Black hole pair creation in de Sitter space: a complete one-loop analysis

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Abstract

We present an exact one-loop calculation of the tunneling process in Euclidean quantum gravity describing creation of black hole pairs in a de Sitter universe. Such processes are mediated by $S^2 \times S^2$ gravitational instantons giving an imaginary contribution to the partition function. The required energy is provided by the expansion of the universe. We utilize the thermal properties of de Sitter space to describe the process as the decay of a metastable thermal state. Within the Euclidean path integral approach to gravity, we explicitly determine the spectra of the fluctuation operators, exactly calculate the one-loop fluctuation determinants in the ζ-function regularization scheme, and check the agreement with the expected scaling behaviour. Our results show a constant volume density of created black holes at late times, and a very strong suppression of the nucleation rate for small values of Λ.

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1 Introduction

Instantons play an important role in flat space gauge field theory [45]. Being stationary points of the Euclidean action, they give the dominant contribution to the Euclidean path integral thus accounting for a variety of important phenomena in QCD-type theories. In addition, self-dual instantons admit supersymmetric extensions, which makes them an important tool for verifying various duality conjectures like the AdS/CFT correspondence [4]. More generally, the Euclidean approach has become the standard method of quantum field theories in flat space.

Since the theory of gravity and Yang-Mills theory are somewhat similar, it is natural to study also gravitational instantons. An impressive amount of work has been done in this direction, leading to a number of important discoveries. A thorough study of instanton solutions of the vacuum Einstein equations and also those with a Λ-term has been carried out [20,17,24]. These solutions dominate the path integral of Euclidean quantum gravity, leading to interesting phenomena like black hole nucleation and quantum creation of universes. Perhaps one of the most spectacular achievements of the Euclidean approach is the derivation of black hole entropy from the action of the Schwarzschild instanton [22]. In addition, gravitational instantons are used in the Kaluza-Klein reductions of string theory.

Along with these very suggestive results, the difficulties of Euclidean quantum gravity have been revealed. Apart from the usual problem of the non-renormalizability of gravity, which can probably be resolved only at the level of a more fundamental theory like string theory, the Euclidean approach presents other challenging problems. In field theories in flat space the correlation functions of field operators are holomorphic functions of the global coordinates in a domain that includes negative imaginary values of the time coordinate, $t = -i\tau$, where τ is real and positive [48]. This allows one to perform the analysis in the Euclidean section and then analytically continue the functions back to the Lorentzian sector to obtain the physical predictions. In curved space the theorems that would ensure the analyticity of any quantities arising in quantum gravity are not known. As a result, even if Euclidean calculations make sense, it is not in general clear how to relate their result to the Lorentzian physics.

This difficulty is most strikingly illustrated by the famous problem of the conformal sector in Euclidean quantum gravity. If one tries evaluating the path integral over Riemannian metrics, then one discovers that it diverges because the Euclidean gravitational action is not bounded from below and can be made arbitrarily large and negative by conformal rescaling of the metric [25]. Such a result is actually expected, for if the integral did converge (with some regularization), then one could give a well-defined meaning to the canonical ensemble of the quantum gravitational field. However, the possibility of having a black hole causes the canonical ensemble to break down – since the degeneracy of black hole states grows faster than the Boltzmann factor decreases. One can, 'improve' the Euclidean gravitational action by analytically continuing the conformal modes, let us call them h, via $h \to ih$, and this improves the convergence of the integral [25]. This shows that if there is a well-defined Euclidean path integral for the gravitational field, then the relation to the Lorentzian sector is more complicated than just via $t \to -i\tau$.

Unfortunately, it is unknown at present whether one can in the general case find a physically well-defined and convergent path integral for the gravitational field. At the same time, the idea of constructing it is conceptually simple [46]: one should start from the Hamiltonian path integral over the physical degrees of freedom of the gravitational field. Such an integral certainly makes sense physically and is well-convergent, since the Hamiltonian is positive – at least in the asymptotically flat case. The Hamiltonian approach is not covariant, but one can covariantize it by changing the integration variables, which leads to a manifestly covariant and convergent path integral for gravity. The main problem with this program is that in the general case it is unclear how to isolate the physical degrees of freedom of the gravitational field. For this reason, so far the program has been carried out only for weak fields in the asymptotically flat case [46]. Remarkably, the result has been shown to exactly correspond to the the standard Euclidean path integral with the conformal modes being complex-rotated via $h \to ih$. This lends support to the Euclidean approach in gravity and allows one to hope that the difficulties of the method can be consistently resolved; (see, for example, [7,6] for the recent new developments within the lattice approach).

One can adopt the viewpoint that Euclidean quantum gravity is a meaningful theory within its range of applicability, at least at one-loop level, by assuming that a consistent resolution of its difficulties exists. Then already in its present status the theory can be used for calculating certain processes, most notably for describing tunneling phenomena, in which case the Euclidean amplitude directly determines the probability. The analytic continuation to the Lorentzian sector in this case is not necessary, apart from when the issue of the interpretation of the corresponding gravitational instanton is considered. The important example of a tunneling process in quantum gravity is the creation of black holes in external fields. Black holes are created whenever the energy pumped into the system is enough in order to make a pair of virtual black holes real [33]. The energy can be provided by the heat bath [30,38,5], by the background magnetic field [21,19,16,15], by the expansion of the universe [28,10,41], by cosmic strings [37], domain walls [11], etc; (see also [43,35,36]). Besides, one can consider pair creation of extended multidimensional objects like p-branes due to interaction with the background supergravity fields [14].

In all these examples the process is mediated by the corresponding gravitational instanton, and the semiclassical nucleation rate for a pair of objects on a given background is given by

$$\Gamma = A \exp\{-(I_{\rm obj} - I_{\rm bg})\} \,. \tag{1.1}$$

Here $I_{\rm obj}$ is the classical action of the gravitational instanton mediating creation of the objects, $I_{\rm bg}$ is the action of the background fields alone, and the prefactor A includes quantum corrections.

In most cases the existing calculations of black hole pair creation processes consider only the classical term in (1.1). This is easily understood, since loop calculations in quantum gravity for non-trivial backgrounds are extremely complicated. To our knowledge, there is only one example of a next-to-leading-order computation, which was undertaken in [30] by Gross, Perry, and Yaffe for the Schwarzschild instanton background. The aim of the present paper is to consider one more example of a complete one-loop computation in quantum gravity.

The problem we are interested in is the quantum creation of black holes in de Sitter space. This problem was considered by Ginsparg and Perry [28], who identified the instanton responsible for this process, which is the $S^2 \times S^2$ solution of the Euclidean Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for $\Lambda > 0$. Ginsparg and Perry noticed that this solution has one negative mode in the physical sector, which renders the partition function complex, thus indicating the quasi-classical instability of the system. This instability leads to spontaneous nucleation of black holes in the rapidly inflating universe. This is the dominant instability of de Sitter space, since classically the space is stable [28]. The energy necessary for the nucleation is provided by the Λ-term, which drives different parts of the universe apart thereby drugging the members of a virtual black hole pair away from each other. The typical radius of the created black holes is $1/\sqrt{\Lambda}$, while the the nucleation rate is of the order of $\exp(-\pi/\Lambda G)$, where G is Newton's constant. As a result, for $\Lambda G \sim 1$ when inflation is fast, the black holes are produced in abundance but they are small and presumably almost immediately evaporate. Large black holes emerge for $\Lambda G \ll 1$ when inflation slows down, and these can probably exist for a long time, but the probability of their creation is exponentially small. This scenario was further studied in Refs.[10,9,18] (see also references in [9]), where the generalization to the charged case was considered and also the subsequent evolution of the created black holes was analyzed. However, the one-loop contribution so far has not been computed.

A remarkable feature of the $S^2 \times S^2$ instanton is its high symmetry. In what follows, we shall utilize this symmetry in order to explicitly determine spectra of all relevant fluctuation operators in the problem. We shall use the ζ-function regularization scheme in order to compute the one-loop determinants, which will give us the partition function $Z[S^2 \times S^2]$ for the small fluctuations around the $S^2 \times S^2$ instanton. We shall then need to normalize this result. The normalization coefficients is $Z[S^4]$, the partition function for small fluctuations around the S^4 instanton, which is the Euclidean version of the de Sitter space. The one-loop quantization around the S^4 instanton was considered by Gibbons and Perry [27], and by Christensen and Duff [13], but unfortunately in none of these papers the analysis was completed. We shall therefore reconsider the problem by rederiving the spectra of fluctuations around S^4 and computing the determinants within the ζ-function scheme, thereby obtaining a closed one-loop expression for $Z[S^4]$.

In our treatment of the path integral we follow the approach of Gibbons and Perry [27]; (see also [42]). In order to have control over the results, we work in a one-parameter family of covariant gauges and perform the Hodge decomposition of the fluctuations. These are then expanded with respect to the complete sets of basis harmonics, and the perturbative path integration measure is defined as the square root of the determinant of the metric on the function space of fluctuations. To insure the convergence of the integral over the conformal modes, which enter the action with the wrong sign, we essentially follow the standard recipe $h \to ih$ [25]; (see also Ref.[42], where a slightly disguised form of the same prescription was advocated). The subtle issue is that the conformal operator $\tilde{\Delta}_0 = -3\nabla_\mu\nabla^\mu - 4\Lambda$ has a finite number, $\mathcal{N}$, of negative modes, and these enter the action with the correct sign from the very beginning. Our treatment of these special modes is different from that by Hawking [32], who suggests that such modes should be complex-rotated twice, the partition function then acquiring the overall factor of $i^{\mathcal{N}}$. However, the presence of this factor in the partition function would lead to unsatisfactory results, and on these grounds we are led to not rotating the special conformal modes at all.

The path integral is computed by integrating over the Fourier expansion coefficients, which leads to infinite products over the eigenvalues. The only conformal modes giving contribution to the result are the special negative modes discussed above. We carefully analyze the resulting products to make sure that all modes are taken into account and that the dependence of the gauge-fixing parameter cancels thereby indicating the correctness of the procedure. We give a detailed consideration to the zero modes of the Faddeev-Popov operator, which arise due to the background isometries. The integration over these modes requires a non-perturbative extension of the path-integration measure, and we find such a non-perturbative measure in the zero mode sector to be proportional to the Haar measure of the isometry group. Collecting all terms yields the partition function for small fluctuations around a background instanton configuration in terms of infinite products over eigenvalues of the gauge-invariant operators. We then use the explicitly known spectra of fluctuations around the $S^2 \times S^2$ and S^4 backgrounds in order to calculate the partition functions.

The rest of the paper is organized as follows. In Sec.2 we present our derivation of the black hole nucleation rate within the finite temperature approach. In Sec.3 the path integration procedure is considered. The spectra of small fluctuations around the $S^2 \times S^2$ instanton are computed in Sec.4 via a direct solving of the differential equations in the eigenvalue problems. The spectra of the fluctuations around the S^4 instanton are rederived in Sec.5 with the use of group theoretic arguments. The partition functions are computed in Sec.6, and Sec.7 contains the final expression for the black hole nucleation rate together with some remarks. We present a detailed analysis of the ζ-functions in the Appendix. We use units where $c = \hbar = k_{\mathrm{B}} = 1$.

2 Black hole nucleation rate

In this section we shall derive the basic formula for the black hole nucleation rate in de Sitter space, whose different parts will be evaluated in the next sections. The existing derivations of the nucleation rate [28,10] recover only the classical factor in (1.1). In addition, it is not always clear to which volume the rate refers. We argue that our formula (2.15) gives the nucleation probability per Hubble volume and unit time as measured by a freely falling observer. The basic idea of our approach is to utilize the relation between the inflation and thermal properties of de Sitter space. This will allow us to use the standard theory of decay of metastable thermal states [39,40,3].

Let us consider the partition function for the gravitational field

$$Z = \int D[g_{\mu\nu}]\, \mathrm{e}^{-I}\,, \tag{2.2}$$

where the integral is taken over Riemannian metrics, and $I = I[g_{\mu\nu}]$ is the Euclidean action for gravity with a positive Λ terms; see Eq.(3.1) below. The path integration procedure will be considered in detail in the next section. At present let us only recall that in the semiclassical approximation the integral is approximated by the sum over the classical extrema of the action I, that is

$$Z \approx \sum_l Z_l\,. \tag{2.3}$$

Here $Z_l = Z[\mathcal{M}_l]$ is the partition function for the small gravitational fluctuations around a background manifold $\mathcal{M}_l$ with a metric $g^l_{\mu\nu}$ subject to the

Euclidean Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Schematically one has

$$Z \approx \sum_l \frac{\exp(-I_l)}{\sqrt{\mathrm{Det}\Delta_l}}\,, \tag{2.4}$$

where I_l is the classical action for the l-th extremum, and Δ_l is the operator for the small fluctuations around this background.

The dominant contribution to the sum in (2.4) is given by the S^4 instanton, which is the four-dimensional sphere with the radius $\sqrt{3/\Lambda}$ and the standard metric. Since this is a maximally symmetry space, its action $I = -3\pi/\Lambda G$ is less than that of any other instanton. Hence,

$$Z \approx Z[S^4] = \frac{\exp(3\pi/\Lambda G)}{\sqrt{\mathrm{Det}\Delta}}\,. \tag{2.5}$$

On the other hand, the S^4 instanton describes the thermal properties of de Sitter space [22,23], since it can be obtained by an analytic continuation via $t \to \tau = it$ of the region of the de Sitter solution

$$ds^2 = -(1 - \frac{\Lambda}{3}r^2)\, dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{2.6}$$

contained inside the event horizon, $r < \sqrt{3/\Lambda}$. Let us call this region a Hubble region. Its boundary, the horizon, has the area $\mathcal{A} = 12\pi/\Lambda$. The temperature associated with this horizon is $T = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$, the entropy $S = \mathcal{A}/4G = 3\pi/\Lambda G$ and the free energy $F = -TS$. The same values can be obtained by writing the partition function for the S^4 instanton as

$$Z[S^4] = \mathrm{e}^{-\beta F} \tag{2.7}$$

with $\beta = 1/T$. Indeed, since S^4 is periodic in all four coordinates, any of them can be chosen to be the 'imaginary time'. The corresponding period, $\beta = 2\pi\sqrt{\frac{3}{\Lambda}}$, can be identified with the proper length of a geodesic on S^4, all of which are circles with the same length. This gives the correct de Sitter temperature. Comparing (2.7) and (2.5) one obtains $\beta F = -3\pi/\Lambda G + \ldots$, the dots denoting the quantum corrections, and this again agrees with the result for the de Sitter space. To recapitulate, the partition function of quantum gravity with $\Lambda > 0$ is approximately

$$Z \approx \mathrm{e}^{-\beta F}\,, \tag{2.8}$$

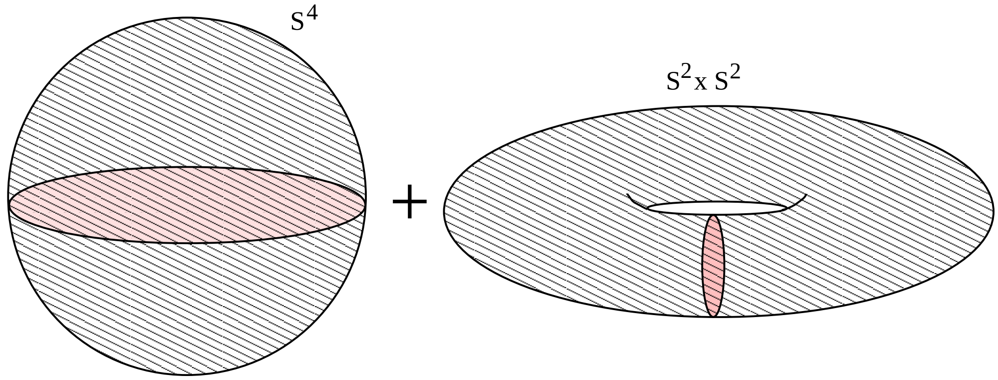

Fig. 1. *The leading contribution to the partition function comes from the S^4 and $S^2 \times S^2$ gravitational bubbles, the effect of the latter being purely imaginary.*

where $1/\beta$ is the de Sitter temperature and F is the free energy in the Hubble region.

Let us now consider the contribution of the other instantons. One has

$$Z \approx \mathrm{e}^{-\beta F}\left(1 + {\sum_l}' \frac{Z[\mathcal{M}_l]}{Z[S^4]}\right) , \tag{2.9}$$

where the prime indicates that $\mathcal{M}_l \neq S^4$. Now, for $\Lambda G \ll 1$ all terms in the sum are exponentially small and can safely be neglected as compared to the unity, if only they are real. If there are complex terms, then they will give an exponentially small imaginary contribution. The $S^2 \times S^2$ instanton is distinguished by the fact that its partition function is purely imaginary due to the negative mode in the physical sector [28]. This is the only solution for $\Lambda > 0$ which is not a local minimum of the action in the class of metrics with constant scalar curvature [20]. Hence (see Fig.1),

$$Z \approx \mathrm{e}^{-\beta F}\left(1 + \frac{Z[S^2 \times S^2]}{Z[S^4]}\right) \approx \exp\left(-\beta\left(F - \frac{Z[S^2 \times S^2]}{\beta Z[S^4]}\right)\right) , \tag{2.10}$$

where $Z[S^2 \times S^2]$ is purely imaginary. As a result, the partition function can still be represented as $Z \approx \mathrm{e}^{-\beta F}$, where the real part of F is the free energy of the Hubble region, and the exponentially small imaginary part is given by

$$\Im(F) = -\frac{Z[S^2 \times S^2]}{\beta Z[S^4]} . \tag{2.11}$$

It is natural to relate this imaginary quantity also to the free energy. We are therefore led to the conclusion that the free energy of the Hubble region has a small imaginary part, thus indicating that the system is metastable. The decay of this metastable state will lead to a spontaneous nucleation of a black hole in the Hubble region, which can be inferred from the geometrical properties of the $S^2 \times S^2$ instanton.

The $S^2 \times S^2$ instanton can be obtained via the analytic continuation of the Schwarzschild-de Sitter solution [26,28,10]

$$ds^2 = -N\,dt^2 + \frac{dr^2}{N} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,. \tag{2.12}$$

Here $N = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$, and for $9M^2\Lambda < 1$ this function has roots at $r = r_+ > 0$ (black hole horizon) and at $r = r_{++} > r_+$ (cosmological horizon). One has $N > 0$ for $r_+ < r < r_{++}$, and only this portion of the solution can be analytically continued to the Euclidean sector via $t \to \tau = it$. The conical singularity at either of the horizons can be removed by a suitable identification of the imaginary time. However, since the two horizons have different surface gravities, the second conical singularity will survive. The situation improves in the extreme limit, $r_+ \to r_{++} \to \frac{1}{\sqrt{\Lambda}}$, since the surface gravities are then the same and both conical singularities can be removed at the same time. Although one might think that the Euclidean region shrinks to zero when the two horizons merge, this is not so. The limit $r_+ \to r_{++}$ implies that $9M^2\Lambda = 1 - 3\epsilon^2$ with $\epsilon \to 0$. One can introduce new coordinates ϑ_1 and φ_1 via $\cos\vartheta_1 = (\sqrt{\Lambda}r - 1)/\epsilon + \epsilon/6$ and $\varphi_1 = \sqrt{\Lambda}\,\epsilon\,\tau$. Passing to the new coordinates and taking the limit $\epsilon \to 0$, the result is

$$ds^2 = \frac{1}{\Lambda}\Big(d{\vartheta_1}^2 + \sin^2\vartheta_1\, d{\varphi_1}^2 + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\Big)\ , \tag{2.13}$$

and this $S^2 \times S^2$ metric fulfills the Einstein equations. Since the instanton field determines the initial value for the created real time configuration, one concludes that the $S^2 \times S^2$ instanton is responsible for the creation of a black hole in the Hubble region. This black hole fills the whole region, since its size is equal to the radius of the cosmological horizon.

It is well known that the region $r < \sqrt{3/\Lambda}$ of the static coordinate system in (2.6) covers only a small portion of the de Sitter hyperboloid [47]; (see Fig.2). In order to cover the whole space, one can introduce an infinite number of freely falling observers and associate the interior of the static coordinate system with each of them. Hence, the spacetime contains infinitely many Hubble regions. It is also instructive to use global coordinates covering the whole de Sitter space,

$$ds^2 = \frac{3}{\Lambda\cos^2\xi}\Big(- d\xi^2 + d\chi^2 + \sin^2\chi\,(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\Big)\ , \tag{2.14}$$

where $\xi \in [-\pi/2, \pi/2]$ and $\chi \in [0, \pi]$. The trajectory of a freely falling observer

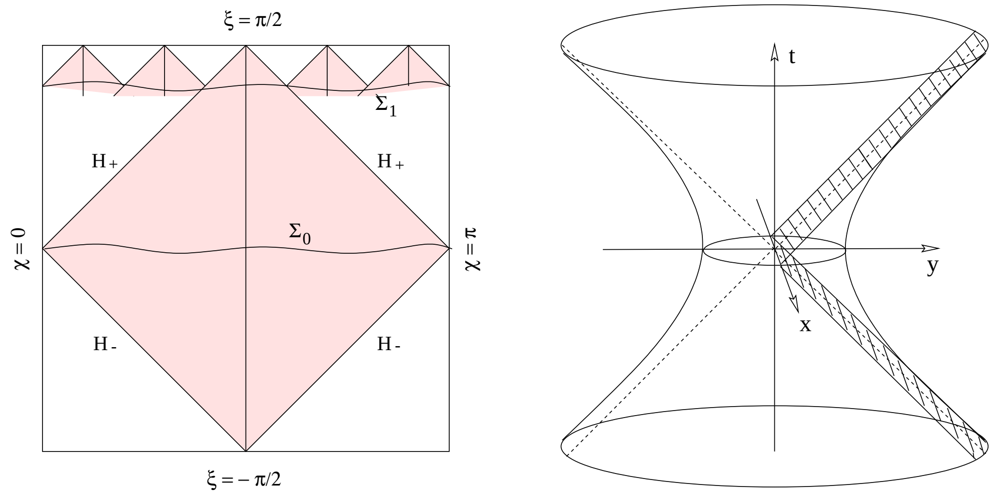

Fig. 2. Left: *The conformal diagram of de Sitter space in coordinates (2.14). The trajectories χ=const. are timelike geodesics. The diamond-shaped region in the center is the Hubble region of the geodesic observer at $\chi = \pi/2$. Although this region completely covers the hypersurface Σ_0, at later times one needs more observers to cover the hypersurface Σ_1 with the interiors of their horizons – the Hubble regions proliferate.* Right: *The de Sitter hyperboloid in the embedding Minkowski space (with two dimensions suppressed). The Hubble region of the inertial observer moving along the hyperbola $x = 0$, $y > 0$ is the portion of the hyperboloid lying to the right from the two shaded strips. This corresponds to the interior region of the observer's static coordinate system.*

is $\chi = \chi_0$ (and also $\vartheta = \vartheta_0$, $\varphi = \varphi_0$), and the domain of the associated static coordinate system, the Hubble region, is the intersection of the interiors of the observer's past and future horizons [34]. Let Σ be a spacelike hypersurface, say $\xi = \xi_0$. If $\xi_0 = 0$ then Σ is completely contained inside the Hubble region of a single observer with $\chi = \pi/2$ (see Fig.2). However, for late moments of time, $\xi \to \pi/2$, one needs more and more independent observers in order to completely cover Σ by the union of their Hubble regions. One can say that the Hubble regions proliferate with the expansion of the universe.

Since de Sitter space consists of infinitely many Hubble regions, the black hole nucleation will lead to some of the regions being completely filled by a black hole, but most of the regions will be empty. The number of the filled regions divided by the number of those without a black hole is the probability for a black hole nucleation in one region. This is proportional to $\Im(F)$ in (2.11).

One can argue that the black holes are actually created in pairs [33,36], where the two members of the pair are located at the antipodal points of the de Sitter hyperboloid. This can be inferred from the conformal diagram of the Schwarzschild-de Sitter solution, which contains an infinite sequence of black hole singularities and spacelike infinities; see Fig.3. One can identify the asymptotically de Sitter regions in the diagram related by a horizontal shift, and the spacetime will then consist of two black holes at antipodal points of the closed universe. This agrees with the standard picture of particles in external fields being created in pairs.

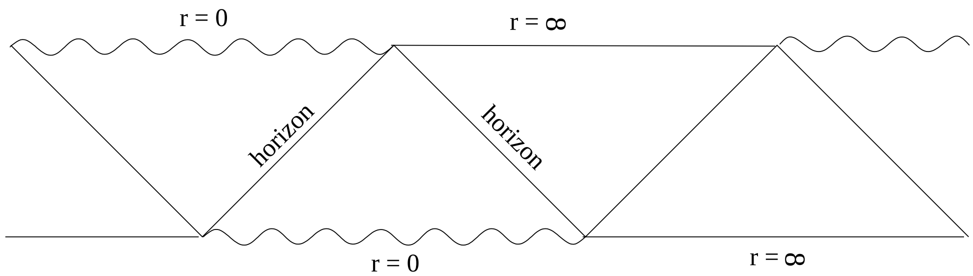

Fig. 3. *The conformal digram for the extreme Schwarzschild-de Sitter solution.*

The surface gravity of the extreme Schwarzschild-de Sitter solution is finite when defined with respect to the suitably normalized Killing vector [10]. This gives a non-zero value for the temperature of the nucleated black holes, which can be read off also from the $S^2 \times S^2$ metric: it is the inverse proper length of the equator of any of the two spheres, $T_{\rm BH} = \frac{\sqrt{\Lambda}}{2\pi}$. How can it be that this is different from the temperature of the heat bath, which is the de Sitter space with $T_{\rm dS} = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$? For example, in the hot Minkowski space the nucleated black holes have the same temperature as the heat bath [30]. However, the global structure of de Sitter space is different from that of Minkowski space. The fluctuations cannot *absorb* energy from *and emit* energy into the whole of de Sitter space, but can only exchange energy with the Hubble region. Thus the energy exchange is restricted. As a result, the local temperature in the vicinity of a created defect may be different from that of the heat bath, but reduces to the latter in the asymptotic region far beyond the cosmological horizon.

The relation of the imaginary part of the free energy to the rate of decay of a metastable thermal state Γ was considered in [39,40,3]. If the decay is only due to tunneling then $\Gamma = 2\Im(F)$. Suppose that there is an additional possibility to classically jump over the potential barrier. In this case on top of the barrier there is a classical saddle point configuration whose real time decay rate is determined by the saddle negative mode ω_-. At low temperatures the tunneling formula is then still correct, while for $T > \frac{|\omega_-|}{2\pi}$ one has $\Gamma = \frac{|\omega_-|}{\pi T}\Im(F)$. In our problem the saddle point configuration also exists, the $S^2 \times S^2$ instanton, but its real time analog, the Schwarzschild-de Sitter black hole, is stable. It seems therefore that there is no classical contribution to the process and the black hole nucleation is a purely quantum phenomenon. [2] [One can imagine that the effective potential barrier is infinitely high, such that a classical transition is

[2] We do not understand the classical interpretation of the Euclidean saddle point solution suggested in [30]. The argument uses a family of non-normalizable deformations of the instanton, and the action is finite as long as they are 'static'.

forbidden, but at the same time so narrow that the tunneling rate is non-zero.] As a result, the rate of quasiclassical decay of the de Sitter space is given by $\Gamma = 2\Im(F)$. Using Eq.(2.11),

$$\Gamma = -2T\,\frac{Z[S^2\times S^2]}{Z[S^4]}\,. \tag{2.15}$$

Here $T = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$ is the temperature of the de Sitter heat bath, which was originally defined with respect to the analytically continued Killing vector $\frac{\partial}{\partial t}$. Since t is the proper time of the geodesic observer resting at the origin of the static coordinate system (2.6), we conclude that the formula gives the probability of a black hole nucleation per Hubble volume and unit time of a freely falling observer.

In order to use the formula (2.15), we should be able to compute the one-loop partition functions $Z[S^2\times S^2]$ and $Z[S^4]$. Now we shall calculate them within the path integral approach.

3 The path integration procedure

In this section we shall consider the path integral for fluctuations around an instanton solution of the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in the stationary phase approximation. We shall largely follow the approach of Gibbons and Perry [27].

3.1 *The second variation of the action*

Our starting point is the action for the gravitational field on a compact Riemannian manifold $\mathcal{M}$,

$$I[g_{\mu\nu}] = -\frac{1}{16\pi G}\int\limits_{\mathcal{M}}(R-2\Lambda)\sqrt{g}\,d^4x\,, \tag{3.1}$$

whose extrema, $\delta I = 0$, are determined by the equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu}\,. \tag{3.2}$$

However, if one considers a time evolution along such a family then the action will be infinite, which shows that the classical picture does not apply. Even if one uses the classical formula for Γ in this case, one arrives at the quantum result, since $|\omega_-|/T$=const.~ 1.

Let $g_{\mu\nu}$ be an arbitrary solution, and consider small fluctuations around it, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$. The action expands as

$$I[g_{\mu\nu} + h_{\mu\nu}] = I[g_{\mu\nu}] + \delta^2 I + \dots \,, \tag{3.3}$$

where $\delta^2 I$ is quadratic in $h_{\mu\nu}$ and dots denote the higher order terms. One can express $\delta^2 I$ directly in terms if $h_{\mu\nu}$. However, it is convenient to use first the standard decomposition of $h_{\mu\nu}$,

$$h_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{4} h \, g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \xi^\sigma \,. \tag{3.4}$$

Here $\phi_{\mu\nu}$ is the transverse tracefree part, $\nabla_\mu \phi^\mu_\nu = \phi^\mu_\mu = 0$, h is the trace, and the piece due to ξ_μ is the longitudinal tracefree part. Under the gauge transformations (general diffeomorphisms) generated by ξ_μ one has $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. The TT-tensor $\phi_{\mu\nu}$ is gauge-invariant, while the trace h changes as $h \to h + 2\nabla_\sigma \xi^\sigma$. It follows that

$$\tilde{h} = h - 2\nabla_\sigma \xi^\sigma \tag{3.5}$$

is gauge-invariant. For further references we note that ξ_μ can in turn be decomposed into its coexact part η_μ, for which $\nabla_\mu \eta^\mu = 0$, the exact part $\nabla_\mu \chi$, and the harmonic piece ξ^{H}_μ,

$$\xi_\mu = \eta_\mu + \nabla_\mu \chi + \xi^{\mathrm{H}}_\mu \,. \tag{3.6}$$

The number of square-integrable harmonic vectors is a topological invariant, which is equal to the first Betti number of the manifold $\mathcal{M}$. Since the latter is zero if $\mathcal{M}$ is simply-connected, which is the case for $\Lambda > 0$, we may safely ignore the harmonic contribution in what follows.

With the decomposition (3.4) the second variation of the action in (3.3) is expressed in terms of the gauge-invariant quantities $\phi_{\mu\nu}$ and $\tilde{h}$ alone,

$$\delta^2 I = \frac{1}{2} \langle \phi^{\mu\nu}, \Delta_2 \phi_{\mu\nu} \rangle - \frac{1}{16} \langle \tilde{h}, \tilde{\Delta}_0 \tilde{h} \rangle \,. \tag{3.7}$$

Here and below we consider the following second order differential operators: the operator for the TT-tensor fluctuations

$$\Delta_2 \phi_{\mu\nu} = -\nabla_\sigma \nabla^\sigma \phi_{\mu\nu} - 2R_{\mu\alpha\nu\beta}\phi^{\alpha\beta} \,, \tag{3.8}$$

the vector operator acting on coexact vectors η_μ

$$\Delta_1 = -\nabla_\sigma \nabla^\sigma - \Lambda\,, \tag{3.9}$$

and the scalar operators for h, $\tilde{h}$, and χ

$$\begin{aligned}\Delta_0 &= -\nabla_\sigma \nabla^\sigma\,,\\ \tilde{\Delta}_0 &= 3\Delta_0 - 4\Lambda\,,\\ \tilde{\Delta}_0^\gamma &= \gamma\tilde{\Delta}_0 - \Delta_0\,,\end{aligned} \tag{3.10}$$

with γ being a real parameter. Since for $\Lambda > 0$ the manifold $\mathcal{M}$ is compact, these operators are (formally) self-adjoint with respect to the scalar product

$$\langle \phi_{\mu\nu}, \phi^{\mu\nu} \rangle = \frac{1}{32\pi G} \int\limits_{\mathcal{M}} \phi_{\mu\nu}\phi^{\mu\nu} \sqrt{g}\, d^4x\,; \tag{3.11}$$

similarly for vectors $\langle \eta_\mu, \eta^\mu \rangle$ and scalars $\langle \chi, \chi \rangle$.

The action $\delta^2 I$ in (3.7) contains only the gauge-invariant amplitudes $\phi_{\mu\nu}$ and $\tilde{h}$, while the dependence on the gauge degrees of freedom ξ_μ cancels. Pure gauge modes are thus zero modes of the action. Fixing of the gauge is therefore necessary in order to carry out the path integration. To fix the gauge we pass from the action $\delta^2 I$ to the gauge-fixed action

$$\delta^2 I_{gf} = \delta^2 I + \delta^2 I_g\,, \tag{3.12}$$

where, following [27], we choose the gauge-fixing terms as

$$\delta^2 I_g = \gamma \left\langle \nabla_\sigma h^\sigma_\rho - \frac{1}{\beta}\,\nabla_\rho h, \nabla^\alpha h^\rho_\alpha - \frac{1}{\beta}\,\nabla^\rho h \right\rangle, \tag{3.13}$$

with γ and β being real parameters. We shall shortly see that it is convenient to choose [27]

$$\beta = \frac{4\gamma}{\gamma + 1}\,. \tag{3.14}$$

This choice, however, implies that $\delta^2 I_g$ does not vanish for $\gamma \to 0$. It is often convenient to set $\gamma = 1$, in which case $\beta = 2$. However, we shall not fix the value of γ, since this will provide us with a check of the gauge-invariance of our results.

Using the decompositions (3.4), (3.6) the gauge-fixing term reads

$$\delta^2 I_g = \gamma\langle\eta_\mu, \Delta_1^2\eta^\mu\rangle + \frac{1}{16\gamma}\langle(\tilde{h} + 2\tilde{\Delta}_0^\gamma\chi), \Delta_0(\tilde{h} + 2\tilde{\Delta}_0^\gamma\chi)\rangle\,. \tag{3.15}$$

Adding this up with $\delta^2 I$ in (3.7) one obtains the gauge-fixed action $\delta^2 I_{gf}$. It is now convenient to pass from the gauge-invariant variable $\tilde{h}$ defined in (3.5) back to the trace h, since with the choice in (3.14) the resulting action then becomes diagonal:

$$\begin{aligned}\delta^2 I_{gf} = &\frac{1}{2}\,\langle\phi^{\mu\nu}, \Delta_2\phi_{\mu\nu}\rangle + \gamma\langle\eta_\mu, \Delta_1^2\eta^\mu\rangle \\ &+ \frac{1}{4}\,\langle\chi, \Delta_0\tilde{\Delta}_0\tilde{\Delta}_0^\gamma\chi\rangle - \frac{1}{16\gamma}\,\langle h, \tilde{\Delta}_0 h\rangle\,.\end{aligned} \tag{3.16}$$

This action generically has no zero modes, but it depends on the arbitrary parameter γ, which reflects the freedom of choice of gauge-fixing. In order to cancel this dependency, the compensating ghost term is needed.

3.2 The mode decomposition of the action

We wish to calculate the path integral

$$Z[g_{\mu\nu}] = \mathrm{e}^{-I}\int D[h_{\mu\nu}]\,\mathcal{D}_{\mathrm{FP}}\exp\left(-\delta^2 I_{gf}\right), \tag{3.17}$$

where $I = I[g_{\mu\nu}]$ is the classical action, and the Faddeev-Popov factor is obtained from

$$1 = \mathcal{D}_{\mathrm{FP}}\int D[\xi_\mu]\,\exp\left(-\delta^2 I_g\right). \tag{3.18}$$

In order to perform the path integration, we introduce the eigenmodes associated with the operators Δ_2, Δ_1 and Δ_0:

$$\begin{aligned}\Delta_2\,\phi_{\mu\nu}^{(k)} &= \varepsilon_k\,\phi_{\mu\nu}^{(k)}, \\ \Delta_1\,\eta_\mu^{(s)} &= \sigma_s\,\eta_\mu^{(s)}, \\ \Delta_0\,\alpha^{(p)} &= \lambda_p\,\alpha^{(p)}.\end{aligned} \tag{3.19}$$

Throughout this paper we shall denote the eigenvalues and eigenfunctions of the tensor operator Δ_2 by ε_k and $\phi_{\mu\nu}^{(k)}$, and those for the vector operator Δ_1 by σ_s and $\eta_\mu^{(s)}$, respectively. [Later we shall use the symbol s also for the argument of the ζ-functions, and this will not lead to any confusion]. Eigenvalues of the scalar operator will be denoted by λ_p, and it will be convenient to split the set $\{\lambda_p\}$ into three subsets, $\{\lambda_p\} = \{\lambda_0, \lambda_i, \lambda_n\}$, where $\lambda_0 = 0$, $\lambda_i = \frac{4}{3}\Lambda$, and $\lambda_n > \frac{4}{3}\Lambda$; see Eqs.(3.25)–(3.27) below. Accordingly, the set of the scalar eigenfunctions will be split as $\{\alpha^{(p)}\} = \{\alpha^{(0)}, \alpha^{(i)}, \alpha^{(n)}\}$.

Since the manifold is compact, we choose the modes to be orthonormal. This allows us to expand all fields in the problem as

$$\phi_{\mu\nu} = \sum_k C_k^\phi \phi_{\mu\nu}^{(k)}\,, \quad \eta_\mu = \sum_s C_s^\eta \eta_\mu^{(s)}\,, \tag{3.20}$$

and

$$\chi = \sum_p C_p^\chi \alpha^{(p)}\,, \quad h = \sum_p C_p^h \alpha^{(p)}\,, \quad \tilde{h} = \sum_p C_p^{\tilde{h}} \alpha^{(p)}\,. \tag{3.21}$$

As a result, the action decomposes into the sum over modes, and the path integral reduces to integrals over the Fourier coefficients.

a) Vector and tensor modes.– Let us consider the mode decomposition for the gauge-fixed action in (3.16). This action is the sum of four terms. For the first two terms we obtain

$$\frac{1}{2}\langle\phi^{\mu\nu}, \Delta_2\phi_{\mu\nu}\rangle = \frac{1}{2}\sum_k \varepsilon_k\,(C_k^\phi)^2\,, \tag{3.22}$$

$$\gamma\,\langle\eta_\mu, \Delta_1^2\eta^\mu\rangle = \gamma\,\sum_s (\sigma_s)^2 (C_s^\eta)^2\,. \tag{3.23}$$

These quadratic forms should be positive definite, since otherwise the integrals over the coefficients C would be ill-defined. We can see that the quadratic form in (3.23) for the vector modes is indeed non-negative definite. Next, the expression in (3.22) for the gauge-invariant tensor modes is positive-definite if all eigenvalues ε_k are positive. If there is a negative eigenvalue, $\varepsilon_- < 0$, as in the case of the $S^2 \times S^2$ instanton background, then it is physically significant. The integration over C_-^ϕ is performed with the complex contour rotation, which renders the partition function imaginary thus indicating the quasiclassical instability of the system.

Let us consider now the contribution of the longitudinal vector piece to the action (3.16). We obtain

$$\frac{1}{4}\langle\chi, \Delta_0\tilde{\Delta}_0\tilde{\Delta}_0^\gamma\chi\rangle = \frac{1}{4}\sum_p \lambda_p\tilde{\lambda}_p\tilde{\lambda}_p^\gamma\,(C_p^\chi)^2\,, \tag{3.24}$$

where $\tilde{\lambda}_p = 3\lambda_p - 4\Lambda$ and $\tilde{\lambda}_p^\gamma = \gamma\tilde{\lambda}_p - \lambda_p$ are the eigenvalues of $\tilde{\Delta}_0$ and $\tilde{\Delta}_0^\gamma$, respectively. We note that while $\lambda_p \geq 0$, the $\tilde{\lambda}_p$ and $\tilde{\lambda}_p^\gamma$ can be negative and should therefore be treated carefully. Let us split the scalar modes into three groups according to the sign of $\tilde{\lambda}_p$:

$$\Delta_0\,\alpha^{(0)} = 0, \qquad \Rightarrow \quad \tilde{\lambda}_0 = -\frac{4}{3}\,\Lambda\,, \tag{3.25}$$

$$\Delta_0\,\alpha^{(i)} = \frac{4}{3}\,\Lambda\,\alpha^{(i)}, \quad \Rightarrow \quad \tilde{\lambda}_i = 0\,, \tag{3.26}$$

$$\Delta_0\,\alpha^{(n)} = \lambda_n\,\alpha^{(n)}, \quad \Rightarrow \quad \tilde{\lambda}_n > 0\,. \tag{3.27}$$

First we consider the constant mode $\alpha^{(0)}$ in (3.25). This exists for any background, and for compact manifolds without boundary this is the only normalizable zero mode of Δ_0. Since this mode is annihilated by Δ_0, it does not contribute to the sum in (3.24).

Consider now the scalar modes with the eigenvalue $4\Lambda/3$ in (3.26). In view of the Lichnerowicz-Obata theorem [49], the lowest non-trivial eigenvalue of Δ_0 for $\Lambda > 0$ is bounded from below by $4\Lambda/3$, and the equality is attained if only the background is S^4. Hence the modes in (3.26) exist only for the S^4 instanton, and there can be no modes 'in between' (3.25) and (3.26). In the S^4 case there are five scalar modes with the eigenvalue $4\Lambda/3$, and their gradients are the five conformal Killing vectors that do not correspond to infinitesimal isometries. If $R_{\mu\nu} = \Lambda g_{\mu\nu}$, a theorem of Yano an Nagano [49] states that such vectors exist only in the S^4 case. Let us call these five scalar modes 'conformal Killing modes'. Notice that these also do not contribute to the sum in (3.24).

To recapitulate, the lowest lying modes in the scalar spectrum are the constant conformal mode in (3.25), which exists for any background, and also 5 'conformal Killing modes' in (3.26) which exist only for the S^4 instanton and generate the conformal isometries. As we shall see, these 1+5 lowest lying modes are physically distinguished, since they are the only scalar modes contributing to the partition function. However, they do not enter the sum in (3.24).

For the remaining infinite number of scalar modes in (3.27) (these are labeled by n) the eigenvalues λ_n and $\tilde{\lambda}_n$ are positive, and it is not difficult to see that all the $\tilde{\lambda}_n^\gamma$'s are also positive, provided that the gauge parameter γ is positive and large enough. To recapitulate, the contribution of the longitudinal vector modes to the action is given by

$$\frac{1}{4}\,\langle\chi, \Delta_0\tilde{\Delta}_0\tilde{\Delta}_0^\gamma\chi\rangle = \frac{1}{4}\,\sum_n \lambda_n\tilde{\lambda}_n\tilde{\lambda}_n^\gamma\,(C_n^\chi)^2\,, \tag{3.28}$$

which is positive definite. We shall see that all modes contributing to this sum are unphysical and cancel from the path integral.

b) Conformal modes.– We now turn to the last term in the gauge-fixed action (3.16). Using (3.25)–(3.27) we obtain

$$-\frac{1}{16\gamma}\,\langle h,\tilde{\Delta}_0 h\rangle = \frac{\Lambda}{4}\,(C^h_0)^2 + \frac{\Lambda}{12\gamma}\,\sum_i (C^h_i)^2 - \frac{1}{16\gamma}\,\sum_n \tilde{\lambda}^\gamma_n\,(C^h_n)^2\,. \tag{3.29}$$

The expression on the right has a finite number of positive terms, corresponding to the distinguished lowest lying modes, and infinitely many negative ones. As a result, an increase in the coefficients C^h_n makes it arbitrarily large and negative, thus rendering the path integral divergent. This represents the well-known problem of conformal modes in Euclidean quantum gravity [25]. A complete solution of this problem is lacking at present, but the origin of the trouble seems to be understood [46]. In brief, the problem is not related to any defects of the theory itself, but arises as a result of the bad choice of the path integral. If one starts from the fundamental Hamiltonian path integral over the physical degrees of freedom of the gravitational field, then one does not encounter this problem. The Hamiltonian path integral, however, is non-covariant and difficult to work with. One can 'covariantize' it by adding gauge degrees of freedom, and this leads to the Euclidean path integral described above, up to the important replacement [25]

$$h \to ih\,. \tag{3.30}$$

The effect of this is to change the overall sign in (3.29), such that the infinite number of negative modes become positive. Unfortunately, such a consistent derivation of the path integral has only been carried out for weak gravitational fields [46] (and for $\Lambda = 0$), since otherwise it is unclear how to choose the physical degrees of freedom. Nevertheless, the rule (3.30) is often used also in the general case [25], and it leads to the cancellation of the unphysical conformal modes. However, some subtle issues can arise.

For $\Lambda > 0$ the expression in (3.29) contains, apart from infinitely many negative terms, also a finite number of positive ones, which are due to the distinguished lowest lying scalar modes. If we apply the rule (3.30) and change the overall sign of the scalar mode action, then the negative modes will become positive, but the positive ones will become negative. As a result, the path integral will still be divergent. It was therefore suggested by Hawking that the contour for these extra negative modes should be rotated back, the partition function then acquiring the factor $i^{\mathcal{N}}$, where $\mathcal{N}$ is the number of such modes [32]. As we know, $\mathcal{N} = 6$ for the S^4 instanton, and $\mathcal{N} = 1$ for any other solution of $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda > 0$.

Unfortunately, this prescription to rotate the contour twice leads in some cases to physically meaningless results; the examples will be given in a moment. We suggest therefore a slightly different scheme: not to touch the positive modes in (3.29) at all and to change the sign only for the negative modes. The whole expression then becomes

$$-\frac{1}{16\gamma}\langle h,\tilde{\Delta}_0 h\rangle = \frac{\Lambda}{4}\,(C_0^h)^2 + \frac{\Lambda}{12\gamma}\sum_i (C_i^h)^2 + \frac{1}{16\gamma}\sum_n \tilde{\lambda}_n^\gamma\,(C_n^h)^2\,. \tag{3.31}$$

We make no attempt to rigorously justify such a rule. We note, however, that it is essentially equivalent to the standard recipe (3.30) – up to a finite number of modes which we handle differently as compared to Hawking's prescription. We shall now comment on this difference.

When compared to Hawking's recipe [32], the ultimate effect of our prescription is to remove the factor $i^{\mathcal{N}}$ from the partition function. We are unaware of any examples where it would be necessary to insist on this factor being present in the final result. On the contrary, the examples are in favour of the factor being absent. For the S^4 instanton one has $\mathcal{N}=6$, such that $i^{\mathcal{N}}=-1$, and this would render the partition function for hot gravitons in a de Sitter universe negative, which would be physically meaningless. Next, for the $S^2\times S^2$ instanton, which already has one negative mode in the spin-2 sector, one has $\mathcal{N}=1$. As a result, the factor $i^{\mathcal{N}}$ would make the partition function real instead of being imaginary, and there would be no black hole pair creation !

These arguments suggest that Hawking's rule should be somehow modified, and we therefore put forward the prescription (3.31). Let us also note that our rule leads to gauge invariant results – the dependence on the gauge parameter γ cancels after the integration. Finally we note that the lowest lying scalar modes are physically distinguished, and since they are positive, they should be treated similarly to the physical tensor modes.

To recapitulate, the mode expansion of the gauge-fixed action $\delta^2 I_{gf}$ is given by the sum of (3.22), (3.23), (3.28), and (3.31):

$$\begin{aligned}\delta^2 I_{gf} = \frac{1}{2}\sum_k \varepsilon_k\,(C_k^\phi)^2 + \gamma\sum_s (\sigma_s)^2 (C_s^\eta)^2 + \frac{1}{4}\sum_n \lambda_n\tilde{\lambda}_n\tilde{\lambda}_n^\gamma\,(C_n^\chi)^2 \\ +\frac{\Lambda}{4}\,(C_0^h)^2 + \frac{\Lambda}{12\gamma}\sum_i (C_i^h)^2 + \frac{1}{16\gamma}\sum_n \tilde{\lambda}_n^\gamma\,(C_n^h)^2\,.\end{aligned} \tag{3.32}$$

In a similar way we obtain the following mode expansion for the gauge-fixing term $\delta^2 I_g$ in (3.15):

$$
\begin{aligned}
\delta^2 I_g = \gamma \sum_s (\sigma_s)^2 (C_s^\eta)^2 + \frac{16}{27\gamma}\Lambda^3 \sum_i \left(C_i^\chi - \frac{3}{8\Lambda} C_i^{\tilde{h}}\right)^2 \\
+ \frac{1}{16\gamma} \sum_n \lambda_n \left(2\tilde{\lambda}_n^\gamma C_n^\chi + C_n^{\tilde{h}}\right)^2 .
\end{aligned} \tag{3.33}
$$

This expression is non-negative definite.

3.3 The path integration measure

In order to compute the path integrals in (3.17),(3.18) we still need to define the path-integration measure. The perturbative measure is defined as the square root of the determinant of the metric on the function space of fluctuations:

$$
D[h_{\mu\nu}] \sim \sqrt{\mathrm{Det}(\langle dh_{\mu\nu}, dh^{\mu\nu}\rangle)} \,, \qquad D[\xi_\mu] \sim \sqrt{\mathrm{Det}(\langle d\xi_\mu, d\xi^\mu\rangle)} \,. \tag{3.34}
$$

Here it is assumed that the fluctuations are Fourier-expanded and the differentials refer to the Fourier coefficients, while the meaning of the proportionality sign will become clear shortly. Let us first consider $D[\xi_\mu]$. It follows from (3.4),(3.6) that

$$
\begin{aligned}
\langle h_{\mu\nu}, h^{\mu\nu}\rangle = \langle \phi_{\mu\nu}, \phi^{\mu\nu}\rangle + 2\langle \eta_\mu, \Delta_1 \eta^\mu\rangle + \langle \chi, \Delta_0 \tilde{\Delta}_0 \chi\rangle + \frac{1}{4}\langle h, h\rangle \,, \\
\langle \xi_\mu, \xi^\mu\rangle = \langle \eta_\mu, \eta^\mu\rangle + \langle \chi, \Delta_0 \chi\rangle \,.
\end{aligned} \tag{3.35}
$$

Expanding the fields on the right according to (3.20),(3.21) and differentiating with respect to the Fourier coefficients we obtain the metric for the vector fluctuations

$$
\langle d\xi_\mu, d\xi^\mu\rangle = \langle d\eta_\mu, d\eta^\mu\rangle + \langle d\chi, \Delta_0 d\chi\rangle = \sum_s (dC_s^\eta)^2 + {\sum_p}' \lambda_p \,(dC_p^\chi)^2 \,, \tag{3.36}
$$

which yields

$$
\sqrt{\mathrm{Det}(\langle d\xi_\mu, d\xi^\mu\rangle)} = \left(\prod_s dC_s^\eta\right)\left({\prod_p}' \sqrt{\lambda_p}\, dC_p^\chi\right) . \tag{3.37}
$$

Here the prime indicates that terms with $\lambda_p = 0$ do not contribute to the sum in (3.36), and should therefore be omitted in the product in (3.37). To obtain the measure $D[\xi_\mu]$ we endow each term in the products in (3.37) with the weight factor $\mu_o^2/\sqrt{\pi}$:

$$D[\xi_\mu] = \left(\prod_s \frac{\mu_o^2}{\sqrt{\pi}}\, dC_s^\eta\right)\left(\prod_i \frac{\mu_o^2}{\sqrt{\pi}}\sqrt{\frac{4\Lambda}{3}}\, dC_i^\chi\right)\left(\prod_n \frac{\mu_o^2}{\sqrt{\pi}}\sqrt{\lambda_n}\, dC_n^\chi\right). \quad (3.38)$$

Such a normalization implies that

$$1 = \int D[\xi_\mu] \exp\left(-\mu_o^4 \left\langle \xi_\mu, \xi^\mu\right\rangle\right) . \quad (3.39)$$

Here μ_o is a parameter with the dimension of an inverse length. In a similar way we obtain the measure $D[h_{\mu\nu}]$, which is normalized as

$$1 = \int D[h_{\mu\nu}] \exp\left(-\frac{\mu_o^2}{2}\left\langle h_{\mu\nu}, h^{\mu\nu}\right\rangle\right) ; \quad (3.40)$$

we shall shortly comment on the relative normalization of $D[h_{\mu\nu}]$ and $D[\xi_\mu]$. The result is

$$\begin{aligned} D[h_{\mu\nu}] = & \left(\prod_k \frac{\mu_o}{\sqrt{2\pi}}\, dC_k^\phi\right)\left(\prod_s{}' \frac{\mu_o}{\sqrt{2\pi}}\sqrt{2\sigma_s}\, dC_s^\eta\right)\left(\prod_n \frac{\mu_o}{\sqrt{2\pi}}\sqrt{\lambda_n\tilde{\lambda}_n}\, dC_n^\chi\right) \\ & \times \left(\frac{\mu_o}{\sqrt{2\pi}}\frac{1}{2}\, dC_0^h\right)\left(\prod_i \frac{\mu_o}{\sqrt{2\pi}}\frac{1}{2}\, dC_i^h\right)\left(\prod_n \frac{\mu_o}{\sqrt{2\pi}}\frac{1}{2}\, dC_n^h\right). \end{aligned} \quad (3.41)$$

Here the prime indicates that the zero modes of the vector fluctuation operator do not contribute to the product. Notice, however, that these modes do contribute to the measure $D[\xi_\mu]$.

The following remarks are in order. We use units where all fields and parameters have dimensions of different powers of a length scale l. One has $[1/G] = [\Lambda] = [\mu_o^2] = [l^{-2}]$. Eigenvalues of all fluctuation operators have the dimension $[l^{-2}]$. The coordinates x^μ are dimensionless, while $[g_{\mu\nu}] = [h_{\mu\nu}] = [l^2]$. For the vectors, $[\eta_\mu] = [\xi_\mu] = [l^2]$, and for the scalars $[h] = [l^0]$ and $[\chi] = [l^2]$. We assume that the scalar, vector and tensor eigenfunctions in (3.19) are orthonormal with respect to the scalar product in (3.11). As a result, the dimensions of the eigenfunctions are $[\phi_{\mu\nu}^{(k)}] = [l]$, $[\eta_\mu^{(s)}] = [l^0]$, $[\alpha^{(p)}] = [l^{-1}]$, which gives for the Fourier coefficients in (3.20),(3.21) $[C^\phi] = [C^h] = [l]$, $[C^\eta] = [l^2]$, and $[C^\chi] = [l^3]$.

The normalization of $D[h_{\mu\nu}]$ can be arbitrary, which is reflected in the presence of the arbitrary parameter μ_o in the above formulas. However, the relative normalization of $D[h_{\mu\nu}]$ and $D[\xi_\mu]$, which is defined by Eqs.(3.39) and (3.40) is fixed by gauge invariance. Had we chosen instead a different relative normalization, say dividing each mode in (3.38) by 2, then the path integral would acquire a factor of $2^{\tilde{N}_0^\gamma}$, where $\tilde{N}_0^\gamma$ is the 'number of eigenvalues' of the non-gauge-invariant operator $\tilde{\Delta}_0^\gamma$. [The issue of relative normalization of the fluctuation and Faddeev-Popov determinants seldom arises, since in most cases the absolute value of the path integral is irrelevant].

3.4 Computation of the path integral

Now we are ready to compute the path integrals in (3.17),(3.18). Let us illustrate the procedure on the example of Eq.(3.18), which determines the Faddeev-Popov factor $\mathcal{D}_{FP}$. Using $\delta^2 I_g$ from Eq.(3.33) and the measure $D[\xi_\mu]$ from (3.38) we obtain

$$
\begin{aligned}
(\mathcal{D}_{FP})^{-1} = & \prod_s \int \frac{\mu_o^2}{\sqrt{\pi}}\, dC_s^\eta \, \exp\left(-\gamma(\sigma_s)^2 (C_s^\eta)^2\right) \\
\times & \prod_i \int \frac{\mu_o^2}{\sqrt{\pi}} \sqrt{\frac{4\Lambda}{3}}\, dC_i^\chi \, \exp\left(-\frac{16}{27\gamma}\,\Lambda^3 \left(C_i^\chi - \frac{3}{8\Lambda} C_i^{\tilde{h}}\right)^2\right) \\
\times & \prod_n \int \frac{\mu_o^2}{\sqrt{\pi}} \sqrt{\lambda_n}\, dC_n^\chi \, \exp\left(-\frac{1}{16\gamma} \sum_n \lambda_n \left(2\tilde{\lambda}_n^\gamma\, C_n^\chi + C_n^{\tilde{h}}\right)^2\right),
\end{aligned}
\tag{3.42}
$$

which gives

$$
(\mathcal{D}_{FP})^{-1} = \Omega_1 \left(\prod_s{}' \frac{\mu_o^2}{\sqrt{\gamma}\sigma_s}\right) \left(\prod_i \frac{3\sqrt{\gamma}\mu_o^2}{2\Lambda}\right) \left(\prod_n \frac{2\sqrt{\gamma}\mu_o^2}{\tilde{\lambda}_n^\gamma}\right). \tag{3.43}
$$

3.4.1 Zero modes

The factor Ω_1 in (3.43) arises due to the gauge zero modes, for which $\sigma_s \equiv \sigma_0 = 0$ and the integral is non-Gaussian:

$$
\Omega_1 = \int \prod_j \frac{\mu_o^2}{\sqrt{\pi}}\, dC_{0j}^\eta\,, \tag{3.44}
$$

with the product taken over all such modes. The existence of zero modes of the Faddeev-Popov operator indicates that the gauge is not completely fixed. This can be related to the global aspects of gauge fixing procedure known as the Gribov ambiguity. However, Gribov's problem is usually not the issue in the perturbative calculations, where zero modes arise rather due to background symmetries. This will be the case in our analysis below. Specifically, the isometries of the background manifold $\mathcal{M}$ form a subgroup $\mathcal{H}$ of the full diffeomorphism group. Sometimes $\mathcal{H}$ is called the stability group; for the S^4 and $S^2 \times S^2$ backgrounds $\mathcal{H}$ is SO(5) and SO(3)×SO(3), respectively. Since the isometries do not change $h_{\mu\nu}$ (in the linearized approximation), their generators, which are the Killing vectors K^μ, are zero modes of the Faddeev-Popov operator.

We therefore conclude that the integration in (3.44) is actually performed over the stability group $\mathcal{H}$. Since the latter is compact in the cases under consideration, the integral is finite. In order to actually compute the integral, some further analysis is necessary, in which we shall adopt the approach of Osborn [44]. First of all, let us recall that all eigenmodes in our analysis have unit norm. If we now rescale the zero modes such that the Killing vectors $K_j \equiv K_j^\mu \frac{\partial}{\partial x^\mu}$ become dimensionless (remember that the coordinates x^μ are also dimensionless), then the expression in Eq.(3.44) reads

$$\Omega_1 = \int \prod_j \frac{\mu_o^2}{\sqrt{\pi}} \, ||K_j|| \, dC_j \,, \tag{3.45}$$

where now $[||K_j||] = [l^2]$ and $[C_j] = [l^0]$. For small values of the parameters C_j they can be regarded as coordinates on the group manifold $\mathcal{H}$ in the vicinity of the unit element. Since $\mathcal{H}$ acts on $\mathcal{M}$ via $x^\mu \to x^\mu(C_j)$, one has $K_j = \frac{\partial}{\partial C_j} \equiv \frac{\partial x^\mu}{\partial C_j}\frac{\partial}{\partial x^\mu}$. However, strictly speaking C_j are not coordinates on the group manifold $\mathcal{H}$ but rather on its tangent space at the group unity, such that their range is infinite. We wish to restrict the range of C_j, and for this we should integrate not over the tangent space but over $\mathcal{H}$ itself. In other words, to render the integral in (3.45) convergent we must treat the zero modes non-perturbatively, and for this we should replace the perturbative measure $\prod_j dC_j$ by a non-perturbative one, $d\mu(C)$.

In general it is a difficult issue to construct the non-perturbative path integration measure. However, in the zero mode sector this can be done. We note that the measure should be invariant under the group multiplications, $d\mu(CC') = d\mu(C)$, and this uniquely requires that $d\mu(C)$ should be the Haar measure for $\mathcal{H}$. The normalization is fixed by the requirement that for $C_j \to 0$ the perturbative result (3.45) is reproduced. This unambiguously gives

$$\Omega_1 = \int \left(\prod_j \frac{\mu_o^2}{\sqrt{\pi}} \left|\left| \frac{\partial}{\partial C_j} \right|\right| \right) d\mu(C) \,, \tag{3.46}$$

where $\frac{\partial}{\partial C_j}$ is computed at $C_j = 0$ and $d\mu(C)$ is the Haar measure of the isometry group $\mathcal{H}$ normalized such that $d\mu(C) \to \prod_j dC_j$ as $C_j \to 0$.

Now, using (3.32) and the measure (3.41), we compute the path integral in (3.17) – first without the Faddeev-Popov factor $\mathcal{D}_{FP}$:

$$\int D[h_{\mu\nu}] \exp\left(-\delta^2 I_{gf}\right) = \Omega_2 \left(\prod_k{}' \frac{\mu_o}{\sqrt{\epsilon_k}}\right) \left(\prod_s{}' \frac{\mu_o}{\sqrt{\gamma\sigma_s}}\right) \left(\prod_n \frac{\sqrt{2}\mu_o}{\sqrt{\tilde{\lambda}_n^\gamma}}\right) \\ \times \frac{\mu_o}{\sqrt{2\Lambda}} \left(\prod_i \frac{\sqrt{3\gamma}\,\mu_o}{\sqrt{2\Lambda}}\right) \left(\prod_n \frac{\sqrt{2\gamma}\mu_o}{\sqrt{\tilde{\lambda}_n^\gamma}}\right). \tag{3.47}$$

Here the primes indicate that zero and negative modes should be omitted from the products. Zero vector modes do not contribute since they are not present in the path-integration measure (3.41), and we assume that there are no negative vector modes, since otherwise the metric on the space of fluctuations would not be positive definite. For tensor fluctuations negative and zero modes are present in the measure (3.41), and their overall contribution is collected in the factor Ω_2 in (3.47). Let us further assume that there are no zero tensor modes, which is the case for the manifolds of interest. If negative modes are also absent then $\Omega_2 = 1$. If there is one negative tensor mode with eigenvalue $\varepsilon_- < 0$, then

$$\Omega_2 = \frac{\mu_o}{\sqrt{2\pi}} \int dC_-^\phi \, \exp\left(-\frac{1}{2}\,\varepsilon_-\,(C_-^\phi)^2\right). \tag{3.48}$$

The integral is computed via the deformation of the contour to the complex plane, which gives the purely imaginary result

$$\Omega_2 = \frac{\mu_o}{2i\,\sqrt{|\varepsilon_-|}}, \tag{3.49}$$

with the factor of 1/2 arising in the course of the analytic continuation [12].

Both the Faddeev-Popov factor in (3.43) and the path integral in (3.47) depend on the gauge parameter γ. However, the γ-dependence exactly cancels in their product, which provides a very good consistency check. In particular, the relative normalization of the integration measures fixed by Eqs.(3.39) and (3.40) is important. If we had divided each factor in the mode products in (3.38) by $a \neq 1$, then the resulting path integral would be proportional to $(\prod_n a) \sim a^{\zeta(0)}$ with ζ being the ζ-function of the γ-dependent operator $\tilde{\Delta}_0^\gamma$. Thus, unless $a = 1$, the result would be gauge-dependent.

We therefore finally obtain the following expression for the path integral in (3.17):

$$Z[g_{\mu\nu}] = \frac{\Omega_2}{\Omega_1}\,\frac{\mu_o}{\sqrt{2\Lambda}}\left(\prod_i \sqrt{\frac{2\Lambda}{3}}\frac{1}{\mu_o}\right)\left(\prod_s{}^{'}\,\frac{\sqrt{\sigma_s}}{\mu_o}\right)\left(\prod_k{}^{'}\,\frac{\mu_o}{\sqrt{\epsilon_k}}\right)\mathrm{e}^{-I} \tag{3.50}$$

Here Ω_2 is the contribution of the negative tensor mode, and Ω_1 is the isometry factor. As we expected, the contribution of all unphysical scalar modes has canceled from the result. The only scalar modes which do contribute are the several lowest lying modes which seem to be physically distinguished. These are the constant conformal mode giving rise to the factor $\mu_o/\sqrt{2\Lambda}$, and the 5 'conformal Killing scalars' which exist only in the S^4 case and give rise to the product over i. The next two factors in (3.50) is the contribution of the transversal vector modes and the TT-tensor modes. Finally, $I = I[g_{\mu\nu}]$ is the classical action.

In order to apply the above formula for $Z[g_{\mu\nu}]$ we need the eigenvalues of the fluctuation operators. Now we shall determine the latter for the manifolds $S^2 \times S^2$ and S^4.

4 Spectra of fluctuation operators

In this section we derive the spectra of small fluctuations around the $S^2 \times S^2$ and S^4 instantons. In the $S^2 \times S^2$ case the problem is tackled via solving the differential equations. It turns out that in a suitable basis the system of 10 coupled equations for the gravity fluctuations splits into 10 independent equations. The latter are solved in terms of spin-weighted spherical harmonics. In the S^4 case the equations do not decouple and the direct approach is not so transparent. However, the problem can be conveniently analyzed with group theoretic methods, which was done some time ago by Gibbons and Perry [27]. We shall describe the group theory approach in some detail – with the same principal result as in [27].

4.1 Fluctuations around the $S^2 \times S^2$ instanton

Let us consider the metric of the $S^2 \times S^2$ instanton background,

$$ds^2 = \frac{1}{\Lambda}\Big(\,(d\vartheta_1)^2 + \sin^2\vartheta_1\,(d\varphi_1)^2 + (d\vartheta_2)^2 + \sin^2\vartheta_2\,(d\varphi_2)^2\Big)\,. \tag{4.1}$$

It is convenient to set $\Lambda = 1$ for the time being; at the end of calculations the Λ-dependence is restored by multiplying all eigenvalues with Λ. We introduce the complex tetrad

$$e^1 = \frac{1}{\sqrt{2}}\left(d\vartheta_1 + \frac{i}{\sin\vartheta_1}\, d\varphi_1 \right), \quad e^2 = \frac{1}{\sqrt{2}}\left(d\vartheta_1 - \frac{i}{\sin\vartheta_1}\, d\varphi_1 \right),$$
$$e^3 = \frac{1}{\sqrt{2}}\left(d\vartheta_2 + \frac{i}{\sin\vartheta_2}\, d\varphi_2 \right), \quad e^4 = \frac{1}{\sqrt{2}}\left(d\vartheta_2 - \frac{i}{\sin\vartheta_2}\, d\varphi_2 \right). \tag{4.2}$$

The metric in (4.1) splits as $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$, where the tetrad metric is

$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

4.1.1 Tensor modes

First we consider the eigenvalue problem

$$-\nabla_\alpha \nabla^\alpha \phi_{\mu\nu} - 2R_{\mu\alpha\nu\beta}\, \phi^{\alpha\beta} = \varepsilon \phi_{\mu\nu}\,, \tag{4.4}$$

where

$$\nabla_\mu \phi^\mu_\nu = 0, \quad \phi^\mu_\mu = 0\,. \tag{4.5}$$

We expand $\phi_{\mu\nu}$ with respect to the complex basis (4.2),

$$\phi_{\mu\nu} = e^a{}_\mu e^b{}_\nu \Phi_{ab}\,, \tag{4.6}$$

insert this into (4.4) and project the resulting equation onto the basis (4.2) again. Remarkably, the system of 10 coupled equations splits then into 10 independent equations for the 10 tetrad projections Φ_{ab}. A partial explanation of this fact is the existence of two different parity symmetries acting independently on the two spheres.

It is convenient to introduce the operator

$$\hat{\mathcal{D}}[\mathbf{s}, \vartheta, \varphi] = \frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\, \frac{\partial}{\partial\vartheta} + 2i\mathbf{s}\, \frac{\cot\vartheta}{\sin\vartheta} + \frac{1}{\sin^2\vartheta}\, \frac{\partial^2}{\partial\phi^2} - \mathbf{s}^2 \cot\vartheta\,, \tag{4.7}$$

whose eigenfunctions are the spin-weighted spherical harmonics ${}_{\mathbf{s}}Y_{jm}$ [29],

$$\hat{\mathcal{D}}[\mathbf{s},\vartheta,\varphi]\,{}_{\mathbf{s}}Y_{jm}(\vartheta,\phi) = (\mathbf{s}^2 - j(j+1))\;{}_{\mathbf{s}}\,Y_{jm}(\vartheta,\phi). \tag{4.8}$$

Here j and m are such that $j \geq |\mathbf{s}|$ and $-j \leq m \leq j$. One has ${}_{\mathbf{s}}Y_{jm} = 0$ for $j < |\mathbf{s}|$. [Notice that we use the bold-faced $\mathbf{s}$ for the spin weight.] The following relations between harmonics with different values of the spin weight $\mathbf{s}$ are useful:

$$\hat{\mathcal{L}}^{\pm}[\mathbf{s},\vartheta,\varphi]\;{}_{\mathbf{s}}Y_{jm} = \pm\sqrt{(j\pm\mathbf{s})(j\mp\mathbf{s}+1)}\;{}_{\mathbf{s}\mp1}Y_{jm}\,, \tag{4.9}$$

where

$$\hat{\mathcal{L}}^{\pm}[\mathbf{s},\vartheta,\varphi] = \frac{\partial}{\partial\vartheta} \mp \frac{i}{\sin\vartheta}\frac{\partial}{\partial\phi} \pm \mathbf{s}\cot\vartheta\,. \tag{4.10}$$

The harmonics for a fixed $\mathbf{s}$ form an orthonormal set on S^2.

Using the above definitions, Eqs. (4.4) can be represented as

$$\Big(\hat{\mathcal{D}}[\mathbf{s}^1_{ab},\vartheta_1,\varphi_1] + \hat{\mathcal{D}}[\mathbf{s}^2_{ab},\vartheta_2,\varphi_2] - (\mathbf{s}^1_{ab})^2 - (\mathbf{s}^2_{ab})^2 + 2 + \varepsilon\Big)\;\Phi_{ab} = 0\,, \tag{4.11}$$

where $1 \leq a, b \leq 4$ (no summation over a, b). Here the nonzero elements of the symmetric matrices $\mathbf{s}^1_{ab}$ and $\mathbf{s}^2_{ab}$ are

$$\begin{aligned} &\mathbf{s}^1_{11} = -\mathbf{s}^1_{22} = 2, \quad \mathbf{s}^1_{13} = \mathbf{s}^1_{14} = -\mathbf{s}^1_{23} = -\mathbf{s}^1_{24} = 1\,, \\ &\mathbf{s}^2_{33} = -\mathbf{s}^2_{44} = 2, \quad \mathbf{s}^2_{13} = \mathbf{s}^2_{23} = -\mathbf{s}^2_{14} = -\mathbf{s}^2_{24} = 1\,. \end{aligned} \tag{4.12}$$

The solution to Eqs. (4.11) reads

$$\Phi_{ab} = C_{ab}\;{}_{\mathbf{s}^1_{ab}}Y_{j_1m_1}(\vartheta_1,\varphi_1)\;{}_{\mathbf{s}^2_{ab}}Y_{j_2m_2}(\vartheta_2,\varphi_2), \tag{4.13}$$

with C_{ab} being integration constants. The eigenvalue, ε, is the same for all Φ_{ab}:

$$\varepsilon = j_1(j_1+1) + j_2(j_2+1) - 2\,. \tag{4.14}$$

This is essentially the sum of squares of the two SO(3) angular momentum operators acting independently on the two spheres.

Let us now count the degeneracy of the modes. For this one should take into account the additional conditions (4.5), which gives algebraic constraints for the coefficients C_{ab}. We consider first the trace condition $\phi^\mu_\mu = 0$. In the language of the tetrad projections this reduces to $\Phi_{12}+\Phi_{34} = 0$, or equivalently to

$$C_{12} + C_{34} = 0\,. \tag{4.15}$$

Hence only 9 out of the 10 constants C_{ab} are independent.

Next, we consider the Lorenz condition $\nabla_\sigma \phi^\sigma_\mu = 0$. This implies

$$\begin{aligned} &\hat{\mathcal{L}}^-[\mathbf{s}^1_{a1}, \vartheta_1, \varphi_1]\,\Phi_{a1} + \hat{\mathcal{L}}^+[\mathbf{s}^1_{a2}, \vartheta_1, \varphi_1]\,\Phi_{a2} \\ &+ \hat{\mathcal{L}}^-[\mathbf{s}^2_{a3}, \vartheta_2, \varphi_2]\,\Phi_{a3} + \hat{\mathcal{L}}^+[\mathbf{s}^2_{a4}, \vartheta_2, \varphi_2]\,\Phi_{a4} = 0 \end{aligned} \tag{4.16}$$

(no summation over a). Inserting the solution (4.13) and using the recurrence relations in (4.9), these conditions reduce to algebraic constraints

$$\begin{aligned} \kappa_1\, C_{11} - \alpha_1 C_{12} + \alpha_2\,(C_{13} - C_{14}) = 0\,, \\ \alpha_1\, C_{12} - \kappa_1 C_{22} + \alpha_2\,(C_{23} - C_{24}) = 0\,, \\ \alpha_1\,(C_{13} - C_{23}) + \kappa_2\, C_{33} - \alpha_2\, C_{34} = 0\,, \\ \alpha_1\,(C_{14} - C_{24}) + \alpha_2\, C_{34} - \kappa_2\, C_{44} = 0\,. \end{aligned} \tag{4.17}$$

Here $\alpha_\iota = \sqrt{j_\iota(j_\iota+1)}$ (with $\iota = 1, 2$) and $\kappa_\iota = \sqrt{(j_\iota+2)(j_\iota-1)}$ for $j_\iota \geq 1$ while $\kappa_\iota = 0$ for $j_\iota = 0$.

For $j_\iota \geq 2$ (which corresponds to quadrupole or higher deformations of each sphere) none of the coefficients α_ι, κ_ι vanish, and the algebraic constraints (4.17) reduce the number of independent coefficients C_{ab} to five. This gives the degeneracy d:

$$j_1 \geq 2,\; j_2 \geq 2, \qquad d = 5\,(2j_1+1)(2j_2+1)\,. \tag{4.18}$$

The situation is different for small values of j_ι. Consider, for example, the $j_1 = j_2 = 0$ sector. Since the harmonics ${}_{\mathbf{s}}Y_{jm}$ vanish for $j < |\mathbf{s}|$, we must set in the solution (4.13) all C_{ab}'s to zero, apart from $C_{12} = -C_{34}$. The Lorenz condition (4.16) is then fulfilled. As a result, there is only one independent integration constant, which yields

$$j_1 = j_2 = 0, \qquad d = 1. \tag{4.19}$$

Notice that in this case $\varepsilon = -2$.

In a similar way one can consider the sector where $j_1 = 0$ and $j_2 = 1$ (or $j_1 = 1$ and $j_2 = 0$), in which case $\varepsilon = 0$. One discovers then that the Lorenz constraints (4.16) require that all non-trivial coefficients C_{ab} mush vanish. As a result,

$$j_1 = 0,\ j_2 = 1,\ \text{or}\ \ j_2 = 1,\ j_1 = 0, \qquad d = 0, \tag{4.20}$$

which shows that there are no zero modes.

Next,

$$j_1 = j_2 = 1, \qquad d = (2j_1 + 1)(2j_2 + 1) = 9; \tag{4.21}$$

and finally

$$\begin{aligned} &j_1 \geq 2,\ j_2 = 0, \qquad d = 2j_1 + 1; \\ &j_1 \geq 2,\ j_2 = 1, \qquad d = 9(2j_1 + 1), \end{aligned} \tag{4.22}$$

which conditions are symmetric under interchanging j_1 and j_2.

To recapitulate, the spectrum of the tensor fluctuations contains one negative mode and no zero modes.

4.1.2 Vector modes

Let us now consider the eigenvalue problem

$$(-\nabla_\alpha \nabla^\alpha - \Lambda)\eta_\mu = \sigma\, \eta_\mu \tag{4.23}$$

subject to the condition

$$\nabla_\mu \eta^\mu = 0 \tag{4.24}$$

for the vector fluctuations on the $S^2 \times S^2$ background. We again expand the fluctuations with respect to the basis (4.2),

$$\eta_\mu = e^a{}_\mu \Psi_a, \tag{4.25}$$

insert this into (4.23), and project back to the tetrad. Similarly as in the tensor case, the equations decouple to give

$$(\hat{\mathcal{D}}[\mathbf{s}^1_a, \vartheta_1, \varphi_1] + \hat{\mathcal{D}}[\mathbf{s}^2_a, \vartheta_2, \varphi_2] + 1 + \sigma)\, \Psi_a = 0\,, \tag{4.26}$$

where $1 \leq a \leq 4$ (no summation over a), and nonzero coefficients read $\mathbf{s}_1^1 = -\mathbf{s}_2^1 = \mathbf{s}_3^2 = -\mathbf{s}_4^2 = 1$. The solution is

$$\Psi_a = C_a \; {}_{\mathbf{s}_a^1}Y_{j_1 m_1}(\vartheta_1, \varphi_1) \; {}_{\mathbf{s}_a^2}Y_{j_2 m_2}(\vartheta_2, \varphi_2), \tag{4.27}$$

with C_a being integration constants, and the eigenvalue is the same for all Ψ_a:

$$\sigma = j_1(j_1+1) + j_2(j_2+1) - 2\,. \tag{4.28}$$

The Lorenz condition, $\nabla_\sigma \eta^\sigma = 0$, reads

$$\begin{aligned} &\hat{\mathcal{L}}^-[\mathbf{s}_1^1, \vartheta_1, \varphi_1]\,\Psi_1 + \hat{\mathcal{L}}^+[\mathbf{s}_2^1, \vartheta_1, \varphi_1]\,\Psi_2 \\ &+ \hat{\mathcal{L}}^-[\mathbf{s}_3^2, \vartheta_2, \varphi_2]\,\Psi_3 + \hat{\mathcal{L}}^+[\mathbf{s}_4^2, \vartheta_2, \varphi_2]\,\Psi_4 = 0\,, \end{aligned} \tag{4.29}$$

which reduces upon inserting (4.27) to the algebraic condition

$$\sqrt{j_1(j_1+1)}\,(C_1 - C_2) + \sqrt{j_2(j_2+1)}\,(C_3 - C_4) = 0. \tag{4.30}$$

This allows one to count the degeneracies:

$$j_1 \geq 1,\; j_2 \geq 1 \quad (\sigma > 0), \quad d = 3\,(2j_1+1)(2j_2+1); \tag{4.31}$$

and also

$$\begin{array}{lll} j_1 \geq 2,\; j_2 = 0 & (\sigma > 0), & d = j_1(j_1+1); \\ j_1 = 1,\; j_2 = 0 & (\sigma = 0), & d = 3; \\ j_1 = 0,\; j_2 = 0 & (\sigma = -2), & d = 0\,. \end{array} \tag{4.32}$$

These results are symmetric under $j_1 \leftrightarrow j_2$, hence there are no negative modes, there are six zero modes corresponding to the six Killing vectors of $S^2 \times S^2$, and the rest of the spectrum is positive.

4.1.3 Scalar modes and the orthogonality conditions

The eigenvalue problem for the scalar modes,

$$-\nabla_\alpha \nabla^\alpha h = \lambda\, h, \tag{4.33}$$

reduces to the equation

$$(\hat{\mathcal{D}}[0, \vartheta_1, \varphi_1] + \hat{\mathcal{D}}[0, \vartheta_2, \varphi_2] + \lambda)\, h = 0\,, \tag{4.34}$$

whose solutions are

$$h = Y_{j_1 m_1}(\vartheta_1, \varphi_1)\, Y_{j_2 m_2}(\vartheta_2, \varphi_2) \tag{4.35}$$

(for $\mathbf{s} = 0$ the spin-weighted spherical harmonics coincide with the usual spherical harmonics). The eigenvalues are just

$$\lambda = j_1(j_1 + 1) + j_2(j_2 + 1)\,, \tag{4.36}$$

and the degeneracies are

$$j_1 \geq 0,\ j_2 \geq 0, \qquad d = (2j_1 + 1)(2j_2 + 1). \tag{4.37}$$

We have obtained the spectra of all relevant fluctuation operators. Although the eigenfunctions are complex, one can pick up their real part in a way that is consistent with the orthogonality conditions. For example, for the scalar modes one considers

$$\Re(h) = \frac{1+i}{\sqrt{2}}\, Y_{j_1 m_1}\, Y_{j_2 m_2} + c.c\,, \tag{4.38}$$

and one can easily see that the modes $\Re(h)$ with different quantum numbers $(j_1 m_1 j_2 m_2)$ are orthogonal with respect to the scalar product defined in Eq. (3.11).

For the vector modes Ψ_a the procedure is slightly more complicated, since the tetrad metric η_{ab} is not diagonal. In addition, harmonics ${}_{\mathbf{s}}Y_{jm}$ for different values of the spin weight are not orthogonal. Consider, however, the real combinations

$$\Re(\eta_\mu) = \frac{1+i}{\sqrt{2}}\, e^a{}_\mu \Psi_a + c.c\,, \tag{4.39}$$

where Ψ_a has quantum numbers $(j_1 m_1 j_2 m_2)$. Consider $\Re(\eta_\mu^{(1)})$ and $\Re(\eta_\mu^{(2)})$ with different quantum numbers. Their scalar product (defined in Eq. (3.11)) can be computed using the relations

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu\, g_{\mu\nu}, \qquad \delta_{ab} = e_a{}^\mu (e_b{}^\nu)^*\, g_{\mu\nu}\,, \tag{4.40}$$

which gives

$$\begin{aligned} \langle \Re(\eta_\mu^{(1)}), \Re(\eta^{(2)\mu}) \rangle &= \sum_a \langle \Psi_a^{(1)}, \Psi_a^{(2)*} \rangle \\ &+ i\langle \Psi_1^{(1)}, \Psi_2^{(2)} \rangle + i\langle \Psi_3^{(1)}, \Psi_4^{(2)} \rangle + c.c. \end{aligned} \tag{4.41}$$

Table 1
Spectra of fluctuations around the $S^2 \times S^2$ instanton

operator	eigenvalue	degeneracy	
Δ_2	-2Λ	1	
	2Λ	9	
	$(j(j+1)-2)\Lambda$	$2(2j+1)$	$j \geq 2$
	$j(j+1)\Lambda$	$18(2j+1)$	$j \geq 2$
	$(j_1(j_1+1)+j_2(j_2+1)-2)\Lambda$	$5(2j_1+1)(2j_2+1)$	$j_1, j_2 \geq 2$
Δ_1	0	6	
	$(j(j+1)-2)\Lambda$	$2(2j+1)$	$j \geq 2$
	$(j_1(j_1+1)+j_2(j_2+1)-2)\Lambda$	$3(2j_1+1)(2j_2+1)$	$j_1, j_2 \geq 1$
Δ_0	$(j_1(j_1+1)+j_2(j_2+1))\Lambda$	$(2j_1+1)(2j_2+1)$	$j_1, j_2 \geq 0$

Here each term in the sum $\sum_a \langle \Psi_a^{(1)}, \Psi_a^{(2)*} \rangle$ is a bilinear combination of spin-weighted harmonics with the same value of the spin weight, such that the orthogonality relation holds. Next, integrating by parts and using the recurrence relations (4.9) one can show that the remaining term in the scalar product, $i\langle \Psi_1^{(1)}, \Psi_2^{(2)} \rangle + i\langle \Psi_3^{(1)}, \Psi_4^{(2)} \rangle + c.c$, vanishes. This shows that vector modes $\Re(\eta_\mu)$ with different quantum numbers are orthogonal.

A similar procedure can be carried out for the tensor modes. Hence for all eigenmodes considered above one can choose the real part in such a way that the orthogonality condition holds. This is a manifestation of the fact that the fluctuation operators are self-adjoint. We finally restore the dependence on Λ and summarize the results of this section in Tab.1. There is one negative mode in the spectrum, and this plays a crucial role in our analysis. The corresponding deformation increases the radius of one of the spheres, shrinking at the same time the second one.

4.2 Fluctuations around the S^4 instanton

The S^4 instanton can be viewed as the four-dimensional sphere with radius $\sqrt{3/\Lambda}$ in five-dimensional Euclidean space E^5. Although the corresponding eigenvalue problem for fluctuations was considered in [27], we have reanalyzed it for the sake of completeness (with the same result) and shall present below the key steps of our analysis. The problem essentially reduces to studying representations of SO(5) [31,1,2,8], whose Casimir operator can be related to the invariant Laplacians on S^4 with the help of the projection formalism

[34]. We shall therefore first outline the group theory part by summarizing the relevant facts about representations of $SO(5)$. We shall work on the unit 4-sphere rescaling at the end the eigenvalues by the factor $\Lambda/3$.

4.2.1 Representations of SO(5)

The unit sphere S^4 in E^5 is defined in Cartesian coordinates by the equation $\sum_{a=1}^{5}(x^a)^2 = 1$. It is convenient to use the complex coordinates $\xi^{\pm 1} = (x^1 \pm ix^2)/\sqrt{2}$, $\xi^{\pm 2} = (x^3 \pm ix^4)/\sqrt{2}$, $\xi^0 = x^5$. We shall not distinguish between lower and upper indices, $\xi_i = \xi^i$. In these new coordinates the defining quadratic form reads $\sum_{i=-2}^{2} \xi^i \xi^{-i} = 1$, which is annihilated by the generators of SO(5):

$$Y^i_j = \xi^i \frac{\partial}{\partial \xi^j} - \xi^{-j} \frac{\partial}{\partial \xi^{-i}}, \tag{4.42}$$

whose commutation relations are

$$[Y^i_j, Y^k_l] = \delta^k_j Y^i_l - \delta^i_l Y^k_j + \delta^{-l}_j Y^i_{-k} - \delta^k_{-i} Y^{-j}_l \equiv C^{pq}_{ij,kl} Y^p_q \,. \tag{4.43}$$

Since $Y^i_j = -Y^j_i$, the independent generators can be chosen to be those for $-i < j$. Y^1_1 and Y^2_2 generate the Cartan subalgebra, while Y^i_j and Y^j_i for $-i < j < i$ are the raising and lowering operators, respectively. One has

$$[Y^i_i, Y^k_l] = \alpha^k_l(i) Y^k_l \,, \tag{4.44}$$

where

$$\alpha^k_l(i) = \delta^i_k - \delta^i_l + \delta^i_{-l} - \delta^i_{-k} \tag{4.45}$$

determine the root vectors with the components $\alpha^k_l \equiv (\alpha^k_l(1), \alpha^k_l(2))$. The roots corresponding to the four raising operators are $\alpha^2_1 = (-1,1)$, $\alpha^2_0 = (0,1)$, $\alpha^2_{-1} = (1,1)$, and $\alpha^1_0 = (1,0)$.

Irreducible representations of SO(5) are characterized by two numbers denoted by $m \equiv (m_1, m_2)$, where $m_2 \geq m_1$ and both m_1 and m_2 are either integer or half-integer. The highest weight vector ψ_m is annihilated by all raising operators, $Y^i_j \psi_m = 0$ for $i > j > -i$, and it is an eigenvector of the Cartan subalgebra generators, $Y^i_i \psi_m = m_i \psi_m$, $i = 1, 2$. Using these properties and also $[Y^i_j, Y^j_i] = Y^i_i - Y^j_j - 2\delta^i_{-j} Y^i_{-j}$, one finds the eigenvalues of the Casimir operator,

$$\hat{C}\psi_m \equiv \frac{1}{2} \sum_{i,j} Y^i_j Y^j_i \psi_m = C_m \psi_m \,, \tag{4.46}$$

where

$$C_m = m_1(m_1+1) + m_2(m_2+3) \tag{4.47}$$

is the same for all vectors of the representation. The dimension of representations is given by

$$\dim(m) = \prod_\alpha \langle \alpha, r+m\rangle / \prod_\alpha \langle \alpha, r\rangle\,. \tag{4.48}$$

Here the product is over the four root vectors described above, and $r = \frac{1}{2}\sum_\alpha \alpha = (\frac{1}{2}, \frac{3}{2})$. One has $r+m = (\frac{1}{2}+m_1, \frac{3}{2}+m_2)$, and $\langle\, ,\, \rangle$ is the scalar with respect to the Cartan metric $g_{ij} = -C^{pq}_{ii,kl}C^{kl}_{jj,pq} = 6\delta_{ij}$ (here $i,j=1,2$). As a result,

$$\dim(m) = \frac{1}{6}\,(2m_1+1)(2m_2+3)(m_2-m_1+1)(m_1+m_2+2)\,. \tag{4.49}$$

4.2.2 Scalar modes

Using Eqs.(4.47),(4.49) one can find the spectra of the relevant fluctuation operators. It is now convenient to pass back to the Cartesian coordinates $x^a = x_a$ $(a = 1, \ldots 5)$, such that the sphere S^4 is determined by the condition $r \equiv \sqrt{x^a x_a} = 1$. Let $n^a \equiv x^a/r$ be the unit normal to the sphere. The (anti-hermitean) generators of SO(5) in Cartesian coordinates are given by $M_{ab} = n_a\partial_b - n_b\partial_a$, and the Casimir operator is $\hat{C} = -\frac{1}{2}(M_{ab})^2 \equiv -\frac{1}{2}\sum_{ab}(M_{ab})^2$.

Let us define the projection operator $P_{ab} = \delta_{ab} - n_a n_b = P^{ab} = P^a_b$, which can be thought of as the induced metric on the sphere. In what follows we shall use the projection formalism [34] to describe geometrical 4-objects tangent to the sphere in terms of 5-objects of the embedding space. For example, a 4-vector η_μ can be described as a 5-vector η_a subject to the condition $n^a\eta_a = 0$. The covariant derivative of a tensor is obtained by taking the partial derivative and then projecting all the indices down to the sphere. For example, $\nabla_a\eta_b = (\partial_p\eta_q)P^p_a P^q_b$. One has $n_a = n^a$, while for objects tangent to the sphere 5-indices can be raised and lowered either with P_{ab} or with δ_{ab}. The curvature tensor is given by $R_{psqt} = P_{pq}P_{st} - P_{pt}P_{sq}$.

Consider first scalar fluctuations. The covariant Laplacian for a scalar field h can be expressed in terms of the angular momentum operator as

$$\Box h \equiv P^{ab}\partial_a(P^c_b\partial_c h) = \frac{1}{2}(M_{ab})^2 h = -\hat{C}h\,. \tag{4.50}$$

Scalars transform according to the $(0,j)$ representations, which correspond to the Young tableaux $\boxed{1}\boxed{2}\boxed{\ldots}\boxed{j}$ and can be represented in terms of homogeneous polynomials on S^4 as

$$h = h_{(a_1 \ldots a_j)} n^{a_1} \ldots n^{a_j} \, . \tag{4.51}$$

Hence, the eigenvalues of the Casimir operator in Eq. (4.50) are $C_{(0,j)} = j(j+3)$, which gives the spectrum of the scalar eigenvalue problem, $\Delta_0 h = \lambda h$ with $\Delta_0 \equiv -\frac{\Lambda}{3}\Box$:

$$\lambda = \frac{\Lambda}{3}\, j(j+3), \qquad d = \frac{1}{6}\,(2j+3)(j+2)(j+1), \qquad j \geq 0. \tag{4.52}$$

4.2.3 Vector modes

Consider a tangent vector field $\eta_s = P^a_s \eta_a$. The invariant Laplacian reads

$$\Box\eta_s \equiv P^{ab}\partial_a (P^c_b \partial_c \eta_p P^p_q) P^q_s = \frac{1}{2}(M_{ab})^2 \eta_s + 2(\partial_a \eta^a) n_s + \eta_s \, . \tag{4.53}$$

Here the last two terms on the right can be related to the contribution of the spin operator. Under general SO(5) rotations a vector $\eta(x)$ transforms into $\tilde{\eta}(x) = R\,\eta(R^{-1}x)$, where $R = \exp(\omega^{ab} S_{ab})$ with $\omega^{ab} = -\omega^{ba}$ being the rotation parameters and $S_{ab} \equiv (S_{ab})^{pq} = \delta^p_a \delta^q_b - \delta^p_b \delta^q_a$. For $|\omega_{ab}| \ll 1$ one obtains $\tilde{\eta} - \eta = \omega^{ab}(M_{ab} + S_{ab})\eta$, such that S_{ab} can be identified with the spin operator: $(S_{ab}\eta)_s = (S_{ab})_s{}^p \eta_p$. As a result,

$$\Box\eta_s = \left(\frac{1}{2}(M_{ab} + S_{ab})^2 + 3\right)\eta_s \equiv (-\hat{C} + 3)\,\eta_s \, , \tag{4.54}$$

where the Casimir operator is now the square of the total angular momentum. The general vector harmonics on S^4 can be obtained by considering the product of a constant vector in E^5 with scalar harmonics on S^4. Such a product decomposes into three irreducible pieces, $(0,1) \otimes (0,j) = (1,j) \oplus (0,j+1) \oplus (0,j-1)$, which can be visualized as

$$\boxed{} \otimes \boxed{1}\boxed{2}\boxed{\ldots}\boxed{j} = \begin{array}{l} \boxed{1}\boxed{2}\boxed{\ldots}\boxed{j} \\ \boxed{} \end{array} \oplus \boxed{0}\boxed{1}\boxed{\ldots}\boxed{j} \oplus \boxed{1}\boxed{2}\boxed{\ldots}\boxed{j\text{-}1} \, . \tag{4.55}$$

The first term on the right here is the $(1,j)$-piece, and in the language of homogeneous polynomials it reads

$$\eta_s = \eta_{[sa](a_1 \ldots a_{j-1})} n^a n^{a_1} \ldots n^{a_{j-2}} n^{a_{j-1}} \, , \tag{4.56}$$

where $\eta_{[sa](a_1 \ldots a_{j-1})}$ is traceless with respect to any pair of indices. This is manifestly tangential and coexact. As a result, the eigenvalues of the Casimir operator are $C_{(1,j)} = j(j+3)+2$, and the spectrum of the vector eigenvalue problem $\Delta_1 \eta_s \equiv (-\frac{\Lambda}{3}\Box - \Lambda)\eta_s = \sigma \eta_s$ in the sector where $\partial_a \eta^a = n^a \eta_a = 0$ is given by

$$\sigma = \frac{\Lambda}{3}\,(j(j+3)-4), \qquad d = \frac{1}{2}\,j(j+3)(2j+3), \qquad j \geq 1\,. \tag{4.57}$$

One can also directly verify that the harmonic η_s in Eq.(4.56) fulfills the condition $\frac{1}{2}(M_{ab})^2\eta_s = -j(j+3)\eta_s$. It follows then from Eq. (4.53) that $\Box\eta_s = -(j(j+3)-1)\eta_s$, and this again yields the spectrum in Eq. (4.57). The correct degeneracy can be obtained by counting the independent components of $\eta_{[sa](a_1\ldots a_{j-1})}$ [31].

The remaining two pieces in (4.55), when represented in terms of the polynomials on S^4, can be related to the exact tangential and the normal components of the vector field.

4.2.4 Tensor modes

For a symmetric tensor $h_{pq} = P^a_p P^b_q h_{ab}$ a direct calculation gives

$$\begin{aligned}\Box h_{pq} + 2R_{psqt}h^{st} &\equiv P^{ab}\partial_a(P^c_b(\partial_c h_{\bar{p}\bar{q}})P^{\bar{p}}_{\underline{p}}P^{\bar{q}}_{\underline{q}})P^{\underline{p}}_p P^{\underline{q}}_q + 2(P_{pq}P_{st} - P_{pt}P_{sq})h^{st} \\ &= \frac{1}{2}(M_{ab})^2 h_{pq} + 2n_p(\partial^a h_{aq}) + 2n_q(\partial^a h_{ap}) + 2\delta_{pq}h^a_a \\ &= \left(\frac{1}{2}(M_{ab}+\Sigma_{ab})^2 + 6\right) h_{pq} \equiv (-\hat{C}+6)h_{pq}\,. \end{aligned} \tag{4.58}$$

Here the spin operator is defined in the same way as for vectors, which gives $(\Sigma_{ab}h)_{pq} = (S_{ab})_p{}^s h_{sq} + (S_{ab})_q{}^s h_{sp}$. The general tensor harmonics on S^4 are obtained by the direct products $(0,2)\otimes(0,j) = (0,j+2)\oplus(1,j+1)\oplus(0,j)\oplus(2,j)\oplus(0,j+1)\oplus(1,j-1)\oplus(0,j-2)$. Again this can be visualized by Young's diagrams and represented in the language of homogeneous polynomials. The transverse and tracefree harmonics tangent to the sphere correspond to the $(2,j)$ piece, whose explicit representation is

$$h_{pq} = h_{[pa][qb](a_1\ldots a_{j-2})}n^a n^b n^{a_1}\ldots n^{a_{j-2}}\,. \tag{4.59}$$

Here $h_{[pa][qb](a_1\ldots a_{j-2})}$ is traceless with respect to any pair of indices and is symmetric under interchange of the $[pa]$ and $[qb]$ pairs. As a result, the eigenvalues of the Casimir operator are $C_{(2,j)} = j(j+3)+6$. This gives the spectrum of the tensor eigenvalue problem $\Delta_2 h_{pq} = \varepsilon h_{pq}$ in the sector where

Table 2
Spectra of fluctuations around the S^4 instanton

operator	eigenvalue	degeneracy	
Δ_2	$\frac{\Lambda}{3}j(j+3)$	$\frac{5}{6}(j-1)(j+4)(2j+3)$	$j \geq 2$
Δ_1	$\frac{\Lambda}{3}(j(j+3)-4)$	$\frac{1}{2}j(j+3)(2j+3)$	$j \geq 1$
Δ_0	$\frac{\Lambda}{3}j(j+3)$	$\frac{1}{6}(j+1)(j+2)(2j+3)$	$j \geq 0$

$\partial^a h_{ab} = n^a h_{ab} = h^a_a = 0$:

$$\varepsilon = \frac{\Lambda}{3}\, j(j+3), \qquad d = \frac{5}{6}\,(j-1)(j+4)(2j+3), \qquad j \geq 2\,. \tag{4.60}$$

The same result can be obtained by directly verifying that h_{pq} in Eq. (4.59) fulfills the condition $\frac{1}{2}(M_{ab})^2 h_{pq} = -j(j+3)h_{pq}$.

The other tensor harmonics in the expansion of $(0,2) \otimes (0,j)$ correspond to the exact and coexact pieces of the longitudinal vector part of the 4-metric, to those of the 4-vector $h_{5\mu}$, and to the trace [27].

We summarize the results of our analysis in this section in Tab.2. Notice that the scalar and tensor eigenvalues are the same (for $j \geq 2$), while the vector spectrum is shifted by a constant.

5 Partition function

Now we are able to derive the explicit expressions for the one-loop partition functions for fluctuations around the $S^2 \times S^2$ and S^4 instantons. The corresponding formula was obtained in Eq.(3.50) above. It is convenient to pass from μ_o to the dimensionless normalization parameter μ_0 via the rescaling

$$\mu_o = \sqrt{\Lambda}\mu_0\,. \tag{5.1}$$

The one-loop partition function for gravity fluctuations around an Euclidean background then reads

$$Z[g_{\mu\nu}] = \frac{\mu_0}{\sqrt{2}}\,\Omega_0\,\frac{\Omega_2}{\Omega_1}\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}}\,\mathrm{e}^{-I}\,. \tag{5.2}$$

Here the first two factors on the right are the contributions of the scalar modes. The factor $\mu_0/\sqrt{2}$ is due to the constant conformal mode, which is always present, and Ω_0 is the contribution of the 5 scalar modes with eigenvalue $4\Lambda/3$ which exist only for the S^4 instanton (see Tab.2):

$$\Omega_0 = \left(\sqrt{\frac{2}{3}}\frac{1}{\mu_0}\right)^5 . \tag{5.3}$$

For any background other than S^4 one has $\Omega_0 = 1$. As was discussed above, other scalar modes do not contribute to the partition function.

The factor Ω_2 in (5.2) is the contribution of the negative tensor mode,

$$\Omega_2 = \frac{\mu_0}{2i\sqrt{|\varepsilon_-|}} . \tag{5.4}$$

For the $S^2 \times S^2$ instanton there is one such mode with $\varepsilon = -2$, while in the S^4 case all tensor modes are positive and $\Omega_2 = 1$. Next,

$$\Omega_1 = \left(\prod_j \frac{\mu_0^2}{\sqrt{\pi}}\Lambda \left|\left|\frac{\partial}{\partial C_j}\right|\right|\right) Vol(\mathcal{H}) , \tag{5.5}$$

is the isometry factor. If the background has no isometries then $\Omega_1 = 1$.

The determinants in Eq. (5.2) are the contributions of the positive vector and tensor modes. One has

$$\sqrt{\mathrm{Det}'\Delta_1} = \left(\prod_s{}' \sqrt{\frac{\sigma_s}{\Lambda\mu_0^2}}\right) = \exp\left(-\frac{1}{2}\zeta_1'(0) - \frac{1}{2}(\ln\mu_0^2)\,\zeta_1(0)\right) , \tag{5.6}$$

where the ζ-function for the positive, transverse vector modes is

$$\zeta_1(z) = \sum_s{}' \left(\frac{\Lambda}{\sigma_s}\right)^z . \tag{5.7}$$

Similarly for the positive, transverse traceless tensor modes:

$$\sqrt{\mathrm{Det}'\Delta_2} = \left(\prod_k{}' \sqrt{\frac{\epsilon_k}{\Lambda\mu_0^2}}\right) = \exp\left(-\frac{1}{2}\zeta_2'(0) - \frac{1}{2}(\ln\mu_0^2)\,\zeta_2(0)\right) \tag{5.8}$$

with

$$\zeta_2(z) = \sum_k{}' \left(\frac{\Lambda}{\epsilon_k}\right)^z . \tag{5.9}$$

The last factor in Eq. (5.2) is the classical contribution, with I being the action for the background. Let us now apply these formulas.

5.1 The $S^2 \times S^2$ instanton

The classical action is $I[S^2 \times S^2] = -2\pi/\Lambda G$, and according to Tab.1,

$$\Omega_0 = 1\,, \qquad \Omega_2 = \frac{\mu_0}{2i\sqrt{2}}\,. \tag{5.10}$$

Consider now the isometry factor Ω_1 in (5.5), which is due to the background SO(3)×SO(3) symmetry. Each of the two SO(3) groups can be parameterized by matrices $U_{ik} = \exp(\varepsilon_{ikj}C_j)$. The invariant metric on the SO(3) space is $g_{ik} = \frac{1}{2}\mathrm{tr}(\partial_i U \partial_k U^{-1}) \to \delta_{ik}$ for $C_j \to 0$. The Haar measure is $d\mu(C) = \sqrt{\det g_{ik}}\, dC_1 dC_2 dC_3$, and the volume Vol(SO(3))$= \int d\mu(C) = 8\pi^2$. For later use, we reproduce this result in a different way. The measure for a (compact, semi-simple) Lie group $\mathcal{G}$ can be represented as the product of the measure for the maximal subgroup $\mathcal{H}$ and that for the coset $\mathcal{G}/\mathcal{H}$. This implies that

$$Vol(\mathcal{G}) = Vol(\mathcal{H}) \times Vol(\mathcal{G}/\mathcal{H})\,. \tag{5.11}$$

In particular, Vol(SO(3))=Vol(SO(2))×$Vol(S^2)$, where Vol(SO(2))$= 2\pi$, and the volume of the S^2 coset with unit (due to the normalization of the measure) radius is $Vol(S^2) = 4\pi$. As a result, Vol(SO(3))$=2\pi \times 4\pi = 8\pi^2$.

When acting on S^2, the SO(3) generators $\frac{\partial}{\partial C_j}$ generate rotations in the three orthogonal planes of the embedding Euclidean 3-space. Let $\frac{\partial}{\partial C_3}$ be the generator of rotations in the XY-plane, such that the azimuthal angle of the spherical coordinate system changes as $\varphi \to \varphi + C_3$. Then the norm $||\frac{\partial}{\partial C_3}||$ is the square root of

$$\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \rangle = \frac{1}{32\pi G} \int\limits_{S^2 \times S^2} g_{\varphi\varphi}\sqrt{g}\, d^4x = \frac{\pi}{3\Lambda^3 G}\,. \tag{5.12}$$

Obviously, the norms $||\frac{\partial}{\partial C_1}||$ and $||\frac{\partial}{\partial C_2}||$ and those of the generators of the second SO(3) factor are the same. Hence,

$$\Omega_1 = \left(\frac{\mu_0^2}{\sqrt{\pi}}\, \Lambda \left|\left| \frac{\partial}{\partial\varphi} \right|\right| \right)^6 (Vol(\mathrm{SO}(3)))^2 = \frac{64\pi^4(\mu_0)^{12}}{27(\Lambda G)^3}\,. \tag{5.13}$$

Consider now the positive modes. The ζ-function associated with the positive vector modes is (see Tab.1)

$$\zeta_1(s) = \sum_{j=2}^{\infty} \frac{2(2j+1)}{\{j(j+1)-2\}^s} + \sum_{j_1=2}^{\infty}\sum_{j_2=2}^{\infty} \frac{3(2j_1+1)(2j_2+1)}{\{j_1(j_1+1)+j_2(j_2+1)-2\}^s}. \quad (5.14)$$

This can be represented as

$$\zeta_1(s) = 4^s \left(2\,\zeta(2,-9|s) + 3\,Z(1,-10|s)\right), \quad (5.15)$$

where the following two functions have been introduced:

$$\zeta(k,\nu|s) = \sum_{j=k}^{\infty} \frac{2j+1}{\{(2j+1)^2+\nu\}^s}\,, \quad (5.16)$$

$$Z(k,\nu|s) = \sum_{j_1=k}^{\infty}\sum_{j_2=k}^{\infty} \frac{(2j_1+1)\,(2j_2+1)}{\{(2j_1+1)^2+(2j_2+1)^2+\nu\}^s}\,. \quad (5.17)$$

These functions are studied in detail in the Appendix. Similarly, using the results of Tab.1 one obtains the ζ-function for the positive tensor modes

$$\zeta_2(s) = 9\times 2^{-s} + 4^s \left(2\,\zeta(2,-9|s) + 18\,\zeta(2,-1|s) + 5\,Z(2,-10|s)\right). \quad (5.18)$$

The following relation implied by the definitions in (5.16), (5.17), will be useful: $Z(1,-10|s) = Z(2,-10|s) + 6\zeta(2,-1|s) + 9\times 8^{-s}$.

5.1.1 *The scaling behaviour*

Before we proceed further, it is very instructive to pause and check whether the expressions above agree with the general formulas for the scaling behaviour of effective actions. We shall follow the approach of Christensen and Duff [13], who relate this scaling behaviour to

$$N_0 = \frac{1}{180\,(4\pi)^2}\int (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 636\Lambda^2)\sqrt{g}\,d^4x\,,$$
$$N_1 = \frac{1}{180\,(4\pi)^2}\int (-11R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 984\Lambda^2)\sqrt{g}\,d^4x\,,$$
$$N_2 = \frac{1}{180\,(4\pi)^2}\int (189R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 756\Lambda^2)\sqrt{g}\,d^4x\,. \quad (5.19)$$

Here N_0 is the 'number of eigenvalues' of the scalar operator $\Delta_0 - 2\Lambda$ acting on a manifold with $R_{\mu\nu} = \Lambda g_{\mu\nu}$. N_1 is the number of eigenvalues of the vector operator Δ_1 acting in the space of all vectors, that is, including both transverse and longitudinal fluctuations. Finally, N_2 counts both transverse and longitudinal eigenstates of the tensor operator Δ_2, with the only requirement that the fluctuations must be traceless.

Let us apply these formulas to the $S^2 \times S^2$ background. The volume of the manifold is $V_{S^2\times S^2} = (4\pi)^2/\Lambda^2$, while $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 8\Lambda^2$. As a result,

$$N_0 = \frac{161}{45}\,, \quad N_1 = \frac{224}{45}\,, \quad N_2 = \frac{21}{5}\,. \tag{5.20}$$

Now let us obtain the same result via a direct evaluation of the ζ-functions. First we consider the scalar case. Using the results of Tab.1, the operator $\Delta_0 - 2\Lambda$ has one negative mode, six zero modes, while the rest of the spectrum is positive and gives rise to the ζ-function

$$\zeta_0(s) = 4^s\left(2\,\zeta(2,-9|s) + Z(1,-10|s)\right). \tag{5.21}$$

Hence the number of all eigenvalues is $7 + \zeta_0(0)$. In order to compute $\zeta_0(0)$, we use the results of the Appendix, where the following values are obtained:

$$\zeta(k,\nu|0) = \frac{1}{12} - \frac{1}{4}\,\nu - k^2\,, \tag{5.22}$$

$$Z(k,\nu|0) = \frac{1}{32}\,\nu^2 - \frac{1}{24}\,\nu + 2k^4 + (\frac{1}{2}\,\nu - \frac{2}{3})\,k^2 + \frac{13}{360}\,. \tag{5.23}$$

This gives for the ζ-functions in (5.15), (5.18), (5.21)

$$\zeta_0(0) = -\frac{154}{45}\,, \quad \zeta_1(0) = -\frac{18}{5}\,, \quad \zeta_2(0) = \frac{38}{9}. \tag{5.24}$$

Using these, the number of scalar eigenvalues is $N_0 = 7 - \frac{154}{45} = \frac{161}{45}$, which agrees with (5.20).

Next, the vector operator Δ_1 has 6 zero modes, such that the number of its eigenvalues in the transverse sector is $6 + \zeta_1(0)$. Now, one should take into account also the longitudinal vectors, which are gradients of scalars. It is not difficult to see that if $\nabla_\mu\chi$ is an eigenvector of Δ_1, such that $\Delta_1\nabla_\mu\chi = \sigma\nabla_\mu\chi$, then $(\Delta_0 - 2\Lambda)\chi = \sigma\chi$. We see that the eigenfunctions of $\Delta_0 - 2\Lambda$ are in one-to-one correspondence with the longitudinal vectors. The number of the latter is therefore $N_0 - 1$, where the one is subtracted because the ground state scalar eigenfunction is constant, which vanishes upon differentiation. We therefore conclude that $N_1 = 6 + \zeta_1(0) + N_0 - 1 = 6 - \frac{18}{5} + \frac{161}{45} - 1 = \frac{224}{45}$, which also agrees with (5.20).

Finally, the number of traceless eigenvalues of Δ_2 is $1 + \zeta_2(0)$ (here the one is the contribution of the negative mode) plus the number of longitudinal traceless tensor harmonics $\phi^{\rm L}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\rho \xi^\rho$.

Now, if $\Delta_1 \xi_\mu = \sigma \xi_\nu$ then for $\phi^{\rm L}_{\mu\nu}$ associated with ξ_μ one has $\Delta_2 \phi^{\rm L}_{\mu\nu} = \sigma \phi^{\rm L}_{\mu\nu}$. Hence, the number of longitudinal tensors is determined by the number of vectors, which gives $N_2 = 1 + \zeta_2(0) + (N_1 - 6)$. Here six is subtracted because the six Killing vectors do not contribute to the tensor spectrum, since for Killing vectors one has $\phi^{\rm L}_{\mu\nu} = 0$. We therefore obtain $N_2 = 1 + \frac{38}{9} + \frac{224}{45} - 6 = \frac{21}{5}$, which again is in perfect agreement with (5.20).

The overall scale dependence of the partition function is given by the factor $(\mu_0)^{N_2 + N_0 - 2N_1}$. For the $S^2 \times S^2$ instanton one has $N_2 + N_0 - 2N_1 = -\frac{98}{45}$, and we shall shortly see that this agrees with our analysis.

5.1.2 The partition function $Z[S^2 \times S^2]$

It is now a simple task to insert the formulas above into the expression for the partition function. We obtain

$$\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}} = \exp\left(\zeta'(0) + \ln \mu_0^2\, \zeta(0)\right) , \tag{5.25}$$

where

$$\zeta(s) \equiv \frac{1}{2}\left(\zeta_2(s) - \zeta_1(s)\right) = -9 \times 2^{-s} + 4^s\, Z(2, -10|s)\,. \tag{5.26}$$

Using the values $Z(2, -10|0) = \frac{581}{45}$ and $Z'(2, -10|0) \equiv \Upsilon = -18.3118$ (see Eq.(A.51) in the Appendix) we find

$$\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}} = 2^{\frac{1567}{45}}\, \mu_0^{\frac{352}{45}}\, \mathrm{e}^{\Upsilon}\,. \tag{5.27}$$

Finally, taking into account the contributions of the negative, zero, and scalar modes computed in (5.10), together with the classical term, we obtain

$$\begin{aligned} Z[S^2 \times S^2] &= -i\, \frac{27\, (\Lambda G)^3}{256\, \pi^4 \mu_0^{10}} \sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}}\, \mathrm{e}^I \\ &= -i\, 0.3667 \times (\Lambda G)^3 \mu_0^{-\frac{98}{45}} \exp\left(\frac{2\pi}{\Lambda G}\right). \end{aligned} \tag{5.28}$$

This is our final result in the $S^2 \times S^2$ sector.

The classical action is $I[S^4] = -3\pi/\Lambda G$. Using the results in Tab.2 we find

$$\Omega_0 = \left(\sqrt{\frac{2}{3}}\frac{1}{\mu_0}\right)^5 , \qquad \Omega_2 = 1 . \tag{5.29}$$

Let us consider the symmetry factor Ω_1. The isometry group is now $\mathcal{H}$=SO(5), and this can be represented by matrices $U_{ik} = \exp(C_{ik})$, where $C_{ik} = -C_{ki}$, $i, k = 1, \ldots 5$. The 10 generators $\frac{\partial}{\partial C_{ik}}$ generate rotations of S^4 in the 10 orthogonal planes of the embedding Euclidean 5-space. Let $\frac{\partial}{\partial C_{12}}$ be the generator of rotations in the XY-plane, such that the standard azimuthal angle changes as $\varphi \to \varphi + C_{12}$. The norm $||\frac{\partial}{\partial C_{12}}||$ is the square root of

$$\langle\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\rangle = \frac{1}{32\pi G}\int_{S^4} g_{\varphi\varphi}\sqrt{g}\, d^4x = \frac{9\pi}{10\Lambda^3 G} , \tag{5.30}$$

which applies also to the the norms of the remaining 9 generators.

The volume of SO(5) can be computed by directly constructing the invariant metric and the Haar measure with the use of the matrix representation $U_{ik} = \exp(C_{ik})$. The measure should be normalized such that for $C_{ik} \to 0$ it reduces to $\prod_{i<k} dC_{ik}$. However, it is much simpler to use the coset reduction formula (5.11). One has SO(5)/SO(4)=S^4 and SO(4)/SO(3)=S^3, such that Vol(SO(5))=$Vol(S^4)\times Vol(S^3)\times Vol$(SO(3)). We know that Vol(SO(3))=$8\pi^2$, while the volumes of unit S^3 and S^4 are $2\pi^2$ and $8\pi^2/3$, respectively. As a result, Vol(SO(5))=$128\pi^6/3$. Summarizing,

$$\Omega_1 = \left(\frac{\mu_0^2}{\sqrt{\pi}}\Lambda\left|\left|\frac{\partial}{\partial\varphi}\right|\right|\right)^{10} Vol(\mathrm{SO}(5)) = \left(\frac{9}{10}\right)^5 \frac{128\pi^6}{3}\,\frac{(\mu_0)^{20}}{(\Lambda G)^5} . \tag{5.31}$$

Let us consider the positive modes. The ζ-function associated with the positive vector modes is (see Tab.2)

$$\zeta_1(s) = \frac{1}{2}\,3^s\sum_{j=2}^{\infty}\frac{j(j+3)(2j+3)}{\{j(j+3)-4\}^s} . \tag{5.32}$$

This can be written as

$$\zeta_1(s) = \frac{1}{2}\,3^s\,\mathcal{Q}(1,-4,0|s) , \tag{5.33}$$

where the following function has been introduced

$$\mathcal{Q}(k,\nu,c|s) = \sum_{j=k}^{\infty} \frac{(2j+3)(j(j+3)+c)}{\{j(j+3)+\nu\}^s}, \tag{5.34}$$

Similarly, using the results of Tab.2, one obtains the ζ-function for the positive tensor modes

$$\zeta_2(s) = \frac{5}{6}\, 3^s\, \mathcal{Q}(2,0,-4|s). \tag{5.35}$$

Finally, consider the scalar operator $\Delta_0 - 2\Lambda$. According to Tab.2, its eigenvalues, measured in units of Λ, are given by $(j(j+3)-6)/3$, and the degeneracy is $(j+1)(j+2)(2j+3)/6$ with $j \geq 0$. Hence, the ζ-function for the positive scalar modes is

$$\zeta_0(s) = \frac{1}{6}\, 3^s\, \mathcal{Q}(2,-6,-4|s). \tag{5.36}$$

5.2.1 The scaling behaviour

Let us again check the consistency with the general formulas for the scaling behaviour of quantum fields (for fluctuations around S^4 this was done by Christensen and Duff [13]). Applying again the formulas in (5.19), where now the volume of the manifold is $V_{S^4} = 24\pi^2/\Lambda^2$, while $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 8\Lambda^2/3$, one has

$$N_0 = \frac{479}{90}, \quad N_1 = \frac{358}{45}, \quad N_2 = -\frac{21}{10}. \tag{5.37}$$

On the other hand, using the result of the Appendix,

$$\begin{aligned}\mathcal{Q}(k,\nu,c|0) = &-\frac{1}{2}k^4 - 2k^3 - (c+\frac{1}{2})k^2 \\ &+ (3-2c)k + \frac{3}{2}\nu^2 + \frac{1}{3}c - \frac{11}{15},\end{aligned} \tag{5.38}$$

one obtains for the ζ-functions in (5.32), (5.33), (5.35)

$$\zeta_0(0) = -\frac{61}{90}, \quad \zeta_1(0) = -\frac{191}{30}, \quad \zeta_2(0) = -\frac{61}{90}. \tag{5.39}$$

Now, since the spectrum of $\Delta_0 - 2\Lambda$ contains six non-positive modes, one has $N_0 = 6 + \zeta_0(0) = 6 - \frac{61}{90} = \frac{479}{90}$, which agrees with (5.37). Next, Δ_1 has 10 zero modes, such that there are $10 + \zeta_1(0)$ transverse vector eigenstates, plus

(N_0-1) longitudinal ones (the constant scalar mode gives no contribution). As a result, $N_1 = 10 - \frac{191}{30} + \frac{479}{90} - 1 = \frac{358}{45}$, which agrees with (5.37). Finally, there are $N_2 = \zeta_2(0) + N_1 - 15$ traceless tensor modes, where 15 is subtracted because 10 Killing vectors and 5 conformal Killing vectors of S^4 do not contribute to the longitudinal tensor modes. One obtains $N_2 = -\frac{61}{90} + \frac{358}{45} - 15 = -\frac{21}{10}$, which again agrees with (5.37).

The overall scale dependence of the partition functions is expected to be $(\mu_0)^{N_2+N_0-2N_1}$, where $N_2 + N_0 - 2N_1 = -\frac{571}{45}$.

5.2.2 The partition function $Z[S^4]$

Let us now obtain the partition function. One finds

$$\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}} = \exp\left(\zeta'(0) + \ln\mu_0\,\zeta(0)\right)\,, \tag{5.40}$$

where

$$\zeta(s) \equiv \frac{1}{2}\left(\zeta_2(s) - \zeta_1(s)\right) = 3^s\left(\frac{5}{12}\,\mathcal{Q}(2,0,-4|s) - \frac{1}{4}\,\mathcal{Q}(2,-4,0|s)\right). \tag{5.41}$$

One has $\zeta(0) = \frac{509}{90}$ and $\zeta'(0) \equiv \Upsilon_1 = 6.1015$ (see Eq.(A.36) in the Appendix). This yields

$$\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}} = \mu_0^{\frac{509}{45}}\,\mathrm{e}^{\Upsilon_1}\,. \tag{5.42}$$

Finally, collecting the contributions of the negative, zero, and scalar modes computed in (5.10), together with the classical term, we obtain

$$\begin{aligned} Z[S^4] &= \frac{\sqrt{3}\,5^5}{3^{12}\pi^6\mu_0^{24}}\sqrt{\frac{\mathrm{Det}'\Delta_1}{\mathrm{Det}'\Delta_2}}\,\mathrm{e}^I \\ &= 0.0047\times(\Lambda G)^5\mu_0^{-\frac{571}{45}}\exp\left(\frac{3\pi}{\Lambda G}\right). \end{aligned} \tag{5.43}$$

To our knowledge, this formula has been obtained here for the first time, since in Refs.[27,13] a closed expression for $Z[S^4]$ was not achieved. In particular, the isometry factor Ω_1 was not taken into account and the derivative of the ζ-function was not computed.

Our last step is to use the expressions for $Z[S^2\times S^2]$ and $Z[S^4]$ in (5.28) and (5.43) and insert these into Eq.(2.15) to find the decay rate

$$\begin{aligned}\Gamma &= -\frac{1}{\pi}\sqrt{\frac{\Lambda}{3}}\,\frac{\Im Z[S^2\times S^2]}{Z[S^4]}\\ &= 14.338\,\sqrt{\Lambda}\,(G\Lambda)^{-2}(\mu_o\Lambda)^{\frac{473}{45}}\exp\left(-\frac{\pi}{\Lambda G}\right).\end{aligned} \tag{6.1}$$

This is the final result of our analysis. This formula gives the rate of semiclassical decay of de Sitter space due to the spontaneous nucleation of black holes. This is the leading mode of decay, since classically de Sitter space is stable [28]. The numerical coefficient in the formula originates from the fluctuation determinants evaluated in the ζ-function scheme. The factor $\sqrt{\Lambda}$ comes from the heat bath temperature coefficient in (2.15) and gives Γ the correct dimension of an inverse time. The coefficient $(G\Lambda)^{-2}$ arises due to the combined effect of the background isometries. The power of $\mu_o\Lambda$ contains the effect of rescalings, where we have passed again to the dimensionful renormalization parameter μ_o. Since quantum gravity is non-renormalizable, μ_o remains undetermined, and we have nothing to say about this problem. For numerical estimates it is reasonable to assume that $\mu_o\sim G$. The last factor in the formula is the classical term. The formula is obtained in the one-loop approximation, which is good as long as the classical term is large compared to the quantum corrections, that is for $\Lambda G\ll 1$. Under this condition the nucleation rate is exponentially small. Notice that since the overall power of Λ is positive, the quantum corrections provide an additional suppression of the transition rate for small Λ.

The formula gives the probability of black hole nucleation per unit proper time of a freely falling observer in his Hubble region. The latter is the region enclosed inside the observer's cosmological horizon. If a black hole is created, then it has the radius $1/\sqrt{\Lambda}$ and fills the whole Hubble region. This does not mean that the whole space will be eaten by a giant black hole, since de Sitter spacetime consists of many Hubble regions, whose number grows as the universe expands. Some of these regions will contain a black hole but most of them will be empty. The black holes are actually born in pairs, where the two members of the pair are created at the opposite sides of the 3-space. The interesting conclusion is that for $G\Lambda\ll 1$, when inflation is 'slow', the rate of black hole nucleation is strongly suppressed, but the created black holes are large. This can be understood as a consequence of the fact that the black holes are made of the energy contained inside the Hubble region. As the size of the latter is large for small Λ, the created black holes are also large. On the other hand, if one is allowed to extrapolate the formula for $G\Lambda\sim 1$, when inflation is fast, then the created black holes are small, but they are created in abundance.

One can see that for late times the number of black holes per unit physical volume will be constant. Let us choose for de Sitter spacetime the global coordinates associated with the freely falling observers:

$$ds^2 = -d\eta^2 + \frac{3}{\Lambda}\cosh^2\left(\sqrt{\frac{\Lambda}{3}}\,\eta\right)\, d\Omega_3^2\,. \tag{6.2}$$

Here η is the (dimensionful) proper time and $d\Omega_3^2$ is the volume element of the unit 3-sphere. The volume of the global hypersurface Σ_η of constant η is $V(\eta) = 2\pi^2\left(\frac{3}{\Lambda}\right)^{3/2}\cosh^3\left(\sqrt{\frac{\Lambda}{3}}\,\eta\right) \approx \frac{\pi^2}{4}\left(\frac{3}{\Lambda}\right)^{3/2}\exp(\sqrt{3\Lambda}\eta)$. The portion of Σ_η contained inside the future event horizon of any observer has the volume $V_{\mathrm{H}} = \frac{4\pi}{3}\left(\frac{3}{\Lambda}\right)^{3/2}$ (for late η). This is the spatial Hubble volume. [This quantity slightly depends on the choice of the hypersurface. Even though for any given observer one has $\eta = t$, which is the time associated with the observer's coordinate system, one has $\Sigma_\eta \neq \Sigma_t$, unless $\eta = t = 0$, in which case the spatial Hubble volume is $V_{\mathrm{H}} = \pi^2\left(\frac{3}{\Lambda}\right)^{3/2}$]. As a result, the number of Hubble volumes on the hypersurface is $N_{\mathrm{H}}(\eta) = V(\eta)/V_{\mathrm{H}}$. [One has $N_{\mathrm{H}}(0) = 2$: the de Sitter throat consists of two causally disconnected parts belonging to the Hubble regions of two antipodal observers [47].] Multiplying $N_{\mathrm{H}}(\eta)$ by Γ gives the black hole nucleation rate per Σ_η,

$$\frac{dN_{\mathrm{BH}}}{d\eta} = \frac{3\pi}{16}\exp(\sqrt{3\Lambda}\eta)\,\Gamma\,. \tag{6.3}$$

Integrating with respect to η and dividing by $V(\eta)$ yields the average volume density of created black holes on Σ_η,

$$\rho_{\mathrm{BH}} = \frac{\Lambda}{12\pi}\,\Gamma\,, \tag{6.4}$$

which does not depend on η.

The subsequent real time evolution of these black holes is an interesting issue. Presumably most of them will immediately evaporate, unless Λ is very small and the black holes are large, in which case however the nucleation rate is strongly suppressed. It was argued in [9] that this process could dramatically change the global structure of de Sitter space. For more information on this issue we refer to [10,9,18] and to the papers cited in Ref.[9].

The following steps have been essential in our analysis. We have derived Eq.(2.15) for the nucleation rate using the thermal properties of de Sitter space. For this we have approximated the partition function for Euclidean quantum gravity with $\Lambda > 0$ by the semiclassical contributions of the S^4 and $S^2 \times S^2$ instantons, of which the first yields the free energy F in the Hubble volume while the contribution of the second can be regarded as a purely imaginary part of F. In a sense one can think of the created black holes as being the bubbles of the new phase spontaneously created out of thermal fluctuations via quantum tunneling. We have argued that these bubbles may have temperature different from that of the heat bath, since they cannot thermalize via interactions with the whole reservoir and only exchange energy inside the Hubble region.

To compute the one-loop contributions of the S^4 and $S^2 \times S^2$ instantons we have used the standard Faddeev-Popov approach to the path integral. We have worked with a one-parameter family of covariant background gauges and employed the Hodge decomposition of the fluctuations with their subsequent spectral expansion. In our treatment of the conformal modes we have followed the standard recipe of complex rotation, up to several lowest lying modes for which a different prescription has been applied. In order to integrate over zero modes of the Faddeev-Popov operator arising due to the background isometries, we have gone beyond the perturbation theory and showed that the corresponding integration measure is the Haar measure on the isometry group. There are no other zero modes in the problem – for example, the standard rotational zero modes are absent because rotations are isometries of the backgrounds under consideration.

We have explicitly determined the spectra of the fluctuation operators. For fluctuations around the $S^2 \times S^2$ instanton the spectrum was obtained by directly solving the differential equations, while in the S^4 case group theoretic methods have been applied, in which we followed the approach of [27]. These spectra have been used in order to compute the functional determinants within the ζ-function regularization scheme, the corresponding ζ-functions being studied in detail in the Appendix below. We have checked that our results agree with the general formulas for the anomalous scaling behaviour. Finally, we have obtained in (5.28), (5.43) the one-loop partition functions for fluctuations around the S^4 and $S^2 \times S^2$ backgrounds. To our knowledge, in both cases such closed expressions have been obtained for the first time. The last step has been to use the resulting partition functions in order to calculate the nucleation rate Γ. This describes a constant density of created black holes per unit physical volume of the expanding 3-space.

After the work of Gross, Perry and Jaffe [30], our analysis presents the second example of a complete one-loop computation on a non-trivial background. [3]

[3] Note also that the analysis in [30] was not quite complete, since the spectrum is unknown and the ζ-functions have not been computed, even though the undetermined quantities can be absorbed into the renormalization parameter. We also

One may hope that our results can lend further support to the Euclidean approach to quantum gravity.

Acknowledgments

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Note added in proof. We would like to thank Dima Vassilevich for bringing to our attention a number of relatively recent papers considering one-loop Euclidean quantum gravity on S^4. Although in none of these papers a closed expression for the one-loop partition function $Z[S^4]$ is achieved, it is worth mentioning the work by Allen [50], by Polchinski [51], and by Taylor and Veneziano [52]. We refer to the paper by Vassilevich [53] for more references. Not all papers agree on the scaling behaviour of the partition function. The reason is that some authors do not take into account the contribution of the 10 zero modes due to the background isometries, thereby obtaining $Z[S^4]$ to be proportional to $\mu_0^{+\frac{329}{45}}$ instead of $\mu_0^{-\frac{571}{45}}$ [52]. However, since these zero modes are in the path integration measure, they do contribute to the anomalous scaling on equal footing with all other modes. In fact, the example of flat space gauge theories [45] shows that the background symmetry zero modes, when treated non-perturbatively as was done above, are of vital importance for obtaining the correct running behaviour of the coupling constant. Our result for the scaling behaviour agrees with that of Christensen and Duff [13] and with the general analysis of Fradkin and Tseytlin [54].

Appendix. Calculation of ζ-functions.

In this Appendix we shall study the ζ-function

$$Z(k,\nu|s) = \sum_{n=k}^{\infty}\sum_{m=k}^{\infty} \frac{(2n+1)\,(2m+1)}{\{(2n+1)^2+(2m+1)^2+\nu\}^s}\,, \tag{A.1}$$

which is used in the main text for computing the one-loop fluctuation term on the $S^2\times S^2$ instanton background. Here ν is real while k is a positive integer

do not understand their treatment of the background isometries and that of the non-normalizable deformations of the instanton.

such that $2(2k+1)^2+\nu>0$. It is assumed that $\Re(s)$ is positive and large enough to ensure the convergence of the series. Despite its apparent simplicity, the analysis of this ζ-function is lacking in the literature. This is probably due to the fact that the summation in (A.1) cannot be extended to all integers and the standard Poisson resummation techniques do not apply. For this reason we use other methods, which are unfortunately rather lengthy. However we think that it is necessary to describe the basic steps, especially in view of other possible applications of our results.

In what follows we shall perform the analytic continuation by finding the integral representation for $Z(k,\nu|s)$ that is valid for any s. This will be used to compute the values of $Z(k,\nu|0)$ and $\frac{d}{ds}Z(k,\nu|s)$ at $s=0$. As a first step, we shall consider the related ζ-function:

$$\zeta(k,\nu|s)=\sum_{n=k}^{\infty}\frac{(2n+1)}{\{(2n+1)^2+\nu\}^s} \tag{A.2}$$

with $(2k+1)^2+\nu>0$. The integral representation for this function will be useful. In addition, we shall study the ζ-function

$$\mathcal{Q}(k,\nu,c|s)=\sum_{j=k}^{\infty}\frac{(2j+3)(j(j+3)+c)}{\{j(j+3)+\nu\}^s}\,, \tag{A.3}$$

where $k(k+3)+\nu>0$, and shall find its value and its s-derivative at $s=0$. This function is needed in the analysis of fluctuations around the S^4 instanton.

A.1 Computation of $Z(k,\nu|0)$ and $\zeta(k,\nu|0)$.

First we shall compute the values of these functions at $s=0$ using the standard heat kernel technique. These values determine the scaling properties of the system. Later we shall rederive the same values by using the integral representations for $Z(k,\nu|s)$ and $\zeta(k,\nu|s)$, and this will provide us with a good consistency check. For $\mathcal{Q}(k,\nu,c|s)$ we shall consider only the integral representation, since the values of $\mathcal{Q}(k,\nu,c|0)$ have been computed in [13].

A ζ-function related to a second order elliptic operator with a positive spectrum can be expressed as

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\Theta(t)\,dt\,. \tag{A.4}$$

On compact spaces the heat kernel $\Theta(t)$ vanishes exponentially fast for large

t, while for small t there is the asymptotic expansion

$$\Theta(t) \sim \sum_r C_r \, t^r \,, \tag{A.5}$$

with r assuming in general both integer and half-integer values. It is not difficult to see that

$$\zeta(0) = C_0 \,. \tag{A.6}$$

The problem therefore reduces to determining the asymptotic expansion of the heat kernel. The heat kernels in our problem are given by

$$\Theta(k,\nu|t) = \Big(\theta(t) - \xi(k|t)\,\Big)^2 \, \mathrm{e}^{-\nu t} \tag{A.7}$$

for $Z(k,\nu|s)$ and

$$\theta(k,\nu|t) = \Big(\theta(t) - \xi(k|t)\,\Big) \, \mathrm{e}^{-\nu t} \tag{A.8}$$

for $\zeta(k,\nu|s)$, where

$$\theta(t) = \sum_{n=0}^{\infty} (2n+1) \, \mathrm{e}^{-t\,(2n+1)^2} \tag{A.9}$$

and

$$\xi(k|t) = \sum_{n=0}^{k-1} (2n+1) \, \mathrm{e}^{-t\,(2n+1)^2} \,. \tag{A.10}$$

The only difficulty is to find the asymptotic expansions for small t for the function $\theta(t)$ in (A.9) [4]. $\theta(t)$ is a partition function for a two-dimensional rotator at temperature $1/t$. We wish therefore to find its high-temperature expansion, and for this we shall construct the integral representation for $\theta(t)$.

Let us consider the "generating function"

$$\chi(t,\alpha) = \sum_{n=0}^{\infty} \mathrm{e}^{-t\,(2n+1)^2 + i\alpha\,(2n+1)} \tag{A.11}$$

[4] We note that $\theta(t)$ cannot be expressed in terms of theta-functions in a simple way, and that the Poisson resummation formula does not directly apply.

such that

$$\theta(t) = -i \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \chi(t, \alpha) . \tag{A.12}$$

$\chi(t, \alpha)$ fulfills the differential equation

$$\frac{\partial \chi}{\partial t} = \frac{\partial^2 \chi}{\partial \alpha^2} . \tag{A.13}$$

This has the special solution

$$\tilde{\chi}(t, \alpha) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(\alpha - \alpha_0)^2}{4t}\right) \tag{A.14}$$

with the property $\tilde{\chi}(0, \alpha) = \delta(\alpha - \alpha_0)$, which allows us to represent the general solution of (A.13) as

$$\chi(t, \alpha) = \int_{-\infty}^{\infty} \tilde{\chi}(t, \alpha_0)\, \chi(0, \alpha_0)\, d\alpha_0 . \tag{A.15}$$

The initial value $\chi(0, \alpha_0)$ is obtained directly from the definition (A.11):

$$\chi(0, \alpha_0) = \sum_{n=0}^{\infty} \mathrm{e}^{i\alpha_0\,(2n+1)} = \frac{i}{2 \sin \alpha_0} , \tag{A.16}$$

where we assume that α_0 has a small positive imaginary part in order to ensure convergence of the geometrical series. We can now insert this into (A.15) and the result into (A.12). Introducing the new variable $x = \alpha_0^2/4$ we obtain the sought for integral representation

$$\theta(t) = \frac{1}{\sqrt{4\pi t^3}} \int_0^{\infty} \mathrm{e}^{-x/t} \frac{dx}{\sin(2\sqrt{x})} . \tag{A.17}$$

Here we should remember that x has a small imaginary part, such that the integration is actually performed along a contour parallel to the positive real axis and approaching it from above.

It is now a straightforward task to find the asymptotic expansion of the integral in (A.17) for small t, since the only non-trivial contribution comes from a small neighbourhood of $x = 0$:

$$\theta(t) \sim \frac{1}{4t} \left(1 + \frac{1}{3}\, t + \frac{7}{30}\, t^2 + O(t^3)\right) . \tag{A.18}$$

Inserting this into (A.7) and (A.8) gives the asymptotic expansions for the heat kernels $\Theta(k,\nu|t)$ and $\theta(k,\nu|t)$, whose coefficients C_0 determine the ζ-functions at $s=0$:

$$Z(k,\nu|0) = \frac{1}{32}\,\nu^2 - \frac{1}{24}\,\nu + \frac{1}{2}\,k^2\nu + 2k^4 - \frac{2}{3}\,k^2 + \frac{13}{360}\,, \tag{A.19}$$

and

$$\zeta(k,\nu|0) = \frac{1}{12} - \frac{1}{4}\,\nu - k^2\,. \tag{A.20}$$

To check these results we note that the definitions in (A.1) and (A.2) imply that

$$\begin{aligned} Z(k_1,\nu|s) = {} & Z(k_2,\nu|s) + 2\sum_{m=k_1}^{k_2-1}(2m+1)\zeta(k_2,\nu+(2m+1)^2|s) \\ & + \sum_{n=k_1}^{k_2-1}\sum_{m=k_1}^{k_2-1}\frac{(2n+1)\,(2m+1)}{\{(2n+1)^2+(2m+1)^2+\nu\}^s}\,, \end{aligned} \tag{A.21}$$

with $k_2 > k_1$. Setting here $s=0$ we obtain a non-trivial relation for $Z(k,\nu|0)$ and $\zeta(k,\nu|0)$, and this is fulfilled by the expressions in (A.19) and (A.20). Finally we use (A.19), (A.20) to obtain the values used in the main text:

$$Z(2,-10|0) = \frac{581}{45}\,, \quad \zeta(2,-9|0) = -\frac{5}{3}\,, \quad \zeta(2,-1|0) = -\frac{11}{3}\,. \tag{A.22}$$

A.2 Computation of $\zeta(k,\nu|s)$, and $\mathcal{Q}(k,\nu,c|s)$.

It is usually more difficult to determine the derivative of a ζ-function at $s=0$ than the value of the function itself, since the knowledge of its behaviour in a neighbourhood of $s=0$ is required. We shall perform the analytic continuation of the ζ-functions defined by Eqs. (A.1), (A.2) and (A.3) to arbitrary values of s with the use of the relation sometimes called

A.2.1 The Abel-Plan formula.

This can be derived using the obvious relation

$$\sum_{n=k}^{\infty} f(n) = \int_C \frac{f(z)}{e^{2\pi i z}-1}\,dz\,, \tag{A.23}$$

where the contour C encompasses the part of the real axis with $Re(z) \geq k$ (see Fig.4) and $f(z)$ is analytic for $Re(z) \geq k$. The idea is to split C into three parts, $C_1 + C_2 + C_3$, as shown in Fig.4. For the first part, C_1, the integral can be written as

$$\int\limits_{C_1} \left(\frac{1}{1 - \mathrm{e}^{-2\pi i z}} - 1 \right) f(z)\, dz = \int\limits_{k}^{\infty} f(t)\, dt + \int\limits_{\bar{C}_1} \frac{f(z)}{1 - \mathrm{e}^{-2\pi i z}}\, dz\,, \tag{A.24}$$

where in the integral over C_1 on the right the contour is then rotated to the position $\bar{C}_1$ as shown in Fig.4. Such a rotation is possible if only $f(z)$ tends to zero fast enough for $Re(z) \geq k$ and $|z| \to \infty$.

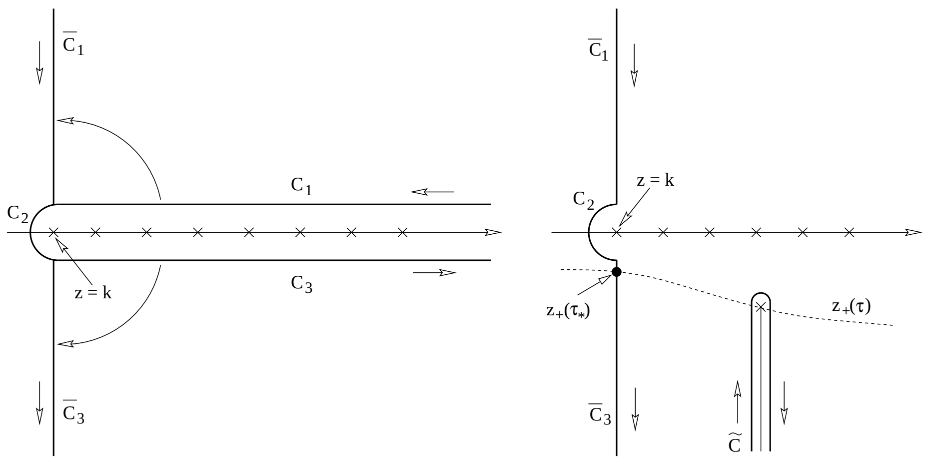

Fig. 4. Left*: Starting from the contour $C_1 + C_2 + C_3$ and rotating we arrive at $\bar{C}_1 + C_2 + \bar{C}_3$.* Right*: the same when a branching point at $z = z_+(\tau)$ is present. The contour C_1 will then wrap around the cut leading to the additional contribution due to $\tilde{C}$. The point $z = z_+(\tau)$ is in the region of interest for $\tau \geq \tau_*$.*

The integral over the second portion of the contour, C_2, is equal to $\frac{1}{2} f(k)$, while in the integral over C_3 the contour is rotated to the position $\bar{C}_3$ as shown in Fig.4. As a result, we arrive at the Abel-Plan formula

$$\sum_{n=k}^{\infty} f(n) = \frac{1}{2}\, f(k) + \int\limits_{k}^{\infty} f(t) dt + i \int\limits_{0}^{\infty} \frac{f(k+it) - f(k-it)}{\mathrm{e}^{2\pi t} - 1}\, dt\,. \tag{A.25}$$

This formula can be used for analytic continuation of ζ-functions, in which case $f(t)$ depends also on s, $f = f(t,s)$. The analytic continuation to small values of s is performed in the first integral on the right in (A.25). This usually converges only for $\Re(s)$ large and positive, but can often be computed in a closed form, and then one can continue the result to arbitrary s. The second integral on the right in (A.25) usually cannot be computed in a closed form, but it converges for any s. Let us first apply the Abel-Plan formula to the ζ-functions in (A.2) and (A.3).

A.2.2 Analytic continuation of $\zeta(k,\nu|s)$.

Applying (A.25) to the series for $\zeta(k,\nu|s)$ in (A.2), we have

$$f(z) = \frac{2z+1}{\{(2z+1)^2+\nu\}^s}\,, \tag{A.26}$$

which is analytic for $\Re(z) > -1/2$ and decays fast enough for $|z| \to \infty$ provided that $\Re(s)$ is large enough. As a result, we can use the Abel-Plan formula, which gives

$$\zeta(k,\nu|s) = \left(k+\frac{1}{2}\right)\frac{1}{\{(2k+1)^2+\nu\}^s} + \frac{1}{4(s-1)}\frac{1}{\{(2k+1)^2+\nu\}^{s-1}}$$
$$+\int_0^\infty \frac{idt}{e^{2\pi t}-1}\left(\frac{2k+1+2it}{\{(2k+1+2it)^2+\nu\}^s} - \frac{2k+1-2it}{\{(2k+1-2it)^2+\nu\}^s}\right). \tag{A.27}$$

This representation is finite for all s, apart from $s=1$, where the pole is located. The remaining integral here converges uniformly for $|s|<\infty$, which allows us to differentiate with respect to s. If we set $s=0$, then the integral can be easily computed. We find $\zeta(k,\nu|0) = \frac{1}{12} - \frac{1}{4}\nu - k^2$, and this agrees with the value obtained above in (A.20).

Similarly, we can differentiate (A.27) with respect to s and then set $s=0$. This gives

$$\zeta'(k,\nu|0) = \frac{1}{4}W(\ln W - 1) - \left(k+\frac{1}{2}\right)\ln W$$
$$+2\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left(t\,\ln\mathcal{A} + (2k+1)\Psi\right), \tag{A.28}$$

where $W = (2k+1)^2+\nu$ and

$$\mathcal{A} = (W-4t^2)^2 + 16(2k+1)^2t^2\,, \qquad \Psi = \arctan\frac{4(2k+1)\,t}{W-4t^2}\,. \tag{A.29}$$

For any k and ν the integral in (A.28) is convergent and can be evaluated numerically. Notice that $\zeta'(k,\nu|0)$ is not needed in the main body of the paper, and for this reason we do not quote the actual number here.

A.2.3 Analytic continuation of $Q(k,\nu,c|s)$.

The procedure is exactly the same as above. Denoting

$$f(z)=\frac{(2z+3)(z(z+3)+c)}{\{z(z+3)+\nu\}^s} \tag{A.30}$$

the direct application of the Abel-Plan formula (A.25) yields

$$\begin{aligned} Q(k,\nu,c|s) &= \left(k+\frac{3}{2}\right)(k(k+3)+c)\,W^{-s}+\frac{1}{s-2}\,W^{2-s} \\ &+\frac{c-\nu}{s-1}\,W^{1-s}+\int_0^\infty\frac{dt}{\mathrm{e}^{2\pi t}-1}\,i(f(k+it)-f(k-it))\,, \end{aligned} \tag{A.31}$$

where $W=k(k+3)+\nu$. Setting $s=0$ the integral can be easily computed leading to

$$\begin{aligned} \mathcal{Q}(k,\nu,c|0) &= -\frac{1}{2}\,k^4-2k^3-\left(c+\frac{1}{2}\right)k^2 \\ &+(3-2c)\,k+\frac{1}{2}\,\nu^2+\left(\frac{4}{3}-\nu\right)c-\frac{11}{15}\,. \end{aligned} \tag{A.32}$$

Next, differentiating (A.30) with respect to s and setting $s=0$ gives

$$\begin{aligned} Q'(k,\nu,c|0) &= -\left(k+\frac{3}{2}\right)(W+c-\nu)\ln W \\ &+\frac{1}{2}\left(\ln W-\frac{1}{2}\right)W^2+(c-\nu)\,(\ln W-1)\,W+\mathcal{G}\,. \end{aligned} \tag{A.33}$$

Here

$$\begin{aligned} \mathcal{G} &= \int_0^\infty\frac{dt}{\mathrm{e}^{2\pi t}-1}\,\{t\,(6k(k+3)+2c+9-2t^2)\ln\mathcal{A} \\ &+(4k^3+18k^2+(18+4c)\,k+6c-6\,(2k+3)\,t^2)\Psi\} \end{aligned} \tag{A.34}$$

with

$$\mathcal{A}=t^4+(2W-4\nu+9)\,t^2+W^2\,,\qquad \Psi=\arctan\frac{(2k+3)\,t}{W-t^2}\,. \tag{A.35}$$

Evaluating the integral numerically, the two values used in the main text are

$$Q'(2,0,-4|0)=3.72344\,,\qquad Q'(2,-4,0|0)=6.65246\,. \tag{A.36}$$

Finally, for the function $\zeta(s) = 3^s(\frac{5}{12}Q(2,0,-4|s) - \frac{1}{4}Q(2,-4,0|s))$ used in Eq.(5.41) in the main text one obtains with the help of (A.32) and (A.36)

$$\zeta(0) = \frac{509}{90}\,, \qquad \zeta'(0) \equiv \Upsilon_1 = 6.10158\,. \tag{A.37}$$

A.3 Computation of $Z(k,\nu|s)$

Let us not turn to our main task – the evaluation of the double-sum function $Z(k,\nu|s)$, which has been defined for large values of $\Re(s)$ by (A.1). The idea is to express it in terms of the single-sum function $\zeta(k,\nu|s)$.

It follows from the definitions (A.1) and (A.2) that

$$Z(k,\nu|s) = \sum_{n=k}^{\infty} (2n+1)\zeta(k,\nu+(2n+1)^2|s)\,. \tag{A.38}$$

Here we can use the integral representation (A.27) for $\zeta(k,\nu+(2n+1)^2|s)$. Indeed, if ν is real and $(2k+1)^2+\nu > 0$ then the same remains true upon replacement $\nu \to \nu+(2n+1)^2$, and the formula (A.27) therefore applies. Now, replacing in (A.27) ν by $\nu+(2n+1)^2$ and assuming for a moment that $\Re(s)$ is large and positive, the integral in (A.27) converges uniformly with respect to n for $n \to \infty$. This allows us, upon insertion of (A.27) into (A.38), to interchange the orders of summation and integration. The result then can be extended to any s by analytic continuation. This gives

$$\begin{aligned} Z(k,\nu|s) = \left(k+\frac{1}{2}\right)\sum_{n=k}^{\infty}\frac{2n+1}{\{(2k+1)^2+\nu+(2n+1)^2\}^s} + \\ +\frac{1}{4(s-1)}\sum_{n=k}^{\infty}\frac{2n+1}{\{(2k+1)^2+\nu+(2n+1)^2\}^{s-1}} \\ +\int_0^{\infty}\frac{id\tau}{e^{2\pi\tau}-1}\Bigg((2k+1+2i\tau)\sum_{n=k}^{\infty}\frac{2n+1}{\{(2k+1+2i\tau)^2+\nu+(2n+1)^2\}^s} \\ -(2k+1-2i\tau)\sum_{n=k}^{\infty}\frac{2n+1}{\{(2k+1-2i\tau)^2+\nu+(2n+1)^2\}^s}\Bigg)\,. \end{aligned} \tag{A.39}$$

One can see that all sums here are exactly the same as in the definition of $\zeta(k,\nu|s)$ in (A.2) – up to the replacements $\nu \to \nu+(2k+1)^2$ and $\nu \to \nu(\tau) \equiv \nu+(2k+1+2i\tau)^2$. Since the definition in (A.2) makes sense for arbitrary values of ν (the series always converges for $\Re(s)$ big enough), we can express the sums in (A.39) in terms of $\zeta(k,\nu+(2k+1)^2|s)$ and $\zeta(k,\nu(\tau)|s)$. This leads to the the following formula:

$$Z(k,\nu|s) = \left(k+\frac{1}{2}\right)\zeta(k,\nu+(2k+1)^2|s) \tag{A.40}$$
$$+\frac{1}{4(s-1)}\zeta(k,\nu+(2k+1)^2|s-1) + \int\limits_0^\infty \frac{i\,d\tau}{\mathrm{e}^{2\pi\tau}-1}\left\{\mathcal{F}(\tau)-\mathcal{F}(-\tau)\right\},$$

with $\mathcal{F}(\tau) = (2k+1+2i\tau)\,\zeta(k,\nu(\tau)|s)$. In this formula the first two terms on the right are determined by the integral representation (A.27) for arbitrary s. We are left with computing the remaining integral over τ. The problem here is that the parameter $\nu(\tau)$ is complex, and for this reason we cannot directly apply the integral representation (A.27) to compute $\zeta(k,\nu(\tau)|s)$.

Let us recall that the formula (A.27) was derived assuming that the function $f(z)$ in Eq.(A.26) had no poles for $\Re(z) > k$. This allowed us to rotate the integration contour as shown in the left part of Fig.4 without intersecting singularities. Let us now replace ν by $\nu(\tau) \equiv \nu + (2k+1+2i\tau)^2$. As a result, $f(z)$ in Eq.(A.26) is replaced by

$$f(z) = \frac{2z+1}{\left\{(2z+1)^2+\nu(\tau)\right\}^s} = \frac{2z+1}{\left\{4(z-z_+(\tau))(z-z_-(\tau))\right\}^s}, \tag{A.41}$$

with $z_\pm(\tau) = \frac{1}{2}(-1 \pm i\sqrt{\nu(\tau)})$. For $\tau = 0$ one has $\Re(z_\pm(0)) = -\frac{1}{2}$. As τ increases, the point $z_+(\tau)$ moves to the right in the complex plane (while $z_-(\tau)$ moves to the left), but as long as $\Re(z_+(\tau)) < k$ one can still use the formula (A.27). However, for large enough values of τ the pole at $z = z_+(\tau)$ enters the region of interest, that is the part of the complex plane with $\Re(z) > k$, and we can no longer use the formula (A.27).

To tackle the problem we notice that the pole of $f(z)$ at $z = z_+(\tau)$ is a branching point, and one can choose the cut in the complex plane as shown in the right part of Fig.4. We then repeat the steps leading to the Abel-Plan formula and the additional problem we encounter is the following: when we rotate the integration contour as we did before, the contour will wrap around the cut as shown in Fig.4. The resulting contour will then consist of two disconnected pieces. The first piece will be the same as the old contour $\bar{C}_1 + C_2 + \bar{C}_3$ (see Fig.4). The second piece is the contour $\tilde{C}$ wrapping around the cut. Integrating around such a combined contour, the result will consist of two parts,

$$\zeta(k,\nu(\tau)|s) = \zeta_{\text{old}}(k,\nu(\tau)|s) + \theta(\tau-\tau_*)\int\limits_{\tilde{C}} \frac{f(z)}{\mathrm{e}^{2\pi iz}-1}dz\,. \tag{A.42}$$

Here the first term on the right, $\zeta_{\text{old}}(k,\nu(\tau)|s)$, is the function given by the previous expression in (A.27) with ν being replaced by $\nu(\tau)$. The second term, with $f(z)$ given by (A.41) and the contour $\tilde{C}$ as shown in the right part of Fig.4, is the contribution of the cut. The step function $\theta(\tau - \tau_*)$ reflects the fact that the cut contributes only for large enough τ when the pole enters the region $\Re(z) > k$. Here $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$, and $\Re(z_+(\tau_*)) = k$.

The representation (A.42) applies for all values of s and for any $\tau > 0$. Similarly, one can obtain $\zeta(k, \nu(-\tau)|s)$ (the cut then resides in the upper half-plane). As a result, the function $\mathcal{F}(\tau) - \mathcal{F}(-\tau)$ in the integrand in Eq.(A.40) is defined for any $\tau > 0$, and the integral converges due to the damping exponential factor. This finally gives $Z(k, \nu|s)$ for any s.

Let us first check our result by computing $Z(k, \nu|0)$. For $s = 0$ the function $f(z)$ has no poles and the contribution of the cut vanishes. The remaining integrals then can be easily computed, which gives for $Z(k, \nu|0)$ exactly the same expression as in Eq.(A.19).

Let us now compute $Z'(k, \nu|0)$. Since all integrals in (A.40),(A.41) converge uniformly with respect to s (at least for $|s| < \infty$), we can differentiate the integrands with respect to s and then set $s = 0$. The result can be represented in the following form:

$$
\begin{aligned}
Z'(k, \nu|0) &= \mathcal{H} + 2 \int_0^\infty \frac{dt}{\mathrm{e}^{2\pi t} - 1}\, \mathcal{G}(t) \\
&+ \int_0^\infty \frac{d\tau}{\mathrm{e}^{2\pi\tau} - 1} \int_0^\infty \frac{dt}{\mathrm{e}^{2\pi t} - 1}\, \mathcal{W}(\tau, t) + \mathcal{S}.
\end{aligned} \tag{A.43}
$$

Here

$$
\begin{aligned}
\mathcal{H} &= \Big(-2k^4 + (2 - \frac{\nu}{2})\, k^2 + k + \frac{1}{32}\,(4 + 4\nu - \nu^2)\Big) \ln(\, 2\,(2k+1)^2 + \nu) \\
&+ 3k^4 + 2k^3 + \frac{3}{4}\,(\nu - 2)\, k^2 + \frac{1}{4}\,(\nu - 6)\, k + \frac{1}{64}\,(3\nu^2 - 4\nu - 20)\,.
\end{aligned} \tag{A.44}
$$

In addition,

$$
\begin{aligned}
\mathcal{G}(t) = &\frac{t}{2}\,(4t^2 - 16k^2 - 12k - 2 - \nu) \ln P \\
&+ (2k+1)(6t^2 - 2k\,(2k+1) - \frac{\nu}{2})\,\Phi\,,
\end{aligned} \tag{A.45}
$$

where we have used

$$
P = \nu^2 + (4(2k+1)^2 - 8t^2)\,\nu + 4(2k+1)^4 + 16t^4\,,
$$

$$\Phi = \text{Phase}[(2k+1)^2 + \nu/2 - 2t^2 + i\,2(2k+1)t]\,, \tag{A.46}$$

and $-\pi <$Phase$[x+iy] \leq \pi$ is the phase of the complex number. Next,

$$\mathcal{W}(\tau,t) = \{((2k+1)^2 - 4\,t\,\tau)\ln Q \tag{A.47}$$
$$+ 4\,(t-\tau)(2k+1)\Psi\} - \{(t,\tau) \leftrightarrow (t,-\tau)\} \tag{A.48}$$

with

$$Q = 16\,(t^2+\tau^2)^2 + (\nu+2)^2 - 8\nu\,(t^2+\tau^2) + 128\,(k^2+k+\frac{1}{4})\,t\,\tau$$
$$+ 16\,k(k+1)\,\nu + 32\,k(k+1)(2k^2+2k+1),$$
$$\Psi = \text{Phase}[(2k+1)^2 + \nu/2 - 2\,(t^2+\tau^2) + i\,2\,(2k+1)(t-\tau)]\,. \tag{A.49}$$

Finally, the contribution of the cut is

$$\mathcal{S} = 4\pi \int_{\tau_*}^{\infty} \frac{d\tau}{\mathrm{e}^{2\pi\tau}-1} \int_0^{\infty} \Im\, \frac{(2k+1-2i\tau)(2z(\tau)+1+2it)}{\mathrm{e}^{2\pi(t-iz(\tau))}-1}\,dt\,, \tag{A.50}$$

where $z(\tau) = -\frac{1}{2} + \sqrt{4\tau^2 - (2k+1)^2 - \nu + 4i(2k+1)\tau}$, and $\Re(z(\tau_*)) = k$.

We now use the formulas above in order to evaluate $Z'(2,-10|0)$, which value is needed in the main text. Setting $k = 2$ and $\nu = -10$ we obtain for the first term on the right in (A.43) $\mathcal{H} = 1.9445$. The second term, containing the integral over t, is evaluated numerically to give -19.9469. The numerical value of the term containing the double integral is -0.1294. As for the last term, $\mathcal{S}$, it is exponentially small and is of the order of 10^{-12}. This is because, as one can see from (A.50), the value of $\mathcal{S}$ is suppressed by the factor of $\exp\{-2\pi(\tau_* + \Im(z(\tau_*))\} = \exp\{-4\pi\sqrt{5}\}$.

Summing everything up, we obtain

$$\Upsilon \equiv Z'(2,-10|0) = -18.3118\,. \tag{A.51}$$

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