation (40).

[4] Physical situations represented by a) and b), as well as by c) and d) panels at Fig.1, are in fact equivalent but can be distinguished by a particular observer.

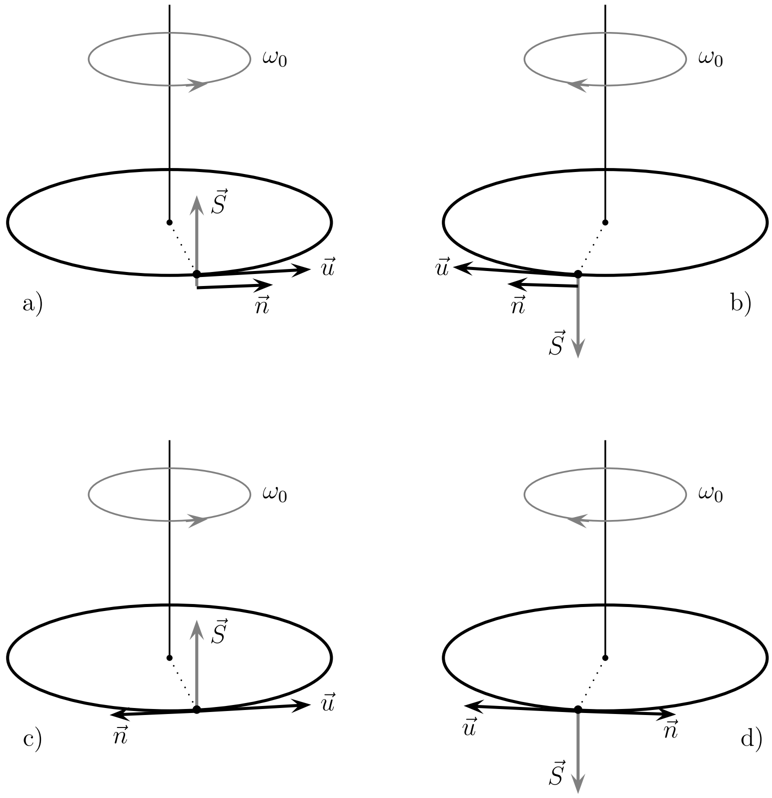

Figure 1: Relative orientations of the three-velocity $\vec{u}$, the direction three-vector $\vec{n}$ and the spin three-vector $\vec{S}$ under uniform internal rotation in the "free" case and in the centre-of-mass reference frame with definite orientation.

and when the value of the spin is set equal to one half of the Planck's constant, $S = \hbar/2$, then *the diameter of the circle of rotation is exactly equal to the Compton length*. Thus, the particle itself is not located at its own center of mass but circles around the latter. This is a common feature of different classical spin models [13]; however, the radius therein is usually quite arbitrary and is fixed ("quantized") *ad hoc*.

Moreover, only in the considered model one comes across the stochastic-type *zittebewegung* phenomenon in full sense of the word. Indeed, the rotation frequency (35) defined according to (8) through the ζ-variable, remains indefinite or, uncontrolled. Indeed, calculating ζ from the defining formula (8) and making use of the expressions (38),(39) and (41) for the corresponding variables, one obtains an identity. Thus, the time variation of the parameter ζ and frequency ω_0 can indeed be arbitrary.

It is noteworthy to point out that independently on the particular form of the considered model, the simple equation (36) for the unit three-vector $\vec{n}$ turns to describe, in account of its general solution (38), an important dynamical system, namely, *a plane rotator with arbitrary varying frequency* $\omega(\tau)$. It is also of interest that introducing a new "internal" time scale $\sigma(\tau)$ (more precisely, *phase* scale),

$$d\sigma = \omega_0(\tau) d\tau \tag{43}$$

one reduces (36) to the equations

$$\frac{d^2\vec{n}}{d\sigma^2} = -\vec{n}, \tag{44}$$

with an unique solution corresponding to a plane rotation (clock- or anticlockwise) with *constant* unit frequency (in the specially adopted time scale). Of course, the trajectory of the particle, a circle of fixed radius (42), is itself invariant under any time scale reparametrizations. On the other hand, we shall see below that in realistic situations, when a constant external field is present, the frequency of rotation is as a rule constant and fixed in value.

In the simplest case one can consider the value of $\sqrt{-\zeta}$ to be a constant, with its range for the choice $f = f_-$ of the function (24) being restricted by the requirement $0 < -\zeta < 1$. Then, according to (38),(39), our "free" particle accomplishes a *uniform rotation* with constant in time (but arbitrary!) frequency $\omega_0 = c\sqrt{-\zeta}/lf(\zeta)$ (the direction of rotation can be arbitrary whereas the relative orientations of the three-vectors $\vec{u}$ and $\vec{n}$ in view of (33) are different for the two different generating functions $f = f_\pm$).

For the temporal component of the four-velocity one gets then:

$$u_0 = \sqrt{1+u^2} = \sqrt{1+\left(\frac{\omega_0 l}{2c}\right)^2}. \tag{45}$$

One can easily check that the same expression for u_0 follows from (34). This allows one to pass from the proper time parameter τ to the laboratory time,

$$d\tau = cdt/u_0, \tag{46}$$

and to obtain the three-velocity $v^a = dx^a/dt$ and physical frequency of rotation ω. One finds

$$\vec{v} = \frac{\omega l}{2}\{\cos\omega t, \sin\omega t, 0\}, \quad \omega := \frac{\omega_0}{\sqrt{1+(\omega_0 l/2c)^2}}. \tag{47}$$

One can see that the linear velocity never exceeds the speed of light, $v^2 = \vec{v}\vec{v} < c^2$. It is also interesting (and even striking!) that the spin vector does not depend on the speed of rotation of the particle. In particular, the frequency ω_0 (and ω) can be arbitrarily small and close to zero, so that in the limit one has an indefinite direction of the spin vector but its modulus remains fixed.

Mathematically, this corresponds to an uncertainty of the type 0/0 in the expression for the second Casimir invariant (20). When the sign of frequency and direction of rotation are changed, one has a *spin-flop transition* at this critical point. In the next section we shall see that the same phenomenon also takes place for the particle in a homogeneous magnetic field and thus may be of universal nature. In the "free" case, however, *the law of motion may be arbitrary.* Particularly, when ζ and thus ω are constant, the position vector will move (as it usually takes place in the classical spin models) along a *helix-like* curve.

4 Frequency shift and spin-flop transitions in magnetic field

Here we consider the classical problem of a relativistic charged particle in a *constant and homogeneous magnetic field.* The action (2) of the spinning particle contains the coupling of the particle to the magnetic field and one of the two generating functions $f = f_\pm$ in (24). The energy $\mathcal{E}$ is then conserved, so that instead of (31) one gets

$$f(u_0 - \frac{y}{Z}) = \frac{\mathcal{E}}{mc^2} = \delta = \text{const} > 1, \tag{48}$$

whereas the principal equation (30) for three-momentum $\vec{P}$ takes the form

$$\frac{d}{d\tau}\left(\frac{\vec{P}}{mc}\right) = \frac{d}{d\tau}\left(f\vec{u} - f\frac{y}{Z}\vec{n}\right) = -\Omega\vec{u}\times\vec{h}. \tag{49}$$

Here $Z = f(u_0 - \vec{u}\vec{n})$ has been defined in (29), y is given by

$$y = \pm\frac{\sqrt{-\zeta}}{2f_\pm} = \pm\frac{\omega_0 l}{2c}, \tag{50}$$

and Ω stands for the canonical *cyclotron frequency* (with respect to the proper time scale) of a particle in a homogeneous magnetic field of strength H,

$$\Omega = \frac{eH}{mc}. \tag{51}$$

Here we assume that the charge of the particle is elementary and *negative* (electron), whereas $\vec{h}$ is the unit vector in the direction of the magnetic field which is aligned with the z-axis.

It follows from (49) that the formerly considered "free" case, corresponding to $H = 0$ and $\vec{P} = 0$, is not a well defined limit of the motion in a homogeneous magnetic field. The limiting "free state" resulting from the limiting process will actually depend on the *ratio* of the external parameters Ω and $\mathcal{P} = |\vec{P}|$ as the two approach zero, and this points to the degeneracy of the "free state".

The component of the three-momentum along $\vec{h}$ is conserved, so that we can get rid of the corresponding uniform motion by setting $P_z = 0$. However, for the spinning particle this condition does not imply $u_z = 0$; instead one gets $u_z = (y/Z)n_z$.

The resulting motion in z-direction could correspond to a *precession* of the spin vector. Here, however, we shall only consider the simplest case of a *plane* motion and hence set $u_z = n_z = 0$.

From the identity (27) for the case of a pure magnetic field it follows

$$\vec{h} \cdot (\vec{u} \times \vec{n}) = 0, \tag{52}$$

so that the vectors $\vec{h}$, $\vec{u}$, and $\vec{n}$ *lie instantaneously in one and the same plane.* For the considered motion in the plane orthogonal to magnetic field this means that the vectors $\vec{u}$ and $\vec{n}$ are either parallel or antiparallel, that is,

$$\vec{u} = (\vec{u}\vec{n})\vec{n}. \tag{53}$$

Now the equations (28) for the unit vector $\vec{n}$ can be simplified to the form

$$\frac{d}{d\tau}\left(\frac{\dot{\vec{n}}}{\omega_0 Z^2}\right) = -\omega_0\vec{n}, \tag{54}$$

strongly resembling the evolution equation in the "free" case (36).

Contracting the equations (49) with $\vec{n}$, one easily gets

$$f(\vec{u}\vec{n} - \frac{y}{Z}) = \pm\frac{\mathcal{P}}{mc} = \gamma = \mathrm{const}, \tag{55}$$

where $\mathcal{P}$ stands for the conserved *modulus* of the three-momentum. By virtue of the fixed mass invariant one has then the usual energy-momentum relation

$$\delta = \sqrt{1+\gamma^2}, \tag{56}$$

where γ may be either positive or negative. In fact, it is the projection of three-momentum onto the $\vec{n}$-direction whose sign depends on relative orientation of $\vec{n}$ and $\vec{u}$, see below.

Combining now equations (48) and (49), one finds

$$Z = \delta - \gamma = \mathrm{const} > 0, \tag{57}$$

so that the considered system of equations simplifies to

$$\gamma\dot{\vec{n}} = -(\vec{u}\vec{n})\Omega\,\vec{n}\times\vec{h}, \tag{58}$$

$$\frac{d}{d\tau}\left(\frac{\dot{\vec{n}}}{\omega}\right) = -\omega\vec{n}, \quad \omega = \omega_0(\delta-\gamma) \equiv \frac{\sqrt{-\zeta}c}{fl}(\delta-\gamma), \tag{59}$$

and two additional (though not independent) relations

$$\delta = fu_0 - \frac{yf}{\delta-\gamma}, \tag{60}$$

$$\gamma = f(\vec{u}\vec{n}) - \frac{yf}{\delta-\gamma}. \tag{61}$$

Equations (59) may be solved similarly as in the "free" case leading to the solution

$$\vec{n} = \{\cos\kappa, \sin\kappa, 0\}, \quad \dot{\kappa} = \pm\omega. \tag{62}$$

Substituting this into the first set of equations (58), one obtains

$$\dot{\kappa} = \Omega\frac{(\vec{u}\vec{n})}{\gamma}, \tag{63}$$

and, because of (62), finds for the *rotational frequency*

$$\omega = \Omega\frac{u}{|\gamma|}. \tag{64}$$

For the *radius* R of the circular orbit one obtains the same expression

$$R = \frac{cu}{\omega} = \frac{c|\gamma|}{\Omega}, \tag{65}$$

as for a canonical spinless particle. Analogously, the expression (40) for the spin vector

$$\vec{S} = \{W^a\} = \pm\frac{1}{2}mcl\vec{e}_z \tag{66}$$

is still valid in the considered case of a *plane* motion in a magnetic field. We conclude that the spin is always *polarized* along the z-direction, with its sign being dependent on the direction of rotation (which may differ from the canonical one, see below).

From (61) one concludes now that the *frequency of rotation* ω *does not coincide with the canonical cyclotron frequency* Ω,

$$\frac{\omega}{\Omega} = \frac{u}{|\gamma|} \neq 1. \tag{67}$$

Note that both frequencies are measured here with respect to *proper* time of the particle; however, the ratio (67) is invariant under any change of time scale.

Substituting the expressions for $(\vec{u}\vec{n})$ from (61) and for ω from (59) one arrives at the following exact equation for the value of the "hidden parameter" $\sqrt{-\zeta}$:

$$\sqrt{-\zeta}\left(\pm\frac{\delta-\gamma}{\alpha} \mp \frac{1}{2\gamma(\delta-\gamma)}\right) = 1, \tag{68}$$

where the following *small parameter* α appears:

$$\alpha = \frac{\Omega l}{c} = \frac{\hbar\Omega}{mc^2}. \tag{69}$$

To get the last relation, we assumed $l = \hbar/mc$ so that the value of spin would be equal to that of a quantum mechanical electron projected onto the z-axis. Note that all signs in (68) are independent, the left pair corresponds to opposite directions of rotation and the right pair – to the possible choices $f_\pm$ for the generating function (24). In addition one should take into account that the sign of γ is not determined. Later on we shall consider various possible combinations of these signs in more detail.

Observe that for the particular case of an *electron*, even in the strongest magnetic fields achieved in laboratories ($H \sim 10^4$ Gauss), the numerical value of α does not exceed 10^{-15}. Moreover, another dimensionless parameter $\alpha/|\gamma|$ seems to be of fundamental importance in the model. In fact, a *macroscopic* radius of rotation $R = c|\gamma|/\Omega$ greatly exceeds the Compton wavelength l, $R \gg l$, so that for the above parameter one gets,

$$\frac{\alpha}{|\gamma|} \ll 1. \tag{70}$$

Thus, the two external quantities governing the classical behavior of the electron in a homogeneous magnetic field, namely, the strength $H \sim \alpha$ of the field and the three-momentum $\mathcal{P} \sim |\gamma|$, define two small parameters. We shall see that the latters determine small *corrections* to the rotational characteristics of a spinning particle in a homogeneous magnetic field.

For such a *macroscopic* motion, the direction of rotation is defined by the orientation of the ordinary Lorentz force towards the centre of rotation, see Fig.2, The canonical motion corresponds to a positive

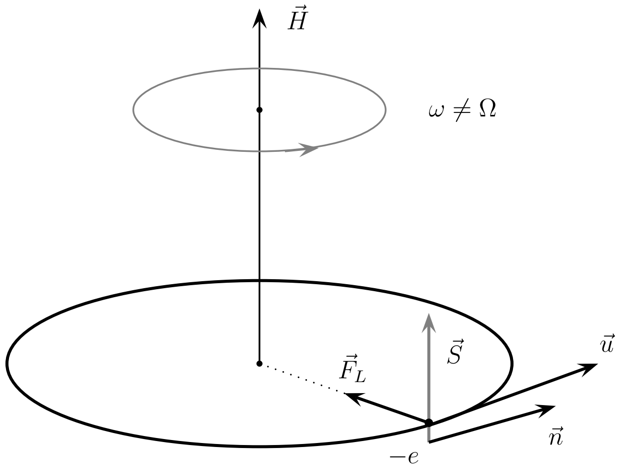

Figure 2: "Canonical" orientation of the three-velocity $\vec{u}$, magnetic field $\vec{H}$ and Lorentz force $\vec{F}_L$ in the "macroscopic" case $R \gg l$.

sign in the expression for the frequency (62), so that from the principal equation (68) one gets for ζ the following expression:

$$\sqrt{-\zeta} = \frac{\alpha}{\delta - \gamma}\left(1 \mp \frac{(\alpha/\gamma)}{2(\delta-\gamma)^2}\right)^{-1}, \tag{71}$$

and this yields then the three-velocity and frequency of rotation,

$$\frac{u}{|\gamma|} = \frac{\omega}{\Omega} = \frac{(\delta-\gamma)\sqrt{-\zeta}}{\alpha f_\pm} = \frac{1}{f_\pm}\left(1 \mp \frac{(\alpha/\gamma)}{2(\delta-\gamma)^2}\right)^{-1}, \tag{72}$$

where the sign in parentheses correlates with the choice of generating function $f_\pm$.

The last expression is exact and general, provided the direction of rotation is the canonical one. It describes *four different configurations* corresponding to the choice of $f = f_\pm$ and the different orientations of $\vec{n}$ relative to $\vec{u}$. They are all non-degenerate and define different three-velocities and frequencies for the same total three-momentum $|\gamma|$. In particular, for the relative *frequency shift*

$$\Delta = (\omega - \Omega)/\Omega \ll 1$$

one obtains in leading order in the small parameters α and $\alpha/|\gamma|$:

$$a)\;\; f = f_+, \quad -\vec{n} \parallel \vec{u} \;\; (\gamma < 0): \qquad \Delta \approx -\frac{(\alpha/|\gamma|)}{2(\delta + |\gamma|)^2} - \frac{\alpha}{2(\delta + |\gamma|)}; \;\; \Delta < 0 \tag{73}$$

$$b)\;\; f = f_-, \quad \vec{n} \parallel \vec{u} \;\; (\gamma > 0): \qquad \Delta \approx -\frac{(\alpha/\gamma)}{2(\delta - \gamma)^2} + \frac{\alpha}{2(\delta - \gamma)}; \tag{74}$$

$$c)\;\; f = f_+, \quad \vec{n} \parallel \vec{u} \;\; (\gamma > 0): \qquad \Delta \approx \frac{(\alpha/\gamma)}{2(\delta - \gamma)^2} - \frac{\alpha}{2(\delta - \gamma)}; \tag{75}$$

$$d)\;\; f = f_-, \quad -\vec{n} \parallel \vec{u} \;\; (\gamma < 0): \qquad \Delta \approx \frac{(\alpha/|\gamma|)}{2(\delta + |\gamma|)^2} + \frac{\alpha}{2(\delta + |\gamma|)}; \;\; \Delta > 0. \tag{76}$$

Note that for the configurations with negative γ the sign of the frequency shift is fixed; this is easy to understand taking in account that the addition to the canonical expression for the three-momentum in (49) is positive for configuration a) and negative for configuration d) so that the period of rotation has to increase or decrease, respectively. In the other two situations the effect depends on the relative magnitudes of magnetic field and three-momentum.

In the *ultra-relativistic* case $\mathcal{E}/mc^2 = \delta \gg 1$, the parameter $(\delta - \gamma)$, for *configurations with parallel* $\vec{n}$ *and* $\vec{u}$ *and* $\gamma \sim \delta > 0$, decreases as $1/2\delta$, and one gets for (75) and (74) respectively:

$$f = f_\pm, \quad \vec{n} \parallel \vec{u}: \quad \Delta \approx \pm\alpha\delta \equiv \pm \left(\frac{\hbar\Omega}{mc^2}\right)\left(\frac{\mathcal{E}}{mc^2}\right). \tag{77}$$

Even in this case, however, the frequency shift is hardly detectable in a laboratory. Indeed, for achievable field strengths of 10^4 Gauss, the parameter α is approximately 10^{-15} for electrons, whereas for an energy of 100 GeV one gets $\delta \sim 2 \cdot 10^5$ such that the relative frequency shift is less than $\Delta \sim 10^{-10}$. This seems to be too small to be detectable on account of the required constancy of energy and homogeneity of the magnetic field. We shall, however, postpone the detailed discussion of this problem to a forthcoming publication.

Let us now turn to the more complicated and questionable non-relativistic case of small three-momenta $|\gamma|$. In the model, this corresponds to a *microscopic* radius of the electron's gyration $R \sim l = \hbar/mc \sim 2.5 \cdot 10^{-12} m$ and such microscopic orbits, as generally accepted, should be described by quantum theory. Nonetheless, we shall now consider solutions corresponding to the microscopic orbits in a magnetic field from a purely classical point of view.

For small three-momenta one can set $\delta - \gamma \approx 1$ in the general formula (68) and obtain for the $\sqrt{-\zeta}$ parameter

$$\sqrt{-\zeta} \simeq \alpha(\pm 1 \mp \frac{\alpha}{2\gamma})^{-1}, \tag{78}$$

and for the frequency of gyration

$$\frac{\omega}{\Omega} \simeq \frac{1}{f_\pm}(\pm 1 \mp \frac{\alpha}{2\gamma})^{-1}, \tag{79}$$

assuming for the expression in parentheses to be non-negative on solutions.

In particular, for $|\gamma| > \alpha/2$, the parameter $\sqrt{-\zeta}$ is only positive for the canonical direction of gyration (this corresponds to the positive sign for the left pair in parentheses). In this case, the two solutions for which

$$\sqrt{-\zeta} \simeq \alpha(1 + \alpha/2|\gamma|)^{-1}$$

are well defined even for very small values of three-momentum $|\gamma|$. They correspond to the following configurations:

$$\begin{array}{llll} a) & f = f_+ \quad , & \gamma < 0 & (-\vec{n} \parallel \vec{u}) \\ b) & f = f_- \quad , & \gamma > 0 & (\vec{n} \parallel \vec{u}), \end{array}$$

see Fig.3. For $|\gamma| \to 0$ and small but fixed α, both $\sqrt{-\zeta}$ and ω approach zero as $\sim |\gamma|$, and in the limit

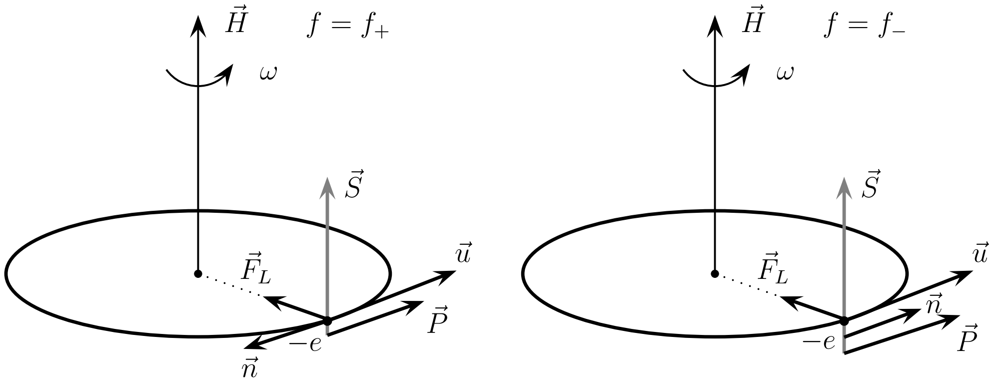

Figure 3: Configurations for the spinning particle rotating in magnetic field that exist for any value of the particle's momentum. On the left a) $f = f_+$ and on the right b) $f = f_-$.

of a "particle at rest in a magnetic field" the internal motion is "frozen", whereby the magnitude of the spin vector (66) is preserved.

For the two other configurations corresponding, for $|\gamma| > \alpha/2$, to the canonical direction of gyration and to

$$\begin{array}{llll} c) & f = f_+ \quad , & \gamma > 0 & (\vec{n} \parallel \vec{u}) \\ d) & f = f_- \quad , & \gamma < 0 & (-\vec{n} \parallel \vec{u}), \end{array}$$

the parameters $\sqrt{-\zeta}$, ω and three-velocity u behave as

$$\sqrt{-\zeta} \simeq \alpha(1 - \frac{\alpha}{2|\gamma|})^{-1}, \quad \omega \simeq \Omega\sqrt{\alpha}(1 - \frac{\alpha}{2|\gamma|})^{-1/2}, \quad u = \frac{\alpha\omega}{2\Omega},$$

and diverge at the *critical value* of three-momentum $|\gamma| = \alpha/2$. Interestingly, the singular behavior happens when the radius of gyration $R = l/2$ is equal to that in the "free" case. Note also that for the second kind of function $f_- = \sqrt{1 - \sqrt{-\zeta}}$ the solution disappears before the critical value is reached, namely, at $\sqrt{-\zeta} = 1$; however, the difference in the values for the critical parameters is very small, and we neglect it for sake of simplicity.

One should further take into account that *physical observables* should be defined according to the laboratory time scale, not to the proper time one. In particular, the *physical* three-velocity v and frequency of rotation $\omega_{\rm ph}$ are defined as follows:

$$\frac{v}{c} = \frac{u}{\sqrt{1 + u^2}}, \quad \omega_{\rm ph} = \omega\sqrt{1 - (v/c)^2} \equiv \frac{\omega}{\sqrt{1 + u^2}}, \tag{80}$$

so that near the critical point $|\gamma| = \alpha/2$, where the four-velocity u and proper-time frequency ω increase without limits, the physical velocity approaches the *speed of light*, $v \to c$, and for the physical frequency of gyration one obtains

$$\omega_{\rm ph} = \frac{\omega}{\sqrt{1+u^2}} \simeq \frac{\omega}{u} = \frac{\Omega}{|\gamma|} \simeq \frac{2\Omega}{\alpha} = \frac{2mc^2}{\hbar}. \tag{81}$$

Thus, near the critical point the frequency of gyration remains finite and close to one half of the internal *de Broglie frequency* Ω_{DB} associated with the particle,

$$\omega_{\rm ph} = \frac{1}{2}\Omega_{DB}, \qquad \hbar\Omega_{DB} = mc^2. \tag{82}$$

In order to better understand what could happen with a particle when its three-momentum approaches the critical value, let us examine the opposite range for the three-momentum, $|\gamma| < \alpha/2$. In this range, apart from the two formerly considered configurations a) and b) with canonical direction of rotation, two unusual and at macro-level forbidden configurations with *direction of rotation opposite to the canonical one* can occur (for $f = f_+$ see Fig.4, right panel). The particle is accelerated in the direction opposite to the Lorentz force acting on it, and to explain this formally one has to prescribe a *negative effective mass* ($m_{eff} < 0$) to the particle. For such unusual configurations, the parameters

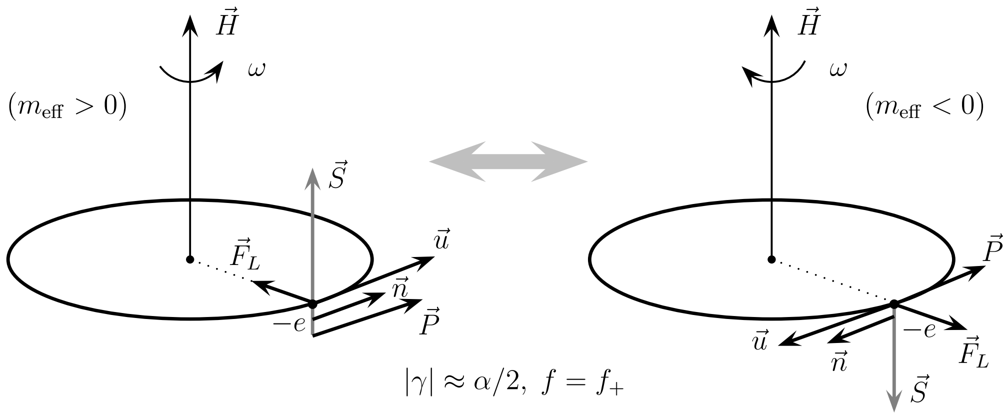

Figure 4: The rotation in a magnetic field is discontinuous at a critical value of the three-momentum. Shown are the canonical direction of rotation ($m_{\rm eff} > 0$) and the rotation in direction opposite to the force ($m_{\rm eff} < 0$). In both cases $f = f_+$ and $|\gamma| \approx \alpha/2$.

$\sqrt{-\zeta}$, ω and u behave in the critical range as

$$\sqrt{-\zeta} \simeq \alpha(\frac{\alpha}{2|\gamma|} - 1)^{-1}, \quad \omega \simeq \Omega\sqrt{\alpha}(\frac{\alpha}{2|\gamma|} - 1)^{-1/2}, \quad u = \frac{\alpha\omega}{2\Omega},$$

and thus diverge at the critical momentum, similarly as for the cases c) and d) considered above. Again the corresponding *physical* quantities remain finite, $v \to c$ and $\omega_{\rm ph} \to \frac{1}{2}\Omega_{DB}$. These "exotic" configurations correspond to the following conditions: cc) $f = f_+$, $\gamma < 0$, or dd) $f = f_-$, $\gamma > 0$. Note, however, that for the opposite direction of rotation $\gamma > 0$ corresponds to $-\vec{n} \parallel \vec{u}$ and vice versa. Thus, near the critical value of the three-momentum $|\gamma| = \alpha/2$ the configurations c) and cc) (as well as d) and dd)) have opposite directions of rotation, opposite directions of unit three-vector $\vec{n}$ and of the spin vector $\vec{S}$ (see Fig.4 for $f = f_+$).

We can thus conjecture that, when the three-momentum of a particle rotating in a magnetic field approaches (say, through radiation) the critical value $|\gamma| = \alpha/2$, its dynamics becomes *irregular*, and a *phase-like* transition to another rotational state may happen, under preservation of energy, three-momentum and magnitude of the spin vector, but through a flip of the directions of the spin vector, unit three-vector and vector of three-velocity. This very much resembles a quantum transition with *spin flop* and, perhaps, can be thought of as its classical counterpart. However, a lot of work has still to be done to comprehend such a complicated and unusual dynamics, even within the framework of a "textbook" problem of motion of a charged particle in homogeneous magnetic field.

5 Conclusion

We are accustomed to consider spin as a purely quantum property of particles. In quantum theory, discretization of spin is associated with the finite dimensional irreducible representations of the Lorentz group. In the classical limit, this should be consistent with universality of the values of second Casimir invariant. As a rule, however, this is not supported by the equations of motion and is usually taken to be a constant number "by hands".

The same is true even for a weaker requirement of *preservation* of a particle's spin *in external fields* demonstrated by real elementary particles but having no evident counterpart in external or internal symmetries (contrary to preservation of mass and charge). To ensure the preservation at the classical level, one usually [1, 3] needs to impose an additional "Frenkel-like" orthogonality condition on the solutions of the equations of motion.

In this respect, the model proposed by Staruszkiewicz is quite remarkable since the modulus of the spin 4-vector becomes therein universal, one and the same for any solution, even under the presence of arbitrary electromagnetic field (section 2). Contrary to the situation in other models (see, e.g. [1, 2, 3, 4]), this property is here a direct consequence of the equations of motion themselves and need not to be postulated via imposition of an additional constraint or obtained from the classical dynamics via a not yet well defined quantization procedure.

From a general point of view, the proposed deformation of the standard action seems to be quite natural, motivated by deep symmetry and other physical considerations and, in a sense, unambiguous. The two permitted forms for the generating function found above (section 2) may be thought of as corresponding to two types of charged elementary particles of different mass but the same spin (if, instead of universal l the constant $\hbar/mc$ with universal $\hbar$ is introduced into the action). However, at present all such conjectures seem to be speculative.

In the model, the original *four-orthogonality condition* (27) is always fulfilled "on shell" and coordinates the spatial and spin dynamics allowed by the equations of motions. In particular, the model demonstrates a number of quite remarkable properties with respect to the particle's dynamics in the "free" case and in the presence of a homogeneous magnetic field (sections 3 and 4 respectively). Some of these (fixed radius and indefinite alteration of the frequency of zitterbewegung, spin-flop "phase-like" transitions) strongly resemble quantum phenomena, others (cyclotron frequency shift) can help to experimentally test the model. Of course, the model requires deep rebuilding of the presently accepted paradigm when one claims to relate it to real physics.

It is also noteworthy that some properties peculiar for the model under consideration (plane character of the Zitterbewegung, magnitude of its radius equal to half a Compton wavelength etc.) draw it nearer to some other models, in particular to that developed by Rivas [13, 16] from rather general symmetry considerations. It would be interesting to find out whether his approach and the Staruszkiewicz model can in fact be cooordinated and united in the starting principles as well as in the predicted spin dynamics.

In any case, however, mathematical beauty and simplicity of the considered model, its correspondence with the canonical electrodynamics in the spinless case, together with experimental verifiability, make it a nice “toy model” to interpret some properties of real spinning particles at a purely classical and visual level of consideration.

6 Afterword

After the paper [20] has been accepted for publication, we became informed that the Staruszkiewicz model had been in fact proposed as long as 15 years ago in a distinguished paper of Kuzenko, Lyakhovich and Segal [22], on the base of elegant geometrical considerations. The authors had realized at least two profound ideas reproduced later in the note of Staruszkiewicz [18]: introduction of a light-like variable responsible for additional spin degrees of freedom and determination of the action for a spinning particle from the requirement of universality (“strong conservation”) of the two Casimir invariants of particle’s mass and spin.

Specifically, they consider the $\mathbb{M}\times S^2$ geometry of the configuration space being a direct product of the Minkowski space and the two-sphere. The latter is the $SO(3,1)$-transformation space of minimal dimension, with (nonlinear) Lorentz action represented by fractional linear transformations of the component $z \in \mathbb{C}$ of the projective spinor $\lambda = \{1, z\}$. The z-component is canonically obtained via stereographic projection $S^2 \mapsto C_*$ of the two-sphere onto the Argand complex plane (for this see, e.g., [21]).

To connect the spatial dynamics in $\mathbb{M}$ and the spin dynamics, the standard metric on S^2,

$$ds^2 \sim \frac{dz d\bar{z}}{(1+z\bar{z})^2}, \tag{83}$$

has been endowed with a conformal factor in which the tangent 4-vector $\dot{x}_\mu$ to the particle’s world line in $\mathbb{M}$ does enter. Finally, they arrive at the S^2-metric induced Lorentz invariant

$$\Upsilon = \frac{4|\dot{z}|^2}{(\xi\cdot\dot{x})^2}, \tag{84}$$

where a null 4-vector

$$\xi_\mu = \lambda^+\sigma_\mu\lambda, \quad (\xi\cdot\xi) \equiv 0 \tag{85}$$

naturally comes into play.

Together with the usual invariant $(\dot{x}\cdot\dot{x})$ corresponding to the metric in $\mathbb{M}$, (84) unambiguously predicts the form of the action for a free relativistic spinning particle which, after the elimination of the Lagrange multipliers, takes the following form (in the notations that we have introduced in this article):

$$S = \int L d\tau, \quad L = -mc\sqrt{(\dot{x}\cdot\dot{x})\left(1 - \frac{4\Delta}{m^2c^2}\frac{|\dot{z}|}{(\xi\cdot\dot{x})}\right)} \equiv -mc\sqrt{(\dot{x}\cdot\dot{x})\left(1 - \frac{\Delta}{m^2c^2}\sqrt{\Upsilon}\right)}, \tag{86}$$

where the constant Δ was assumed to be $\Delta = \hbar mc\sqrt{s(s+1)}$, s being *arbitrary* (at the classical level of consideration) spin of the particle [5].

The action (86) leads to the constant and universal values of both Casimir invariants and strongly resembles the Staruszkiewicz action (2). To proof their equivalence, let us recall that any null 4-vector k_μ can be represented in the spinorial form

$$k_\mu = \omega^+\sigma_\mu\omega. \tag{87}$$

[5] This interpretation is different from that accepted in our article. We note that in experiments one measures only the *projection* of the spin vector and not its whole length, so the above identification of Δ seems doubtful.

The spinor $\omega \in \mathbb{C}^2$ is defined by (87) up to a phase factor and, in account of the projective invariance of the principle ζ-variable (1) of Staruszkiewicz approach (see Sec. 1), can be reduced to the form $\omega = \{1, \tilde{z}\}, \quad \tilde{z} \in \mathbb{C}$. If one now makes evident identifications

$$\tilde{z} \equiv z, \quad \omega \equiv \lambda, \quad k_\mu \equiv \xi_\mu, \tag{88}$$

then, using rules of the 2-spinor algebra, it is an easy exercise to show that the ζ-variable (1) can be expressed in the spinorial form,

$$\zeta = l^2 \frac{(\dot{k} \cdot \dot{k})}{(k \cdot \dot{x})^2} = -4l^2 \frac{|\dot{z}|^2}{(\xi \cdot \dot{x})^2}, \tag{89}$$

and, up to a dimensional factor, coincides with the S^2-invariant (84),

$$-\zeta = l^2 \Upsilon. \tag{90}$$

The action (2), up to a sign of the additional term [6], also takes the form (86) of the $\mathbb{M} \times S^2$ action.

Thus, the Staruszkiewich model is indeed equivalent to that proposed earlier in [22]. In the latter paper, the authors even accomplished a thorough Hamiltonian analysis and carried out canonical quantization of the free model. As for the classical part, in the free case the plane character and fixed radius of Zitterbewegung have been already established in [22]. The only difference with the above presented results (Sec. 3) is in the frequency of particle's revolution which in [22] was obtained arbitrary but constant. We think that this fact is due to the therein used time gauge, just as it takes place in our study of free motion under the special choice of time scale (43). Nonetheless, in the physical laboratory time scale Zitterbewegung frequency turns out to be capable of variation in an arbitrary manner.

It is noteworthy to mention that the Kuzenko-Lyakhovich-Segal (KLS) model has been later generalized to the supersymmetrical case [23] and to the Anti-de-Sitter background geometry [24]. In the paper [25] some additional possibilities to construct the action for a (both massive and massless) spinning particle in $\mathbb{M} \times S^2$ exploiting spinor structure of the two-sphere have been proposed, and the coupling with external electromagnetic and gravitational field formally considered. However, contrary to the original paper [22] and to the Staruszkiewicz approach, the constructions developed in [25] require the introduction of new parameters with vague physical sense and, in general, seem rather cumbrous and ambigous.

After the afore-presented short comparative analysis of the KLS-model and Staruszkiewich approach it is noteworthy to underline that nearly all the results obtained above (and in our paper [20]) are new in the respect when an external electromagnetic field is present. This concerns to the universality ("strong conservation") of the mass-spin invariants, to the "on-shell" orthogonality (27) of the null four-vector and four-force vector and the exact results related to the plane motion of a spinning particle in constant magnetic field (corrections to cyclotron frequency, possible spin-flop transitions etc.). As to the free case, our results mainly reproduce those obtained in [22] (distinctions in the frequency of Zitterbewegung and in interpretation of the spin value were mentioned above and reqiure additional analysis). On the other hand, the two different approaches and the results obtained here go "hand by hand" and provide essential support to the Kuzenko-Lyakhovich-Segal-Staruszkiewich model demonstrating its really exceptional character.

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[6]It seems that the possibility of two opposite signs revealed in the above presented paper, see (24), has been overlooked in both preceding approaches.

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