Finite Temperature Schwinger Model with Chirality Breaking Boundary Conditions

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Abstract

The N_f-flavour Schwinger Model on a finite space $0 \leq x^1 \leq L$ and subject to bag-type boundary-conditions at $x^1 = 0$ und $x^1 = L$ is solved at finite temperature $T = 1/\beta$. The boundary conditions depend on a real parameter θ and break the axial flavour symmetry. We argue that this approach is more appropriate to study the broken phases than introducing small quark masses, since all calculations can be performed analytically.

In the imaginary time formalism we determine the thermal correlators for the fermion-fields and the determinant of the Dirac-operator in arbitrary background gauge-fields. We show that the boundary conditions induce a CP-odd θ-term in the effective action.

The chiral condensate, and in particular its T- and L- dependence, is calculated for N_f fermions. It is seen to depend on the order in which the two lengths $\beta = 1/T$ and L are sent to infinity.

1 Introduction

Over the past decades the Schwinger model [1] has proved to be an excellent laboratory for field theory because it turned out to shed some light on a couple of questions which naturally arise in realistic gauge-field theories, but lead to immense difficulties as soon as one tries to attack them directly. Longstanding problems of this type are the wellknown $U(1)_A$-problem [30], the question whether QCD in the chiral limit shows a spontaneous breakdown of the chiral symmetry and the question about the nature of the chiral phase transition at ~ 200 MeV [2, 31].

The Schwinger model is known to be the most simple model field theory which exhibits chiral symmetry breaking. However the quantization on the plane suffers from the deficit, that a naive calculation of the condensates $\langle \overline{\psi}\psi \rangle$ and $\langle \overline{\psi}\gamma_5\psi \rangle$ gives zero results and the correct values can be derived only a posteriori by using the clustering theorem [6]. Whenever a symmetry is expected to be broken it is most recommendable to break it explicitly and to try to determine how the system behaves in the limit when the external trigger is softly removed. Thus it is most natural for both the Schwinger model and QCD to break explicitly the axial flavour symmetry and to investigate how observables do behave in the limit where the symmetry is restored.

The most direct way to do this is to introduce small fermion masses and to try to determine how the chiral correlators behave in the limit where these fermion masses tend to be negligible as compared to the intrinsic energy scale of the gauge-interaction. Once these calculations do predict nonvanishing chiral condensates in the thermodynamic limit one can be sure that a spontaneous breaking of the axial flavour symmetry $SU(N_f)_A$ really takes place - however this is not a conditio sine qua non. There is however a technical obstacle to this approach: the value of the chiral condensates is related to the mean level density of the eigenvalues of the Dirac operator [7] in the infrared. Unfortunately, the spectral density of the massive Dirac operator is only known for very special background gauge-fields.

In this paper we shall break the chiral symmetry explicitly by boundary conditions for the fermions instead of giving them a small mass. Although this version seems at first sight less natural, it has many advantages - both conceptual and calculational in nature. The most important point is clearly the fact that it allows for an entirely analytical treatment. In a previous paper [5] we investigated QCD-type theories with N_f massless flavours on an even-dimensional ($d = 2n$) euclidean manifold M with boundary ∂M on which the boundary conditions studied by Hrasko and Balog [4] have been applied. These chirality-breaking-(CB-) boundary-conditions relate the different spin components of one flavour on ∂M and are neutral with respect to vector-flavour transformations - so that the (gauge-invariant) fermionic determinant is the same for all flavours. For a simply connected M, e.g. a ball, the instanton number, which in four dimensions takes the form

$$q = \frac{1}{32\pi^2} \int F_{\mu\nu}^a \, \tilde{F}_{\mu\nu}^a \, d^4x, \tag{1}$$

is *not quantized* and may take any real value [5]. Contrary to the situation on a compact manifold without boundary, on which q is integer [10], the configuration space is *topologically trivial* (i.e. without disconnected instanton sectors) [5]. In addition there are *no fermionic zero modes* [5, 11] which usually tend to complicate the quantization considerably [3].

Our previously cited work focused on the euclidean N_f-flavour $U(N_c)$- or $SU(N_c)$- gauge-theories inside $2n$-dimensional balls of radius R. We computed that part of the effective

action reflecting the interaction of the particles with the boundary S_R^{2n-1}. Here we investigate whether the approach of breaking the $SU(N_f)_A$ symmetry by boundary conditions can be extended to gauge-systems in thermal equilibrium states. In the imaginary time formalism spacetime is then a cylindrical manifolds $M = [0, \beta] \times \{space\}$ and (anti)periodic boundary conditions for the (fermi)bose-fields in the euclidean time x^0 with period $\beta = 1/T$ are imposed.

Note that at finite temperature it is only the boundary of space and not of space-time where chirality is broken and it is a priori an open question whether this is sufficient to trigger a chiral symmetry breaking even in the one flavour case. In addition there is a technical obstacle to extending the CB-boundary-condition approach to non-simply connected manifolds, e.g a cylinder. On cylinders the standard decomposition for the Dirac operator, on which the analytic treatment heavily relies, must be modified. The present paper is a technical one - mostly devoted to show how this difficulty can be overcome. From the physical point of view it is our aim to investigate how the breakdown of the chiral symmetry - when triggered by boundary conditions - in the one- and the multi-flavour cases is affected by finite temperature effects.

Here we shall quantize the Schwinger Model with action

$$S[A, \psi^\dagger, \psi] = S_B[A] + S_F[A, \psi^\dagger, \psi]$$

$$S_B = \frac{1}{4} \int_M F_{\mu\nu} F_{\mu\nu} \quad , \quad S_F = \sum_{n=1}^{N_f} \int_M \psi_n^\dagger i\not{D} \psi_n \tag{2}$$

on the manifold

$$M = [0, \beta] \times [0, L] \quad \ni \quad (x^0, x^1) \tag{3}$$

with volume $V = \beta L$. At finite temperature the fields A and ψ are periodic and antiperiodic in euclidean time with period β and hence $x^0 = 0$ and $x^0 = \beta$ are identified. This means that $[0, \beta] \times [0, L]$ is a cylinder with circumference $\beta = 1/T$. At the spatial ends of the cylinder (i.e. at $x^1 = 0$ and $x^1 = L$) specific CB-boundary-conditions are applied. Then there are no fermionic zero modes (see next section) and the generating functional for the fermions in a given gauge-field background A is given by the textbook formula

$$Z_F[A, \eta^\dagger, \eta] = \det(i\not{D}) \; e^{\int \eta^\dagger (i\not{D})^{-1} \eta}. \tag{4}$$

We shall see that these CB-boundary-conditions indeed generate chiral condensates for any finite length L of the cylinder. However in the limit $\beta \to \infty, L \to \infty$ the condensates will only survive for the one flavour case and this only if the limit $\beta \to \infty$ is taken before the limit $L \to \infty$.

During the calculations the following abbreviations are used for notational simplicity:

$$\tau = \frac{\beta}{2L} \quad , \qquad \eta = \frac{x^1 + y^1}{L} \quad , \qquad \xi = \frac{x^1}{L}. \tag{5}$$

This paper is organized as follows : In section 2 we discuss the CB-boundary-conditions to be applied together with some immediate consequences for the spectrum of the Dirac operator $i\not{D}$. Section 3 is devoted to the question of how to decompose an arbitrary gauge-field on a cylinder. In section 4 we compute the fermionic Green's function with respect to CB-boundary-conditions in arbitrary external fields. In section 5 we determine the effective action after the fermions have been integrated out. Using the results of the two previous steps

the chiral condensates are calculated in section 6. In section 7 we show that the value of the chiral condensate crucially depends not only on the number of flavours but also on the order in which the two limits $\beta \to \infty$ and $L \to \infty$ are performed. Finally we compare our result with the condensate generated by fractons on a torus of identical size and with analogous results of noncommutativity of the limits $m \to 0$, $L \to \infty$ in the the usual small-quark-mass approach. In the appendices we derive the boundary Seeley-DeWitt coefficient used in the body of the paper.

2   Chirality Breaking Boundary Conditions

In this section we shall shortly review the boundary conditions as discussed by Hrasko and Balog [4] together with their most important consequences [5, 26].

Since Z_F should be real we want $i\slashed{D}$ to be symmetric under the scalar product

$$(\chi, \psi) := \int_M \chi^\dagger \psi$$

from which we get the condition

$$(\chi, i\slashed{D}\psi) - (i\slashed{D}\chi, \psi) = i \oint_{\partial M} \chi^\dagger \gamma_n \psi \equiv 0 \ . \tag{6}$$

Imposing local linear boundary conditions which ensure this requirement amounts to have $\chi^\dagger \gamma_n \psi = 0$ on ∂M for each pair, which is achieved by

$$\psi = B\psi \quad \text{on} \quad \partial M \qquad \text{with} \qquad B^\dagger \gamma_n B = -\gamma_n \quad , \quad B^2 = 1, \tag{7}$$

where $\gamma_n = (\gamma, n) = n_\mu \gamma_\mu = \slashed{n}$ and n_μ is the outward oriented normal vectorfield on ∂M. We shall choose the one-parametric family of boundary operators [4]

$$B \equiv B_\theta := i\gamma_5 e^{\theta\gamma_5} \gamma_n \tag{8}$$

which is understood to act as the identity in flavour space. These CB-boundary-conditions break the γ_5 invariance of the theory, making the N_f flavour theory invariant under $SU(N_f)_V$ instead of $SU(N_f)_L \times SU(N_f)_R$. Later they will be supplemented by suitable boundary conditions for the gauge-field. These boundary conditions imply that there is no net $U(1)$-current leaking through the boundary, since $n \cdot j = \psi^\dagger \gamma_n \psi = 0$ on ∂M.

In the following we shall make use of a Feynman Hellmann [16] boundary formula, which may be derived from (6,7,8) [5]

$$\frac{d}{d\theta}\lambda_k = \frac{i}{2} \oint \psi_k^\dagger (\gamma \cdot n)\gamma_5 \psi_k = -\lambda_k (\psi_k, \gamma_5 \psi_k) \ , \tag{9}$$

where the λ_k denote the eigenvalues of $i\slashed{D}$.

We choose the chiral representation $\gamma_0 = \sigma_1, \gamma_1 = \sigma_2$ and $\gamma_5 = \sigma_3$. Then the boundary operators at the two ends of the cylinder read

$$B_L = -\begin{pmatrix} 0 & e^\theta \\ e^{-\theta} & 0 \end{pmatrix} \text{ (at } x^1\!=\!0) \quad \text{and} \quad B_R = +\begin{pmatrix} 0 & e^\theta \\ e^{-\theta} & 0 \end{pmatrix} \text{ (at } x^1\!=\!L) \ . \tag{10}$$

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The most important properties of these boundary conditions are summarized as follows [5]:
(1) The Dirac operator has a *purely discrete* real spectrum which is *not symmetric* with respect to zero.
(2) The Dirac operator has *no zero modes.*
(3) The instanton number $q = \frac{1}{4\pi} \int \epsilon_{\mu\nu} F_{\mu\nu} = \frac{1}{2\pi} \int E$ is *not quantized* . The second property allows us to calculate expectation values of gauge-invariant operators as

$$\langle O \rangle \;\; = \;\; \int \langle O \rangle_A \; d\mu_\theta[A], \quad \text{where} \tag{11}$$

$$d\mu_\theta[A] \;\; = \;\; \frac{1}{Z_F} \, \det{}_\theta(i\slashed{D}) \; e^{-S_B[A]} \; D[A]. \tag{12}$$

Here $D[A]$ is assumed to contain the gauge-fixing factor including the corresponding Fadeev-Popov determinant and $\langle O \rangle_A$ denotes the expectation value of O in a fixed background

$$\langle O \rangle_A = \frac{1}{\det_\theta(i\slashed{D})} \cdot \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; O \; e^{-\int \psi^\dagger i\slashed{D} \psi}. \tag{13}$$

Throughout θ is the free parameter in boundary operators (10). We shall see that the θ-dependence of the fermionic determinant $\det_\theta(i\slashed{D})$ can be calculated analytically.

3 Decomposition and Deformation techniques

In this section we present the decomposition and deformation techniques needed to determine the functional determinant of the Dirac operator on the cylinder with CB-boundary-conditions as given by (7) and (10) .

   On simply connected regions we have the decomposition $eA_\mu = -\epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\chi$ such that $eF_{01} = \triangle\phi$. On the cylinder there is a one to one correspondence between ϕ and eF_{01} if ϕ obeys Dirichlet boundary conditions at the two ends of the cylinder. But cylinders are not simply connected, $\pi_1(M) = \mathbf{Z}$, and as a result the Polyakov-loop operators

$$e^{ie \int_0^\beta A_0 \; dx^0} =: e^{2\pi i c},$$

i.e. $e \int_0^\beta A_0$ mod 2π, are gauge-invariant. On the other hand, using the β-periodicity of χ and the Dirichlet boundary conditions on ϕ, the above decomposition would imply that $\int_0^\beta \int_0^L A_0 = 0$, a condition which does not hold in general (take a constant A_0). This simple observation already indicates, that the correct decomposition of A_μ on the cylinder reads

$$\begin{aligned} eA_0 &= \quad -\partial_1\phi + \partial_0\chi \quad +\tfrac{2\pi}{\beta}c \\ eA_1 &= \quad +\partial_0\phi + \partial_1\chi, \end{aligned} \tag{14}$$

where ϕ obeys Dirichlet boundary conditions at $x^1 = 0, L$ and χ fulfills $\chi(0) + \chi(L) = 0$ and $c \in [0, 1[$ is the constant harmonic part. To prove (14) one Fourier decomposes the various fields and carefully handles the zero-modes of the Laplacian. The harmonic part can then be reconstructed from its values on the boundaries.
The Dirac operator $i\slashed{D} = i\gamma_\mu(\partial_\mu - ieA_\mu)$ may be factorized according to

$$i\slashed{D} = G^\dagger i\slashed{D}_0 G, \quad \text{where} \quad i\slashed{D}_0 = \gamma^0(i\partial_0 + 2\pi c/\beta) + \gamma^1 i\partial_1, \tag{15}$$

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and

$$G = \begin{pmatrix} g^{*-1} & 0 \\ 0 & g \end{pmatrix}, \quad g :\equiv e^{-(\phi+i\chi)}. \tag{16}$$

The prepotential g is an element of the complexified gauge-group $U(1)^* = S^1 \times R_+$. Now we deform the prepotential and Dirac operator as

$$g_\alpha :\equiv e^{-\alpha(\phi+i\chi)} \quad \text{and} \quad i\slashed{D}_\alpha = G_\alpha^\dagger \, i\slashed{D}_0 \, G_\alpha \tag{17}$$

such the deformed operator interpolates between the free and full ones: $i\slashed{D}_{\alpha=1} = i\slashed{D}$ and $i\slashed{D}_{\alpha=0} = i\slashed{D}_0$. By using

$$\frac{d}{d\alpha} G_\alpha = -G_\alpha H \quad , \quad H = \begin{pmatrix} -h^* & 0 \\ 0 & h \end{pmatrix} = -\phi\gamma_5 + i\chi I \ . \tag{18}$$

one finds for the α-variation of the integrated heatkernel of $(i\slashed{D}_\alpha)^2$

$$\frac{d}{d\alpha}(\operatorname{tr}\{e^{t\slashed{D}_\alpha^2}\}) = 2t \operatorname{tr}\{e^{t\slashed{D}_\alpha^2}(H+H^\dagger)(i\slashed{D}_\alpha)^2\} = 2t\frac{d}{dt}(\operatorname{tr}\{e^{t(\slashed{D}_\alpha)^2}2\phi\gamma_5\}) \tag{19}$$

and this formula will prove to be useful in section 5.

4 Fermionic Propagator w.r.t. Boundary Conditions

In order to calculate the condensates we need the Green's function S_θ of the Dirac operator $i\slashed{D}$ on the cylinder subject to the CB-boundary conditions. This Green's function obeys

$$
\begin{align}
(i\slashed{D}\, S_\theta)(x,y) &= \delta(x-y) \tag{20}\\
S_\theta(x^0+\beta, x^1, y) &= -S_\theta(x,y) \tag{21}\\
(B_L\, S_\theta)(x^0, x^1=0, y) &= S_\theta(x^0, x^1=0, y) \tag{22}\\
(B_R\, S_\theta)(x^0, x^1=L, y) &= S_\theta(x^0, x^1=L, y) \tag{23}
\end{align}
$$

plus the adjoint relations with respect to y. The dependence of the gauge-potential has not been made explicit and the boundary operator $B_{L/R}$ is the one defined in (10). From the factorization property (15) for the Dirac operator it follows at once, that S_θ is related to the Green's function $\tilde{S}_\theta$ of $i\slashed{D}_0$ as

$$S_\theta(x,y) = G^{-1}(x)\tilde{S}_\theta(x,y)G^{\dagger\,-1}(y) \ . \tag{24}$$

Indeed, since the field ϕ obeys Dirichlet boundary conditions at the ends of the cylinder, g is unitary there and the boundary conditions (21-23) transform into the identical ones for

$$\tilde{S}_\theta(x,y) = \begin{pmatrix} \tilde{S}_{++} & \tilde{S}_{+-} \\ \tilde{S}_{-+} & \tilde{S}_{--} \end{pmatrix},$$

where the indices refer to chirality.

The free Green's function on the cylinder of infinite length

$$\tilde{S}_{ther}(x,y) = \frac{1}{2\pi i}\sum_{n\in Z}(-1)^n e^{2\pi i c(\xi^0 - n\beta)/\beta} \cdot \begin{pmatrix} 0 & \dfrac{1}{\xi^0 + i\xi^1 - n\beta} \\ \dfrac{1}{\xi^0 - i\xi^1 - n\beta} & 0 \end{pmatrix},$$

where $\xi^\mu = x^\mu - y^\mu$, is purely off-diagonal and thus chirality preserving, as expected, since thermal boundary conditions are chirality-neutral. To implement the chirality-breaking boundary conditions at the ends of the cylinder on can either augment $\tilde{S}_{ther}$ by pieces built from the zero modes (which themselves cannot obey the L/R conditions simultaneously) or by exploiting analyticity arguments. In either case we end up with

$$\tilde{S}_\theta(x, y) = \frac{i}{2\pi} \cdot \sum_{m,n \in Z \times Z} (-1)^{(m+n)} \cdot e^{2\pi i c(\xi^0/\beta - n)} \cdot \begin{pmatrix} e^\theta/r_{nm} & -(1/s_{nm}) \\ -(1/\bar{s}_{nm}) & e^{-\theta}/\bar{r}_{nm} \end{pmatrix}, \qquad (25)$$

where $r_{nm} = \xi^0 + i\eta - (n\beta + 2imL)$ and s_{nm} is the same expression with $\eta \equiv x^1 + y^1$ replaced by ξ^1. From this explicit expression one sees at once that the off-diagonal elements only depend on $x^\mu - y^\mu$ and become singular for $x \to y$, whereas the diagonal elements depend on both x^μ and y^μ separately but are regular at coinciding points inside the cylinder. The sum over m respectively n in (25) can be performed by using [27]

$$\sum_Z (-1)^m \frac{e^{imx}}{m + ia} = -\frac{i\pi}{\sinh a\pi} e^{ax} \qquad (-\pi \leq x \leq \pi) \qquad (26)$$

with the results

$$\tilde{S}_\theta(x, y) = \frac{i e^{2\pi i c\xi^0/\beta}}{4L} \cdot \sum_Z (-1)^n e^{-2\pi i nc} \cdot \begin{pmatrix} \dfrac{e^\theta}{\sinh(\pi r_{n0}/2L)} & -\dfrac{1}{\sinh(\pi s_{n0}/2L)} \\ -\dfrac{1}{\sinh(\pi \bar{s}_{n0}/2L)} & \dfrac{e^{-\theta}}{\sinh(\pi \bar{r}_{n0}/2L)} \end{pmatrix}, \qquad (27)$$

or

$$\tilde{S}_\theta(x, y) = \frac{i e^{2\pi i c\xi^0/\beta}}{2\beta} \cdot \sum_Z (-1)^m \cdot \begin{pmatrix} \dfrac{e^\theta e^{-2\pi i c r_{0m}/\beta}}{\sin(\pi r_{0m}/\beta)} & -\dfrac{e^{-2\pi i c s_{0m}/\beta}}{\sin(\pi s_{0m}/\beta)} \\ -\dfrac{e^{-2\pi i c\bar{s}_{om}/\beta}}{\sin(\pi \bar{s}_{0m}/\beta)} & \dfrac{e^{-\theta} e^{-2\pi i c\bar{r}_{om}/\beta}}{\sin(\pi \bar{r}_{0m}/\beta)} \end{pmatrix} . \qquad (28)$$

valid for $c \in [-\frac{1}{2}, \frac{1}{2}]$. For calculating the chiral condensates we shall need the $++$ and $--$ elements at coinciding points inside the cylinder. From (27) we find the expression

$$\tilde{S}_\theta(x, x)_{\pm\pm} = \pm\frac{e^{\pm\theta}}{4L} \sum_{n \in Z} (-)^n \frac{e^{\pm 2in\pi c}}{\sin(\pi[\xi - in\tau])}. \qquad (29)$$

which rapidly converges for low temperature, and from (28) the alternative form

$$\tilde{S}_\theta(x, x)_{\pm\pm} = \pm\frac{e^{\pm\theta}}{2\beta} \sum_{m \in Z} (-)^m \frac{e^{\pm 2\pi c(\xi + m)/\tau}}{\sinh(\pi[\xi + m]/\tau)} \qquad (30)$$

which is adequate for high temperature.

With (24) we end up with the following expressions for chirality violating entries of the fermionic Green's function on the diagonal

$$S_\theta(x; x)_{\pm\pm} = e^{\mp 2\phi(x)} \tilde{S}_\theta(x; x)_{\pm\pm} \qquad . \qquad (31)$$

The free Green's functions $\tilde{S}_{\pm\pm}$ have been computed in (29) and (30). They depend only on the harmonic part c in the decomposition (14) of the gauge-potential.

5 Fermionic Determinant w.r.t. Boundary Conditions

In this section we shall compute the θ-dependence of the fermionic determinant. We shall see that the scattering of the fermions off the boundary generates a CP-odd θ-term in the effective action for the gauge-bosons.

5.1 Zetafunction Definition

The Dirac operator and the boundary conditions are both flavour neutral. Thus the determinant is the same for all flavours and it is sufficient to calculate it for one flavour. For the explicit calculations we shall use the gauge-invariant ζ-function definition of the determinant [17, 18]

$$\log \det{}_\theta(i\slashed{D}) :\equiv \frac{1}{2}\log \det{}_\theta(-\slashed{D}^2) :\equiv -\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \zeta_\theta(-\slashed{D}^2, s) \tag{32}$$

and calculate the θ-dependence of the ζ-function by means of the boundary Feynman Hellmann formula (9). Denoting $\{\mu_k | k \in \mathbf{N}\}$ the (positive) eigenvalues of $-\slashed{D}^2$, the corresponding ζ-function is defined and rewritten as a Mellin transform in the usual way

$$\zeta_\theta(s) :\equiv \zeta_\theta(-\slashed{D}^2, s) :\equiv \sum_k \mu_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}{}_\theta(e^{-t(-\slashed{D}^2)})\, dt \tag{33}$$

for $\operatorname{Re}(s) > d/2 = 1$ and its analytic continuation to $\operatorname{Re}(s) \leq 1$.

5.2 Stepwise Calculation

We will study how $\det_\theta(i\slashed{D}_{\alpha,c})$ varies with θ, α and c to compute the normalized determinant

$$\frac{\det{}_\theta(i\slashed{D})}{\det{}_0(i\slashed{\partial})} \equiv \frac{\det{}_\theta(i\slashed{D}_{\alpha=1,c})}{\det{}_0(i\slashed{D}_{\alpha=0,0})}\ . \tag{34}$$

The calculation is done in three steps. We shall calculate all three factors in

$$\frac{\det{}_\theta(i\slashed{D}_{\alpha=1,c})}{\det{}_0(i\slashed{D}_{\alpha=0,0})} \equiv \frac{\det{}_\theta(i\slashed{D}_{\alpha=1,c})}{\det{}_0(i\slashed{D}_{\alpha=1,c})} \cdot \frac{\det{}_0(i\slashed{D}_{\alpha=1,c})}{\det{}_0(i\slashed{D}_{\alpha=0,c})} \cdot \frac{\det{}_0(i\slashed{D}_{\alpha=0,c})}{\det{}_0(i\slashed{D}_{\alpha=0,0})} \tag{35}$$

in turn.

From the generalized Feynman-Hellmann formula (9) and the fact that $i\slashed{D}$ has no zeromodes so that the various partial integrations are justified, the θ-variation of (33) is found to be

$$\frac{d}{d\theta}\zeta_\theta(s) = \frac{2s}{\Gamma(s)} \int_0^\infty t^{s-1}\ \operatorname{tr}{}_\theta(e^{t\slashed{D}^2}\gamma_5)\ . \tag{36}$$

Now one can use the asymptotic small-t-expansion for $e^{t\slashed{D}^2}f$, where f is a testfunction,

$$\operatorname{tr}{}_\theta(e^{t\slashed{D}^2}f) = \frac{1}{2\pi t} \sum_{m=0,1,\dots} t^{m/2}\ \operatorname{tr}{}_\theta\left(\int a_{m/2}(f) + \oint b_{m/2}(f)\right)\ , \tag{37}$$

and where the $a_{m/2}, b_{m/2}$ denote the corresponding volume and boundary Seeley DeWitt coefficients respectively. Plugging this into the expression (36) yields [19, 20, 21]

$$\frac{d}{d\theta}\frac{1}{2}\log \det{}_\theta(-\slashed{D}^2) = -\frac{1}{4\pi}\int \operatorname{tr}(a_1(\gamma_5)) - \frac{1}{4\pi}\oint \operatorname{tr}(b_1(\gamma_5))\ . \tag{38}$$

For the squared Dirac operator $-\not{D}^2$ that part of a_1 which leads to a nonvanishing γ_5-trace is known [22] to be $eF_{01}/2\pi$, i.e. independent of θ. On the other hand $\oint b_1(.)$, which depends on the boundary conditions, is calculated explicitly in the appendix to be

$$\oint b_1(f) = \oint \frac{1}{2}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\log(e^\theta)}{\sinh(\theta)} \begin{pmatrix} e^\theta & -1 \\ -1 & e^{-\theta} \end{pmatrix} \right\} \partial_n f \tag{39}$$

and does not contribute for $f = \gamma_5$.

Integrating with respect to θ yields the following first factor in (35)

$$\frac{\det_\theta(i\not{D}_{\alpha=1,c})}{\det_0(i\not{D}_{\alpha=1,c})} = \exp\{-\frac{\theta}{2\pi}\int F_{01}\} = \exp\{-\frac{\theta}{2\pi}\int \triangle\phi\}. \tag{40}$$

To find the α-variation leading to the second factor we use (19) in (33) with the result

$$\frac{d}{d\alpha}\zeta_\theta(-\not{D}^2_{\alpha,c},s) = -\frac{2s}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{tr}_\theta(e^{t\not{D}^2_\alpha}2\phi\gamma_5), \tag{41}$$

where we integrated by parts. Again we use the small-t-expansion (37) of the heat kernel, but now with test function $f = 2\phi\gamma_5$. Thus

$$\frac{d}{d\alpha}\frac{1}{2}\log\det_\theta(-\not{D}^2_{\alpha,c}) = \frac{1}{4\pi}\int \mathrm{tr}\,(a_1(2\phi\gamma_5)) + \frac{1}{4\pi}\oint \mathrm{tr}\,(b_1(2\phi\gamma_5)) \tag{42}$$

where the universal $a_1(.)$ yields the wellknown Schwinger term [22]. Since now the normal derivative of the testfunction on the boundary in non-zero, the last surface term contributes. Using (39) we end up with

$$\frac{d}{d\alpha}\frac{1}{2}\log\det_\theta(-\not{D}^2_{\alpha,c}) = \frac{1}{4\pi}\int_M 2\phi\triangle\phi - \frac{1}{2\pi}\oint_{\partial M}\log(e^\theta)\,\partial_n\phi. \tag{43}$$

Setting $\theta = 0$ and integrating with respect to α yields the following second factor in (35):

$$\frac{\det_0(i\not{D}_{\alpha=1,c})}{\det_0(i\not{D}_{\alpha=0,c})} = e^{\frac{1}{2\pi}\int\phi\triangle\phi}. \tag{44}$$

We are left with the task to calculate the third factor

$$\log\frac{\det_0(i\not{D}_{\alpha=0,c})}{\det_0(i\not{D}_{\alpha=0,0})} = -\frac{1}{2}\int_0^c \frac{d}{ds}\Big|_{s=0}\frac{d}{d\tilde{c}}\zeta_0(-\not{D}^2_{\alpha=0,\tilde{c}},s)\,d\tilde{c}\quad. \tag{45}$$

For that we computed the heatkernel of the operator

$$-\not{D}^2_{\alpha=0,\tilde{c}} = -\Big((\partial_0 - 2\pi i\tilde{c}/\beta)^2 + \partial_1^2\Big)I_2$$

for $\theta = 0$. The explicit result is

$$K(t,x,y) = \frac{1}{4\pi t}\sum_{Z\times Z}(-1)^{m+n}e^{-(\xi^0-n\beta)^2/4t}e^{2\pi i\tilde{c}(\xi^0-n\beta)/\beta}$$
$$\begin{pmatrix} e^{-(\xi^1-2mL)^2/4t} & -e^{-(\eta-2mL)^2/4t} \\ -e^{-(\eta+2mL)^2/4t} & e^{-(\xi^1+2mL)^2/4t} \end{pmatrix} \tag{46}$$

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which results in the trace $(V = \beta L)$

$$\int_M \mathrm{tr}_{\theta=0}\,\left(K(t,x,x)\right) = \frac{V}{2\pi t}\left(1 + {\sum}'(-1)^{m+n} e^{-\frac{(n\beta)^2+(2mL)^2}{4t}} \cos(2\pi n\tilde{c})\right) , \qquad (47)$$

where the prime denotes the omission of the $(m,n) = (0,0)$ term. The $\tilde{c}$-derivative of the Mellin transform, after substituting $t \to 1/t$, reads

$$\frac{d}{d\tilde{c}}\zeta_0\big(-\slashed{D}^2_{\alpha=0,\tilde{c}}\,,s\big) = \frac{V}{2\pi\Gamma(s)}\int_0^\infty {\sum}'(-1)^{m+n} e^{-t[(n\beta/2)^2+(mL)^2]}\,\frac{d}{d\tilde{c}}\cos(2\pi n\tilde{c})\,t^{-s}\,dt \qquad (48)$$

which may be integrated by parts (for $s > 0$) to give

$$\frac{d}{d\tilde{c}}\zeta_0\big(-\slashed{D}^2_{\alpha=0,\tilde{c}}\,,s\big) = -\frac{2V}{\pi}\frac{s}{\Gamma(s)}\int_0^\infty {\sum}'(-1)^{m+n}\frac{e^{-t\,[(n\beta/2)^2+(mL)^2]}}{[(n\beta)^2+4(mL)^2]}\frac{d}{d\tilde{c}}\cos(2\pi n\tilde{c})\,t^{-s-1}\,dt .$$

Only the pole of order one of the integral at $s = 0$ can contribute to the s-derivative at $s = 0$ of the ζ-function. Since this pole entirely stems from the lower limit of the integral we may split the latter into two parts

$$\begin{aligned}\frac{d}{d\tilde{c}}\zeta_0\big(-\slashed{D}^2_{\alpha=0,\tilde{c}}\,,s\downarrow 0\big) &= -\frac{2V}{\pi}\frac{s}{\Gamma(s)}\Big(\int_0^\epsilon \ldots + \int_\epsilon^\infty \ldots\Big)\\ &= \frac{2V}{\pi}(s+\gamma s^2+\ldots)\cdot{\sum}'(-1)^{m+n}\frac{\frac{d}{d\tilde{c}}\cos(2\pi n\tilde{c})}{[(n\beta)^2+4(mL)^2]}\epsilon^{-s} + \ldots\end{aligned}$$

to obtain

$$\left.\frac{d}{ds}\right|_{s=0}\frac{d}{d\tilde{c}}\zeta_0\big(-\slashed{D}^2_{\alpha=0,\tilde{c}},s\big) = \frac{2V}{\pi}{\sum}'(-1)^{m+n}\frac{\frac{d}{d\tilde{c}}\cos(2\pi n\tilde{c})}{(n\beta)^2+(2mL)^2} . \qquad (49)$$

Plugging this result into (45) we end up with the expression

$$-\Gamma(c) \equiv \log\frac{\det_0(i\slashed{D}_{\alpha=0,c})}{\det_0(i\slashed{D}_{\alpha=0,0})} = -\frac{V}{\pi}{\sum}'(-1)^{m+n}\frac{\cos(2\pi nc)-1}{(n\beta)^2+(2mL)^2} \qquad (50)$$

for the third factor of the functional determinant (34) in the factorization (35). With the help of

$$\frac{\mathrm{Im}(\tau)}{\pi}{\sum}'\frac{e^{2\pi i(ma_1+na_2)}}{|m+\tau n|^2} = -2\log\left|\frac{1}{\eta(\tau)}\theta\left[\begin{array}{c}\frac{1}{2}+a_1\\a_2\end{array}\right]\right|$$

this result can be rewritten as [29]

$$e^{-\Gamma(c)} = \frac{\det_0(i\slashed{D}_{\alpha=0,c})}{\det_0(i\slashed{D}_{\alpha=0,0})} = \begin{cases}\frac{\theta_3(c,i\tau)}{\theta_3(0,i\tau)}\\[2ex] e^{-\pi c^2/\tau}\frac{\theta_3(ic/\tau,i/\tau)}{\theta_3(0,i/\tau)}\end{cases} . \qquad (51)$$

These two equivalent forms will be useful in the low- and high- temperature expansion of the condensates.

5.3 Effective Action

Now we can combine the classical (euclidean) action of the photon field, rewritten in the new variables (14)

$$\frac{1}{4}F_{\mu\nu}F_{\mu\nu} = \frac{1}{2e^2}\triangle\phi\triangle\phi \equiv : S_B[\phi] \tag{52}$$

with our explicit result for the functional determinant (34). Collecting the contributions (40,44,51) and adding the classical action (52) we end up with the effective action

$$\Gamma \equiv \Gamma_\theta[c,\phi] \equiv N_f\Gamma(c) + \Gamma_\theta[\phi] \tag{53}$$

where $\Gamma(c)$ has been given in (51) and $\Gamma_\theta[\phi]$ is

$$\Gamma_\theta[\phi] \equiv \frac{1}{2e^2}\left\{\int_M \phi\triangle^2\phi - \mu^2\int_M \phi\triangle\phi + \theta\cdot\mu^2\int_M \triangle\phi\right\} \tag{54}$$

and

$$\mu \; :\equiv \; \sqrt{\frac{N_f e^2}{\pi}} \tag{55}$$

is the analog of the η'-mass in QCD. We have used that the functional determinant is the same for all flavours. The functional measure takes the form

$$d\mu_\theta[A] = \frac{1}{Z_\theta}\; e^{-\Gamma_\theta[c,\phi]}\; dc\; D\phi\; \delta(\chi)\; D\chi \; . \tag{56}$$

We have taken into account that the gauge-variation of the Lorentz gauge-condition

$$F :\equiv \partial_\mu A^\mu = \triangle\chi$$

and the Jacobian of the transformation from $\{A\}$ to the new variables $\{\phi, c, \chi\}$ are independent of the fields. Actually, the corresponding determinants cancel each other.

We conclude that the expectation value of any gauge-invariant operator O (which will not depend on χ) is given by

$$\langle O\rangle = \frac{\int dc\; D\phi\; O\; e^{-\Gamma_\theta[c,\phi]}}{\int dc\; D\phi\; e^{-\Gamma_\theta[c,\phi]}} \tag{57}$$

with $\Gamma_\theta[c,\phi]$ from (53,54).

6 Chiral Condensates

Our result (57) may be applied to calculate the chiral condensates as

$$\langle\psi^\dagger(x)P_\pm\psi(x)\rangle = \frac{\int dc\,D\phi\; S_\theta(x,x)_{\pm\pm}\; e^{-\Gamma_\theta[c,\phi]}}{\int dc\,D\phi\; e^{-\Gamma_\theta[c,\phi]}} \tag{58}$$

with S_θ from (31) and Γ_θ from (53). Both the (exponentiated) action and the Green's function factorize into parts which only depend on c and on ϕ, respectively. Thus (58) factorizes as

$$\langle\psi^\dagger(x)P_\pm\psi(x)\rangle = C^\pm(x)\cdot D^\pm(x) \tag{59}$$

with x^0-independent factors

$$C^\pm(x^1) = \frac{\int dc\; \tilde{S}_\theta(x,x)_{\pm\pm}\; e^{-N_f\Gamma(c)}}{\int dc\; e^{-N_f\Gamma(c)}} \quad , \quad D^\pm(x^1) = \frac{\int\; D\phi\; e^{\mp 2\phi(x)-\Gamma_\theta[\phi]}}{\int\; D\phi\; e^{-\Gamma_\theta[\phi]}} \tag{60}$$

which depend on the parameters θ, N_f, β, L. Here and below the c-integrals extend over one period, e.g. $[-1/2, 1/2]$.

6.1 Harmonic Integral

Now we shall see, how far we can evaluate the first factor in (59) which contains the integrals over the harmonic part of the gauge-field.

Plugging in the Green's function (25,31) as well as (50) we obtain the unevaluated expression

$$C^{\pm}(x^1) = \pm \frac{e^{\pm\theta}}{4\pi L} \sum_{Z \times Z} (-1)^{m+n} \frac{\xi+m}{(\xi+m)^2+(n\tau)^2} \cdot \frac{\int_{-1/2}^{1/2} \cos(2\pi nc) e^{-\frac{N_f}{2\pi} \sum'(-1)^{k+l} \frac{\cos(2\pi lc)-1}{k^2/\tau+l^2\cdot\tau}} \, dc}{\int_{-1/2}^{1/2} e^{-\frac{N_f}{2\pi} \sum'(-1)^{k+l} \frac{\cos(2\pi lc)-1}{k^2/\tau+l^2\cdot\tau}} \, dc} \ . \tag{61}$$

To investigate the low-temperature expansion we use (29) and the upper line in (51) and arrive at

$$C^{\pm}(x^1) = \pm \frac{e^{\pm\theta}}{4L} \sum_{n\in Z} \frac{(-1)^n}{\sin(\pi[\xi - in\tau])} \cdot \frac{\int dc \, e^{\pm 2\pi inc} \, \theta_3^{N_f}(c, i\tau)}{\int dc \, \theta_3^{N_f}(c, i\tau)} \ , \tag{62}$$

Alternatively, for the high-temperature expansion we use (30) and the lower line in (51), so that

$$C^{\pm}(x^1) = \pm \frac{e^{\pm\theta}}{2\beta} \sum_{m\in Z} \frac{(-1)^m}{\sinh(\pi[\xi + m]/\tau)} \cdot \frac{\int dc \, e^{-\pi c[N_f c \mp 2(\xi+m)]/\tau} \, \theta_3^{N_f}(ic/\tau, i/\tau)}{\int dc \, e^{-\pi c^2 N_f/\tau} \, \theta_3^{N_f}(ic/\tau, i/\tau)} \ . \tag{63}$$

For one flavour the c-integral in (62) is easily calculated and one finds

$$C^{\pm}(x^1) = \pm \frac{e^{\pm\theta}}{4L} \sum_{n\in Z} \frac{(-1)^n e^{-\pi\tau n^2}}{\sin(\pi[\xi - in\tau])} \ . \tag{64}$$

6.2 Nonharmonic Integral

Now we shall compute the second factor in (59) as defined in (60). We recall that the integration extends over fields ϕ, which are periodic in the x^0 and satisfy Dirichlet boundary conditions at the ends of the cylinder, i.e. at $x^1 = 0, L$.

Doing the gaussian integrals one ends up with

$$D^{\pm}(x^1) = \exp\left\{\frac{2\pi}{N_f} K_{\mu^2}(x, x)\right\} \cdot \exp\left\{\pm \frac{\theta}{2} \cdot \int \triangle' K_{\mu^2}(x, x') \pm \frac{\theta}{2} \cdot \int \triangle' K_{\mu^2}(x', x)\right\} \tag{65}$$

where the integration is over x' and the kernel

$$K_{\mu^2}(x, y) = \langle x| \frac{\mu^2}{-\triangle(-\triangle + \mu^2)} |y\rangle = \langle x| \frac{1}{-\triangle} |y\rangle - \langle x| \frac{1}{-\triangle + \mu^2} |y\rangle \tag{66}$$

is with respect to Dirichlet boundary conditions. Being the difference of two Green's functions with the same singular behaviour it is finite at coinciding arguments.

The explicit form of the kernel is

$$K_{\mu^2}(x, y) = \frac{V}{\pi^2} \sum_{m,n\in Z}{}' \left(\frac{1}{(2mL)^2+(n\beta)^2} - \frac{1}{(2mL)^2+(n\beta)^2+(\mu V/\pi)^2}\right) \cdot$$

$$\cos\left(\frac{2\pi m\xi^0}{\beta}\right) \sin\left(\frac{\pi n x^1}{L}\right) \sin\left(\frac{\pi n y^1}{L}\right) \ , \tag{67}$$

where the prime indicates the omission of the term with $m = n = 0$. For coinciding arguments K becomes x^0-independent as required by translational invariance. For performing either the sum over m or over n in (67) one uses the formula

$$\sum_{j \in Z} \frac{\cos(jx)}{j^2 + a^2} = \frac{\pi}{a} \frac{\cosh(a(\pi - x))}{\sinh(a\pi)} \qquad (x \in [0, 2\pi])$$

to end up either with the expression

$$K_{\mu^2}(x, x) = \frac{1}{2\pi} \sum_{n \geq 1} \left(\frac{\mathrm{cth}(n\pi\tau)}{n} - (n \to \sqrt{n^2 + (\mu L/\pi)^2}) \right) \left(1 - \cos(2\pi n\xi) \right) , \qquad (68)$$

which is useful for the low temperature expansion, or alternatively with the expression

$$
\begin{aligned}
K_{\mu^2}(x, x) &= \frac{1}{2\pi} \sum_{m \geq 1} \frac{\cosh(m\pi/\tau) - \cosh(m\pi(1 - 2\xi)/\tau)}{m \ \sinh(m\pi/\tau)} - (m \to \sqrt{m^2 + (\mu\beta/2\pi)^2}) \\
&\quad + \frac{\xi(1 - \xi)}{2\tau} + \frac{\cosh(\mu L(1 - 2\xi)) - \cosh(\mu L)}{2\mu\beta \ \sinh(\mu L)} ,
\end{aligned}
\qquad (69)
$$

which is useful for the high temperature expansion. Both expressions (68) and (69) do indeed vanish as x^1 reaches the boundary in accordance with the imposed boundary conditions.

Once we have the explicit formula (67) at hand we can compute in a straightforward way the expression

$$\int \triangle_z K_{\mu^2}(z, x) \ dz = -\frac{4}{\pi} \sum_{n=1,3,\dots} (\frac{1}{n} - \frac{n}{n^2 + (\mu L/\pi)^2}) \sin(\pi n\xi). \qquad (70)$$

Applying the formula

$$\sum_{n=1,3,\dots} \frac{n \ \sin(nx)}{n^2 + a^2} = \frac{\pi}{4} \frac{\mathrm{sh}(a(\pi - x)) + \mathrm{sh}(ax)}{\mathrm{sh}(a\pi)} \qquad (\ x \in \ ]0, \pi[\ ) \qquad (71)$$

the expression (70) is seen to take the simple form

$$\int \triangle_z K_{\mu^2}(z, x) \ dz = \frac{\sinh(\mu L(1 - \xi)) + \sinh(\mu L\xi)}{\sinh(\mu L)} - 1 . \qquad (72)$$

6.3   Final Result

Now all pieces to compute the chiral condensate (59) have been calculated. For $C^{\pm}$ we have the two alternative forms (62) and (63), and $D^{\pm}$ is given by (65) wherein we can use one of the equivalent representations (68) or (69) for K_{μ^2} together with (72). Thus we have

$$
\begin{aligned}
\langle \psi^\dagger P_\pm \psi \rangle(x^1) &= \pm \frac{1}{4L} \sum_{n \in Z} \frac{(-1)^n}{\sin(\pi[\xi - in\tau])} \cdot \frac{\int dc \ e^{\pm 2\pi inc} \ \theta_3^{N_f}(c, i\tau)}{\int dc \ \theta_3^{N_f}(c, i\tau)} \cdot \\
&\quad \exp\{\frac{1}{N_f} \sum_{n \geq 1} \left(\frac{\mathrm{cth}(n\pi\tau)}{n} - (n \to \sqrt{n^2 + (\frac{\mu L}{\pi})^2}) \right) \left(1 - \cos(2\pi n\xi) \right)\} \cdot
\end{aligned}
$$

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$$\exp\{\pm\theta \cdot \frac{\sinh(\mu L(1-\xi)) + \sinh(\mu L\xi)}{\sinh(\mu L)}\} \tag{73}$$

$$\langle\psi^\dagger P_\pm \psi\rangle(x^1) = \pm\frac{1}{2\beta} \sum_{m\in Z} \frac{(-1)^m}{\sinh(\pi[\xi+m]/\tau)} \cdot \frac{\int dc\, e^{-\pi c[N_f c \mp 2(\xi+m)]/\tau}\, \theta_3^{N_f}(ic/\tau, i/\tau)}{\int dc\, e^{-\pi c^2 N_f/\tau}\, \theta_3^{N_f}(ic/\tau, i/\tau)} \cdot$$

$$\exp\{\frac{1}{N_f} \sum_{m\geq 1} \frac{\cosh(m\pi/\tau) - \cosh(m\pi(1-2\xi)/\tau)}{m\,\sinh(m\pi/\tau)} - (m \to \sqrt{m^2 + (\frac{\mu\beta}{2\pi})^2})\} \cdot$$

$$\exp\{\frac{2\pi}{N_f}\Big(\frac{\xi(1-\xi)}{2\tau} + \frac{\cosh(\mu L(1-2\xi)) - \cosh(\mu L)}{2\mu\beta\,\sinh(\mu L)}\Big)\} \cdot$$

$$\exp\{\pm\theta \cdot \frac{\sinh(\mu L(1-\xi)) + \sinh(\mu L\xi)}{\sinh(\mu L)}\} \tag{74}$$

with excellent convergence properties in the low- and high-temperature regime, respectively.

This result is one of the two main results of this article. To simplify the analysis we shall now study the condensates at the midpoints of the cylinder.

6.4 $\langle\psi^\dagger P_\pm \psi\rangle$ at Midpoints

If a condensate survives at the midpoints when the boundaries are taken to infinity then the chiral symmetry is broken.
For $x^1 = L/2$ the formulas (62), (63) simplify to

$$C^\pm(\frac{L}{2}) = \pm\frac{e^{\pm\theta}}{4L}\left(1 + 2\sum_{n\geq 1}(-1)^n \frac{\int \cos(2\pi nc)\,\theta_3^{N_f}(c, i\tau)dc}{\cosh(n\pi\tau)\int \theta_3^{N_f}(c, i\tau)\,dc}\right) \tag{75}$$

$$C^\pm(\frac{L}{2}) = \pm\frac{e^{\pm\theta}}{\beta}\sum_{m\geq 0}(-1)^m \frac{\int \cosh((2m+1)\pi c/\tau)\,e^{-\pi N_f c^2/\tau}\theta_3^{N_f}(ic/\tau, i/\tau)\,dc}{\sinh((2m+1)\pi/2\tau)\int e^{-\pi N_f c^2/\tau}\theta_3^{N_f}(ic/\tau, i/\tau)\,dc}. \tag{76}$$

The formulas (68) and (69) simplify to

$$K_{\mu^2}(\frac{L}{2}) = \frac{1}{\pi} \sum_{n=1,3,\dots} \left(\frac{\mathrm{cth}(n\pi\tau)}{n} - (n \to \sqrt{n^2 + (\frac{\mu L}{\pi})^2}\,)\right) \tag{77}$$

$$K_{\mu^2}(\frac{L}{2}) = \frac{1}{2\pi} \sum_{m\geq 1} \left(\frac{\mathrm{ch}(m\pi/\tau) - 1}{m\,\mathrm{sh}(m\pi/\tau)} - (m \to \sqrt{m^2 + (\frac{\mu\beta}{2\pi})^2}\,)\right) + \frac{1}{8\tau} - \frac{1}{2\mu\beta}\frac{\mathrm{ch}(\mu L) - 1}{\mathrm{sh}(\mu L)}. \tag{78}$$

Depending on which one of the equivalent forms (68) and (69) for K_{μ^2} on the diagonal is used the factor $D^\pm$ at the midpoints is found to read

$$D^\pm(\frac{L}{2}) = \exp\Big\{\frac{2}{N_f}\sum_{n=1,3,\dots}\frac{\mathrm{cth}(n\pi\tau)}{n} - (n \to \sqrt{n^2 + (\frac{\mu L}{\pi})^2}\,)\Big\} \cdot$$

$$\exp\Big\{\mp\theta\Big(1 - 1/\mathrm{ch}(\mu L/2)\Big)\Big\} \tag{79}$$

$$D^\pm(\frac{L}{2}) = \exp\Big\{\frac{1}{N_f}\sum_{m\geq 1}\frac{\mathrm{th}(m\pi/2\tau)}{m} - (m \to \sqrt{m^2 + (\frac{\mu\beta}{2\pi})^2}\,)\Big\} \cdot$$

$$\exp\Big\{\frac{\pi}{N_f}\Big(\frac{1}{4\tau} - \frac{\mathrm{th}(\mu L/2)}{\mu\beta}\Big)\Big\} \cdot \exp\Big\{\mp\theta\Big(1 - 1/\mathrm{ch}(\mu L/2)\Big)\Big\} \tag{80}$$

which can be used to derive the low and high temperature expansions, respectively.

7 Noncommutativity of the Limits $\beta^{-1} \to 0$ and $L \to \infty$

In this section we show that the condensates at the midpoints, $\langle \psi^\dagger P_\pm \psi \rangle_\beta(\frac{L}{2})$, depend on the order in which the limits $\beta \to \infty$ and $L \to \infty$ are taken; for $N_f = 1$ the condensates survive only if we first let $\beta \to \infty$.

7.1 Limit $\beta \to \infty$ for finite spatial length L

Here we derive the low temperature limit, i.e. the condensates for β large compared to the fixed spatial length L and the induced mass μ.

From the explicit expression (75) we see at once that

$$C^\pm(\frac{L}{2}) = \pm \frac{e^{\pm\theta}}{4L}(1 + O(e^{-2\cdot\pi\beta/2L})) \tag{81}$$

for any number of flavours.

In order to get the corresponding limit for the second factor $D^\pm(\frac{L}{2})$ in (59) we use (79) and perform the asymptotic expansion of the coth to get

$$
\begin{aligned}
D^\pm(\frac{L}{2}) &= \exp\Big\{ \frac{2}{N_f} \sum_{n=1,3,\ldots} \Big(\frac{1}{n} - \frac{1}{\sqrt{n^2 + (\mu L/\pi)^2}} \Big) \Big\} \cdot \\
&\quad \exp\Big\{ \frac{4}{N_f} \sum_{n=1,3,\ldots} \sum_{k \geq 1} \Big(\frac{e^{-2k\cdot n \frac{\pi\beta}{2L}}}{n} - \frac{e^{-2k\sqrt{n^2 + (\mu L/\pi)^2} \frac{\pi\beta}{2L}}}{\sqrt{n^2 + (\mu L/\pi)^2}} \Big) \Big\} \cdot \\
&\quad \exp\Big\{ \mp \theta(1 - 1/\mathrm{ch}(\mu L/2)) \Big\}
\end{aligned}
\tag{82}
$$

adapted to $\beta \gg L$ as an intermediate result. Using the identity [27]

$$\sum_{n=1,3\ldots} \frac{1}{n} - \frac{1}{\sqrt{n^2 + (x/\pi)^2}} = \frac{\gamma}{2} + \frac{1}{2}\ln(\frac{x}{\pi}) - \sum_{j \geq 1}(-)^j K_0(jx) \tag{83}$$

valid for $x > 0$, where γ denotes the Euler Masceroni constant and K_0 the zeroth Bessel function the second factor can be rewritten as

$$
\begin{aligned}
D^\pm(\frac{L}{2}) &= e^{\gamma/N_f} \Big(\mu L/\pi \Big)^{1/N_f} \exp\Big\{ - \frac{2}{N_f} \sum_{j \geq 1}(-1)^j K_0(j\mu L) \Big\} \cdot \\
&\quad \exp\Big\{ \mp \theta(1 - 1/\mathrm{ch}(\mu L/2)) \Big\} \cdot O(\exp(\frac{4}{N_f} e^{-2\cdot\pi\beta/2L}))
\end{aligned}
\tag{84}
$$

Combining (81) and (84) we get the result

$$
\begin{aligned}
\langle \psi^\dagger P_\pm \psi \rangle_\beta(\frac{L}{2}) &= \pm \frac{1}{4L}(\frac{\mu L}{2\pi})^{1/N_f} e^{\gamma/N_f} \exp\Big\{ - \frac{2}{N_f} \sum_{j \geq 1}(-1)^j K_0(j\mu L) \Big\} \cdot \\
&\quad \exp\Big\{ \pm \theta/\mathrm{ch}(\mu L/2) \Big\} \cdot (1 + O(e^{-2\cdot\pi\beta/2L}))
\end{aligned}
\tag{85}
$$

where the θ dependencies are found to cancel up to exponentially small remainders. In particular we have found a nonzero value for $\langle \psi^\dagger P_\pm \psi \rangle$ for midpoints at zero temperature for any N_f for finite spatial length L.

7.2   Limit $L \to \infty$ for finite temperature

Here we give the large volume expansion of (59) valid for length L which is large as compared to the fixed inverse temperature β and μ^{-1}.

The first task is to derive the high temperature asymptotics for the first factor $C^\pm$ in (59). By a variety of manipulations including infinite product representations for the exponential factors constituting the measure we arrived at the asymptotic result [24]

$$
C^\pm(\frac{L}{2}) = \begin{cases}
\pm\frac{e^{\pm\theta}}{\beta} \cdot \frac{\sqrt{\beta}}{\pi\sqrt{2L}} e^{-\frac{5}{2}\frac{\pi L}{\beta}} & (N_f = 1) \\[2ex]
\pm\frac{e^{\pm\theta}}{\beta} \cdot 2 e^{-3\frac{\pi L}{\beta}} & (N_f = 2) \\[2ex]
\pm\frac{e^{\pm\theta}}{\beta} \cdot 4 e^{-2\frac{2N_f-1}{N_f}\frac{\pi L}{\beta}} & (N_f \geq 3)
\end{cases}
\tag{86}
$$

which is an exponential decay which goes faster as the number of flavours increases.

Also, we performed the asymptotic expansions of the hyperbolic functions in (80) and arrived at

$$
\begin{aligned}
D^\pm(\frac{L}{2}) &= \exp\Big\{\frac{1}{N_f}\Big(\gamma + \frac{\pi}{\mu\beta} + \ln(\frac{\mu\beta}{4\pi}) - 2\sum_{j\geq 1} K_0(j\mu\beta)\Big)\Big\} \cdot \\
&\quad \exp\Big\{\frac{2}{N_f}\sum_{m\geq 1}\sum_{l\geq 1}(-1)^l\Big(\frac{e^{-l\,m\pi/\tau}}{m} - \frac{e^{-l\pi\sqrt{m^2+(\mu\beta/2)^2}/\tau}}{\sqrt{m^2+(\mu\beta/2)^2}}\Big)\Big\} \cdot \\
&\quad \exp\Big\{\frac{1}{N_f}\frac{\pi}{4\tau}\Big(1 - \frac{1+2\sum_{l\geq 1}(-1)^l e^{-l\mu L}}{\mu L/2}\Big)\Big\} \cdot \\
&\quad \exp\Big\{\mp\theta \pm 2\theta\sum_{l\geq 0}(-1)^l e^{-(2l+1)\mu L/2}\Big\}
\end{aligned}
\tag{87}
$$

where everything is at least exponentially suppressed as compared to the growing factor in the second-last line.

Combining (86) and (87) we end up with the result

$$
\begin{aligned}
\langle\psi^\dagger P_\pm\psi\rangle(\frac{L}{2}) &= \pm \begin{cases}
\frac{1}{\pi\sqrt{2\beta L}} \cdot e^{-\frac{5}{2}\frac{\pi L}{\beta}} & (N_f = 1) \\[1ex]
\frac{2}{\beta} \cdot e^{-3\frac{\pi L}{\beta}} & (N_f = 2) \\[1ex]
\frac{4}{\beta} \cdot e^{-2\frac{2N_f-1}{N_f}\frac{\pi L}{\beta}} & (N_f \geq 3)
\end{cases} \cdot e^{\frac{1}{N_f}\frac{\pi L}{2\beta}} \cdot e^{\gamma/N_f} \cdot \Big(\frac{\mu\beta}{4\pi}\Big)^{1/N_f} \\
&\quad \exp\Big\{-\frac{2}{N_f}\sum_{j\geq 1}K_0(j\mu\beta)\Big\} \cdot O\Big(\exp\Big\{-\frac{2}{N_f}e^{-2\pi L/\beta}\Big\}\Big) \cdot \\
&\quad \exp\Big\{-\frac{1}{N_f}\frac{2\pi}{\mu\beta}\sum_{l\geq 1}(-1)^l e^{-l\mu L}\Big\} \cdot \exp\Big\{\pm 2\theta\sum_{l\geq 0}(-1)^l e^{-(2l+1)\mu L/2}\Big\}
\end{aligned}
\tag{88}
$$

which gives a decay

$$
\langle\psi^\dagger P_\pm\psi\rangle(\frac{L}{2}) \sim \begin{cases}
\pm\text{const} \cdot \frac{1}{\sqrt{L}} e^{-2\frac{\pi L}{\beta}} & (N_f = 1) \\[2ex]
\pm\text{const} \cdot e^{-\frac{8N_f-5}{2N_f}\frac{\pi L}{\beta}} & (N_f \geq 2)
\end{cases}
\tag{89}
$$

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for spatial lengths L which are large compared to the inverse temperature β and the inverse charge e^{-1}.

7.3 Noncommutativity of the limits $\beta \to \infty$ and $L \to \infty$

Using the results of the previous subsections it is easy to show that the two limits $\beta \to \infty$ and $L \to \infty$ do not commute.

Recall that $\langle \psi^\dagger P_\pm \psi \rangle(\frac{L}{2})$ is a shorthand for $\langle \psi^\dagger P_\pm \psi \rangle_{\theta, N_f, \beta, L}(x^1 := \frac{L}{2})$.

Now the formulas (85), (89) imply

$$
\lim_{L \to \infty} \lim_{\beta \to \infty} \langle \psi^\dagger P_\pm \psi \rangle(\frac{L}{2}) \;=\; \begin{cases} \pm \frac{1}{4\pi} e^\gamma \sqrt{\frac{N_f e^2}{\pi}} & (N_f = 1) \\[2ex] 0 & (N_f \geq 2) \end{cases} \tag{90}
$$

$$
\lim_{\beta \to \infty} \lim_{L \to \infty} \langle \psi^\dagger P_\pm \psi \rangle(\frac{L}{2}) \;=\; 0 \qquad\qquad (\forall\, N_f \geq 1) \tag{91}
$$

respectively, which is the other main result of this paper.

From a physical point of view this means that the system under consideration shows a distinctive hysteresis phenomenon: When both of β and L are sent to infinity, the one-flavour system keeps the knowledge of which limit was performed first in the actual value of its chiral condensate. Obviously there is no such non-commutativity for finite changes of the lengths β and L. We shall further comment on this interesting behaviour in the conclusions.

8 Discussion and Conclusions

In this paper we have performed in a functional framework the quantization of the N_f flavour euclidean Schwinger model inside a finite temperature cylinder with $SU(N_f)_A$ breaking local boundary conditions at the two spatial ends to trigger chiral symmetry breaking. We have determined the effective action for the bosonic subsystem subject to these boundary conditions, which arises after integrating out the fermions. We have shown the way the expectation value of an arbitrary gauge-invariant operator can be computed and in particular we have performed the calculation of the condensates $\langle \psi^\dagger P_\pm \psi \rangle(x)$ (to be used as the most simple order parameters) for any point x inside the cylinder and any value of the inverse temperature β and spatial length L.

The quantization was greatly simplified by the fact that the boundary conditions chosen (the CB-boundary-conditions) completely ban the zero modes. Once more we emphasize the fact that our results have been obtained purely analytically and without doing 'instanton physics'. The technical aspects are rather different as those one encounters when quantizing the theory on a sphere [15] or on a torus [3, 14, 23].

Nevertheless our results are in full agreement with the earlier instanton-type and small-quark-mass calculations. Thus it seems that the CB-boundary-conditions applied at the two spatial ends of the cylinder give a perfect substitute for introducing small quark masses to trigger the chiral symmetry breaking and a real alternative to the study of torons [12] or fractons [8] or singular gauge-fields on S^4 [13]. The real advantage is of course the fact that they constitute almost exactly the border of what can be calculated analytically. The functional integral over the prepotential is gaussian, whereas, in general, the integration over

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the harmonic part of the gauge-potential is not. However the latter reduces to gaussian integrals in the low and high temperature expansions.

In the low temperature limit $\mu^{-1} = 1/\sqrt{N_f e^2/\pi} \ll L \ll \beta = T^{-1}$ we found for the chiral condensate the asymptotic value

$$\langle \psi^\dagger P_\pm \psi \rangle (\frac{L}{2}) = \pm \frac{1}{4L} e^{\gamma/N_f} \left(\frac{\mu L}{\pi} \right)^{1/N_f} = \pm \frac{1}{4L} e^{\gamma/N_f} \left(\frac{\sqrt{N_f e^2/\pi} \; L}{\pi} \right)^{1/N_f} \tag{92}$$

which, when restricted to the two-flavour case reduces to

$$\langle \psi^\dagger P_\pm \psi \rangle (\frac{L}{2}) = \pm \left(\frac{e^\gamma \sqrt{2e^2/\pi}}{16\pi L} \right)^{1/2} . \tag{93}$$

This expression is identical to the result of Shifman and Smilga [8], who allowed for fracton configurations on the torus.

In the high temperature limit $T = \beta^{-1} \gg \sqrt{N_f e^2/\pi} \gg L^{-1}$ we found for the chiral condensate an exponential decay with T.

For intermediate temperatures $T = \beta^{-1} \simeq \sqrt{N_f e^2/\pi}$ and finite L one has to retreat to numerical methods to evaluate the remaining sum and the integrals in (75) and (79) or equivalently in (76) and (80). One realizes that the observable $\langle \psi P_\pm \psi \rangle$ viewed as a function of T strongly resembles the behaviour of an order parameter in a system which suffers a second order phase transition for the case $N_f \geq 2$. However, the chiral condensate does not really vanish at any finite temperature, it is just exponentially close to zero for temperatures larger than the induced mass $\mu = \sqrt{N_f e^2/\pi}$. Thus, in a strict sense, the chiral symmetry remains broken even for $N_f \geq 2$ at all finite temperatures as long as L stays finite, as has been argued to be a general fact by Dolan and Jackiw [9]. However, if L is sent to infinity for finite β, the condensate exponentially drops to zero.

Our main result is the fact that the limits $\beta \to \infty$ and $L \to \infty$ do not commute for the observable $\langle \psi^\dagger P_\pm \psi \rangle$ in the $N_f = 1$ case, since

$$\lim_{L \to \infty} \lim_{\beta \to \infty} \langle \psi^\dagger P_\pm \psi \rangle (\frac{L}{2}) = \pm \frac{1}{4\pi} e^\gamma \sqrt{\frac{e^2}{\pi}} \qquad (N_f = 1) \tag{94}$$

$$\lim_{\beta \to \infty} \lim_{L \to \infty} \langle \psi^\dagger P_\pm \psi \rangle (\frac{L}{2}) = 0 \qquad (\forall N_f \geq 1) \tag{95}$$

which implies that there is no unique infinite volume limit. Thus it seems that the combination of finite-temperature and CB- boundary conditions provides an interesting tool for driving this system either into the true or the wrong vacuum state. The result (94,95) is rather remarkable, since it means that the one-flavour system shows some hysteresis phenomenon: As far as we are aware of the literature, such phenomena are known for spin systems but they are rather untypical for analytically solvable field theories. However one of the interesting new results in this respect is the work by Hetrick, Hosotani and Iso about the massive multi-flavour Schwinger model on the zero temperature cylinder [25]. They analyzed the situation for small quark masses and finite (cyclic) spatial length L. In particular they found that the two limits $m \to 0$ and $L \to \infty$ fail to commute. Thus we conclude that chirality breaking boundary conditions give an interesting alternative to introducing small quark masses.

Acknowledgments

A Explicit Construction of the Fermionic Heat Kernel

In this appendix we sketch the construction of the heat kernel of the squared Dirac operator $(i\not{D}|_{\alpha=0})^2 = (i\not\partial + 2\pi c/\beta\gamma^0)^2$ on a thermal manifold, which allows to compute the relevant Seeley DeWitt coefficient, a task, which itself is postponed to appendix B. For that we construct the heat kernel $\tilde{K}$ on the finite cylinder $\{(x^0, x^1) \mid x^0 \in [0, \beta[\,, x^1 \geq 0 \}$ which obeys (besides the usual heat kernel relations) the boundary conditions

$$(B_\theta \tilde{K})(t, x^0, 0, y) = \tilde{K}(t, x^0, 0, y) \tag{96}$$

$$(B_\theta i\not\partial \tilde{K})(t, x^0, 0, y) = (i\not\partial \tilde{K})(t, x^0, 0, y) \tag{97}$$

$$\tilde{K}(t, x^0 + \beta, x^1, y) = -\tilde{K}(t, x^0, x^1, y) \tag{98}$$

as well as the adjoint relations with respect to y, where B_θ is a shorthand for $B_L(\theta)$ defined in (10).

The trick is to start considerations on the half plane $\{(x^0, x^1) \mid x^1 \geq 0\}$, since here the above squared Dirac operator can be decomposed as

$$(i\not\partial_x + 2\pi c/\beta \cdot \sigma^1)^2 = e^{2\pi icx^0/\beta}(i\not\partial_x)^2 e^{-2\pi icx^0/\beta} \tag{99}$$

and correspondingly the free heat kernel takes the simple form

$$\frac{1}{4\pi t}e^{-((\xi^0)^2+(\xi^1)^2)/4t}e^{2\pi ic\xi^0/\beta} = \frac{1}{4\pi t}e^{-((\xi^0-4\pi ict/\beta)^2+(\xi^1)^2)/4t}e^{-4\pi^2c^2t/\beta^2} \tag{100}$$

where $\xi^0 = x^0 - y^0, \xi^1 = x^1 - y^1$. Using that the kernel can be Fourier transformed and from the mirror principle one is led to consider the expression

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_0^2+k_1^2)t}e^{ik_0\xi^0+ik_1\xi^1} \, dk_0 dk_1$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_0^2+k_1^2)t} \begin{pmatrix} f(k_0, k_1) & g(k_0, k_1) \\ g(k_0, k_1) & h(k_0, k_1) \end{pmatrix} e^{ik_0\xi^0+ik_1\eta} \, dk_0 dk_1$$

as an ansatz for the heat kernel of the operator $(i\not\partial)^2 = -\Delta \cdot I_2$ on the half plane. The boundary condition at $x^1 = 0$ immediately transforms into an algebraic relation among f, g, h which is solved by the expressions

$$f(k_0, k_1) = -\frac{e^{2\theta}(k_0 - ik_1) - (k_0 - ik_1)}{e^{2\theta}(k_0 + ik_1) - (k_0 - ik_1)}$$

$$g(k_0, k_1) = -\frac{2e^\theta \, ik_1}{e^{2\theta}(k_0 + ik_1) - (k_0 - ik_1)}$$

$$h(k_0, k_1) = -\frac{e^{2\theta}(k_0 + ik_1) - (k_0 + ik_1)}{e^{2\theta}(k_0 + ik_1) - (k_0 - ik_1)} \quad .$$

The resulting integrals can be done in two steps. First only the numerators of the functions f, g, h are taken into account and the resulting expressions are integrated over. Second the full expressions have to be read as differential equations in x^0, x^1 in the manner indicated by the previously omitted denominators of the functions f, g, h. There is a unique solution to this procedure which falls off in both x^0 plus the positive x^1 directions (note $\theta \in \mathrm{R}$) :

$$\frac{1}{4\pi t}\, e^{-\frac{(\xi^0)^2+(\xi^1)^2}{4t}} + \frac{1}{4\pi t}\begin{pmatrix} e^\theta \mathrm{sh}\theta & -\mathrm{ch}\theta \\ -\mathrm{ch}\theta & -e^{-\theta}\mathrm{sh}\theta \end{pmatrix} e^{-\frac{(\xi^0)^2+\eta^2}{4t}}$$

$$+\frac{i}{8\pi^{1/2}t^{3/2}}\begin{pmatrix} e^\theta \mathrm{sh}\theta & -\mathrm{sh}\theta \\ -\mathrm{sh}\theta & e^{-\theta}\mathrm{sh}\theta \end{pmatrix} \cdot (\xi^0 \mathrm{ch}\theta + i\eta\, \mathrm{sh}\theta)\ \cdot$$

$$e^{-\frac{(\xi^0 \mathrm{ch}\theta+i\eta\, \mathrm{sh}\theta)^2}{4t}} \cdot \left(1 + \mathrm{erf}(\frac{i\xi^0 \mathrm{sh}\theta-\eta\, \mathrm{ch}\theta}{2t^{1/2}})\right) \quad .$$

Since on the half-plane the operator $(i\partial\!\!\!/ + 2\pi c\sigma^1/\beta)^2$ has the decomposition (99) this immediately yields its heat kernel (to be denoted $\tilde{K}$) by just including a factor $e^{2\pi ic\xi^0/\beta}$ in each term. Finally the finite temperature boundary condition (98) is taken into account by substituting ξ^0 by $\xi^0-n\beta$, including an additional $(-1)^n$ and performing the sum over $n \in \mathrm{Z}$.

The heat kernel $\tilde{K}_{\alpha=0}$ of $(i\!D\!\!\!\!/\,|_{\alpha=0})^2 = (i\partial\!\!\!/ + 2\pi c\sigma^1/\beta)^2$ subject to the boundary conditions (96) - (98) on the half cylinder $\{\ (x^0, x^1)\ |\ x^0 \in [0, \beta[\ , x^1 \geq 0\ \}$ takes the final form

$$\tilde{K} \;=\; \sum (-1)^n \frac{1}{4\pi t}\, e^{-\frac{(\xi^0-n\beta)^2+(\xi^1)^2}{4t}}\ e^{2\pi ic(\xi^0-n\beta)/\beta}$$

$$+\ \sum(-1)^n \frac{1}{4\pi t}\begin{pmatrix} e^\theta \mathrm{sh}\theta & -\mathrm{ch}\theta \\ -\mathrm{ch}\theta & -e^{-\theta}\mathrm{sh}\theta \end{pmatrix} e^{-\frac{(\xi^0-n\beta)^2+\eta^2}{4t}}\ e^{2\pi ic(\xi^0-n\beta)/\beta}$$

$$+\ \sum(-1)^n \frac{i}{8\pi^{1/2}t^{3/2}}\begin{pmatrix} e^\theta \mathrm{sh}\theta & -\mathrm{sh}\theta \\ -\mathrm{sh}\theta & e^{-\theta}\mathrm{sh}\theta \end{pmatrix} \cdot \left((\xi^0-n\beta)\mathrm{ch}\theta + i\eta\, \mathrm{sh}\theta\right) \cdot$$

$$e^{-\frac{\left((\xi^0-n\beta)\mathrm{ch}\theta+i\eta\, \mathrm{sh}\theta\right)^2}{4t}} \cdot \left(1 + \mathrm{erf}(\frac{i(\xi^0-n\beta)\mathrm{sh}\theta-\eta\, \mathrm{ch}\theta}{2t^{1/2}})\right) \qquad (101)$$

where the sums run over $n\in\mathrm{Z}$ and can be seen to converge absolutely and thus uniformely.

B   Extraction of the Relevant Heat Kernel Coefficients

In this appendix we shall compute the surface Seeley DeWitt coefficient b_1 of the operator $-D\!\!\!\!/\,^2$ which enters the calculation of it's functional determinant. We first note that in general $\oint \mathrm{tr}\,(b_m(\varphi))$ with a smooth test function φ on a d dimensional manifold M has the expansion

$$\oint_{\partial M} \mathrm{tr}\,(b_m(\varphi)) = \sum_{p=0}^{d-1} \oint_{\partial M} \mathrm{tr}\,(b_{m.p}(R, \chi, F_{..}) \cdot \partial_n^p \varphi)\ ,$$

where $b_{m.p}$ is a gauge-invariant and Lorentz-covariant local polynomial in the intrinsic and extrinsic curvatures of the boundary as well as in the field strength and its covariant derivatives on the boundary. Here $\partial_n^p \varphi$ denotes the p fold derivative of the test function φ along the

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(outward oriented) normal of the boundary. In the case of a two dimensional manifold with Hrasko Balog boundary conditions (10) the expansion of $\oint \mathrm{tr}\,(b_1(\varphi))$ simplifies to

$$\oint \mathrm{tr}\,(b_1(\varphi)) = \oint \mathrm{tr}\,(b_{1.0}(\theta)\,\chi\cdot\varphi) + \oint \mathrm{tr}\,(b_{1.1}(\theta)\cdot\partial_n\varphi) \ .$$

For our purposes it is sufficient to know the coefficient $b_{1.1}$, since the first term does not contribute to (42) (due to $\varphi :\equiv H + H^\dagger = 0$ on ∂M) and in (38) it would yield an uninteresting constant which finally cancels in expectation values of gauge-invariant operators. The function $b_{1.1}$ can be determined from the heat kernel on the diagonal, $K(t,x,x)$ of $-\slashed{D}^2 = -\slashed{D}^2|_{\alpha=1}$ which is identical to $\tilde{K}(t,x,x)$ of $-\slashed{D}^2|_{\alpha=0}$ by calculating

$$\int_M K(t,x,x)\cdot\varphi(x) = \int_M \tilde{K}(t,x,x)\cdot\varphi(x) \sim \int_0^\infty \tilde{K}(t,x,x)\cdot\left(\varphi(x^0,0) + x^1\cdot\partial_1\varphi(x^0,0) + ...\right)dx^1$$

where $\tilde{K}$ denotes the heat kernel (101) calculated in appendix A. In writing this expansion we have anticipated that for small t the heat kernel on the diagonal is sharply peaked about the boundary whereupon it is justified to expand the test function φ about $x^1 = 0$. Using this result and denoting $\varphi'(x^0,.)$ the first derivative of φ with respect to it's second argument one has to compute an expression whose first few terms in the small t expansion take the form

$$\sum_{n\in Z}(-1)^n\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\quad\cdot\quad\frac{1}{4\pi t}\,e^{-\frac{n^2\beta^2}{4t}}\,e^{-2\pi inc}\cdot\int_0^\infty\varphi(x^0,x)\,dx$$

$$+\sum_{n\in Z}(-1)^n\begin{pmatrix}e^\theta\mathrm{sh} & -\mathrm{ch} \\ -\mathrm{ch} & -e^{-\theta}\mathrm{sh}\end{pmatrix}\quad\cdot\quad\frac{1}{4\pi t}\,e^{-\frac{n^2\beta^2}{4t}}\,e^{-2\pi inc}\cdot\int_0^\infty e^{-x^2/t}dx\cdot\varphi(x^0,0)$$

$$+\sum_{n\in Z}(-1)^n\begin{pmatrix}e^\theta\mathrm{sh} & -\mathrm{ch} \\ -\mathrm{ch} & -e^{-\theta}\mathrm{sh}\end{pmatrix}\quad\cdot\quad\frac{1}{4\pi t}\,e^{-\frac{n^2\beta^2}{4t}}\,e^{-2\pi inc}\cdot\int_0^\infty x\,e^{-x^2/t}dx\cdot\varphi'(x^0,0)$$

$$+\sum_{n\in Z}(-1)^n\begin{pmatrix}e^\theta\mathrm{sh} & -\mathrm{sh} \\ -\mathrm{sh} & +e^{-\theta}\mathrm{sh}\end{pmatrix}\quad\cdot\quad\int_0^\infty 1\,\frac{\mathrm{ch}n\beta - 2i\mathrm{sh}x}{8\pi^{1/2}it^{3/2}}\,e^{-\frac{(\mathrm{ch}n\beta-2i\mathrm{sh}x)^2}{4t}}\ .$$

$$\left(1 - \mathrm{erf}(\frac{\mathrm{ch}2x + i\mathrm{sh}n\beta}{2t^{1/2}})\right)\,e^{-2\pi inc}\,dx\cdot\varphi(x^0,0)$$

$$+\sum_{n\in Z}(-1)^n\begin{pmatrix}e^\theta\mathrm{sh} & -\mathrm{sh} \\ -\mathrm{sh} & +e^{-\theta}\mathrm{sh}\end{pmatrix}\quad\cdot\quad\int_0^\infty x\,\frac{\mathrm{ch}n\beta - 2i\mathrm{sh}x}{8\pi^{1/2}it^{3/2}}\,e^{-\frac{(\mathrm{ch}n\beta-2i\mathrm{sh}x)^2}{4t}}\ .$$

$$\left(1 - \mathrm{erf}(\frac{\mathrm{ch}2x + i\mathrm{sh}n\beta}{2t^{1/2}})\right)\,e^{-2\pi inc}\,dx\cdot\varphi'(x^0,0)$$

where the first line gives the usual a_0 coefficient whereas the remaining four integrals contain information about the $b_{1/2}$ and b_1 coefficients. Here and below we use the abbreviations $\mathrm{sh} = \mathrm{sh}\theta, \mathrm{ch} = \mathrm{ch}\theta$.

The first and second integrals are easily evaluated using the formulas

$$I_1 := \int_0^\infty e^{-\frac{x^2}{t}}\,dx = \frac{\sqrt{\pi t}}{2}\quad,\qquad I_2 := \int_0^\infty x\cdot e^{-\frac{x^2}{t}}\,dx = \frac{t}{2}\quad.$$

The third and fourth integrals are handled using the formulas

$$
I_3 : \quad = \quad \int_0^\infty \frac{\mathrm{ch}n\beta - 2i\mathrm{sh}x}{8\pi^{1/2}it^{3/2}} \, e^{-\frac{(\mathrm{ch}n\beta - 2i\mathrm{sh}x)^2}{4t}} \, (1 - \mathrm{erf}(\frac{\mathrm{ch}2x + i\mathrm{sh}n\beta}{2t^{1/2}})) \, dx
$$

$$
= \quad -\frac{1}{8\pi^{1/2}t^{1/2}} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} + \frac{1}{8\pi^{1/2}t^{1/2}} \frac{1}{\mathrm{sh}} \cdot e^{-\frac{\mathrm{ch}^2 \, n^2\beta^2}{4t}} \mathrm{erfc}(\frac{i\mathrm{sh}n\beta}{2t^{1/2}})
$$

$$
I_4 : \quad = \quad \int_0^\infty x \, \frac{\mathrm{ch}n\beta - 2i\mathrm{sh}x}{8\pi^{1/2}it^{3/2}} \, e^{-\frac{(\mathrm{ch}n\beta - 2i\mathrm{sh}x)^2}{4t}} \, (1 - \mathrm{erf}(\frac{\mathrm{ch}2x + i\mathrm{sh}n\beta}{2t^{1/2}})) \, dx
$$

$$
= \quad -\frac{1}{8\pi} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} + \frac{1}{8\pi^{1/2}t^{1/2}} \frac{1}{\mathrm{sh}} \cdot \int_0^\infty e^{-\frac{(\mathrm{ch}n\beta - 2i\mathrm{sh}x)^2}{4t}} \mathrm{erfc}(\frac{\mathrm{ch}2x + i\mathrm{sh}n\beta}{2t^{1/2}}) \, dx
$$

which result in the small t asymptotics

$$
I_3 \cong \begin{cases} -\frac{1}{8\pi^{1/2}t^{1/2}} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} - \frac{i}{4\pi\mathrm{sh}^2} e^{-\frac{n^2\beta^2}{4t}} (1 + O(t)) & (n > 0) \\[2mm] -\frac{1}{8\pi^{1/2}t^{1/2}} \frac{\mathrm{ch}}{\mathrm{sh}} + \frac{1}{8\pi^{1/2}t^{1/2}} \frac{1}{\mathrm{sh}} & (n = 0) \\[2mm] -\frac{1}{8\pi^{1/2}t^{1/2}} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} + \frac{i}{4\pi\mathrm{sh}^2} e^{-\frac{n^2\beta^2}{4t}} (1 + O(t)) & (n < 0) \end{cases} \tag{102}
$$

$$
I_4 \cong \begin{cases} -\frac{1}{8\pi} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} - \frac{it^{1/2}}{8\pi^{1/2}\mathrm{sh}^2 n\beta} e^{-\frac{n^2\beta^2}{4t}} (1 + O(t^{1/2})) & (n > 0) \\[2mm] -\frac{1}{8\pi} \frac{\mathrm{ch}}{\mathrm{sh}} + \frac{1}{8\pi} \frac{\log(\mathrm{ch}+\mathrm{sh}) - \log(\mathrm{ch}-\mathrm{sh})}{2\mathrm{sh}^2} & (n = 0) \\[2mm] -\frac{1}{8\pi} \frac{\mathrm{ch}}{\mathrm{sh}} e^{-\frac{n^2\beta^2}{4t}} - \frac{it^{1/2}}{8\pi^{1/2}\mathrm{sh}^2 n\beta} e^{-\frac{n^2\beta^2}{4t}} (1 + O(t^{1/2})) & (n < 0) \end{cases} \tag{103}
$$

where the result for I_3 immediately follows from the asymptotic expansion [28]

$$
\sqrt{\pi} \, z \, e^{z^2} \mathrm{erfc}(z) \cong 1 + \sum_{k=1}^\infty (-1)^k \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(2^k z^{2k})} \qquad (z \to \infty, |\mathrm{arg}z| < \frac{3\pi}{4}) \tag{104}
$$

whereas the expression for I_4 results from a computation establishing the asymptotic behaviour

$$
f(w) \quad = \quad \int_0^\infty e^{-(\mathrm{ch}w - i\mathrm{sh}x)^2} \mathrm{erfc}(\mathrm{ch}x + i\mathrm{sh}w) dx
$$

$$
= \quad e^{-w^2} \left(-\frac{i}{2\mathrm{sh}} \cdot \frac{1}{w} + \frac{\mathrm{ch}}{2\pi^{1/2}\mathrm{sh}^2} \cdot \frac{1}{w^2} + \frac{i}{4\mathrm{sh}} \cdot \frac{1}{w^3} + O(\frac{1}{w^4}) \right) \tag{105}
$$

for $w \gg 1$.

Putting everything together we arrive at the small t expansion of the heat kernel

$$
\int_M K(t, x, x)\varphi(x) \, dx - O(t^{1/2}) = \tag{106}
$$

$$+\frac{1}{4\pi t}\cdot\left\{\sum_{\mathbf{Z}}(-1)^n e^{-\frac{n^2\beta^2}{4t}}\cos(2\pi nc)\right\}\cdot\int\varphi(x^0,x^1)\,d^2x$$

$$+\frac{1}{8\pi^{1/2}t^{1/2}}\cdot\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\sum_{\mathbf{Z}}(-1)^n e^{-\frac{n^2\beta^2}{4t}}\cos(2\pi nc)+\begin{pmatrix}e^{\theta} & -1\\ -1 & e^{-\theta}\end{pmatrix}\right\}\cdot\int\varphi(x^0,0)\,dx^0$$

$$+\frac{1}{2\pi}\cdot\left\{\frac{1}{\mathrm{sh}(\theta)}\begin{pmatrix}e^{\theta} & -1\\ -1 & e^{-\theta}\end{pmatrix}\sum_{n\geq 1}(-1)^n e^{-\frac{n^2\beta^2}{4t}}\sin(2\pi nc)\right\}\cdot\int\varphi(x^0,0)\,dx^0$$

$$+\frac{1}{8\pi}\cdot\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\sum_{\mathbf{Z}}(-1)^n e^{-\frac{n^2\beta^2}{4t}}\cos(2\pi nc)+\frac{\ln(e^{\theta})}{\mathrm{sh}(\theta)}\begin{pmatrix}e^{\theta} & -1\\ -1 & e^{-\theta}\end{pmatrix}\right\}\cdot\int\varphi'(x^0,0)\,dx^0$$

which is used to determine the effective action (53). This formula resolves also the apparent paradox that the θ-term in the effective action (53) is linear, whereas the whole model was defined through hyperbolic functions of θ, thus there must be an invariance under the replacement $\theta\to\theta+2\pi i$.

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