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Ward Identities for Invariant Group Integrals[1]

S. Uhlmann, R. Meinel and A. Wipf[2]

Theoretisch-Physikalisches Institut,
Friedrich-Schiller-Universität Jena
Fröbelstieg 1, D-07743 Jena, Germany

Abstract

We derive two types of Ward identities for the generating functions for invariant integrals of monomials of the fundamental characters for arbitrary simple compact Lie groups. The results are applied to the groups $SU(3)$, $\mathrm{Spin}(5)$ and G_2 of rank 2 as well as $SU(4)$.

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[2]e–mail: s.uhlmann@uni-jena.de, meinel and wipf@tpi.uni-jena.de

1 Introduction

Invariant group integrals are encountered in many problems in physics. Examples are the one-link integrals in mean field or strong coupling expansions in Euclidian lattice gauge theories [1] or mass-gap calculations in the Hamiltonian formulation of these theories [2]. They are used in the exact solution of two-dimensional lattice gauge theories [3], the matching of gauge theories to chiral models [4] and the loop formulation of quantum gravity on spin network states [5]; they appear in random matrix theory [6] and its widespread applications in nuclear physics [7], quantum chaos and transport in mesoscopic devices [8] as well as in quantum information theory [9]. Apart from that, they are tightly connected to various enumerative problems in mathematics such as counting the number of invariants in a given tensor product of group representations or the number of Young tableaux of bounded height [10].

We consider invariant integrals over compact Lie groups with left-right invariant Haar measure $d\mu_{\mathrm{Haar}}$. For example, integrals of the form

$$Z_G(j, j^\dagger) = \int d\mu_{\mathrm{Haar}}(g) \exp\left(\mathrm{tr}\,(j^\dagger g) + \mathrm{tr}\,(j g^\dagger)\right) \tag{1}$$

with a *matrix valued* source j can be calculated for the $U(N)$ groups [11]. The solution of the Schwinger-Dyson equations corresponding to the left and right group actions on the source yields the closed form expression in terms of Bessel functions,

$$Z_{U(N)}(j, j^\dagger) = \left(2^{N(N-1)/2} \prod_{m=0}^{N-1} m!\right) \frac{\det\left(z_b^{a-1} I_{a-1}(z_b)\right)}{\det\left((z_b^2)^{a-1}\right)}, \tag{2}$$

where $(z_b/2)^2$ are the left- and right-invariant eigenvalues of $j \cdot j^\dagger$. For the $SU(N)$ groups there are further invariants independent of $j^\dagger$, for example $\det j$, and no comparable simple solution to the Schwinger-Dyson equations are known.

An alternative approach based on an explicit parametrization of the group elements has been proposed in [12]. Here one ends up with series representations of the form

$$Z_G(j, j^\dagger) = \sum_{0 \le n_1, \ldots, n_s \le \infty} a_{n_1, \ldots, n_s}\, x_1^{n_1} \cdots x_s^{n_s}\,, \tag{3}$$

where $x_1, \ldots, x_s$ are the algebraically independent invariants, $x_k\,(gjg') = x_k(j)$. This method is only applicable to small groups with sufficiently simple parametriziations. For example, for $SU(2)$ one obtains the series

$$Z_{SU(2)} = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\mathrm{tr}\,(jj^\dagger) + \det j + \det j^\dagger\right)^n. \tag{4}$$

Already for $SO(3)$, the result is a triple series expansion in powers of the three independent invariants. For $SU(3)$, a convenient parametrization of the group elements leads to the series [12]

$$Z_{SU(3)}(j,j^\dagger) = \sum_{n_1,\dots,n_4} \frac{2}{(n_1+2n_2+3n_3+n_4+2)!(n_2+2n_3+n_4+1)!} \prod_{p=1}^{4} \frac{x^{n_p}}{n_p!} \tag{5}$$

with four left-right invariants

$$x_1 = \mathrm{tr}\,(jj^\dagger), \quad x_2 = \tfrac{1}{2}\left(x_1^2 - \mathrm{tr}\,(jj^\dagger)^2\right), \quad x_3 = \det(jj^\dagger), \quad x_4 = \det j + \det j^\dagger. \tag{6}$$

Since x_4 is not invariant under $U(3)$ transformations one finds a triple series for this group. The corresponding results for $U(2)$ and $U(3)$ actually coincide with the expression in terms of Bessel functions, eq. (2) [12].

Numerous results for so-called n-vector integrals containing only n columns of unitary matrices were derived in [13] and more recently in [14] using an elegant method based solely on the unitary constraint and left-right invariance of the Haar measure. These methods were applied to orthogonal groups in [15].

In (1), Z_G is a generating function for invariant integrals of arbitrary functions on the group. In many applications one only needs integrals of *class functions*. Such functions are constant on conjugacy classes,

$$F(ghg^{-1}) = F(h), \tag{7}$$

such that we can consider them as functions of the maximal Abelian torus in G. By the Peter-Weyl theorem, the group characters form an orthonormal basis on the Hilbert space of square integrable class functions. Characters can be computed with the help of Weyl's character formula; they are polynomials of the 'fundamental characters' χ_p belonging to the fundamental representations with highest weights $\mu_{(p)}$, $p = 1, \dots, r$.

One of the authors was involved in the mean field analysis of effective models for pure gauge theories near their phase transition point [16]. There one is confronted with calculating invariant integrals of the type

$$Z_G(\boldsymbol{u}) = \int d\mu_{\mathrm{red}}(g)\, \exp\Big(\sum_{p=1}^{r} u_p \chi_p(g)\Big), \tag{8}$$

where $d\mu_{\mathrm{red}}$ is the reduced Haar measure on G. Such invariant integrals together with their Ward identities also show up in inverse Monte-Carlo simulations where one calculates the couplings in the Polyakov loop dynamics of gauge theories [17]. Other applications are, e.g., the computation of glueball masses in Hamiltonian $SU(N)$ lattice gauge theories [2, 18] or studies of the strong coupling limit of these theories [19]. The function $Z_G(\boldsymbol{u})$ is the generating function for the *moments* of all fundamental characters,

$$\frac{\partial^{m_1+\ldots+m_r}}{\partial u_1^{m_1}\cdots\partial u_r^{m_r}} Z_G(\boldsymbol{u})\big|_{\boldsymbol{u}=0} = \int d\mu_{\mathrm{red}}(g)\,\chi_1^{m_1}(g)\cdots\chi_r^{m_r}(g) \equiv t_{m_1,\ldots,m_r} \tag{9}$$

for r-tuples $\boldsymbol{u} = (u_1, \ldots, u_r)$.[3]

In the literature there seem to be no suitable Ward identities for $Z_G(\boldsymbol{u})$ for arbitrary groups G. Thus we decided to publish our findings since they could be useful for colleagues confronted with similar invariant integrals.

The paper is organized as follows: The next section gives an overview of the known results for the generating functions $Z_G(\boldsymbol{u})$ for $G = U(N)$ and $G = SU(N)$ and extend those for $SU(N)$ in the special case of a generating function for the defining representation. In section 3 we derive what we call geometric Ward identities, since they are based on the invariance of the Haar measure. These results hold for all compact and simple Lie groups. In section 4 our method is applied to the simple rank 2 groups $SU(3)$, Spin(5) and G_2 and in section 5 to $SU(4)$. In section 6 an alternative and more analytic method is presented which sheds further light on the properties of the reduced Haar measures. It turns out that the square of the Jacobian of the transformation $\boldsymbol{\varphi} \mapsto \boldsymbol{\chi}$ from the angles parametrizing the maximal Abelian torus to the fundamental characters is proportional to the density of the reduced Haar measure. We have no general proof of this conjecture but have checked it for groups with rank 2 and 3. Based on the conjecture we find alternative Ward identities which are applied to the groups with rank 2. In section 7 we use both types of Ward identities to derive recursion relations for the moments $t_{m_1,\ldots,m_r}$ in (9). The appendices contain a detailed description of the solution of the Ward identities derived in section 4.1 as well as tables of the lowest moments for the above rank 2 groups. In the conclusions we comment on possible applications of our Ward identities.

[3] Note that for $j = u_1\mathbb{1}$, eq. (1) yields the generating function for the defining representation of $U(N)$ and its complex conjugate; however, for $N \geq 3$ there is no direct way to derive the generating function $Z_{U(N)}(\boldsymbol{u})$ for all fundamental characters from the generating function $Z_{U(N)}(j, j^\dagger)$.

2 Results for $U(N)$ and $SU(N)$

Let z be an arbitrary element of the *group center*. By Schur's lemma it acts on $\chi_p(g)$ by multiplication with a factor z_p such that for the r-vector $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_r)^t$,

$$\boldsymbol{\chi}'(g) = D(z)\boldsymbol{\chi}(g) \quad \text{with} \quad D(z) = \text{diag}(z_1, \ldots, z_r). \tag{10}$$

By invariance of the Haar measure under $g \mapsto zg$, this implies the symmetry

$$Z_G(\boldsymbol{u}) = Z_G\left(D^{-1}(z)\boldsymbol{u}\right) \quad \forall z \in \text{center}. \tag{11}$$

This observation proves to be crucial for explicit computations of Z_G in the following sections. We now briefly summarize the known results for $G = U(N)$ and extend those for $SU(N)$ in the special case of a generating function for the defining representation.
For $U(N)$ the fact that the reduced Haar measure factorizes into a flat measure times the absolute square of a Vandermonde determinant $\det \Delta(\varphi)$,

$$d\mu_{\text{red}}(g) = \det \Delta(\varphi) \det \Delta^*(\varphi)\, d^r\varphi, \tag{12}$$

facilitates the integration such that the generating function

$$Z_{U(N)}(u, v) = \int d\mu_{\text{red}} \, \exp(u \,\text{tr}\, g + v \,\text{tr}\, g^*) \tag{13}$$

for integrals over the characters of the defining representation and its complex conjugate can be computed explicitly. An integration over the maximal torus parameterized by the angular variables $\varphi_1, \ldots, \varphi_N$ yields the closed expression

$$Z_{U(N)}(u, v) = \det \begin{pmatrix} I_0 & I_1 & \cdots & I_{N-1} \\ I_1 & I_0 & \cdots & I_{N-2} \\ \vdots & \vdots & & \vdots \\ I_{N-1} & I_{N-2} & \cdots & I_0 \end{pmatrix} \left(2\sqrt{uv}\right). \tag{14}$$

In accordance with the general result (11), this function is invariant under center transformations,

$$Z_{U(N)}\left(e^{i\alpha}u, e^{-i\alpha}v\right) = Z_{U(N)}(u, v). \tag{15}$$

It can easily be checked that for $v = u^*$ this function is the limit of (2) for $z_b \to 4u^*u$; this limit has to be taken with caution since both the numerator and denominator in (2) vanish for two or more coinciding eigenvalues such that l'Hôpital's rule and certain Bessel function identities are needed.

In order to calculate the analogous generating function for $SU(N)$ we follow [20] and insert the constraint $\det g = 1$ on the maximal Abelian torus in the form

$$\delta_{\text{per}}(\varphi_1 + \ldots + \varphi_N) = \frac{1}{2\pi} \sum_n e^{in(\varphi_1 + \ldots + \varphi_N)}$$

into the invariant integral (13) and find

$$Z_{SU(N)}(u,v) = \sum_{n \in \mathbb{Z}} \left(\frac{u}{v}\right)^{Nn/2} \det \begin{pmatrix} I_n & I_{n+1} & \cdots & I_{n+N-1} \\ I_{n-1} & I_n & \cdots & I_{n+N-2} \\ \vdots & \vdots & & \vdots \\ I_{n-N+1} & I_{n-N+2} & \cdots & I_n \end{pmatrix} \left(2\sqrt{uv}\right), \tag{16}$$

see [18]. The generating function for $SU(N)$ is left invariant by $\mathbb{Z}_N$ center transformations for which

$$uv \longrightarrow uv, \qquad \frac{u}{v} \to e^{4\pi i k/N}\, \frac{u}{v}, \quad k = 1, \ldots, N. \tag{17}$$

In the case of $SU(2)$, the sum can be worked out and leads to

$$Z_{SU(2)}(u,v) = {}_0F_1\left[\, 2 \,\middle|\, w^2 \right] = I_0(4w) - I_2(4w), \quad w = \frac{u+v}{2}. \tag{18}$$

The case of $SU(3)$ is the subject of section 4.1. We are not aware of similar explicit results for $SU(N)$ with $N \geq 4$. In these cases, the complexity increases since the complex conjugate fundamental representations are inequivalent.

In (18), the hypergeometric function ${}_0F_1$ does not appear accidentally; for all $SU(N)$ the generating function for the defining representation,

$$Z_{SU(N)}(u) = \int d\mu_{\text{red}}(g)\, \exp\left(u \operatorname{tr} g\right), \tag{19}$$

is one of the generalized hypergeometric functions

$${}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, x\right] = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!}, \quad \frac{\alpha_{n+1}}{\alpha_n} = \frac{(n+a_1)\cdots(n+a_p)}{(n+b_1)\cdots(n+b_q)}, \tag{20}$$

with $\alpha_0 = 1$. In the course of the paper, we will mostly identify them as solutions to the generalized hypergeometric differential equation,

$$\left\{\theta \prod_{i=1}^{q}(\theta + b_i - 1) - x \prod_{i=1}^{p}(\theta + a_i)\right\} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, x\right] = 0 \quad \text{with} \quad \theta = x\frac{d}{dx}. \tag{21}$$

For $q = 0$ (or $p = 0$), the first (or second) products of differential operators are to be replaced by the identity operator.
Center symmetry entails that the function $Z_{SU(N)}(u)$ in (19) is in fact only a function of $x = \det(u\mathbb{1}) = u^N$, and $Z_{SU(N)}(u) = Z(x)$ satisfies the differential equation

$$\frac{d^N}{dx^N}\left(x^{N-1}Z(x)\right) = Z(x). \tag{22}$$

The solution is the hypergeometric function ${}_0F_{N-1}\big[\,2,3,\dots,N\big|x\big]$ such that

$$Z_{SU(N)}(u) = {}_0F_{N-1}\big[\,2,3,\dots,N\,\big|u^N\big]. \tag{23}$$

Since for $SU(2)$, $\operatorname{tr} g = \operatorname{tr} g^\dagger$, this generalizes the standard result (18) (with $u = v$) by Arisue [21]. Eq. (23) follows from (16) when v tends to zero so that

$$Z_{SU(N)}(u) = \sum_{n\geq 0} u^{Nn}\det\Delta^{(n)}, \quad \left(\Delta^{(n)}\right)_{pq} = \begin{cases} \frac{1}{(n+q-p)!} & \text{for } n+q-p\geq 0, \\ 0 & \text{else}\end{cases}. \tag{24}$$

If we multiply the p'th row of $\Delta^{(n)}$ with $(n+N-p)!$, it is easy to calculate the determinants of these Toeplitz matrices,

$$\det\Delta^{(n)} = \prod_{p=0}^{N-1}\frac{p!}{(n+p)!} \quad\Longrightarrow\quad \frac{\det\Delta^{(n+1)}}{\det\Delta^{(n)}} = \frac{1}{(n+1)\cdots(n+N)}. \tag{25}$$

This proves eq. (23).

3 Ward identities for generating functions

We denote the left derivative in the direction of the Lie algebra element T_a by L_a, i.e. $L_a f(g) = \frac{d}{dt}|_{t=0} f\big(\exp(itT_a)g\big)$ for some function f on G. The Haar-measure is left (and right) invariant, thus

$$\int d\mu_{\text{Haar}}(g)\,(L_a f)(g) = 0, \quad f\in L_2(G). \tag{26}$$

For class functions F and $\tilde F$ the function

$$\sum_a L_a\left(F\cdot L_a\tilde F\right) \equiv \boldsymbol{L}\left(F\boldsymbol{L}\tilde F\right) = F\boldsymbol{L}^2\tilde F + \boldsymbol{L}F\cdot\boldsymbol{L}\tilde F \tag{27}$$

is a class function as well. In order to see this, we may assume that F and $\tilde F$ are basis elements (i.e., characters χ_μ and χ_ν of some representations with highest weights μ, ν). In

$\sum_a(\chi_\mu \cdot L_a^2\chi_\nu + L_a\chi_\mu \cdot L_a\chi_\nu)$, the first part of the sum is a class function since the quadratic Casimir operator $\sum_a T_a^2$ commutes with all group elements, and the second part is a class function since invariance of the Killing metric $\operatorname{tr} T_a T_b$ under adjoint action by some group element h implies that hT_ah^{-1} can be expanded as $R_a{}^cT_c$ with an orthogonal matrix R. Thus, (26) with $f = F \cdot L_a\tilde{F}$ reduces to an integral over the maximal Abelian torus,

$$0 = \int d\mu_{\text{red}}\, \boldsymbol{L}\left(F\boldsymbol{L}\tilde{F}\right) = \int d\mu_{\text{red}}\left(F\boldsymbol{L}^2\tilde{F} + \boldsymbol{L}F \cdot \boldsymbol{L}\tilde{F}\right). \tag{28}$$

We take $\tilde{F}$ to be a fundamental character χ_p with $p \in \{1, \dots, r\}$. The $\chi_1, \dots, \chi_r$ are good coordinates for the maximal torus such that any class function can be thought of as a function of these characters, $F = F(\chi_1, \dots, \chi_r)$. Then the identity (28) reads

$$0 = \int d\mu_{\text{red}}\left(F(\boldsymbol{\chi})\boldsymbol{L}^2\chi_p + \sum_q(\boldsymbol{L}\chi_p)\frac{\partial F(\boldsymbol{\chi})}{\partial\chi_q}(\boldsymbol{L}\chi_q)\right). \tag{29}$$

Every character χ_μ of a representation V_μ with highest weight μ is an eigenfunction of the Laplacian $\boldsymbol{L}^2$ with eigenvalue $-c_\mu$, where c_μ is the value of the quadratic Casimir in V_μ,

$$\boldsymbol{L}^2\chi_\mu = -c_\mu\chi_\mu. \tag{30}$$

To calculate the last term in (29) we decompose the tensor product of $V_\mu \otimes V_\nu$ into irreducible pieces,[4]

$$V_\mu \otimes V_\nu = \bigoplus_\lambda C_{\mu\nu}^\lambda V_\lambda, \quad \text{such that} \quad \chi_\mu\chi_\nu = \sum C_{\mu\nu}^\lambda \chi_\lambda. \tag{31}$$

Acting with $\boldsymbol{L}^2$ on this relation and using (30) we find the useful relation

$$(\boldsymbol{L}\chi_\mu) \cdot (\boldsymbol{L}\chi_\nu) = \frac{1}{2}(c_\mu + c_\nu)\chi_\mu\chi_\nu - \frac{1}{2}\sum_\lambda C_{\mu\nu}^\lambda\, c_\lambda\chi_\lambda. \tag{32}$$

with Clebsch-Gordan coefficients $C_{\mu\nu}^\lambda$ and second order Casimirs c_μ. Now we may rewrite the Ward identity (29) as follows,

$$\begin{aligned} c_p \int d\mu_{\text{red}}\, \chi_p F(\boldsymbol{\chi}) &= \frac{1}{2}\sum_{q=1}^r (c_p + c_q)\int d\mu_{\text{red}}\, \chi_p\chi_q \frac{\partial F(\boldsymbol{\chi})}{\partial\chi_q} \\ &\quad - \frac{1}{2}\sum_{q,\lambda} C_{pq}^\lambda\, c_\lambda \int d\mu_{\text{red}}\, \chi_\lambda \frac{\partial F(\boldsymbol{\chi})}{\partial\chi_q}, \quad 1 \le p \le r. \end{aligned} \tag{33}$$

[4] The characters χ_λ are polynomials $\chi_\lambda = \chi_\lambda(\chi_1, \dots, \chi_r)$ of the fundamental characters. For groups of higher rank it can be cumbersome to calculate these polynomials.

We choose the class function $F = \exp(\boldsymbol{u} \cdot \boldsymbol{\chi})$ such that

$$\int d\mu_{\rm red}\, F = Z_G(\boldsymbol{u}), \quad \frac{\partial F}{\partial \chi_q} = u_q F \quad \text{and} \quad \frac{\partial F}{\partial u_q} = \chi_q F. \tag{34}$$

Then, the identity (33) translates into the following *master equation* for Z_G,

$$\begin{aligned} c_p \frac{\partial}{\partial u_p} Z_G(\boldsymbol{u}) &= \frac{1}{2} \sum_q (c_p + c_q) u_q \frac{\partial^2}{\partial u_q \partial u_p} Z_G(\boldsymbol{u}) \\ &\quad - \frac{1}{2} \sum_{q,\lambda} C^{\lambda}_{pq} c_\lambda\, u_q\, \chi_\lambda(\boldsymbol{\partial}) Z_G(\boldsymbol{u}), \quad p = 1, \ldots, r, \end{aligned} \tag{35}$$

where $\chi_\lambda(\boldsymbol{\partial})$ is the differential operator obtained by formally evaluating the polynomial $\chi_\lambda(\boldsymbol{\chi})$ at $(\partial/\partial u_1, \ldots, \partial/\partial u_r)$. The properties of the group G enter these Ward identities at three places: via the polynomials $\chi_\lambda(\boldsymbol{\chi})$, the values c_μ of the quadratic Casimir operators and the Clebsch-Gordan coefficients C^λ_{pq}. The coefficient functions of these linear partial differential equations are constant or linear functions of the variables $u_1, \ldots, u_r$. Their complexity depends crucially on the polynomials χ_λ. For $SU(3)$ and Spin(5) all χ_λ are quadratic, and one ends up with second order differential equations.

For explicit calculations, we have to fix our Lie algebra conventions which were chosen to allow for an easy comparison with the computer algebra program LiE [22]. In these conventions, the Cartan matrix is given by

$$K_{pq} = \frac{2(\alpha_{(p)}, \alpha_{(q)})}{(\alpha_{(q)}, \alpha_{(q)})} \tag{36}$$

in terms of the simple roots $\alpha_{(1)}, \ldots, \alpha_{(r)}$. Thus, simple roots and fundamental weights are connected by the relation

$$\alpha_{(p)} = \sum_q K_{pq} \mu_{(q)}. \tag{37}$$

The *shortest* simple root has squared length 2. In particular, for simply laced groups, the Cartan matrix reduces to $K_{pq} = (\alpha_{(p)}, \alpha_{(q)})$. Arbitrary roots and weights are linear combinations of the simple roots and fundamental weights, respectively,

$$\alpha = \sum m_p \alpha_{(p)} \equiv [m_1, \ldots, m_r] \quad \text{and} \quad \mu = \sum n_p \mu_{(p)} \equiv [n_1, \ldots, n_r]. \tag{38}$$

The Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{p=1}^{r} \mu_{(p)} \tag{39}$$

plays an important role in the theory of representations; we will mostly need it for its appearance in the formula for the value of the second order Casimir operator in a representation V_μ,

$$c_\mu = (\mu, \mu + 2\rho). \tag{40}$$

With the help of (37) and $(\alpha_{(p)}, \mu_{(q)}) = \delta_{pq} v_p$ the Casimir of the representation with highest weight $\mu = [n_1, \ldots, n_r]$ can be written as

$$c_\mu = \big(\boldsymbol{n}, K^{-1}\boldsymbol{n}'\big) \quad \text{with} \quad n'_p = v_p(n_p + 2). \tag{41}$$

For simply laced groups all v_p are equal to 1 so that $n'_p = n_p + 2$.

4 Ward indentities for groups of rank 2

In this section, we are going to investigate and exploit the geometric Ward identities (35) for the groups $SU(3)$, Spin(5) and G_2. For these rank 2 groups, the generating function (8) depends on two variables $u_1 \equiv u$ and $u_2 \equiv v$.

4.1 The group $SU(3)$

As a simply-laced group $SU(3)$ has a symmetric Cartan matrix,

$$K_{SU(3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{42}$$

and the quadratic Casimir of the representation with highest weight $\mu = [n_1, n_2]$ is

$$c_\mu = \frac{2}{3}\left(n_1^2 + n_2^2 + n_1 n_2 + 3n_1 + 3n_2\right). \tag{43}$$

The fundamental 3-dimensional representation $3 \equiv [1, 0]$ and its complex conjugate $\bar{3} \equiv [0, 1]$ both have Casimir 8/3. Since

$$3 \otimes 3 = 6 \oplus \bar{3}, \quad \bar{3} \otimes \bar{3} = \bar{6} \oplus 3, \quad 3 \otimes \bar{3} = 1 \oplus 8, \tag{44}$$

the λ-sum in (35) contains both fundamental, two sextet and the octet representations. The singlet representation has vanishing Casimir invariant and does not contribute. The Casimir operator on the sextets $6 = [2, 0]$ and $\bar{6} = [0, 2]$ takes the value 20/3, and the octet $8 = [1, 1]$ has Casimir 6. To derive explicit Ward identities we must express the characters of the representations 6, $\bar{6}$ and 8 in terms of the fundamental characters. By (31), eq. (44) yields

$$\chi_6 = \chi_3^2 - \chi_{\bar{3}}, \quad \chi_{\bar{6}} = \chi_{\bar{3}}^2 - \chi_3, \quad \chi_8 = \chi_3\chi_{\bar{3}} - 1. \tag{45}$$

Thus the differential operators $\chi_\lambda(\boldsymbol{\chi})$ in the Ward identity (35) read

$$\chi_3 = \partial_u, \quad \chi_{\bar{3}} = \partial_v, \quad \chi_6 = \partial_u^2 - \partial_v, \quad \chi_{\bar{6}} = \partial_v^2 - \partial_u, \quad \chi_8 = \partial_u\partial_v - 1, \tag{46}$$

and the two Ward identities in (35) for the generating function

$$Z_{SU(3)}(u,v) = \int d\mu_{\text{red}}(g) \exp\big(u\chi_3(g) + v\chi_{\bar{3}}(g)\big) \quad \text{with} \quad \chi_3(g) = \operatorname{tr} g \tag{47}$$

have the simple form[5]

$$\begin{aligned}\big(2u\partial_u^2 + v\partial_u\partial_v + 8\partial_u - 6u\partial_v - 9v\big)Z_{SU(3)} &= 0, & (48)\\ \big(2v\partial_v^2 + u\partial_v\partial_u + 8\partial_v - 6v\partial_u - 9u\big)Z_{SU(3)} &= 0. & (49)\end{aligned}$$

Note that by the property $d\mu(g) = d\mu(g^{-1})$ of the Haar measure, $Z_{SU(3)}(u,v) = Z_{SU(3)}(v,u)$ is a symmetric function. Thus it is sufficient to study one of the two identities. In the defining representation of $SU(3)$, the center $\mathbb{Z}_3$ acts by multiplication with $\exp(2\pi i/3)\mathbb{1}$, and the symmetry (11) reads

$$Z_{SU(3)}\left(e^{2\pi i/3}u, e^{-2\pi i/3}v\right) = Z_{SU(3)}(u,v). \tag{50}$$

Together with the symmetry in its two arguments this suggests that $Z_{SU(3)}$ is just a function of the combinations $u^3 + v^3$ and uv. In fact, we prove in the appendix that the solution to (48) is given by

$$Z_{SU(3)}(u,v) = \sum_{p,q=0}^{\infty} \frac{2}{(p+q+1)!\,(p+q+2)!\,q!} \binom{3(p+q+1)}{p} (uv)^p \left(u^3 + v^3\right)^q. \tag{51}$$

We have found only one comparably simple series representation for the generating function in the literature [18]. To arrive at their result the authors took the quartic series (5) and calculated 2 of the 4 infinite sums. It seems evident that our method based on geometric Ward identities is more efficient to find simple series representation for generating functions.

[5] The action of left-derivatives on characters can also be worked out concretely in a matrix representation although this requires more computational effort, cf. [23].

As two special cases, we will restrict $Z_{SU(3)}(u,v)$ to the u-axis (i.e., $v=0$) and the diagonal (i.e., $u=v$). Instead of performing a resummation of the general result (51), we derive differential equations for $Z_{SU(3)}$ in these cases. First we solve the Ward identities (48,49) on the u-axis, where

$$\big(2u\partial_u^2 + 8\partial_u Z - 6u\partial_v\big)Z_{SU(3)}|_{v=0} = 0, \quad \big(u\partial_v\partial_u + 8\partial_v - 9u\big)Z_{SU(3)}|_{v=0} = 0. \tag{52}$$

To get rid of the v-derivatives at $v=0$ we act with $u\partial_u$ on the first equation and use (52) to eliminate the terms $\partial_v Z_{SU(3)}$ and $\partial_u\partial_v Z_{SU(3)}$. We find the ordinary differential equation

$$\left(u^2\partial_u^3 + 12u\partial_u^2 + 28\partial_u - 27u^2\right) Z_{SU(3)}(u,0) = 0. \tag{53}$$

Since $Z_{SU(3)}(u,0)$ only depends on u^3 we may equally well use $x=u^3$ as a new variable, $Z_{SU(3)}(u,0) = Y_{SU(3)}(x)$. Then the differential equation takes the simpler form

$$\left(x^2\partial_x^3 + 6x\partial_x^2 + 6\partial_x - 1\right) Y_{SU(3)} = (x^2 Y_{SU(3)})''' - Y_{SU(3)} = 0, \tag{54}$$

which has a solution in terms of a generalized hypergeometric function (cf. (22)), $Y_{SU(3)}(x) = {}_0F_2\big[\,2,3\,\big|x\big]$. This is just the result (23) for the group $SU(3)$.

In order to find a differential equation on the diagonal $u=v$, we act by the operator $(3u\partial_u + 5u\partial_v + 5 + 4u)$ on (48) and evaluate the result at $u=v=:t$. In terms of

$$\begin{aligned}
&\frac{d}{dt}Z_{SU(3)}(t,t) = (\partial_u+\partial_v)Z_{SU(3)}, \qquad \frac{d^2}{dt^2}Z_{SU(3)}(t,t) = (2\partial_{uu}+2\partial_{vv})Z_{SU(3)},\\
&\frac{d^3}{dt^3}Z_{SU(3)}(t,t) = (2\partial_{uuu}+6\partial_{uuv})Z_{SU(3)},
\end{aligned} \tag{55}$$

the resulting ordinary differential equation reads

$$\Big(t^2\frac{d^3}{dt^3} + t(10-t)\frac{d^2}{dt^2} - 2(12t^2+t-10)\frac{d}{dt} - 12t(3t+5)\Big)Z_{SU(3)} = 0. \tag{56}$$

This is solved by a function

$$Z_{SU(3)}(t,t) = 1 + t^2 + \sum_{n=3}^{\infty}\frac{a_n}{n!}t^n \tag{57}$$

where the coefficients a_n satisfy the recursion relation

$$a_{n+1} = \frac{1}{(n+4)(n+5)}\big(n(n+1)a_n - 12n(2n+3)a_{n-1} - 36n(n-1)a_{n-2}\big) = 0. \tag{58}$$

Together with $a_0=1$, $a_1=0$, and $a_2=2$, this determines all coefficients in the expansion (58).

4.2 The group Spin(5)

Since Spin(5) is not simply-laced, the Cartan matrix

$$K_{\mathrm{Spin}(5)} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{59}$$

is not symmetric. With the convention in (36), we take α_1 to be the longer root, $\alpha_1^2 = 4$ and $\alpha_2^2 = 2$. The eigenvalue of the Casimir operator of the representation with highest weight $\mu = [n_1, n_2]$ reads

$$c_\mu = 2n_1^2 + n_2^2 + 2n_1 n_2 + 6n_1 + 4n_2. \tag{60}$$

The fundamental representations are the $SO(5)$ vector representation $5 = [1, 0]$ and the spin representation $4 = [0, 1]$. The center $\mathbb{Z}_2$ is generated by $-\mathbb{1}$ in the spin representation and acts trivially in the vector representation; the center symmetry (11) implies that

$$Z_{\mathrm{Spin}(5)}(u, v) = \int d\mu_{\mathrm{red}}(g) \exp\big(u\chi_5(g) + v\chi_4(g)\big) = Z_{\mathrm{Spin}(5)}(u, -v) \tag{61}$$

is an even function in v. For the geometric Ward identities we need the tensor products

$$5 \otimes 5 = 1 \oplus 10 \oplus 14, \quad 4 \otimes 4 = 1 \oplus 5 \oplus 10, \quad 5 \otimes 4 = 4 \oplus 16. \tag{62}$$

With $10 = [0, 2]$, $14 = [2, 0]$ and $16 = [1, 1]$, we obtain for the Casimir operators (41):

$$c_5 = 8, \quad c_4 = 5, \quad c_{10} = 12, \quad c_{14} = 20, \quad c_{16} = 15. \tag{63}$$

Together with (62), (31) implies that

$$\chi_{10} = \chi_4^2 - \chi_5 - 1, \quad \chi_{14} = \chi_5^2 - \chi_4^2 + \chi_5, \quad \chi_{16} = \chi_5\chi_4 - \chi_4, \tag{64}$$

which leads to the Ward identities

$$\big(u(4\partial_v^2 - 2\partial_u^2 - 4\partial_u + 6) + v(5\partial_v - \partial_u\partial_v) - 8\partial_u\big)\, Z_{\mathrm{Spin}(5)}(u, v) \;=\; 0, \tag{65}$$

$$\big(u(5\partial_v - \partial_u\partial_v) + v(2\partial_u - \partial_v^2 + 6) - 5\partial_v\big)\, Z_{\mathrm{Spin}(5)}(u, v) \;=\; 0. \tag{66}$$

Since the center of Spin(5) is smaller than the center of $SU(3)$ these differential equations are more complicated than the corresponding $SU(3)$ equations in (48) and (49). The characteristics of the second (generally hyperbolic) equation are given by $u =$ const. and $\frac{u}{v} =$ const., respectively. These families of lines coincide for $u = 0$ (where (66) is parabolic), and we may solve the characteristic problem given a solution of both equations for $u = 0$.

The restriction of the Ward identities to the v-axis

$$\big(8\partial_u - v(5\partial_v - \partial_u\partial_v)\big) Z_{\mathrm{Spin}(5)}\big|_{u=0} = 0, \qquad \big(5\partial_v - v(2\partial_u - \partial_v^2 + 6)\big) Z_{\mathrm{Spin}(5)}\big|_{u=0} = 0, \quad (67)$$

can be solved if we differentiate the second equation with respect to v and use the two relations (67) to eliminate the u-derivatives at $u = 0$. We find the following ordinary differential equation for $Z_{\mathrm{Spin}(5)}(0, v)$,

$$\big(v^2\partial_v^3 + 13v\partial_v^2 - 16v^2\partial_v + 35\partial_v - 48v\big)\, Z_{\mathrm{Spin}(5)}(0, v) = 0. \quad (68)$$

Since $Z_{\mathrm{Spin}(5)}$ is an even function in v we set $Z_{\mathrm{Spin}(5)}(0, v) = Z_0\big(x = v^2\big)$ and obtain the simple equation

$$\big(x^2\partial_x^3 + 8x\partial_x^2 - 4x\partial_x + 12\partial_x - 6\big)\, Z_0(x) = 0. \quad (69)$$

The solution is a hypergeometric function ${}_1F_2\big[{}^{3/2}_{3,\,4}\big|4x\big]$ so that

$$Z_{\mathrm{Spin}(5)}(0, v) = \int d\mu_{\mathrm{red}}(g) \exp\big(v\chi_4(g)\big) = {}_1F_2\big[{}^{3/2}_{3,\,4}\big|4v^2\big]. \quad (70)$$

Plugging this into the second Ward identity (66) we obtain a recursive solution

$$Z_{\mathrm{Spin}(5)}(u, v) = \sum_{n=0}^{\infty} \frac{Z_n(x = v^2)}{n!}\, u^n, \quad (71)$$

where

$$\begin{aligned} Z_1(x) &= \big(6\partial_x + 2x\partial_x^2 - 3\big) Z_0(x), \\ Z_n(x) &= \big[(5+n)\partial_x + 2x\partial_x^2 - 3\big] Z_{n-1}(x) - 5(n-1)\partial_x Z_{n-2}(x) \quad \text{for} \quad n \geq 2. \end{aligned} \quad (72)$$

As a special case, let us consider the Spin(5)-Ward identities on the u-axis,

$$\big(u(2\partial_v^2 - \partial_u^2 - 2\partial_u + 3) - 4\partial_u\big) Z_{\mathrm{Spin}(5)}\big|_{v=0} = 0, \quad \big(u(5\partial_v - \partial_u\partial_v) - 5\partial_v\big) Z_{\mathrm{Spin}(5)}\big|_{v=0} = 0. \; (73)$$

We differentiate the first equation with respect to u and obtain

$$\big(2\partial_v^2 - 5\partial_u^2 - 2\partial_u + 3 + u(2\partial_u\partial_v^2 - \partial_u^3 - 2\partial_u^2 + 3\partial_u)\big)\, Z_{\mathrm{Spin}(5)}\big|_{v=0} = 0. \quad (74)$$

Eq. (73) together with the v-derivative of (66) at $v = 0$ can be used to eliminate the v-derivatives,

$$\big(u^2\partial_u^3 + (10u - 3u^2)\partial_u^2 + (20 - 12u - 13u^2)\partial_u - 30u + 15u^2\big)\, Z_{\mathrm{Spin}(5)}(u, 0) = 0. \quad (75)$$

Very probably this cannot be converted into a differential equation for a generalized hypergeometric series. The differential equation is solved by the power series

$$Z_{\mathrm{Spin}(5)}(u,0) = 1 + u^2 + \sum_{n=3}^{\infty} \frac{b_n}{n!}\, u^n \tag{76}$$

provided that the coefficients satisfy the recursion relation

$$b_{n+1} = \frac{1}{(n+4)(n+5)}\Big(3n(n+3)b_n + n(13n+17)b_{n-1} - 15n(n-1)b_{n-2}\Big), \quad n \geq 2. \tag{77}$$

Together with $b_0 = 1$, $b_1 = 0$ and $b_2 = 2$, this determines all coefficients in the expansion (76).

4.3 The group G_2

For this group, the Cartan matrix is

$$K_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{78}$$

Hence, α_1 is a short root and α_2 a long root, $\alpha_1^2 = 2$ and $\alpha_2^2 = 6$. The quadratic Casimir of the representation with highest weight $\mu = [n_1, n_2]$ reads

$$c_\mu = 2n_1^2 + 6n_2^2 + 6n_1n_2 + 10n_1 + 18n_2. \tag{79}$$

The first fundamental representation $7 = [1,0]$ coincides with the subrepresentation of the 8-dimensional spinor representation of Spin(7) leaving an arbitrary spinor fixed, and the second is just the adjoint representation $14 = [0,1]$. The center of G_2 is trivial and we expect no symmetries of the generating function.

For the Ward identity we need the tensor products

$$\begin{aligned} 7 \otimes 7 &= 1 \oplus 7 \oplus 14 \oplus 27, \\ 7 \otimes 14 &= 7 \oplus 27 \oplus 64, \\ 14 \otimes 14 &= 1 \oplus 14 \oplus 27 \oplus 77 \oplus 77', \\ 7 \otimes 27 &= 7 \oplus 14 \oplus 27 \oplus 64 \oplus 77'. \end{aligned} \tag{80}$$

We need the last product in order to express the characters χ_λ as functions of the fundamental characters. Note that two irreducible representations of dimension 77 appear in the decompositions. We identify $27 = [2, 0]$, $64 = [1, 1]$, $77 = [0, 2]$ and $77' = [3, 0]$ so that the quadratic Casimir operators (41) take the following values:

$$c_7 = 12, \quad c_{14} = 24, \quad c_{27} = 28, \quad c_{64} = 42, \quad c_{77} = 60, \quad c_{77'} = 48. \tag{81}$$

Furthermore, we use

$$\begin{aligned} \chi_{27} &= \chi_7^2 - \chi_7 - \chi_{14} - 1, \\ \chi_{64} &= \chi_7\chi_{14} - \chi_7^2 + \chi_{14} + 1, \\ \chi_{77} &= -\chi_7^3 + \chi_{14}^2 + 2\chi_7\chi_{14} + 2\chi_7 + \chi_{14}, \\ \chi_{77'} &= \chi_7^3 - \chi_7^2 - 2\chi_7\chi_{14} - \chi_7 - \chi_{14} \end{aligned} \tag{82}$$

to derive the following G_2-Ward identities

$$\begin{aligned} 0 &= \Big(u(-2\partial_u^2 + 8\partial_u + 2\partial_v + 14) \\ &\quad + v(7\partial_u^2 - 3\partial_u\partial_v + 8\partial_u - 7\partial_v - 7) - 12\partial_u\Big) Z_{G_2}(u, v), \end{aligned} \tag{83}$$

$$\begin{aligned} 0 &= \Big(u(7\partial_u^2 - 3\partial_u\partial_v + 8\partial_u - 7\partial_v - 7) \\ &\quad + v(6\partial_u^3 + 10\partial_u^2 - 6\partial_v^2 - 12\partial_u\partial_v - 22\partial_u - 4\partial_v + 14) - 24\partial_v\Big) Z_{G_2}(u, v) \end{aligned} \tag{84}$$

for the generating function

$$Z_{G_2}(u, v) = \int d\mu_{\text{red}}(g) \exp\big(u\chi_7(g) + v\chi_{14}(g)\big). \tag{85}$$

On the u-axis these equations simplify to

$$0 = \big(u(-\partial_u^2 + 4\partial_u + \partial_v + 7) - 6\partial_u\big)\, Z_{G_2}(u, v)|_{v=0}, \tag{86}$$

$$0 = \big(u(7\partial_u^2 - 3\partial_u\partial_v + 8\partial_u - 7\partial_v - 7) - 24\partial_v\big)\, Z_{G_2}(u, v)|_{v=0}. \tag{87}$$

We solve the first equation and its u-derivative for the v-derivatives at $v = 0$ occurring in the second equation and end up with the third order differential equation

$$\big(u^2\partial_u^3 + (14u - 4u^2)\partial_u^2 + (42 - 18u - 19u^2)\partial_u - 14u^2 - 56u\big)\, Z_{G_2}(u, 0) = 0. \tag{88}$$

Again this is probably not related to a generalized hypergeometric series. It may be solved in terms of a series expansion

$$Z_{G_2}(u, 0) = 1 + u^2 + \sum_{n\geq 3} \frac{g_n}{n!} u^n \tag{89}$$

provided that the coefficients satisfy the recursion relation (for $n \geq 2$)

$$g_{n+1} = \frac{1}{(n+6)(n+7)}\Big(2n(2n+7)g_n + n(19n+37)g_{n-1} + 14n(n-1)g_{n-2}\Big), \tag{90}$$

together with $g_0 = 1$, $g_1 = 0$, $g_2 = 2$. These coefficients are related to the triangulations of n-gones with inner vertices with valences ≥ 6 [24].
Starting from $Z_{G_2}(u,0) = \tilde{Z}_0(u)$, we can now solve the corresponding characteristic problem for the first Ward identity (83) (analogously to the Ward identity for Spin(5)) by means of an expansion

$$Z_{G_2}(u,v) = \sum_{n=0}^{\infty} \tilde{Z}_n(u)v^n \tag{91}$$

with

$$\begin{aligned} u\tilde{Z}_1(u) &= \big(u\partial_u^2 - 4u\partial_u + 6\partial_u - 7u\big)\tilde{Z}_0(u), \\ 2u\tilde{Z}_n(u) &= (1-n)\big(7\partial_u^2 + 8\partial_u - 7\big)\tilde{Z}_{n-2}(u) \\ &\quad + \big(2u\partial_u^2 - 8u\partial_u + 3(n+3)\partial_u - 14u + 7n - 7\big)\tilde{Z}_{n-1}(u) \end{aligned} \tag{92}$$

for all $n \geq 2$. With the help of (90) and (92), one can reproduce the moments given in the appendix (section B) recursively.

5 Results for $SU(4)$

In this section, we will derive a solution to the Ward identities of the rank 3 group $SU(4)$ in a certain range of the parameter values. For $SU(4)$, the quadratic Casimir of the representation with highest weight $\mu = [n_1, n_2, n_3]$ is given by

$$c_\mu = \frac{1}{4}(3n_1^2 + 4n_2^2 + 3n_3^3 + 4n_1n_2 + 2n_1n_3 + 4n_2n_3) + 3n_1 + 4n_2 + 3n_3, \tag{93}$$

and the fundamental representations 4, 6, $\bar{4}$ with highest weights $\mu_1 \equiv [1,0,0]$, $\mu_2 \equiv [0,1,0]$, and $\mu_3 \equiv [0,0,1]$ have Casimirs

$$c_4 = c_{\bar{4}} = \frac{15}{4} \quad \text{and} \quad c_6 = 5. \tag{94}$$

The real representation 6 coincides with the vector representation of $SO(6)$, and $\bar{4}$ is complex conjugated to the defining representation 4; the latter two can be identified with the complex fundamental spinor representations of Spin(6). Their tensor products can be decomposed according to

$$
\begin{aligned}
&4\otimes 4 = 6\oplus 10, \quad 6\otimes 6 = 1\oplus 20\oplus 15, \quad \bar{4}\otimes\bar{4} = 6\oplus\overline{10},\\
&4\otimes\bar{4} = 1\oplus 15, \quad 4\otimes 6 = \bar{4}\oplus 20', \quad \bar{4}\otimes 6 = 4\oplus\overline{20}',
\end{aligned} \tag{95}
$$

where again we denoted the representations by their dimensions,

$$
10 = [2,0,0],\ \overline{10} = [0,0,2],\ 15 = [1,0,1],\ 20 = [0,2,0],\ 20' = [1,1,0],\ \overline{20}' = [0,1,1].
$$

The representations 15 and 20 are real and $\bar{4}$, $\overline{20}'$ are complex conjugate to 4, $20'$. From (95), we find

$$
\chi_{10} = \chi_4^2 - \chi_6, \quad \chi_{15} = \chi_4\bar{\chi}_4 - 1, \quad \chi_{20} = \chi_6^2 - \chi_4\bar{\chi}_4, \quad \chi_{20'} = \chi_4\chi_6 - \bar{\chi}_4.
$$

These data enter the Ward identities (35) for the generating function

$$
Z_{SU(4)}(u,v,w) = \int d\mu_{\text{red}}\ e^{u\chi_4 + v\bar{\chi}_4 + w\chi_6}, \tag{96}
$$

which is center-symmetric, $Z_{SU(4)}(iu,-iv,-w) = Z_{SU(4)}(u,v,w)$. They take the form

$$
\begin{aligned}
0 &= \left\{15\partial_u + u(3\partial_u^2 - 8\partial_w) + 2w(\partial_u\partial_w - 6\partial_v) + v(\partial_u\partial_v - 16)\right\} Z_{SU(4)}(u,v,w), && (97)\\
0 &= \left\{10\partial_w + u(\partial_u\partial_w - 6\partial_v) + 2w(\partial_w^2 - 2\partial_u\partial_v - 4) + v(\partial_v\partial_w - 6\partial_u)\right\} Z_{SU(4)}(u,v,w), && (98)\\
0 &= \left\{15\partial_v + v(3\partial_v^2 - 8\partial_w) + 2w(\partial_v\partial_w - 6\partial_u) + u(\partial_u\partial_v - 16)\right\} Z_{SU(4)}(u,v,w). && (99)
\end{aligned}
$$

In order to find an explicit solution for arbitrary products of 4-dimensional representations (i.e., on the diagonal with $u=v$ and $w=0$), we proceed analogously to section 4.1: We act with the operator $(u\partial_u + v\partial_v + 9)$ on the first and last equations and use (98) to eliminate the w-derivative in the resulting differential equations. At $w=0$, we obtain

$$
\begin{aligned}
&\left(3u^2\partial_u^3 + 4uv\partial_u^2\partial_v + v^2\partial_u\partial_v^2 + 45u\partial_u^2 + 25v\partial_u\partial_v\right) Z_{SU(4)}(u,v,0)\\
&\quad + \left((135 - 64uv)\partial_u - (16v^2 + 48u^2)\partial_v - 160v\right) Z_{SU(4)}(u,v,0) = 0,\\
&\left(3v^2\partial_v^3 + 4uv\partial_u\partial_v^2 + u^2\partial_u^2\partial_v + 45v\partial_v^2 + 25u\partial_u\partial_v\right) Z_{SU(4)}(u,v,0)\\
&\quad + \left((135 - 64uv)\partial_v - (16u^2 + 48v^2)\partial_u - 160u\right) Z_{SU(4)}(u,v,0) = 0.
\end{aligned} \tag{100}
$$

Acting with the operator $(u\partial_u + 2u\partial_v + 7)$ on the first equation in (100) and evaluating the result at $u=v=t$ and $w=0$, we obtain the following differential equation for $Z(t,t,0)$,

$$
\left(t^3\frac{d^4}{dt^4} + 24t^2\frac{d^3}{dt^3} + (165 - 64t^2)t\frac{d^2}{dt^2} + 9(35 - 64t^2)\frac{d}{dt} - 960t\right) Z_{SU(4)} = 0, \tag{101}
$$

where the derivatives with respect to t are given by expressions analogous to (55). Restricting $Z_{SU(4)}$ to the diagonal breaks the $\mathbb{Z}_4$ center symmetry down to $\mathbb{Z}_2$; we identify (101) in terms of the $\mathbb{Z}_2$-invariant coordinate $y = 16t^2$,

$$\big(y^4\partial_y^4 + 15y^3\partial_y^3 + (60y^2 - y^3)\partial_y^2 + (60y - 5y^2)\partial_y - \tfrac{15}{4}y\big) Z_{SU(4)} = 0, \tag{102}$$

as the defining equation (21) for the hypergeometric function

$$Z_{SU(4)}(t,t,0) = \int d\mu_{\text{red}}(g) \exp\big(t(\operatorname{tr} g + \operatorname{tr} g^\dagger)\big) = {}_2F_3\left[\begin{matrix}3/2, 5/2\\ 3,4,5\end{matrix}\,\middle|\, 16t^2\right]. \tag{103}$$

This result proves the conjecture in [18] (which is based on numerical observations). As a by-product, this also leads to the remarkable identity

$$\sum_{n\in\mathbb{Z}} \det\begin{pmatrix} I_n & I_{n+1} & I_{n+2} & I_{n+3}\\ I_{n-1} & I_n & I_{n+1} & I_{n+2}\\ I_{n-2} & I_{n-1} & I_n & I_{n+1}\\ I_{n-3} & I_{n-2} & I_{n-1} & I_n \end{pmatrix}(2t) \;=\; {}_2F_3\left[\begin{matrix}3/2, 5/2\\ 3,4,5\end{matrix}\,\middle|\, 16t^2\right], \tag{104}$$

cf. eq. (16) for $u = v = t$, relating generalized hypergeometric functions and determinants of Bessel functions.

6 On the reduced Haar measure

In this section, we will describe an alternative approach to Ward identities for the generating function Z_G based on a factorization of the reduced Haar measure on the maximal Abelian torus in G. Tangent vectors to this torus are linear combinations $H_\varphi = \sum_p \varphi_p H_p$ of the Cartan generators H_p. The reduced Haar measure $d\mu_{\text{red}} = \rho_{\text{red}}\, d^r\varphi$ on the maximal abelian torus has the product representation [25]

$$\rho_{\text{red}}\left(e^{iH_\varphi}\right) \propto \prod_{\alpha>0} 4\sin^2\left(\frac{1}{2}\,\alpha(H_\varphi)\right) = \prod_{\boldsymbol{m}\in\Phi^+} 4\sin^2\left(\frac{1}{2}(\boldsymbol{m}, K\boldsymbol{\varphi})\right) \tag{105}$$

with one factor for every positive root α. The $\frac{1}{2}\big(\dim(G) - \operatorname{rank}(G)\big)$ positive roots are linear combination of the simple roots,

$$\alpha = m_1\alpha_{(1)} + m_2\alpha_{(2)} + \ldots + m_r\alpha_{(r)}, \quad m_i \in \mathbb{N}_0, \tag{106}$$

and the range Φ^+ for $\boldsymbol{m} = (m_1, \ldots, m_r)^t$ in (105) is chosen in such a way that it parametrizes all positive roots. We may take the square root of the density [25],

$$\rho_{\text{red}}\left(e^{iH_\varphi}\right) \propto |\Delta|^2, \quad \Delta = \prod_{\boldsymbol{m}\in\Phi^+} 2i\sin\left(\frac{1}{2}(\boldsymbol{m}, K\boldsymbol{\varphi})\right) = \sum_{w\in W} \operatorname{sign}(w) e^{iw(\rho)(H_\varphi)}, \tag{107}$$

where the sum runs over the Weyl orbit W of the Weyl vector ρ introduced in (39). Since the Weyl orbit of ρ contains $|W|$ elements the product representation is preferable for large groups.[6] But it is evident from the sum representation that Δ changes sign under Weyl reflections.

The density $\rho_{\rm red} \propto \Delta\bar{\Delta}$ of the reduced Haar measure is a Weyl-invariant function on the maximal Abelian torus and hence a function of the fundamental characters. From (105) we see that it actually is a *polynomial* of the fundamental characters. In contrast, Δ is not Weyl-invariant and hence cannot be written as function of the characters.

Less obvious is the observation that Δ is related to the Jacobian of the transformation $\boldsymbol{\varphi} \mapsto \boldsymbol{\chi}(\boldsymbol{\varphi})$ from the angular variables to the fundamental characters,

$$J(\boldsymbol{\chi}) \equiv \Big| \det\left(\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{\varphi}}\right) \Big| \propto |\Delta|, \quad \text{such that} \quad d\mu_{\rm red} = J^2 d^r\varphi = J(\boldsymbol{\chi}) d^r\chi. \tag{108}$$

This mapping is one-to-one on the fundamental domain $\mathcal{F}$ of the action of the Weyl group on the maximal Abelian torus. This is the closed connected region containing $\boldsymbol{\chi} = 0$ in which $\Delta \geq 0$. We have no proof of the *conjecture* (108) for all compact simple groups, but have checked it for the groups $SU(3)$, Spin(5) and G_2 considered in the following sections as well as for $SU(2)$ and $SU(4)$.

Based on this conjecture we may derive alternative Ward identities from

$$\int_{\mathcal{F}} d^r\chi \, \frac{\partial}{\partial \chi_p} \left(J^3(\boldsymbol{\chi}) F(\boldsymbol{\chi})\right) = 0, \tag{109}$$

where use was made of the fact that the Jacobian vanishes on the boundary of the fundamental domain. This leads to the general and simple looking Ward identities

$$0 = \int_{\mathcal{F}} d\mu_{\rm red} \left(\frac{3}{2}\frac{\partial J^2}{\partial \chi_p} F + J^2 \frac{\partial F}{\partial \chi_p}\right), \qquad p = 1, \dots, r \tag{110}$$

for any regular function $F = F(\boldsymbol{\chi})$ on the fundamental domain. In particular they imply the following differential identities for the generating function $Z_G(\boldsymbol{u})$:

$$\left(\frac{3}{2}\frac{\partial J^2}{\partial \chi_p}(\boldsymbol{\partial}) + u_p J^2(\boldsymbol{\partial})\right) Z_G(\boldsymbol{u}) = 0, \qquad p = 1, \dots, r. \tag{111}$$

These should be compared with the geometric Ward identities (35). In reference [17] by one of the authors, both types of identities were applied to calculate effective Polyakov loop

[6] For example for $SU(N)$ the sum has $N!$ terms, whereas the product has only $\frac{1}{2}N(N-1)$ factors.

dynamics of $SU(3)$ Yang-Mills theories on the lattice. The geometric Ward identities are usually simpler but not necessarily favored in computer simulations.
Ultimately, the two systems of linear partial differential equations (111) and (35) must be equivalent; but a proof is not straightforward at all. For example, for $SU(3)$ the equations (111) are 4th order differential equation whereas (35) are of 2nd order.
Now we apply the general result (111) to all simple compact simply-connected groups of rank 2. As mentioned above, for these group the conjecture that the density of the reduced Haar measure is proportional to the square of the Jacobian J of the transformation $\boldsymbol{\varphi} \mapsto \boldsymbol{\chi}$ can be checked explicitly. The Jacobians for the three groups are computed in the following subsections.

6.1 The group $SU(3)$

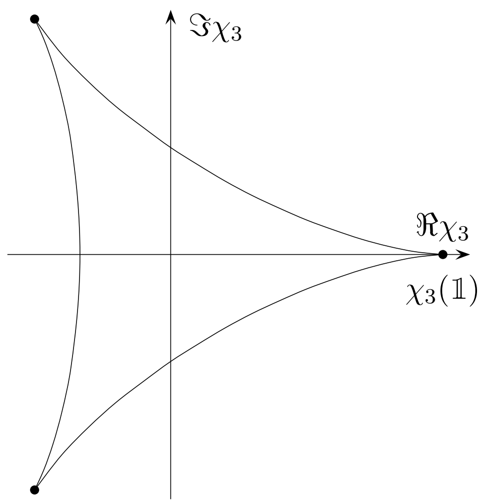
As in section 4.1, $[1,0]$ denotes the defining representation 3 and $[0,1]$ its complex conjugate $\bar{3}$. The reduced Haar measure reads

$$d\mu_{\rm red} = \frac{1}{6\pi^2} J^2\, d\varphi_1 d\varphi_2 = \frac{1}{6\pi^2} J(\boldsymbol{\chi})\, d\chi_3 d\chi_{\bar{3}}, \tag{112}$$

with $J^2 \propto \rho_{\rm red}$ from (105). As a function of the fundamental characters the Weyl-invariant and center-symmetric J^2 reads

$$J^2 = 27 + \chi_3^3 + \chi_{\bar{3}}^3 - \frac{1}{4}\left(9 + \chi_3\chi_{\bar{3}}\right)^2. \tag{113}$$

It can be easily verified that its positive square root J indeed coincides with the Jacobian of the map $(\varphi_1, \varphi_2) \mapsto (\chi_3, \chi_{\bar{3}})$.



The Jacobian vanishes for

$$y^2 = -(9 + 12x + x^2) \pm 2(2x+3)^{3/2},$$

where $x = \Re(\chi_3)$ and $y = \Im(\chi_3)$ are the real and imaginary parts of χ_3. The fundamental domain inside the triangularly shaped region is symmetric under $\mathbb{Z}_3$ center transformations which rotate χ_3 by multiples of $e^{2\pi i/3}$. Its corners are the values of $(\Re\chi_3, \Im\chi_3)$ at the center elements. Here, the identities (111) for the generating function take the form

$$0 = \Big(3\big(6\partial_u^2 - 9\partial_v - \partial_u\partial_v^2\big) + u\big(27 + 4\partial_u^3 + 4\partial_v^3 - 18\partial_u\partial_v - \partial_u^2\partial_v^2\big)\Big) Z_{SU(3)}, \tag{114}$$

$$0 \;=\; \Big(3\big(6\partial_v^2 - 9\partial_u - \partial_v\partial_u^2\big) + v\big(27 + 4\partial_u^3 + 4\partial_v^3 - 18\partial_u\partial_v - \partial_u^2\partial_v^2\big)\Big) Z_{SU(3)}. \qquad (115)$$

In contrast to the geometric identities (48, 49) these are 4th order differential equations.

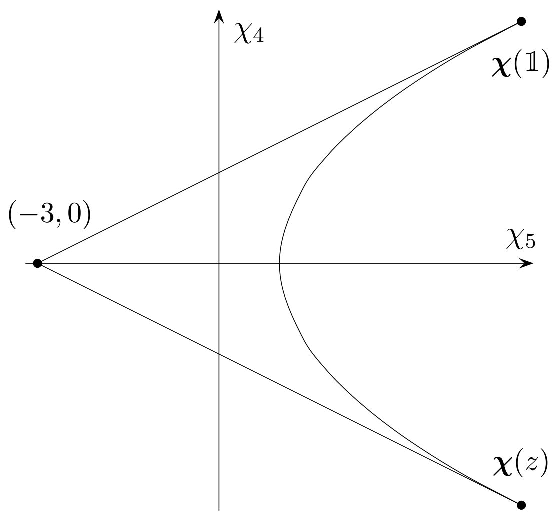
6.2 The group Spin(5)

With the conventions used in section 4.2, $[1,0] = 5$ is the defining representation and $[0,1] = 4$ denotes the spin representation. The reduced Haar measure reads

$$d\mu_{\text{red}} \propto J^2 d\varphi_1 d\varphi_2 = J d\chi_5 d\chi_4 \qquad (116)$$

with Jacobian J such that

$$J^2 = (3 + \chi_5 - 2\chi_4)(3 + \chi_5 + 2\chi_4)(4 - 4\chi_5 + \chi_4^2). \qquad (117)$$



The zero locus of the Jacobian is given by

$$2y = \pm(x+3), \quad y = \pm\sqrt{x-1},$$

where we abbreviated $x = \chi_5$ and $y = \chi_4$. The fundamental domain inside the triangularly shaped region is invariant under the $\mathbb{Z}_2$ center symmetry flipping the sign of χ_4. Two of the corners, $(\chi_5(\mathbb{1}), \chi_4(\mathbb{1})) = (5,4)$ and $(\chi_5(z), \chi_4(z)) = (5,-4)$, respectively, are located at the values of the characters at the center elements. The identities (111) for the generating function take the form

$$0 = \Big(u\big((3+\partial_u)^2 - 4\partial_v^2\big)\big(4 - 4\partial_u + \partial_v^2\big) + 3\big(\partial_u\partial_v^2 - 6\partial_u^2 + 11\partial_v^2 - 20\partial_u - 6\big)\Big) Z_{\text{Spin}(5)}, \qquad (118)$$

$$0 = \Big(v\big((3+\partial_u)^2 - 4\partial_v^2\big)\big(4 - 4\partial_u + \partial_v^2\big) + 3\big(\partial_u^2\partial_v - 8\partial_v^3 + 22\partial_u\partial_v - 7\partial_v\big)\Big) Z_{\text{Spin}(5)}. \qquad (119)$$

Again, these are 4th order differential equations.

6.3 The group G_2

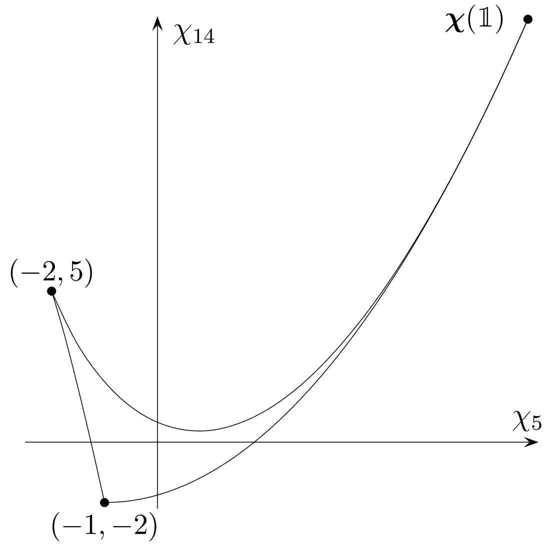
With the conventions of section 4.3, $7 = [1,0]$ denotes the 7-dimensional representation and $14 = [0,1]$ the adjoint representation. The density J^2 of the reduced Haar measure

$$d\mu_{\text{red}} \propto J^2 d\varphi_1 d\varphi_2 = J d\chi_5 d\chi_4 \qquad (120)$$

is a quintic polynomial in χ_7 and a cubic polynomial in χ_{14},

$$J^2 = \left(4\chi_7^3 - \chi_7^2 - 2\chi_7 - 10\chi_7\chi_{14} + 7 - 10\chi_{14} - \chi_{14}^2\right)\left(7 - \chi_7^2 - 2\chi_7 + 4\chi_{14}\right). \quad (121)$$

Since the center of G_2 is trivial, this polynomial shows no symmetries at all. Nevertheless, it is possible to characterize the fundamental domain for the exceptional group G_2 explicitly.



The zero locus of the Jacobian is given by

$$\begin{aligned} y &= \frac{1}{4}(x+1)^2 - 2, \\ y &= -5(x+1) \pm 2(x+2)^{3/2}, \end{aligned}$$

where we introduced $x = \chi_7$ and $y = \chi_{14}$. The fundamental domain is the region bounded by the three curves defined by the above equations. The upper right corner is located at the characters of the unit element, $(\chi_7, \chi_{14}) = (7, 14)$. In this case, the identity (111) for the generating function reads

$$\begin{aligned} 0 &= \Big(u\big(4\partial_u^3 - \partial_u^2 - 2\partial_u - 10\partial_u\partial_v + 7 - 10\partial_v - \partial_v^2\big)\big(7 - \partial_u^2 - 2\partial_u + 4\partial_v\big) \\ &\quad + 3\big(3 - 29\partial_u + 13\partial_u^3 + 13\partial_u^2 - 38\partial_u\partial_v - 21 - 47\partial_v + \partial_u^2\partial_v - 6\partial_v^2\big)\Big) Z_{G_2}, \quad (122) \\ 0 &= \Big(v\big(4\partial_u^3 - \partial_u^2 - 2\partial_u - 10\partial_u\partial_v + 7 - 10\partial_v - \partial_v^2\big)\big(7 - \partial_u^2 - 2\partial_u + 4\partial_v\big) \\ &\quad + \big(3 - 29\partial_u + 13\partial_u^3 + 13\partial_u^2 - 38\partial_u\partial_v - 21 - 47\partial_v + \partial_u^2\partial_v - 6\partial_v^2\big)\Big) Z_{G_2}. \quad (123) \end{aligned}$$

Since J^2 is quintic we arrive at complicated 5th order linear partial differential equations which should be compared with the equivalent geometric Ward identites (83, 84).

7 Recursion relations for the moments

The function $Z_G(\boldsymbol{u})$ generates the moments

$$t_{m_1,\dots,m_r} = \int d\mu_{\text{red}}\, \chi_1^{m_1} \cdots \chi_r^{m_r} \quad (124)$$

by multiple differentiation at $\boldsymbol{u}=\mathbf{0}$, see (9). Due to center symmetry of the Haar measure only center symmetric moments are nonzero, and this selection rule must be respected by any recursion relation for the moments.

One may use the Ward identity for $Z_G(\boldsymbol{u})$ to find such relations. A more direct derivation takes advantage of (33) with $F=\chi_1^{m_1}\cdots\chi_r^{m_r}$. One finds

$$\begin{aligned} 0 &= \Big(2c_p-\sum_q (c_p+c_q)m_q\Big)t_{m_1,\dots,m_p+1,\dots m_r} \\ &\quad +\sum_{\lambda,q} C^\lambda_{pq}c_\lambda m_q\int d\mu_{\rm red}\,\chi_\lambda(\boldsymbol{\chi})\chi_1^{m_1}\cdots\chi_q^{m_q-1}\cdots\chi_r^{m_r}. \end{aligned} \tag{125}$$

Clearly, the complexity of these relations increases with the degree of the polynomials χ_λ in the last sum. Alternatively, we could apply eq. (110) to the same function F, with the result

$$0=\int d\mu_{\rm red}\left(3\,\frac{\partial J^2(\boldsymbol{\chi})}{\partial\chi_p}\chi_1^{m_1}\cdots\chi_r^{m_r}+2m_pJ^2(\boldsymbol{\chi})\chi_1^{m_1}\cdots\chi_p^{m_p-1}\cdots\chi_r^{m_r}\right). \tag{126}$$

These relations are based on the conjecture (108), in contrast to the 'geometric recursion relations' in (125).

For the group SU(3): For this group the recursion relations (125) take the form

$$(8+2m+n)t_{m+1,n}-6mt_{m-1,n+1}-9nt_{m,n-1} = 0, \tag{127}$$

$$(8+2n+m)t_{m,n+1}-6nt_{m+1,n-1}-9mt_{m-1,n} = 0. \tag{128}$$

Since the moments are symmetric the two identities are equivalent. These 'geometric identities' are much simpler than the 'non-geometric' relations (126), which for $p=1$ lead to

$$(18+4m)t_{n,m+2}-(3+m)t_{n+2,m+1}-(27+18m)t_{n+1,m}+27mt_{n,m-1}+4mt_{n+3,m-1}=0.$$

By symmetry of the coefficients t_{mn}, the relation for $p=2$ is again equivalent to this recursion formula. The difference of both leads to the simpler formula

$$\begin{aligned} 0 &= (4k+6)t_{3k+m+3,m}+(4k-6)t_{3k+m,m+3} \\ &\quad +k\left(27t_{3k+m,m}-18t_{3k+m+1,m+1}-t_{3k+m+2,m+2}\right). \end{aligned} \tag{129}$$

All recursion relations are compatible with center symmetry which implies $t_{mn}=0$ unless $m=n$ mod 3. With the moments

$$t_{3m,0}=\frac{2(3m)!}{m!(m+1)!(m+2)!},\quad t_{3m+1,1}=\frac{6(3m+1)!}{m!(m+1)!(m+3)!} \tag{130}$$

one can compute all t_{mn} with the recursion relation (127). For example, for $m = n$, one finds

$$t_{mm} = 2\sum_{k=0}^{m}\binom{2k}{k}\binom{m}{k}^2\frac{3k^2+2k+1-2km-m}{(k+1)^2(k+2)(m-k+1)}. \tag{131}$$

The moments t_{mn} for small m and n are given in the appendix.

For the group Spin(5): For this group the geometric recursion relations (125) read

$$\begin{aligned}(8+2m+n)t_{m+1,n}+(4m-5n)t_{m,n}-6mt_{m-1,n}-4mt_{m-1,n+2} &= 0, &(132)\\ (5+m+n)t_{m,n+1}-5mt_{m-1,n+1}-2nt_{m+1,n-1}-6nt_{m,n-1} &= 0. &(133)\end{aligned}$$

These relations are compatible with center symmetry which implies that $t_{mn} = 0$ for odd n. With the help of the first or second recursion relations, one can determine any t_{mn} given $t_{m,0}$. For the first relation we also need the $t_{0,n}$ which are just the coefficients in the series expansion of $Z(0,v)$ in (70). The moments t_{mn} for small m and n can be found in the appendix.

For the group G$_2$: For this exceptional group the recursion relation (125) are more involved,

$$\begin{aligned}0 &= 2mt_{m-1,n+1}-(12+2m+3n)t_{m+1,n}+(8m-7n)t_{m,n}\\ &\quad +14mt_{m-1,n}+7nt_{m+2,n-1}+8nt_{m+1,n-1}-7nt_{m,n-1}, &(134)\\ 0 &= (24+3m+6n)t_{m,n+1}+7mt_{m-1,n+1}-(7m-12n)t_{m+1,n}-(8m-4n)t_{m,n}\\ &\quad +7mt_{m-1,n}-6nt_{m+3,n-1}-10nt_{m+2,n-1}+22nt_{m+1,n-1}-14nt_{m,n-1}. &(135)\end{aligned}$$

For example, one can calculate all t_{mn} from the $t_{m,0}$ and $t_{m,1}$. The former ones are just the coefficients g_m in (90). The moments t_{mn} for small m and n are given in the appendix.

8 Conclusions

In this paper, we have derived two kinds of Ward identities for the generating functions for integrals over arbitrary polynomials of fundamental characters. One is a consequence of the fact that a left-derivative of any function on the Lie group integrates to zero with the full Haar measure. For a convenient choice of this function, this left-derivative is a class function so that the vanishing of the integral reduces to an identity on the maximal Abelian torus. The other Ward identity derives from an integral of a total derivative of an arbitrary class function over a certain domain. If one chooses this domain as the region where the Jacobian of the change of variables from the angles of the maximal Abelian torus to the fundamental characters is non-negative, one can split powers of the Jacobian from the arbitrary class function so that the result vanishes on the boundary of this fundamental domain. This, however, leads to ultimately more complicated differential equations for the generating functions than the first, geometric, approach. Both furnish generalizations and structural clarifications of identities used in the case of $SU(3)$ in an earlier publication [17] by one of the authors. In this paper, they have been applied to all simple compact simply connected Lie groups of rank two. Furthermore, they have been used to prove several conjectures in the literature concerning explicit solutions for $SU(3)$ and $SU(4)$; beyond that, we derived recursion relations determining all integrals over polynomials of fundamental characters for the above groups.

The derivation of the second kind of Ward identities is based on a conjecture concerning the factorization of the reduced Haar measure density ρ_{red} into the square of this Jacobian. This conjecture has been checked explicitly for several cases under consideration, but so far it lacks a general proof. It might be interesting to clarify this issue from a group theoretical point of view.

Another obvious open question is the equivalence of our two approaches. Since both encode information determining the same generating functions it ultimately should be possible to derive one from the other. Perhaps with a deeper insight into the group theoretical connection between the approaches it might be possible to obtain even simpler identities which might be of use, e.g., in lattice gauge theories or random matrix models. From a mathematical point of view they answer the question how many invariants there are in a given tensor product of fundamental representations.

We found it surprising to note that the generating function for powers of the sum of characters of only the defining and its complex conjugate representation for $SU(2)$ and $SU(4)$ can be expressed in terms of appropriate generalized hypergeometric functions. One might speculate that this generalizes to higher rank $SU(2n)$ as well. Apart from that, this fact for $SU(4)$ leads to a hitherto unknown relation (104) between generalized hypergeometric functions and Bessel functions. Finally, one might note the remarkable fact that all Ward identities (of the first kind), at least for the Lie groups of rank 2, reduce to parabolic differential equations on the locus where their families of characteristics coincide. This allows for a recursive integration starting from a solution on this locus.

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A Generating function for $SU(3)$

In this section, we derive an analytic solution to eq. (48). By invariance of the Haar measure, $d\mu(g) = d\mu(g^{-1})$, we expect $Z_{SU(3)}$ to be a symmetric function of u and v. Thus, our objective is to solve (48) and (49) with the symmetry $Z_{SU(3)}(u,v) = Z_{SU(3)}(v,u)$ and initial condition $Z_{SU(3)}(0,0) = 1$. From (23), we know that

$$Z_{SU(3)}(0,v) = \sum_{n=0}^{\infty} \frac{2}{n!(n+1)!(n+2)!}\, v^{3n}.$$

Multiplying (48) by u and (49) by v, the difference of the resulting equations reads

$$\big(u^2\partial_u^2 - v^2\partial_v^2 + 4\,(u\partial_u - v\partial_v) - 3(u^2\partial_v - v^2\partial_u)\big)Z_{SU(3)} = 0. \tag{136}$$

In principle, it is possible to find a recursive solution analogously to (71) and (91) starting from a solution on the locus where the families of characteristics of the differential equation coincide (here on the u- and the v-axis), where the differential system turns out to be parabolic. However, in the case of $SU(3)$ we are able to give a closed expression of the solution which allows for an easier computation of the moments (9):
Center symmetry (50) as well as the symmetry of $Z_{SU(3)}$ in its two arguments suggests to introduce new coordinates $x = uv$ and $y = u^3 + v^3$. As a function of these variables, $Z_{SU(3)}$ has to satisfy

$$(3y\partial_y^2 + 6\partial_y + 2x\partial_x\partial_y - \partial_x)Z_{SU(3)} = 0, \tag{137}$$

which can easily be solved by a power series expansion,

$$Z_{SU(3)}(x,y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n \tag{138}$$

with recursion relation

$$(3n + 3 + 2m)n a_{mn} = (m+1)a_{m+1,n-1}. \tag{139}$$

The solution to (138) and (139) with the condition

$$Z_{SU(3)}(x=0,y) = \sum_{n=0}^{\infty} \frac{2}{n!(n+1)!(n+2)!)}\, y^n \equiv \sum_{n=0}^{\infty} a_{0n} y^n \tag{140}$$

is given by

$$Z_{SU(3)}(u,v) = \sum_{m,n=0}^{\infty} \frac{2}{n!(m+n+1)!(m+n+2)!} \binom{3(m+n+1)}{m} (uv)^m \left(u^3+v^3\right)^n. \tag{141}$$

This solution also satisfies the original differential equations (48) and (49). It encodes information about how many invariants there are in the tensor product $3^{\otimes p} \otimes \bar{3}^{\otimes q}$ of $SU(3)$ representations. General results for integrals over $SU(3)$ matrix elements can be found in [12].

B Tables of moments for the rank 2 groups

With the recursion relations (127), (128), (132), (133), (134), and (135), we determined the moments

$$t_{mn} = \int_G d\mu_{\text{red}}(g) \chi^m_{[1,0]}(g) \chi^n_{[0,1]}(g), \tag{142}$$

for the groups with rank 2 for small m and n. Here $\chi_{[1,0]}$ and $\chi_{[0,1]}$ are the characters of the fundamental representations with highest weights $\mu_1 \equiv [1,0]$ and $\mu_2 \equiv [0,1]$.

For SU(3) the lowest moments of $\chi_3^m \chi_{\bar{3}}^n$ are

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	1	0	0	5	0	0	42	0
1	0	1	0	0	3	0	0	21	0	0	210
2	0	0	2	0	0	11	0	0	98	0	0
3	1	0	0	6	0	0	47	0	0	498	0
4	0	3	0	0	23	0	0	225	0	0	2709
5	0	0	11	0	0	103	0	0	1173	0	0
6	5	0	0	47	0	0	513	0	0	6529	0
7	0	21	0	0	225	0	0	2761	0	0	38265
8	0	0	98	0	0	1173	0	0	15767	0	0
9	42	0	0	498	0	0	6529	0	0	94359	0
10	0	210	0	0	2709	0	0	38265	0	0	586590

For Spin(5) the lowest moments of $\chi_5^m \chi_4^n$ are

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	1	0	3	0	14	0	84	0	594
1	0	0	1	0	5	0	30	0	210	0	1650
2	1	0	2	0	11	0	75	0	580	0	4917
3	0	0	4	0	27	0	205	0	1714	0	15435
4	3	0	10	0	73	0	600	0	5338	0	50506
5	1	0	26	0	211	0	1852	0	17342	0	171022
6	15	0	75	0	645	0	5970	0	58350	0	596085
7	15	0	225	0	2061	0	19950	0	202230	0	2129719
8	105	0	715	0	6837	0	68730	0	718928	0	7774600
9	190	0	2347	0	23403	0	243050	0	2612796	0	28922112
10	945	0	7990	0	82301	0	879204	0	9681144	0	109404729

Both tables nicely display the constraints from center symmetry.

For $\mathbf{G}_2$ the lowest moments of $\chi_7^m \chi_{14}^n$ are

$m\backslash n$	0	1	2	3	4	5	6	7	8
0	1	0	1	1	5	16	80	436	2786
1	0	0	0	1	6	40	260	1785	12852
2	1	1	3	10	45	236	1421	9444	67852
3	1	2	7	32	170	1016	6637	46656	348553
4	4	9	33	151	817	4984	33357	240181	1835171
5	10	30	126	641	3728	23986	167080	1241285	9727650
6	35	120	545	2932	17827	118945	854135	6511050	52159514
7	120	476	2359	13517	86171	596686	4415055	34500369	282217558
8	455	2002	10626	64078	425194	3041241	23115050	184754906	1540766892

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