Running Surface Couplings

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Abstract

We discuss the renormalization group improved effective action and running surface couplings in curved spacetime with boundary. Using scalar self-interacting theory as an example, we study the influence of boundary effects to effective equations of motion in spherical cap and the relevance of surface running couplings to quantum cosmology and symmetry breaking phenomenon. Running surface couplings in the asymptotically free SU(2) gauge theory are found.

1 Introduction

Boundary terms may play an important role in quantum cosmology and in particular in connection with the quantum state of the universe [1]. That is why, starting from the 80'ties [7], there has been a continued interest to study boundary divergences (see, for example, [2-6,8] and references therein). In [5, 6] some misprints of previous calculations have been corrected and the surface divergences have been found in a form of conformal anomalies for various boundary conditions.

In [9] the running surface couplings have been introduced. The motivation to do it was the fact that in order to make a theory multiplicatively renormalizable in curved spacetime with boundary one has to include the surface Lagrangian with arbitrary coupling constants in the total Lagrangian. When the renormalization group is constructed, each coupling becomes a running effective coupling. A similiar idea has been persued in [10], where running surface couplings have been discussed in spacetime with boundaries and have been related to the finite size effects. It is quite well-known that running couplings have different physical applications. It is the purpose of this work to discuss the running surface couplings for different theories and to look for the consequences to which they may lead.

In the next section we discuss the self-interacting scalar theory on curved spacetime with boundary using Dirichlet boundary conditions. The explicit expressions for the volume and running surface couplings are given. The procedure to construct the RG improved effective action in such a spacetime is discussed. In the section 3 we find the RG improved effective action in a spherical cap and show how boundary terms become relevant in the effective field equations. For the example of a disc we show the possible influence of boundary terms to symmetry breaking phenomena. In section 4, we show how the above discussion can be generalized to arbitrary GUTs, and in particular to the asymptotically free SU(2) gauge theory with scalars and spinors, in curved spacetime with boundary. Some discussions are presented in the last section.

2 Self-interacting scalar theory in curved space with boundary.

Consider the self-interacting scalar theory in curved spacetime M with boundary ∂M. The renormalization of the theory maybe done in close analogy with the renormalization in curved spacetime without boundary (for a general introduction see [11]). The boundary conditions for scalar fields maybe chosen to be of Dirichlet type

$$\phi(x) = 0 \quad , \quad x \in \partial M \tag{1}$$

or Robin type

$$(\psi + n^\mu \nabla_\mu)\,\phi(x) = 0 \quad , \quad x \in \partial M. \tag{2}$$

Here n^μ is the outward normal on ∂M and ψ is an arbitrary scalar function.

The euclidean action corresponding to a massless multiplicatively renormalizable theory maybe written as the following:

$$S = S_M + S_V + S_S, \tag{3}$$

where

$$\begin{aligned} S_M &= \int d^4x\sqrt{g}\Big\{\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}\xi R\varphi^2 + \frac{\lambda\varphi^4}{4!}\Big\}\ , \\ S_V &= \int d^4x\sqrt{g}\Big\{a_1R^2 + a_2C^2_{\mu\nu\alpha\beta} + a_3G + a_4\Box R\Big\}, \end{aligned} \tag{4}$$

and $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, G the Gauss-Bonnet invariant and a_1, a_2, a_3, a_4 are coupling constants in the external fields sector.

In the discussion of the surface action we will limit ourselves to Dirichlet boundary conditions. We use two invariants of dimension L^{-3} expressed in terms of $R_{\mu\nu\alpha\beta}$ and the extrinsic curvature of the boundary $K_{\mu\nu}$ [3, 6]

$$\begin{aligned} q &= \frac{8}{3}K^3 + \frac{16}{3}K_\mu^{\ \nu}K_\nu^{\ \alpha}K_\alpha^{\ \mu} - 8KK_{\mu\nu}K^{\mu\nu} + 4KR \\ &- 8R_{\mu\nu}(Kn^\mu n^\nu + K^{\mu\nu}) + 8R_{\mu\nu\alpha\beta}K^{\mu\alpha}n^\nu n^\beta, \\ g &= K_\mu^{\ \nu}K_\nu^{\ \alpha}K_\alpha^{\ \mu} - KK_{\mu\nu}K^{\mu\nu} + \frac{2}{9}K^3. \end{aligned} \tag{5}$$

Then, the surface action maybe rewritten as

$$S_S = \int\limits_{\partial M} d^3x\,\sqrt{\gamma}\,L_S$$

with

$$L_S = \alpha_D q + \beta_D g + \gamma_D RK + \delta_D n^\mu\nabla_\mu R + \zeta_D C_{\mu\nu\alpha\beta}K^{\mu\alpha}n^\nu n^\beta, \tag{6}$$

where $\gamma_{\alpha\beta}$ is the induced metric of the boundary and $\alpha_D, \ldots, \zeta_D$ are surface coupling constants. In the same way one can write S_S for other boundary conditions.

Now, from the point of view of the renormalization group, each coupling constant has the correspondent effective coupling constant. Using the well-known results for the one-loop divergences of the volume terms one easily finds the running volume couplings:

$$
\begin{aligned}
\lambda(t) &= \frac{\lambda}{\kappa(t)} \qquad , \qquad \xi(t) = \frac{1}{6} + \big(\xi - \frac{1}{6}\big)\kappa(t)^{-\frac{1}{3}} \\
a_1(t) &= a_1 - \frac{1}{2\lambda}\big(\xi - \frac{1}{6}\big)^2\big[\kappa(t)^{\frac{1}{3}} - 1\big] \quad , \quad a_2(t) = a_2 + \frac{t}{120(4\pi)^2} \\
a_3(t) &= a_3 - \frac{t}{360(4\pi)^2} \ , \ a_4(t) = a_4 - \frac{t}{180(4\pi)^2} - \frac{\xi - \frac{1}{6}}{12\lambda}\big[\kappa(t)^{\frac{2}{3}} - 1\big],
\end{aligned}
\tag{7}
$$

where t is renormalization group parameter and

$$
\kappa(t) = 1 - \frac{3\lambda t}{(4\pi)^2}.
$$

Using the explicit results for the boundary conterterms [3, 6] we can write down the explicit expressions for the running surface couplings in theory (3) [9, 10]:

$$
\begin{aligned}
\alpha_D(t) &= \alpha_D - \frac{t}{360(4\pi)^2} \qquad , \qquad \beta_D(t) = \beta_D + \frac{2t}{35(4\pi)^2} \\
\gamma_D(t) &= \gamma_D + \frac{D(t)}{3} \ , \ \delta_D(t) = \delta_D + \frac{D(t)}{2} \ , \ \zeta_D = \zeta_D + \frac{t}{15(4\pi)^2}
\end{aligned}
\tag{8}
$$

where

$$
D(t) = \frac{\xi - \frac{1}{6}}{2\lambda}\big[\kappa(t)^{2/3} - 1\big].
$$

As usually the $t \to \infty$ limit defines the theory at very high energies (strong gravitational field). As we see from Eqs. (8) there is already some mixture of the volume with the surface couplings when they are running.

Now, after this overview of the situation with running surface couplings in curved spacetime, the interesting question is – what new phenomena may be encountered using the renormalization group. In particular, as it was already mentioned, the boundary effects are expected to be important in quantum cosmology. Hence it is interesting to understand the relevance of renormalization group in this respect.

Let us consider the situation where the volume Lagrangian (as well as L_S) is independent of one of the coordinates. Then, in the volume action we may integrate explicitly over this coordinate and as a result we can write the action (assuming that there is only a gravitational background field) as

$$
S_{grav.} = \int d^3\sqrt{g}\,\Big\{l_1 L_V + l_2 L_S\Big\},
\tag{9}
$$

where l_1, l_2 are some dimensionful constants, for example, $l_1 = \int dx$ (where x is the variable on which the Lagrangean does not depend). Due to the fact that the theory is multiplicatively renormalizable, we may now write explicitly the RG equation for effective Lagrangian:

$$(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma_i \phi_i \frac{\delta}{\delta \phi_i})\, L_{\mathrm{eff}}\,(\mu, \lambda_i\,,\, \phi_i) = 0, \tag{10}$$

where μ is a mass parameter, λ_i are volume and surface coupling constants with corresponding beta-functions β_i and ϕ_i are the fields. For an alternative derivation of (10), where μ is replaced by the inverse diameter of the spacetime M see [10].

Solving Eq.(10) by the method of characteristics, with Lagrangean (9) as initial condition at $t = 0$ and assuming a gravitational background field only (the other background fields are set to zero) we find the following contribution to L_{eff}

$$\begin{aligned} L_{eff}(\mu, \lambda_i, \phi_i) &= L_{eff}(\mu e^t, \lambda_i(t), \phi_i(t)) \\ &= l_1 \Big\{ a_1(t) R^2 + a_2(t) C^2_{\mu\nu\alpha\beta} + a_3(t) G + a_4(t) \Box R \Big\} \\ &+ l_2 \Big\{ \alpha_D(t) q + \beta_D(t) g + \gamma_D(t) RK + \delta_D(t) n^\mu \nabla_\mu R \\ &+ \zeta_D(t) C_{\mu\nu\alpha\beta} K^{\mu\alpha} n^\nu n^\beta \Big\}, \end{aligned} \tag{11}$$

where the running volume and surface couplings are given by eqs.(7,8). The above discussion which yielded the RG improved Lagrangian in curved space is very similar to standard RG improvement of the effective potential in flat [12, 13] or in curved space [14, 15]. The problem now is the choice of RG parameter t. Motivated by the one-loop considerations of the theory under discussion, the natural choice is (let R be positive)

$$t = \frac{1}{2} \log \frac{R}{\mu^2}. \tag{12}$$

With this choice, we get the improved effective Lagrangian (the summation over all leading logarithms of perturbation theory). In that sense the result is beyond one-loop order. The important implication of (11,12) is that due to the RG, the surface terms cease to be surface terms. They give contributions to the equations of motion, and hence, they influence quantum cosmology dynamically. Classically the surface terms maybe dropped. On the quantum level, however, these terms are important, as after RG improvement they contribute to the equations of motion. We give an explicit example in the next section.

3 RG improved Lagrangian in scalar theory on four-sphere with boundary

In what follows we will limit ourselves to the spaces of the type $R_{\mu\nu} = \Lambda g_{\mu\nu}$ which are of interest for quantum cosmology as they describe the inflationary Universes. In this case the structure of the initial Lagrangian significally simplifies.

Consider as an example a spherical cap C, i.e. region of the four-sphere with maximum colatitude θ. Then, the RG improved action is

$$\begin{aligned} S_{eff} &= \int\limits_M d^4x\sqrt{g}\, S_{V,eff} + \int\limits_{\partial M} d^3x\sqrt{\gamma}\, S_{S,eff} \\ &= 24\pi^2\Bigg\{\Big[16a_1(t) + \frac{8}{3}a_3(t)\Big]\cdot\Big[\frac{1}{2} - \frac{3}{4}\cos\theta + \frac{1}{4}\cos^3\theta\Big] \\ &+ \cos^3\theta\Big[2\alpha_D(t)\Big] + \frac{9}{2}\cos\theta\sin^2\theta\big\{\gamma_D(t)\big\}\Bigg\}, \end{aligned} \tag{13}$$

where $t = \frac{1}{2}\log\frac{4\Lambda}{\mu^2}$. We supposed Dirichlet boundary conditions for the scalar field. One may consider other conditions as well. The calculation of conformal anomaly in above-described situationes has been given in [6]. For comparison we may give the RG improved action in case of the 4-sphere(for the discussion of the effective action in De Sitter space see, also [16] and [18])

$$S_{eff} = 24\pi^2\Big(16a_1(t) + \frac{8}{3}a_3(t)\Big). \tag{14}$$

The effective equations of motion are given by

$$\frac{\partial S_{eff}}{\partial\Lambda} = 0. \tag{15}$$

Classically a_1 and a_3 are constant and the cosmological constant is not determined. On quantum level we get from from (14,15)

$$8\Big(\xi - \frac{1}{6}\Big)^2\kappa(t)^{-2/3} - \frac{1}{135} = 0,$$

where $\kappa(t)$ has been introduced below (7), the selfconsistent quantum solution

$$\frac{1}{2}\log\frac{4\Lambda}{\mu^2} = \frac{(4\pi)^2}{3\lambda}\Bigg\{1 - \Big[8\cdot 135(\xi - \frac{1}{6})^2\Big]^{3/2}\Bigg\}.$$

Hence, the effective cosmological constant is defined from the back-reaction of the quantum matter on the geometry. The corresponding non-singular universe is a De-Sitter spacetime (for free theory see also[21]).

Let us now consider a universe which is a spherical cap C. Its RG improved gravitational action is given by (13). The effective equation is found to be

$$\begin{aligned}&\Big[8(\xi-\frac{1}{6})^2\kappa(t)^{-2/3}-\frac{1}{135}\Big]\Big[\frac{1}{2}-\frac{3}{4}\cos\theta+\frac{1}{4}\cos^3\theta\Big]\\&-\frac{2\cos^3\theta}{360}-\frac{3}{2}\cos\theta\sin^2\theta(\xi-\frac{1}{6})\kappa(t)^{-1/3}=0.\end{aligned}\tag{16}$$

This effective equation of motion in which the boundary effects have been taken into account, cannot be solved explicitly. Assuming λt (on which κ depends) to be small and keeping only terms which are linear in this parameter we get the quantum solution

$$\begin{aligned}-\frac{1}{2}\log\frac{4\Lambda}{\mu^2}=&\Bigg\{\Big[8(\xi-\frac{1}{6})^2-\frac{1}{135}\Big]\Big[\frac{1}{2}-\frac{3\cos\theta}{4}+\frac{\cos^3\theta}{4}\Big]\\&-\frac{\cos^3\theta}{180}-\frac{3}{2}\cos\theta\sin^2\theta(\xi-\frac{1}{6})\Bigg\}\\\cdot&\Bigg\{\frac{16(\xi-\frac{1}{6})^2\lambda}{(4\pi)^2}\Big[\frac{1}{2}-\frac{3\cos\theta}{4}+\frac{\cos^3\theta}{4}\Big]-\frac{3}{2}\cos\theta\sin^2\theta(\xi-\frac{1}{6})\frac{\lambda}{(4\pi)^2}\Bigg\}^{-1}.\end{aligned}\tag{17}$$

As one sees the boundary terms play an important role. They change the structure of the self-consistent effective equation qualitatively. Our considerations provides an example how through the RG the boundary terms may become relevant in quantum cosmology.

Moreover, this feature is quite general and maybe extended to any renormalizable theory - this only changes the coefficients in (13) and possibly $\gamma(t)$. One may further admit a scalar background field in which case L_{eff} becomes quite complicated and leads to two sets of effective equations of motion.

As another application one can consider the wave function of the Universe [1] which is defined (in our example) as path integral with a spherical cap as boundary surface

$$\psi(\Lambda)=e^{-S_{eff}}.\tag{18}$$

The solution of the field equations is given by (17) and yields the curvature $R=4\Lambda$ of such a spacetime or equivalently its radius $R=\frac{1}{a^2}$. The effective action is the obtained by substituting (17) into (13) and with (18) yields to the wave function of the system and to the probability distribution on the set of boundary conditions.

As an another interesting example let us consider a ball D, i.e. the region in flat spacetime bounded by a three-sphere. We suppose that the scalar background is non-zero and constant. Then we may calculate S_{eff} in (11) as the follows:

$$S_{eff} = V_4 \cdot \Big\{ \frac{\lambda(t)\varphi^4}{4!} - 2c_1\alpha_D(t) \Big\} \tag{19}$$

where V_4 is 'volume' of the ball and c_1 is a dimensionless constant. It is evident that in this case $t = \frac{1}{2}\log\varphi^2/\mu^2$, as in Coleman-Weinberg approach [12]. Now one may discuss the symmetry breaking induced by boundary effects (for the first study of symmetry breaking under external curvature ,see [22]). Solving the equation of motion $\frac{\delta S}{\delta\varphi} = 0$ to first order in λ we get

$$\varphi^4 = \frac{c_1}{120\lambda(4\pi)^2}. \tag{20}$$

Classically $\varphi = 0$, and no symmetry breaking occurs. This simple example shows how boundary effects may trigger the spontaneous symmetry breaking. Now we turn to the discussion of more complicated theories.

4 Running surface constants in GUTs.

Let us show now that one can easily generalize the above picture to the (for simplicity) massless GUT's in curved spacetime. We will consider an arbitrary asymptotically free GUT (for a list of such GUTs, see for example [19]). In this case, we have for running gauge, Yukawa and scalar couplings

$$g^2(t) = \frac{g^2}{1+a^2g^2t} \quad , \quad h^2(t) = k_2 g^2(t) \quad \text{and} \quad f(t) = k_1 g^2(t), \tag{21}$$

where for Yukawa and scalar couplings k_1 and k_2 are constant matrices. The scalar-gravitational running coupling is generally of the form [11]

$$\xi(t) = \frac{1}{6} + (\xi - \frac{1}{6})(1+a^2g^2t)^B, \tag{22}$$

where B maybe positive or negative, depending on the detailed field-content of the theory. The running volume couplings have the structure similar as those in section 1 (powers of terms connected with ξ are changing according to (22)), so we will not present them here (for details, see [11]). As regards to the running surface couplings they maybe easily found using the general results of refs. [3, 6]. To be more specific let us consider the asymptotically free SU(2) gauge theory with one scalar and two spinor triplets [19]. Imposing the boundary conditions of refs. [17, 10] for the fermions and absolute boundary conditions for the scalars and gauge fields and assuming $R_{ab} = \Lambda g_{ab}$ we find now the boundary action

$$
\begin{aligned}
S_S &= \int\limits_{\partial M} d^3x\,\sqrt{\gamma}\,L_S \\
L_S &= \beta_1\Lambda K + \beta_2 K^3 + \beta_3 K K_{\mu\nu}K^{\mu\nu} \\
&\quad + \beta_4 K_\mu^{\ \nu} K_\nu^{\ \alpha} K_\alpha^{\ \mu} + \beta_5 C_{\mu\nu\alpha\beta}K^{\mu\alpha}n^\nu n^\beta,
\end{aligned}
\tag{23}
$$

where the corresponding running couplings are

$$
\begin{aligned}
\beta_1(t) &= \beta_1 - \frac{t}{(4\pi)^2}\left(\frac{62n_A}{135} - \frac{11n_F}{135}\right) - \frac{4(\xi-\frac{1}{6})}{3(4\pi)^2(B+1)a^2}[(1+a^2t)^{B+1}-1] \\
\beta_2(t) &= \beta_2 + \frac{t}{(4\pi)^2}\left(\frac{n_s}{27} + \frac{17n_F}{945} - \frac{338n_A}{945}\right) \\
\beta_3(t) &= \beta_3 + \frac{t}{(4\pi)^2}\left(\frac{n_s}{45} + \frac{13n_F}{315} + \frac{58n_A}{63}\right) \\
\beta_4(t) &= \beta_4 + \frac{t}{(4\pi)^2}\left(\frac{4n_s}{135} - \frac{116n_F}{945} - \frac{436n_A}{945}\right) \\
\beta_5(t) &= \beta_5 + \frac{t}{(4\pi)^2}\left(\frac{2n_s}{45} - \frac{7n_F}{45} - \frac{26n_A}{45}\right),
\end{aligned}
\tag{24}
$$

where for the SU(2) model $n_A = 3, n_s = 3, n_F = 3$ or $n_F = 6$ and [20]

$$
\xi(t) = \frac{1}{6} + \left(\xi - \frac{1}{6}\right)(1 + a^2g^2t)^{-\left(\frac{12-\frac{5}{3}k_1-8k_2}{b^2}\right)}.
$$

Here b^2 is constant and k_1, k_2 can be found in [19]. For $n_F = 3$ we have

$$
\left(12 - \frac{5}{3}k_1 - 8\frac{k_2}{b^2}\right) < 0
$$

and for $n_F = 6$ we have $B > 0$. The running surface couplings in other GUTs can be found similarly as for the scalar theory considered in the previous section. They lead to corrections of the quantum states in quantum cosmology.

5 Conclusion.

We have discussed RG improved effective action in curved spacetime with boundaries. The running surface couplings are getting important in this approach as they maybe relevant in different physical applications. Among examples given in this work we have studied the influence of the boundary terms to the effective field equations, possible application to quantum cosmology and symmetry breaking. Note that we have studied all these questions using the effective action on constant curvature spaces. Nevertheless, one may apply similar technique to the non-local effective action and black hole physics where boundary effects may also play an important role. We hope to return to some of these questions in near future.

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