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ON THE GENERAL STRUCTURE OF HAMILTONIAN REDUCTIONS OF THE WZNW THEORY

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Abstract

The structure of Hamiltonian reductions of the Wess-Zumino-Novikov-Witten (WZNW) theory by first class Kac-Moody constraints is analyzed in detail. Lie algebraic conditions are given for ensuring the presence of exact integrability, conformal invariance and $\mathcal{W}$-symmetry in the reduced theories. A Lagrangean, gauged WZNW implementation of the reduction is established in the general case and thereby the path integral as well as the BRST formalism are set up for studying the quantum version of the reduction. The general results are applied to a number of examples. In particular, a $\mathcal{W}$-algebra is associated to each embedding of $sl(2)$ into the simple Lie algebras by using purely first class constraints. The importance of these $sl(2)$ systems is demonstrated by showing that they underlie the W_n^l-algebras as well. New generalized Toda theories are found whose chiral algebras are the $\mathcal{W}$-algebras belonging to the half-integral $sl(2)$ embeddings, and the $\mathcal{W}$-symmetry of the effective action of those generalized Toda theories associated with the integral gradings is exhibited explicitly.

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1. Introduction

Due to their intimate relationship with Lie algebras, the various one- and two-dimensional Toda systems are among the most important models of the theory of integrable non-linear equations [1-19]. In particular, the standard conformal Toda field theories, which are given by the Lagrangean

$$\mathcal{L}_{\rm Toda}(\varphi) = \frac{\kappa}{2}\Big(\sum_{i,j=1}^{l} \frac{1}{2|\alpha_i|^2} K_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j - \sum_{i=1}^{l} m_i^2 \exp\Big\{\frac{1}{2}\sum_{j=1}^{l} K_{ij}\varphi^j\Big\}\Big) , \qquad (1.1)$$

where κ is a coupling constant, K_{ij} is the Cartan matrix and the α_i are the simple roots of a simple Lie algebra of rank l, have been the subject of many studies [1,3,4,8-13,19]. It has been first shown by Leznov and Saveliev [1,3] that the Euler-Lagrange equations of (1.1) can be written as a zero curvature condition, are exactly integrable, and possess interesting non-linear symmetry algebras [3,4,10,11,13,19]. These symmetry algebras are generated by chiral conserved currents, and are polynomial extensions of the chiral Virasoro algebras generated by the traceless energy-momentum tensor. The chiral currents in question are conformal primary fields, whose conformal weights are given by the orders of the independent Casimirs of the corresponding simple Lie algebra. Polynomial extensions of the Virasoro algebra by chiral primary fields are generally known as $\mathcal{W}$-algebras [20], which are expected to play an important role in the classification of conformal field theories and are in the focus of current investigations [20-29]. The importance of Toda systems in two-dimensional conformal field theory is in fact greatly enhanced by their realizing the $\mathcal{W}$-algebra symmetries.

It has been discovered recently that the conformal Toda field theories can be naturally viewed as Hamiltonian reductions of the Wess-Zumino-Novikov-Witten (WZNW) theory [12,13]. The main feature of the WZNW theory is its affine Kac-Moody (KM) symmetry, which underlies its integrability [30,31]. The WZNW theory provides the most 'economical' realization of the KM symmetry in the sense that its phase space is essentially a direct product of the left $\times$ right KM phase spaces. The WZNW $\rightarrow$ Toda Hamiltonian reduction is achieved by imposing certain first class, conformally invariant constraints on the KM currents, which reduce the chiral KM phase spaces to phase spaces carrying the chiral $\mathcal{W}$-algebras as their Poisson bracket structure [12,13]. Thus the $\mathcal{W}$-algebra is related to the phase space of the Toda theory in the same way as the KM algebra is related to the phase space of the WZNW theory. In the above manner, the $\mathcal{W}$-symmetry of the Toda theories becomes manifest by describing these theories as reduced WZNW theories. This way of looking at Toda theories has also numerous other advantages, described in detail in [13].

The constrained WZNW (KM) setting of the standard Toda theories ($\mathcal{W}$-algebras) allows for generalizations, some of which have already been investigated [14-18,26-29]. An important recent development is the realization that it is possible to associate a generalized $\mathcal{W}$-algebra to every embedding of the Lie algebra $sl(2)$ into the simple Lie algebras [16-18]. The standard $\mathcal{W}$-algebra, occurring in Toda theory, corresponds to the so called principal $sl(2)$. In fact, these generalized $\mathcal{W}$-algebras can be obtained from the KM algebra by constraining the current to the highest weight gauge, which has been originally introduced in [13] for describing the standard case. Another interesting development is the W_n^l-algebras introduced by Bershadsky [26] and further studied in [28]. It is known that the simplest non-trivial case W_3^2, which was originally proposed by Polyakov [27], falls into a special case of the $\mathcal{W}$-algebras obtained by the $sl(2)$ embeddings mentioned above. It has not been clear, however, as to whether the two classes of $\mathcal{W}$-algebras are related in general, or to what extent one can further generalize the KM reduction to achieve new $\mathcal{W}$-algebras.

In the present paper, we undertake the first systematic study of the Hamiltonian reductions of the WZNW theory, aiming at uncovering the general structure of the reduction and, at the same time, try to answer the above question. Various different questions arising from this main problem are also addressed (see Contents), and some of them can be examined on its own right. As this provides our motivation and in fact most of the later developments originate from it, we wish to recall here the main points of the WZNW $\rightarrow$ Toda reduction before giving a more detailed outline of the content.

To make contact with the Toda theories, we consider the WZNW theory*

$$S_{\mathrm{WZ}}(g) = \frac{\kappa}{2}\int d^2x\,\eta^{\mu\nu}\,\mathrm{Tr}\,(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) - \frac{\kappa}{3}\int_{B_3}\mathrm{Tr}\,(g^{-1}dg)^3\ , \tag{1.2}$$

for a simple, maximally non-compact, connected real Lie group G. In other words, we assume that the simple Lie algebra, $\mathcal{G}$, corresponding to G allows for a Cartan decomposition over the field of real numbers. The field equation of the WZNW theory can be written in the equivalent forms

$$\partial_- J = 0 \qquad \text{or} \qquad \partial_+ \tilde{J} = 0\ , \tag{1.3}$$

* The KM level k is $-4\pi\kappa$. The space-time conventions are: $\eta_{00} = -\eta_{11} = 1$ and $x^\pm = \frac{1}{2}(x^0 \pm x^1)$. The WZNW field g is periodic in x^1 with period $2\pi r$.

where

$$J=\kappa\partial_+ g\cdot g^{-1}\ ,\quad \text{and}\quad \tilde{J}=-\kappa g^{-1}\partial_- g\ . \tag{1.4}$$

These equations express the conservation of the left- and right KM currents, J and $\tilde{J}$, respectively. The general solution of the WZNW field equation is given by the simple formula

$$g(x^+,x^-)=g_L(x^+)\cdot g_R(x^-)\ , \tag{1.5}$$

where g_L and g_R are arbitrary G-valued functions, i.e., constrained only by the boundary condition imposed on g.

Let now M_-, M_0 and M_+ be the standard generators of the principal $sl(2)$ subalgebra of $\mathcal{G}$ [32]. By considering the eigenspaces $\mathcal{G}_m$ of M_0 in the adjoint of $\mathcal{G}$, $\mathrm{ad}_{M_0}=[M_0,\ ]$, one can define a grading of $\mathcal{G}$ by the eigenvalues m. Under the principal $sl(2)$ this grading is an integral grading, in fact the spins occurring in the decomposition of the adjoint of $\mathcal{G}$ are the exponents of $\mathcal{G}$, which are related to the orders of the independent Casimirs by a shift by 1. It is also worth noting that the grade 0 part of

$$\mathcal{G}=\mathcal{G}_+ +\mathcal{G}_0+\mathcal{G}_-\ ,\qquad\qquad \mathcal{G}_\pm=\sum_{m=1}^{N}\mathcal{G}_{\pm m}\ , \tag{1.6}$$

is a Cartan subalgebra, and (by using some automorphism of the Lie algebra) one can assume that the generator M_0 is given by the formula $M_0=\frac{1}{2}\sum_{\alpha>0}H_\alpha$, where H_α is the standard Cartan generator corresponding to the positive root α, and the generators $M_\pm$ are certain linear combinations of the step operators $E_{\pm\alpha_i}$ corresponding to the simple roots α_i, $i=1,\ldots,\mathrm{rank}\,\mathcal{G}$.

The basic observation of [12,13] has been that the standard Toda theory can be obtained from the WZNW theory by imposing first class constraints which restrict the currents to take the following form:

$$J(x)=\kappa M_- + j(x),\qquad \text{with}\qquad j(x)\in(\mathcal{G}_0+\mathcal{G}_+)\ , \tag{1.7a}$$

and

$$\tilde{J}(x)=-\kappa M_+ + \tilde{j}(x),\qquad \text{with}\qquad \tilde{j}(x)\in(\mathcal{G}_0+\mathcal{G}_-)\ . \tag{1.7b}$$

(For clarity, we note that one should in principle include some dimensional constants in $M_\pm$ which are dimensionless, but such constants are always put to unity in this paper, for simplicity.) To derive the Toda theory (1.1) from the WZNW theory (1.2), one uses the generalized Gauss decomposition $g=g_+\cdot g_0\cdot g_-$ of the WZNW field g, where $g_{0,\pm}$ are from the subgroups $G_{0,\pm}$ of G corresponding to the Lie subalgebras $\mathcal{G}_{0,\pm}$, respectively. In this framework the Toda fields φ_i are given by the middle-piece of the Gauss decomposition, $g_0 = \exp[\frac{1}{2}\sum_{i=1}^{l}\varphi_i H_i]$, which is invariant under the triangular KM gauge transformations belonging to the first class constraints (1.7). Note that here the elements $H_i \in \mathcal{G}_0$ are the standard Cartan generators associated to the simple roots. In fact, the Toda field equation can be derived directly from the WZNW field equation by inserting the Gauss decomposition of g into (1.3) and using the constraints (1.7). The effective action of the reduced theory, (1.1), can also be obtained in a natural way, by using the Lagrangean, gauged WZNW implementation of the Hamiltonian reduction [13].

In their pioneering work [1,3], Leznov and Saveliev proved the exact integrability of the conformal Toda systems by exhibiting chiral quantities by using the field equation and the special graded structure of the Lax potential $\mathcal{A}_\pm$, in terms of which the Toda equation takes the zero curvature form

$$[\partial_+ - \mathcal{A}_+ \, , \, \partial_- - \mathcal{A}_-] = 0 \; . \tag{1.8}$$

In our framework the exact integrability of Toda systems is seen as an immediate consequence of the obvious integrability of the WZNW theory, which survives the reduction to Toda theory. In other words, the chiral fields underlying the integrability of the Toda equation are available from the very beginning, that is, they come from the fields entering the left $\times$ right decomposition of the general WZNW solution (1.5). Furthermore, the Toda Lax potential itself emerges naturally from the trivial, chiral Lax potential of the WZNW theory. To see this one first observes that the WZNW field equation is a zero curvature condition, since one can write for example the first equation in (1.3) as

$$[\partial_+ - J \, , \, \partial_- - 0] = 0 \; . \tag{1.9}$$

Using the constraints of the reduction, the Toda zero curvature condition (1.8) of [1,3] arises from (1.9) by conjugating this equation by $g_+^{-1}(x^+, x^-)$, namely by the inverse of the upper triangular piece of the generalized Gauss decomposition of the WZNW field g [18].

The $\mathcal{W}$-symmetry of the Toda theory appears in the WZNW setting in a very direct and natural way. Namely, one can interpret the $\mathcal{W}$-algebra as the KM Poisson bracket algebra of the gauge invariant differential polynomials of the constrained currents in (1.7). Concentrating on the left sector, the gauge transformations act on the current according to

$$J(x) \to e^{a(x^+)}\, J(x)\, e^{-a(x^+)} + \kappa (e^{a(x^+)})'\, e^{-a(x^+)}, \tag{1.10}$$

where $a(x^+) \in \mathcal{G}_+$ is an arbitrary chiral parameter function.* The constraints (1.7) are chosen in such a way that the following Virasoro generator

$$L_{M_0}(x) \equiv L_{\rm KM}(x) - {\rm Tr}\,(M_0 J'(x)), \quad \text{where} \quad L_{\rm KM}(x) = \frac{1}{2\kappa}{\rm Tr}(J^2(x)), \tag{1.11}$$

is gauge invariant, which ensures the conformal invariance of the reduced theory.

One obtains an equivalent interpretation of the $\mathcal{W}$-algebra by identifying it with the Dirac bracket algebra of the differential polynomials of the current components in certain gauges, which are such that a basis of the gauge invariant differential polynomials reduces to the independent current components after the gauge fixing. We call the gauges in question Drinfeld-Sokolov (DS) gauges [13], since such gauges has been used also in [5]. They have the nice property that any constrained current $J(x)$ can be brought to the gauge fixed form by a unique gauge transformation depending on $J(x)$ in a differential polynomial way. The most important DS gauge is the highest weight gauge [13], which is defined by requiring the gauge fixed current to be of the following form:

$$J_{\rm red}(x) = \kappa M_- + j_{\rm red}(x)\ , \qquad j_{\rm red}(x) \in {\rm Ker}({\rm ad}_{M_+})\ , \tag{1.12}$$

where ${\rm Ker}({\rm ad}_{M_+})$ is the kernel of the adjoint of M_+. In other words, $j_{\rm red}(x)$ is restricted to be an arbitrary linear combination of the highest weight vectors of the $sl(2)$ subalgebra in the adjoint of $\mathcal{G}$. The special property of the highest weight gauge is that in this gauge the conformal properties become manifest. Of course, the quantity $L_{\rm red}(x)$ obtained by restricting $L_{M_0}(x)$ in (1.11) to the highest weight gauge generates a Virasoro algebra under Dirac bracket. (Note that in our case $L_{\rm red}(x)$ is proportional to the M_+-component of $j_{\rm red}(x)$.) The important point is that, with the exception of the M_+-component, the spin s component of $j_{\rm red}(x)$ is in fact a primary field of conformal weight $(s+1)$ with respect to $L_{\rm red}(x)$ under the Dirac bracket. Thus *the highest weight gauge automatically yields a primary field basis of the $\mathcal{W}$-algebra*, from which one sees that the spectrum of conformal weights is fixed by the $sl(2)$ content of the adjoint of $\mathcal{G}$ [13].

In the above we arrived at the description of the $\mathcal{W}$-algebra as a Dirac bracket algebra by gauge fixing the first class system of constraints corresponding to (1.7). However, it is

* Throughout the paper, the notation $f' = 2\partial_1 f$ is used for every function f, including the spatial δ-functions. For a chiral function $f(x^+)$ one has then $f' = \partial_+ f$.

clear now that it would have been possible to define the $\mathcal{W}$-algebra as the Dirac bracket algebra of the components of $j_{\rm red}$ in (1.12) in the first place. Once this point is realized, a natural generalization arises immediately [16-18]. Namely, one can associate a classical $\mathcal{W}$-algebra to any $sl(2)$ subalgebra $\mathcal{S} = \{M_-,\ M_0,\ M_+\}$ of any simple Lie algebra $\mathcal{G}$, by defining it to be the Dirac bracket algebra of the components of $j_{\rm red}$ in (1.12), where one simply substitutes the generators $M_\pm$ of the *arbitrary* $sl(2)$ subalgebra $\mathcal{S}$ for those of the *principal* $sl(2)$. As we shall see in this paper, this Dirac bracket algebra is a polynomial extension of the Virasoro algebra by primary fields, whose conformal weights are related to the spins occurring in the decomposition of the adjoint of $\mathcal{G}$ under $\mathcal{S}$ by a shift by 1, in complete analogy with the case of the principal $sl(2)$. We shall designate the generalized $\mathcal{W}$-algebra associated to the $sl(2)$ embedding $\mathcal{S}$ as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$.

With the main features of the WZNW $\rightarrow$ Toda reduction and the above definition of the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras at our disposal, now we sketch the philosophy and the outline of the present paper. We start by giving the most important assumption underlying our investigations, which is that we consider those reductions which can be obtained by imposing *first class* KM constraints generalizing the ones in (1.7). To be more precise, our most general constraints restrict the current to take the following form:

$$J(x) = \kappa M + j(x), \qquad \text{with} \qquad j(x) \in \Gamma^\perp\ , \tag{1.13}$$

where M is *some* constant element of the underlying simple Lie algebra $\mathcal{G}$, and $\Gamma^\perp$ is the subspace consisting of the Lie algebra elements trace orthogonal to *some* subspace Γ of $\mathcal{G}$. We note that earlier in (1.7a) we have chosen $\Gamma = \mathcal{G}_+$ and $M = M_-$, but we do not need any $sl(2)$ structure here. The whole analysis is based on requiring the first-classness of the system of linear KM constraints corresponding the pair (Γ, M) according to (1.13). However, this first-classness assumption is not as restrictive as one perhaps might think at first sight. In fact, as far as we know, our first class method is capable of covering all Hamiltonian reductions of the WZNW theory considered to date. The many technical advantages of using purely first class KM constraints will be apparent.

The investigations in this paper are organized according to three distinct levels of generality. At the most general level we only make the first-classness assumption and deduce the following results. First, we give a complete Lie algebraic analysis of the conditions on the pair (Γ, M) imposed by the first-classness of the constraints. We shall see that Γ in (1.13) has to be a subalgebra of $\mathcal{G}$ on which the Cartan-Killing form vanishes, and that every such subalgebra is solvable. The Lie subalgebra Γ will be referred to as the 'gauge algebra' of the reduction. For a given Γ, the first-classness imposes a further condition on the element M, and we shall describe the space of the allowed M's. Second, we establish a gauged WZNW implementation of the reduction, generalizing the one found previously in the standard case [13]. This gauged WZNW setting of the reduction will be first seen classically, but it will be also established in the quantum theory by considering the phase space path integral of the constrained WZNW theory. Third, the gauged WZNW framework will be used to set up the BRST formalism for the quantum Hamiltonian reduction in the general case. Fourth, by making the additional assumption that the left and right gauge algebras are dual to each other with respect to the Cartan-Killing form, we will be able to give a detailed local analysis of the effective theories resulting from the reduction. This duality assumption will also be related to the parity invariance of the effective theories, which is satisfied in the standard Toda case where the left and right gauge algebras are $\mathcal{G}_+$ and $\mathcal{G}_-$ in (1.6), respectively. In general, the WZNW reduction not only allows us to make contact with known theories, like the Toda theory in (1.1), where the simplicity and the large symmetry of the 'parent' WZNW theory are fully exploited for analyzing them, but also leads to new theories which are 'integrable by construction'.

At the next level of generality, we study the conformally invariant reductions. The basic idea here is that one can guarantee the conformal invariance of the reduced theory by exhibiting a Virasoro density such that the corresponding conformal action preserves the constraints in (1.13). Generalizing (1.11), we assume that this Virasoro density is of the form

$$L_H(x) = L_{\mathrm{KM}}(x) - \mathrm{Tr}\,(HJ'(x)) \ , \tag{1.14}$$

where H is *some* Lie algebra element, to be determined from the condition that L_H weakly commutes with the first class constraints. We shall describe the relations which are imposed on the triple of quantities (Γ, M, H) by this requirement, and thereby obtain a Lie algebraic sufficient condition for conformal invariance.

At the third level of generality, we deal with polynomial reductions and $\mathcal{W}$-algebras. The above mentioned sufficient condition for conformal invariance is a guarantee for L_H being a gauge invariant differential polynomial. We shall provide an additional condition on the triple of quantities (Γ, M, H) which allows one to construct out of the current in (1.13) a complete set of gauge invariant differential polynomials by means of a polynomial gauge fixing algorithm. The KM Poisson bracket algebra of the gauge invariant differential polynomials yields a polynomial extension of the Virasoro algebra generated by L_H. The most important application of our sufficient condition for polynomiality concerns the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras mentioned previously.

Let us remember that, for an arbitrary $sl(2)$ subalgebra $\mathcal{S}$ of $\mathcal{G}$, the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra can be defined as the Dirac bracket algebra of the highest weight current in (1.12) realized by purely second class constraints. However, we shall see in this paper that these second class constraints can be replaced by purely first class constraints even in the case of arbitrary, *integral or half-integral*, $sl(2)$ embeddings. Since the first class constraints satisfy our sufficient condition for polynomiality, we can realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra as the KM Poisson bracket algebra of the corresponding gauge invariant differential polynomials. After having our hands on first class KM constraints leading to the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras, we shall immediately apply our general construction to exhibiting reduced WZNW theories realizing these $\mathcal{W}$-algebras as their chiral algebras for arbitrary $sl(2)$-embeddings. In the non-trivial case of half-integral $sl(2)$-embeddings, these generalized Toda theories represent a new class of integrable models, which will be studied in some detail. It is also worth noting that realizing the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra as a KM Poisson bracket algebra of gauge invariant differential polynomials should in principle allow for quantizing it through the KM representation theory, for example by using the general BRST formalism which will be set up in this paper. As a first step, we shall give a concise formula for the Virasoro centre of this algebra in terms of the level of the underlying KM algebra.

The existence of purely first class KM constraints leading to the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra might be perhaps surprizing to the reader, since earlier in [16] it was claimed to be inevitably necessary to use at least some second class constraints from the very beginning, when reducing the KM algebra to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ in the case of a half-integral $sl(2)$ embedding. Contrary to their claim, we will demonstrate that it is possible and in fact easy to obtain the appropriate first class constraints which lead to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$. Roughly speaking, this will be achieved by discarding 'half' of those constraints which form the second class part in the mixed system of the constraints imposed in [16]. The mixed system of constraints can be recovered by a partial gauge fixing of our purely first class KM constraints. Similarly, Bershadsky's constraints [26], used to define the W^l_n-algebra, are also a mixed system in the above sense, i.e., it contains both first and second class parts. We can also replace these constraints by purely first class ones without changing the final reduced phase space. In this procedure we shall uncover the hidden $sl(2)$ structure of the W^l_n-algebras, namely, we shall identify them in general as further reductions of particular $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras.

The study of WZNW reductions embraces various subjects, such as integrable models, $\mathcal{W}$-algebras and their field theoretic realizations. We hope that the readers with different interests will find relevant results throughout this paper, and find an interplay of general considerations and investigations of numerous examples.

2. General structure of KM and WZNW reductions

The purpose of this chapter is to investigate the general structure of those reductions of the KM phase space and corresponding reductions of the full WZNW theory which can be defined by imposing *first class constraints setting certain current components to constant values.* In the rest of the paper, we assume that the WZNW group, G, is a connected real Lie group whose Lie algebra, $\mathcal{G}$, is a non-compact real form of a complex simple Lie algebra, $\mathcal{G}_c$. We shall first uncover the Lie algebraic implications of the constraints being first class, and also discuss a sufficient condition which may be used to ensure their conformal invariance. In particular, we shall see why the compact real form is outside our framework. We then set up a gauged WZNW theory which provides a Lagrangean realization of the WZNW reduction, for the case of general first class constraints. Finally, we shall describe the effective field theories resulting from the reduction in some detail in an important special case, namely when the left and right KM currents are constrained for such subalgebras of $\mathcal{G}$ which are dual to each other with respect to the Cartan-Killing form.

2.1. First class and conformally invariant KM constraints

Here we analyze the general form of the KM constraints which will be used subsequently to reduce the WZNW theory. The analysis applies to each current J and $\tilde{J}$ separately so we choose one of them, J say, for definiteness. To fix the conventions, we first note that the KM Poisson bracket reads

$$\{\langle u, J(x)\rangle\,,\,\langle v, J(y)\rangle\}_{|x^0=y^0} = \langle [u,v], J(x)\rangle \delta(x^1-y^1) + \kappa\langle u,v\rangle \delta'(x^1-y^1), \tag{2.1}$$

where u and v are arbitrary generators of $\mathcal{G}$ and the inner product $\langle u\,,\,v\rangle = \mathrm{Tr}\,(u\cdot v)$ is normalized so that the long roots of $\mathcal{G}_c$ have length squared 2. This normalization means that in terms of the adjoint representation one has $\langle u\,,\,v\rangle = \frac{1}{2g}\mathrm{tr}\,(\mathrm{ad}_u\cdot \mathrm{ad}_v)$, where g is the dual Coxeter number. It is worth noting that $\langle u\,,\,v\rangle$ is the usual matrix trace in the defining, vector representation for the classical Lie algebras A_l and C_l, and it is $\frac{1}{2}\times$ trace in the defining representation for the B_l and D_l series. We also wish to point out that the KM Poisson bracket together with all the subsequent relations which follow from it hold *in the same form* both on the usual canonical phase space and on the space of the classical solutions of the theory. This is the advantage of using equal time Poisson brackets and spatial δ-functions even on the latter space, where $J(x)$ depends on $x=(x^0,x^1)$ only through x^+ (see the footnote on page 7).

The KM reduction we consider is defined by requiring the constrained current to be of the following special form:

$$J(x)=\kappa M+j(x)\ ,\qquad \text{with}\qquad j(x)\in\Gamma^{\perp}\ , \tag{2.2}$$

where Γ is some linear subspace and M is some element of $\mathcal{G}$. Equivalently, the constraints can be given as

$$\phi_\gamma(x)=\langle\gamma\,,\,J(x)\rangle-\kappa\langle\gamma\,,\,M\rangle=0\ ,\qquad\qquad \forall\ \gamma\in\Gamma\ . \tag{2.3}$$

In words, our constraints set the current components corresponding to Γ to constant values. It is clear both from (2.2) and (2.3) that M can be shifted by an arbitrary element from the space $\Gamma^{\perp}$ without changing the actual content of the constraints. This ambiguity is unessential, since one can fix M, for example, by requiring that it is from some given linear complement of $\Gamma^{\perp}$ in $\mathcal{G}$, which can be chosen by convention.

In our method we assume that the above system of constraints is *first class*, and now we analyze the content of this condition. Immediately from (2.1), we have*

$$\{\phi_\alpha(x),\phi_\beta(y)\}=\phi_{[\alpha,\beta]}(x)\delta(x^1-y^1)+\omega_M(\alpha,\beta)\delta(x^1-y^1)+\langle\alpha,\beta\rangle\delta'(x^1-y^1), \tag{2.4}$$

where the second term contains the restriction to Γ of the following anti-symmetric 2-form of $\mathcal{G}$:

$$\omega_M(u,v)\equiv\langle M\,,\,[u\,,\,v]\rangle\ ,\qquad\quad \forall u\,,\,v\in\mathcal{G}\ . \tag{2.5}$$

It is evident from (2.4) that the constraints are first class if, and only if, we have

$$[\alpha\,,\,\beta]\in\Gamma,\qquad \langle\alpha\,,\,\beta\rangle=0\quad\text{and}\quad\omega_M(\alpha\,,\,\beta)=0,\quad\text{for}\quad\forall\alpha\,,\,\beta\in\Gamma. \tag{2.6}$$

This means that the linear subspace Γ has to be *a subalgebra on which the Cartan-Killing form and ω_M vanish.* It is easy to see that the three conditions in (2.6) can be equivalently written as

$$[\Gamma\,,\,\Gamma^{\perp}]\subset\Gamma^{\perp},\qquad \Gamma\subset\Gamma^{\perp}\qquad\text{and}\qquad[M\,,\,\Gamma]\subset\Gamma^{\perp}\ , \tag{2.7}$$

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* For simplicity, we set κ to 1 in the rest of the paper, except in Chapter 5, where κ occurs in the formula of the Virasoro centre.

respectively. Subalgebras Γ satisfying $\Gamma \subset \Gamma^\perp$ exist in every real form of the complex simple Lie algebras except the compact one, since for the compact real form the Cartan-Killing inner product is (negative) definite.

We note that for a given Γ the third condition and the ambiguity in choosing M can be concisely summarized by the (equivalent) statement that

$$M \in [\Gamma\,,\,\Gamma]^\perp / \Gamma^\perp\ . \tag{2.8}$$

The constraints defined by the zero element of this factor-space are in a sense trivial. It is clear that, for a subalgebra Γ such that $\Gamma \subset \Gamma^\perp$, the above factor-space contains non-zero elements if and only if $[\Gamma, \Gamma] \neq \Gamma$. Actually this is always so because $\Gamma \subset \Gamma^\perp$ implies that Γ is a *solvable* subalgebra of $\mathcal{G}$. To prove this, we first note that if Γ is not solvable then, by Levi's theorem [33], it contains a semi-simple subalgebra, in which one can find either an $so(3, R)$ or an $sl(2, R)$ subalgebra. From this one sees that there exists at least one generator λ of Γ for which the operator ad_λ is diagonalizable with real eigenvalues. It cannot be that all eigenvalues of ad_λ are 0 since $\mathcal{G}$ is a simple Lie algebra, and from this one gets that $\langle \lambda\,,\,\lambda\rangle \neq 0$, which contradicts $\Gamma \subset \Gamma^\perp$. Therefore one can conclude that Γ is necessarily a solvable subalgebra of $\mathcal{G}$.

The second condition in (2.6) can be satisfied for example by assuming that every $\gamma \in \Gamma$ is a nilpotent element of $\mathcal{G}$. This is true in the concrete instances of the reduction studied in Chapters 3 and 4. We note that in this case Γ is actually a nilpotent Lie algebra, by Engel's theorem [33]. However, the nilpotency of Γ is not necessary for satisfying $\Gamma \subset \Gamma^\perp$. In fact, a solvable but not nilpotent Γ can be found in Appendix A.

The current components constrained in (2.3) are the infinitesimal generators of the KM transformations corresponding to the subalgebra Γ, which act on the KM phase space as

$$J(x) \longrightarrow e^{a^i(x^+)\gamma_i}\, J(x)\, e^{-a^i(x^+)\gamma_i} + (e^{a^i(x^+)\gamma_i})'\, e^{-a^i(x^+)\gamma_i}\ , \tag{2.9}$$

where the $a^i(x^+)$ are parameter functions and there is a summation over some basis γ_i of Γ. Of course, the first class conditions are equivalent to the statement that the constraint surface, consisting of currents of the form (2.2), is left invariant by the above transformations. From the point of view of the reduced theory, these transformations are to be regarded as gauge transformations, which means that the reduced phase space can be identified as the space of gauge orbits in the constraint surface. Taking this into account, we shall often refer to Γ as the gauge algebra of the reduction.

We next discuss a sufficient condition for the conformal invariance of the constraints. We assume that $M \notin \Gamma^\perp$ from now on. The standard conformal symmetry generated by the Sugawara Virasoro density $L_{\rm KM}(x)$ is then broken by the constraints (2.3), since they set some component of the current, which has spin 1, to a non-zero constant. The idea is to circumvent this apparent violation of conformal invariance by changing the standard action of the conformal group on the KM phase space to one which does leave the constraint surface invariant. One can try to generate the new conformal action by changing the usual KM Virasoro density to the new Virasoro density

$$L_H(x) = L_{\rm KM}(x) - \langle H, J'(x)\rangle, \tag{2.10}$$

where H is some element of $\mathcal{G}$. The conformal action generated by $L_H(x)$ operates on the KM phase space as

$$\begin{aligned} \delta_{f,H}\, J(x) &\equiv - \int dy^1\, f(y^+)\, \{L_H(y)\,,\, J(x)\} \\ &= f(x^+) J'(x) + f'(x^+)\big(J(x) + [H, J(x)]\big) + f''(x^+) H\ , \end{aligned} \tag{2.11}$$

for any parameter function $f(x^+)$, corresponding to the conformal coordinate transformation $\delta_f\, x^+ = -f(x^+)$. In particular, $j(x)$ in (2.2) transforms under this new conformal action according to

$$\delta_{f,H}\, j(x) = f(x^+) j'(x) + f''(x^+) H + f'(x^+)\big(j(x) + [H, j(x)] + ([H,M] + M)\big), \tag{2.12}$$

and our condition is that this variation should be in $\Gamma^\perp$, which means that this conformal action preserves the constraint surface. From (2.12), one sees that this is equivalent to having the following relations:

$$H \in \Gamma^\perp, \qquad [H, \Gamma^\perp] \subset \Gamma^\perp \qquad \text{and} \qquad ([H,M] + M) \in \Gamma^\perp\ . \tag{2.13}$$

In conclusion, the existence of an operator H satisfying these relations is a *sufficient condition* for the conformal invariance of the KM reduction obtained by imposing (2.3). The conditions in (2.13) are equivalent to $L_H(x)$ being a gauge invariant quantity, inducing a corresponding conformal action on the reduced phase space. Obviously, the second relation in (2.13) is equivalent to

$$[H, \Gamma] \subset \Gamma\ . \tag{2.14}$$

An element $H \in \mathcal{G}$ is called *diagonalizable* if the linear operator ${\rm ad}_H$ possesses a complete set of eigenvectors in $\mathcal{G}$. By the eigenspaces of ${\rm ad}_H$, such an element defines a grading of $\mathcal{G}$, and below we shall refer to a diagonalizable element as a *grading operator* of $\mathcal{G}$. In the examples we study later, conformal invariance will be ensured by the existence of a grading operator subject to (2.13).

If H is a grading operator satisfying (2.13) then it is always possible to shift M by some element of $\Gamma^\perp$ (i.e., without changing the physics) so that the new M satisfies

$$[H, M] = -M \ , \tag{2.15}$$

instead of the last condition in (2.13). It is also clear that if H is a grading operator then one can take graded bases in Γ and $\Gamma^\perp$, since these are invariant subspaces under ad_H. On re-inserting (2.15) into (2.12) it then follows that all components of $j(x)$ are primary fields with respect to the conformal action generated by $L_H(x)$, with the exception of the H-component, which also survives the constraints according to the first condition in (2.13).

As an example, let us now consider some arbitrary grading operator H and denote by $\mathcal{G}_m$ the eigensubspace corresponding to the eigenvalue m of ad_H. Then the graded subalgebra $\mathcal{G}_{\geq n}$, which is defined to be the direct sum of the subspaces $\mathcal{G}_m$ for all $m \geq n$, will qualify as a gauge algebra Γ for any $n > 0$ from the spectrum of ad_H. In this case $\Gamma^\perp = \mathcal{G}_{>-n}$ and the factor space $[\Gamma, \Gamma]^\perp / \Gamma^\perp$, which is the space of the allowed M's, can be represented as the direct sum of $\mathcal{G}_{-n}$ and that graded subspace of $\mathcal{G}_{<-n}$ which is orthogonal to $[\Gamma, \Gamma]$. It is easy to see that one obtains *conformally invariant* first class constraints by choosing M to be any *graded element* from this factor space. Indeed, if the grade of M is $-m$ then $L_{H/m}$ yields a Virasoro density weakly commuting with the corresponding constraints.

In summary, in this section we have seen that one can associate a first class system of KM constraints to any pair (Γ,M) subject to (2.6) by requiring the constrained current to take the form (2.2), and that the conformal invariance of this system of constraints is guaranteed if one can find an operator H such that the triple (Γ,M,H) satisfies the conditions in (2.13).

2.2. Lagrangean realization of the Hamiltonian reduction

We shall exhibit here a gauged WZNW theory providing the Lagrangean realization of those Hamiltonian reductions of the WZNW theory which can be defined by imposing first class constraints of the type (2.3) on the KM currents J and $\tilde{J}$ of the theory. It should be noted that, in the rest of this chapter, we do not assume that the constraints are conformally invariant.

To define the WZNW reduction, we can choose left and right constraints completely independently. We shall denote the pairs consisting of an appropriate subalgebra and a constant matrix corresponding to the left and right constraints as (Γ, M) and $(\tilde{\Gamma}, -\tilde{M})$, respectively. The reduced theory is obtained by first constraining the WZNW phase space by setting

$$\phi_i = \langle \gamma_i \, , \, J \rangle - \langle \gamma_i \, , \, M \rangle = 0, \qquad \text{and} \qquad \tilde{\phi}_i = -\langle \tilde{\gamma}_i \, , \, \tilde{J} \rangle - \langle \tilde{\gamma}_i \, , \, \tilde{M} \rangle = 0, \tag{2.16}$$

where the γ_i and the $\tilde{\gamma}_i$ form bases of Γ and $\tilde{\Gamma}$, respectively, and then factorizing the constraint surface by the canonical transformations generated by these constraints. One can apply this reduction either to the usual canonical phase space or to the space of solutions of the classical field equation. These are equivalent procedures since the two spaces in question are isomorphic. For later purpose we note that the constraints generate the following *chiral* gauge transformations on the space of solutions:

$$g(x^+, x^-) \longrightarrow e^{\gamma(x^+)} \cdot g(x^+, x^-) \cdot e^{-\tilde{\gamma}(x^-)} \; , \tag{2.17}$$

where $\gamma(x^+)$ and $\tilde{\gamma}(x^-)$ are arbitrary Γ and $\tilde{\Gamma}$ valued functions.

For completeness, we wish to mention here how the above way of reducing the WZNW theory fits into the general theory of Hamiltonian (symplectic) reduction of symmetries [34]. In general, the Hamiltonian reduction is obtained by setting the phase space functions generating the symmetry transformations through Poisson bracket (in other words, the components of the momentum map) to some constant values. The reduced phase space results by factorizing this constraint surface by the subgroup of the symmetry group respecting the constraints. The symmetry group we consider is the left $\times$ right KM group generated by $\Gamma \times \tilde{\Gamma}$ and our Hamiltonian reduction is special in the sense that the full symmetry group preserves the constraints. Of course, the latter fact is just a reformulation of the first-classness of our constraints.

We now come to the main point of the section, which is that the reduced WZNW theory, defined in the above by using the Hamiltonian picture, can be identified as the gauge invariant content of a corresponding gauged WZNW theory. This gauged WZNW interpretation of the reduction was pointed out in the concrete case of the WZNW $\rightarrow$ standard Toda reduction in [13], and we below generalize that construction to the present situation.

The gauged WZNW theory we are interested in is given by the following action functional:

$$I(g, A_-, A_+) \equiv S_{\rm WZ}(g) + \int d^2x \left(\langle A_-, \partial_+ g g^{-1} - M \rangle + \langle A_+, g^{-1}\partial_- g - \tilde{M} \rangle + \langle A_-, g A_+ g^{-1} \rangle \right), \tag{2.18}$$

where the gauge fields $A_-(x)$ and $A_+(x)$ vary in Γ and $\tilde{\Gamma}$, respectively. The main property of this action is that it is invariant under the following *non-chiral* gauge transformations:

$$g \to \alpha g \tilde{\alpha}^{-1}; \quad A_- \to \alpha A_- \alpha^{-1} + \alpha \partial_- \alpha^{-1}; \quad A_+ \to \tilde{\alpha} A_+ \tilde{\alpha}^{-1} + (\partial_+ \tilde{\alpha}) \tilde{\alpha}^{-1} , \tag{2.19a}$$

where

$$\alpha = e^{\gamma(x^+, x^-)} \qquad \text{and} \qquad \tilde{\alpha} = e^{\tilde{\gamma}(x^+, x^-)} , \tag{2.19b}$$

for any $\gamma(x^+, x^-) \in \Gamma$ and $\tilde{\gamma}(x^+, x^-) \in \tilde{\Gamma}$. The proof of the invariance of (2.18) under (2.19) can proceed along the same lines as for the special case in [13]. In the proof one rewrites $S_{\rm WZ}(\alpha g \tilde{\alpha}^{-1})$ by using the well-known Polyakov-Wiegmann identity [35], and in this step one uses the fact that the WZNW action *vanishes* for fields in the subgroups of G with Lie algebras Γ or $\tilde{\Gamma}$. This is an obvious consequence of the relations $\Gamma \subset \Gamma^\perp$ and $\tilde{\Gamma} \subset \tilde{\Gamma}^\perp$. The other crucial point is that the terms in (2.18) containing the constant matrices M and $\tilde{M}$ are separately invariant under (2.19). It is easy to see that this follows from the third condition in (2.6). For example, under an infinitesimal gauge transformation belonging to $\alpha \simeq 1 + \gamma$, the term $\langle A_-, M \rangle$ changes by

$$\delta \langle A_-, M \rangle = -\langle \partial_- \gamma, M \rangle + \omega_M(\gamma, A_-) , \tag{2.20}$$

which is a total divergence since the second term vanishes, as both A_- and γ are from Γ.

The Euler-Lagrange equation derived from (2.18) by varying g can be written equivalently as

$$\partial_-(\partial_+ g g^{-1} + g A_+ g^{-1}) + [A_-, \partial_+ g g^{-1} + g A_+ g^{-1}] + \partial_+ A_- = 0 , \tag{2.21a}$$

or

$$\partial_+(g^{-1}\partial_- g + g^{-1} A_- g) - [A_+, g^{-1}\partial_- g + g^{-1} A_- g] + \partial_- A_+ = 0 , \tag{2.21b}$$

and the field equations obtained by varying A_- and A_+ are given by

$$\langle \gamma , \partial_+ g g^{-1} + g A_+ g^{-1} - M \rangle = 0, \qquad \forall\, \gamma \in \Gamma , \tag{2.21c}$$

and

$$\langle \tilde{\gamma}\,,\, g^{-1}\partial_- g + g^{-1}A_- g - \tilde{M}\rangle = 0, \qquad \forall\, \tilde{\gamma} \in \tilde{\Gamma}\,, \tag{2.21d}$$

respectively. We now note that by making use of the gauge invariance, A_+ and A_- can be set equal to zero simultaneously. The important point for us is that, as is easy to see, in the $A_\pm = 0$ gauge one recovers from (2.21) both the field equations (1.3) of the WZNW theory and the constraints (2.16). Furthermore, one sees that setting $A_\pm$ to zero is not a complete gauge fixing, the residual gauge transformations are exactly the chiral gauge transformations of equation (2.17).

The above arguments tell us that the space of gauge orbits in the space of classical solutions of the gauged WZNW theory (2.18) can be naturally identified with the reduced phase space belonging to the Hamiltonian reduction of the WZNW theory determined by the first class constraints (2.16). It can be also shown that the Poisson bracket induced on the reduced phase space by the Hamiltonian reduction is the same as the one determined by the gauged WZNW action (2.18). In summary, we see that the gauged WZNW theory (2.18) provides a natural Lagrangean implementation of the WZNW reduction.

2.3. Effective field theories from left-right dual reductions

The aim of this section is to describe the effective field equations and action functionals for an important class of the reduced WZNW theories. This class of theories is obtained by making the assumption that the left and right gauge algebras Γ and $\tilde{\Gamma}$ are *dual to each other* with respect to the Cartan-Killing form, which means that one can choose bases $\gamma_i \in \Gamma$ and $\tilde{\gamma}_j \in \tilde{\Gamma}$ so that

$$\langle \gamma_i, \tilde{\gamma}_j\rangle = \delta_{ij}\ . \tag{2.22}$$

This *technical* assumption allows for having a simple general algorithm for disentangling the constraints:

$$\phi_i = \langle \gamma_i,\, \partial_+ g\, g^{-1} - M\rangle = 0, \quad \text{and} \quad \tilde{\phi}_i = \langle \tilde{\gamma}_i,\, g^{-1}\partial_- g - \tilde{M}\rangle = 0, \tag{2.23}$$

which define the reduction. We shall comment on the physical meaning of the assumption at the end of the section, here we only point out that it holds, e.g., if one chooses Γ and

$\tilde{\Gamma}$ to be the images of each other under a Cartan involution* of the underlying simple Lie algebra.

For concreteness, let us consider the maximally non-compact real form which can be defined as the real span of a Chevalley basis H_i, $E_{\pm\alpha}$ of the corresponding complex Lie algebra $\mathcal{G}_c$, and in the case of the classical series A_n, B_n, C_n and D_n is given by $sl(n+1,R)$, $so(n,n+1,R)$, $sp(2n,R)$ and $so(n,n,R)$, respectively. In this case the Cartan involution is $(-1)\times$ transpose, operating on the Chevalley basis according to

$$H_i \longrightarrow -H_i \qquad E_{\pm\alpha} \longrightarrow -E_{\mp\alpha} \ . \tag{2.24}$$

It is obvious that $\langle v\,,\, v^t\rangle > 0$ for any non-zero $v \in \mathcal{G}$ and from this one sees that Γ^t is dual to Γ with respect to the Cartan-Killing form, i.e., (2.22) holds for $\tilde{\Gamma} = \Gamma^t$. It should also be mentioned that there is a Cartan involution for every non-compact real form of the complex simple Lie algebras, as explained in detail in [36].

Equation (2.22) implies that the left and right gauge algebras do not intersect, and thus we can consider a direct sum decomposition of $\mathcal{G}$ of the form

$$\mathcal{G} = \Gamma + \mathcal{B} + \tilde{\Gamma}\ , \tag{2.25a}$$

where $\mathcal{B}$ is some linear subspace of $\mathcal{G}$. Here $\mathcal{B}$ is in principle an arbitrary complementary space to $(\Gamma + \tilde{\Gamma})$ in $\mathcal{G}$, but one can always make the choice

$$\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp}\ , \tag{2.25b}$$

which is natural in the sense that the Cartan-Killing form is non-degenerate on this $\mathcal{B}$. Choosing $\mathcal{B}$ according to (2.25b) is especially well-suited in the case of the parity invariant effective theories discussed at the end of the section. We note that it might also be convenient if one can take the space $\mathcal{B}$ to be a subalgebra of $\mathcal{G}$, but this is not necessary for our arguments and is not always possible either.

We can associate a 'generalized Gauss decomposition' of the group G to the direct sum decomposition (2.25), which is the main tool of our analysis. By 'Gauss decomposing' an element $g \in G$ according to (2.25), we mean writing it in the form

$$g = a\cdot b\cdot c\,, \quad \text{with} \quad a = e^{\gamma}, \quad b = e^{\beta} \quad \text{and} \quad c = e^{\tilde{\gamma}}, \tag{2.26}$$

* A Cartan involution σ of the simple Lie algebra $\mathcal{G}$ is an automorphism for which $\sigma^2 = 1$ and $\langle v, \sigma(v)\rangle < 0$ for any non-zero element v of $\mathcal{G}$.

where γ, β and $\tilde{\gamma}$ are from the respective subspaces in (2.25).

There is a neighbourhood of the identity in G consisting of elements which allow a unique decomposition of this sort, and in this neighbourhood the pieces a, b and c can be extracted from g by algebraic operations. (Actually it is also possible to define b as a product of exponentials corresponding to subspaces of $\mathcal{B}$, and we shall make use of this freedom later, in Chapter 4.) We make the assumption that every G-valued field we encounter is decomposable as g in (2.26). It is easily seen that in this 'Gauss decomposable sector' the components of $b(x^+, x^-)$ provide a complete set of gauge invariant local fields, which are the local fields of the reduced theory we are after. Below we explain how to solve the constraints (2.23) in the Gauss decomposable sector of the WZNW theory. More exactly, for our method to work, we restrict ourselves to considering those fields which vary in such a Gauss decomposable neighbourhood of the identity where the matrix

$$V_{ij}(b) = \langle \gamma_i, b\tilde{\gamma}_j b^{-1} \rangle \tag{2.27}$$

is invertible. Due to the assumptions, the analysis given in the following yields a *local* description of the reduced theories. It is clear that for a global description one should use patches on G obtained by multiplying out the Gauss decomposable neighbourhood of the identity, but we do not deal with this issue here.

First we derive the field equation of the reduced theory by implementing the constraints directly in the WZNW field equation $\partial_-(\partial_+ g g^{-1}) = 0$. (This is allowed since the WZNW dynamics leaves the constraint surface invariant, i.e., the WZNW Hamiltonian weakly commutes with the constraints.) By inserting the Gauss decomposition of g into (2.23) and making use of the constraints being first class, the constraint equations can be rewritten as

$$\begin{aligned} &\langle \gamma_i,\, \partial_+ b b^{-1} + b(\partial_+ c c^{-1}) b^{-1} - M \rangle = 0\,, \\ &\langle \tilde{\gamma}_i,\, b^{-1}\partial_- b + b^{-1}(a^{-1}\partial_- a) b - \tilde{M} \rangle = 0\,. \end{aligned} \tag{2.28}$$

With the help of the inverse of $V_{ij}(b)$ in (2.27), one can solve these equations for $\partial_+ c c^{-1}$ and $a^{-1}\partial_- a$ in terms of b,

$$\partial_+ c c^{-1} = b^{-1} T(b) b, \qquad \text{and} \qquad a^{-1}\partial_- a = b\tilde{T}(b) b^{-1}, \tag{2.29a}$$

where

$$\begin{aligned} T(b) &= \sum_{ij} V_{ij}^{-1}(b) \langle \gamma_j,\, M - \partial_+ b b^{-1} \rangle b\tilde{\gamma}_i b^{-1}, \\ \tilde{T}(b) &= \sum_{ij} V_{ij}^{-1}(b) \langle \tilde{\gamma}_i,\, \tilde{M} - b^{-1}\partial_- b \rangle b^{-1}\gamma_j b\,. \end{aligned} \tag{2.29b}$$

It is easy to obtain the effective field equation for the field $b(x^+, x^-)$ by using this explicit form of the constraints. This can be achieved for example by noting that, by applying the operator $\mathrm{Ad}_{a^{-1}}$ to equation (1.9) (i.e., by conjugating it by a^{-1}) the WZNW field equation can be written in the form

$$[\partial_+ - \mathcal{A}_+ \, , \, \partial_- - \mathcal{A}_-] = 0 \tag{2.30}$$

with

$$\mathcal{A}_+ = \partial_+ b\, b^{-1} + b(\partial_+ c c^{-1}) b^{-1} \quad \text{and} \quad \mathcal{A}_- = -a^{-1}\partial_- a \ . \tag{2.31}$$

Thus, by inserting the constraints (2.29) into the above form of the WZNW equation, we see that the field equation of the reduced theory is the zero curvature condition of the following Lax potential:

$$\mathcal{A}_+(b) = \partial_+ b\, b^{-1} + T(b) \quad \text{and} \quad \mathcal{A}_-(b) = -b\tilde{T}(b) b^{-1} \ . \tag{2.32}$$

More explicitly, the effective field equation reads

$$\partial_-(\partial_+ b b^{-1}) + [b\tilde{T}(b)b^{-1}, \, T(b)] + \partial_- T(b) + b(\partial_+ \tilde{T}(b))b^{-1} = 0. \tag{2.33}$$

The expression on the left-hand-side of (2.33) in general varies in the full space $\mathcal{G}$, but not all the components represent independent equations. The number of the independent equations is the number of the independent components of the WZNW field equation minus the number of the constraints in (2.23), since the constraints automatically imply the corresponding components of the WZNW equation. Thus there are exactly as many independent equations in (2.33) as the number of the reduced degrees of freedom. In fact, the *independent field equations* can be obtained by taking the Cartan-Killing inner product of (2.33) with a basis of the linear space $\mathcal{B}$ in (2.25), and the inner product of (2.33) with the γ_i and the $\tilde{\gamma}_i$ vanishes as a consequence of the constraints in (2.23) together with the independent field equations. To see this one first recalls that the left-hand-side of (2.33) is, upon imposing the constaints, equivalent to $a^{-1}(\partial_- J)a$. Thus the inner product of this with Γ, and similarly that of $c(\partial_+ \tilde{J})c^{-1}$ with $\tilde{\Gamma}$, vanishes as a consequence of the constraints. From this, by using the identity $a^{-1}(\partial_- J)a = -bc(\partial_+ \tilde{J})c^{-1}b^{-1}$, one can conclude that the inner product of $a^{-1}(\partial_- J)a$ with $\tilde{\Gamma}$ also vanishes as a consequence of the constraints and the independent field equations.

At this point we would like to mention certain special cases when the above equations simplify. First we note that if one has

$$[\mathcal{B} \, , \, \Gamma] \subset \Gamma \qquad \text{and} \qquad [\mathcal{B} \, , \, \tilde{\Gamma}] \subset \tilde{\Gamma} \ , \tag{2.34}$$

then

$$T(b) = M - \pi_{\tilde{\Gamma}}(\partial_+ b b^{-1}) \quad \text{and} \quad \tilde{T}(b) = \tilde{M} - \pi_{\Gamma}(b^{-1}\partial_- b) \ , \tag{2.35}$$

where we introduced the operators

$$\pi_{\Gamma} = \sum_i |\gamma_i\rangle\langle\tilde{\gamma}_i| \qquad \text{and} \qquad \pi_{\tilde{\Gamma}} = \sum_i |\tilde{\gamma}_i\rangle\langle\gamma_i| \ , \tag{2.36}$$

which project onto the spaces Γ and $\tilde{\Gamma}$, and assumed that $M \in \tilde{\Gamma}$ and $\tilde{M} \in \Gamma$. (The latter assumption can be done without loss of generality due to the duality condition (2.22)). One obtains (2.35) from (2.29) by taking into account that in this case $V_{ij}(b)$ in (2.27) is the matrix of the operator Ad_b acting on $\tilde{\Gamma}$, and thus the inverse is given by $\mathrm{Ad}_{b^{-1}}$. The nicest possible situation occurs when $\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp}$ is a *subalgebra* of $\mathcal{G}$ and also satisfies (2.34). In this case one simply has $T = M$ and $\tilde{T} = \tilde{M}$ and thus (2.33) simplifies to

$$\partial_-(\partial_+ b b^{-1}) + [b\tilde{M}b^{-1} \, , \, M] = 0 \ . \tag{2.37}$$

The derivative term is now an element of $\mathcal{B}$ and by combining the above assumptions with the first class conditions $[M, \Gamma] \subset \Gamma^{\perp}$ and $[\tilde{M}, \tilde{\Gamma}] \subset \tilde{\Gamma}^{\perp}$ one sees that the commutator term in (2.37) also varies in $\mathcal{B}$, which ensures the consistency of this equation.

The effective field equation (2.33) is in general a non-linear equation for the field $b(x^+, x^-)$, and we can give a procedure which can in principle be used for producing its *general solution*. We are going to do this by making use of the fact that the space of solutions of the reduced theory is the space of the constrained WZNW solutions factorized by the chiral gauge transformations, according to equation (2.17). Thus the idea is to find the general solution of the effective field equation by first parametrizing, in terms of arbitrary chiral functions, those WZNW solutions which satisfy the constraints (2.23), and then extracting the b-part of those WZNW solutions by algebraic operations. In other words, we propose to derive the general solution of (2.33) by looking at the origin of this equation, instead of its explicit form.

To be more concrete, one can start the construction of the general solution by first Gauss-decomposing the chiral factors of the general WZNW solution $g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-)$ as

$$g_L(x^+) = a_L(x^+) \cdot b_L(x^+) \cdot c_L(x^+), \quad g_R(x^-) = a_R(x^-) \cdot b_R(x^-) \cdot c_R(x^-). \tag{2.38}$$

Then the constraint equations (2.23) become

$$\partial_+ c_L c_L^{-1} = b_L^{-1} T(b_L) b_L \quad \text{and} \quad a_R^{-1} \partial_- a_R = b_R \tilde{T}(b_R) b_R^{-1} \ . \tag{2.39}$$

In addition to the the purely algebraic problems of computing the quantities T and $\tilde{T}$ and extracting b from $g = g_L \cdot g_R = a \cdot b \cdot c$, these first order systems of ordinary differential equations are all one has to solve to produce the general solution of the effective field equation. If this can be done by quadrature then the effective field equation is also integrable by quadrature. In general, one can proceed by trying to solve (2.39) for the functions $c_L(x^+)$ and $a_R(x^-)$ in terms of the arbitrary 'input functions' $b_L(x^+)$ and $b_R(x^-)$. Clearly, this involves only a *finite number of integrations whenever the gauge algebras* Γ *and* $\tilde{\Gamma}$ *consist of nilpotent elements of* $\mathcal{G}$. Thus in this case (2.33) is exactly integrable, i.e., its general solution can be obtained by quadrature.

We note that in concrete cases some other choice of input functions, instead of the chiral b's, might prove more convenient for finding the general solutions of the systems of first order equations on g_L and g_R given in (2.39) (see for instance the derivation of the general solution of the Liouville equation given in [12]).

It is natural to ask for the action functional underlying the effective field theory obtained by imposing the constraints (2.23) on the WZNW theory. In fact, the effective action is given by the following formula:

$$I_{\rm eff}(b) = S_{\rm WZ}(b) - \int d^2x\, \langle b\tilde{T}(b)b^{-1}\,,\, T(b)\rangle. \tag{2.40}$$

One can derive the following condition for the extremum of this action:

$$\langle \delta b b^{-1}, \partial_-(\partial_+ b b^{-1}) + [b\tilde{T}(b)b^{-1},\, T(b)] + \partial_- T(b) + b(\partial_+\tilde{T}(b))b^{-1}\rangle = 0. \tag{2.41}$$

It is straightforward to compute this, the only thing to remember is that the objects $b\tilde{T}b^{-1}$ and $b^{-1}Tb$ introduced in (2.29) vary in the gauge algebras Γ and $\tilde{\Gamma}$. The arbitrary variation of $b(x)$ is determined by the arbitrary variation of $\beta(x) \in \mathcal{B}$, according to $b(x) = e^{\beta(x)}$, and thus we see from (2.41) that the Euler-Lagrange equation of the action (2.40) yields exactly the independent components of the effective field equation (2.33), which we obtained previously by imposing the constraints directly in the WZNW field equation.

The effective action given above can be derived from the gauged WZNW action $I(g, A_-, A_+)$ given in (2.18), by eliminating the gauge fields $A_\pm$ by means of their Euler-Lagrange equations (2.21c-d). By using the Gauss decomposition, these Euler-Lagrange equations become equivalent to the relations

$$a^{-1}D_- a = b\tilde{T}(b)b^{-1}\ , \qquad \text{and} \qquad cD_+ c^{-1} = -b^{-1}T(b)b\ , \tag{2.42}$$

where the quantities $T(b)$ and $\tilde{T}(b)$ are given by the expressions in (2.29b) and $D_\pm$ denotes the gauge covariant derivatives, $D_\pm = \partial_\pm \mp A_\pm$. Now we show that $I_{\rm eff}(b)$ in (2.40) can indeed be obtained by substituting the solution of (2.42) for $A_\pm$ back into $I(g, A_-, A_+)$ with $g = abc$. To this first we rewrite $I(abc, A_-, A_+)$ by using the Polyakov-Wiegmann identity [35] as

$$\begin{aligned} I(abc\,,\, A_-, A_+) = S_{\rm WZ}(b) - \int d^2x\, \Big(& \langle a^{-1}D_- a\,,\, b(cD_+c^{-1})b^{-1}\rangle \\ & + \langle b^{-1}\partial_- b\,,\, cD_+c^{-1}\rangle - \langle \partial_+ bb^{-1}\,,\, a^{-1}D_- a\rangle + \langle A_-, M\rangle + \langle A_+, \tilde{M}\rangle \Big). \end{aligned} \tag{2.43}$$

This equation can be regarded as the gauge covariant form of the Polyakov-Wiegmann identity, and all but the last two terms are manifestly gauge invariant. The effective action (2.40) is derived from (2.43) together with (2.42) by noting, for example, that $\langle \partial_- aa^{-1}\,,\, M\rangle$ is a total derivative, which follows from the facts that $a(x) \in e^\Gamma$ and $M \in [\Gamma\,,\,\Gamma]^\perp$, by (2.8).

Above we have used the field equations to eliminate the gauge fields from the gauged WZNW action (2.18) on the ground that A_- and A_+ are not dynamical fields, but 'Lagrange multiplier fields' implementing the constraints. However, it should be noted that without further assumptions the Euler-Lagrange equation of the action resulting from (2.18) by means of this elimination procedure *does not* always give the effective field equation, which can always be obtained directly from the WZNW field equation. One can see this on an example in which one imposes constraints *only on one of the chiral sectors* of the WZNW theory. From this point of view, the role of our assumption on the duality of the left and right gauge algebras is that it guarantees that the effective action underlying the effective field equation can be derived from $I(g, A_-, A_+)$ in the above manner. To end this discussion, we note that for $g = abc$ the non-degeneracy of $V_{ij}(b)$ in (2.27) is equivalent to the non-degeneracy of the quadratic expression $\langle A_-\,,\, gA_+g^{-1}\rangle$ in the components of $A_- = A^i_-\gamma_i$ and $A_+ = A^i_+\tilde{\gamma}_i$. This quadratic term enters into the gauged WZNW action given by (2.18), and its non-degeneracy is clearly important in the quantum theory, which we consider in Chapter 5.

We mentioned at the beginning of the section that, considering a maximally non-compact $\mathcal{G}$, one can make sure that the duality assumption expressed by (2.22) holds by choosing Γ and $\tilde{\Gamma}$ to be the *transposes* of each other. Here we point out that this particular left-right related choice of the gauge algebras can also be used to ensure the *parity invariance* of the effective field theory. To this first we notice that, in the case of a maximally non-compact connected Lie group G, the WZNW action $S_{\rm WZ}(g)$ is invariant under any of the following two 'parity transformations' $g \longrightarrow Pg$:

$$(P_1 g)(x^0, x^1) \equiv g^t(x^0, -x^1) \ , \quad \text{and} \quad (P_2 g)(x^0, x^1) \equiv g^{-1}(x^0, -x^1). \tag{2.44}$$

If one chooses $\tilde{\Gamma} = \Gamma^t$ and $\tilde{M} = M^t$ to define the WZNW reduction then the parity transformation P_1 simply interchanges the left and right constraints, ϕ and $\tilde{\phi}$ in (2.23), and thus the corresponding effective field theory is invariant under the parity P_1. The space $\mathcal{B} = (\Gamma + \tilde{\Gamma})^\perp$, i.e., the choice in (2.25b), is invariant under the transpose in this case, and thus the gauge invariant field b transforms in the same way under P_1 as g does in (2.44). Of course, the parity invariance can also be seen on the level of the gauged action $I(g, A_-, A_+)$. Namely, $I(g, A_-, A_+)$ is invariant under P_1 if one extends the definition in (2.44) to include the following parity transformation of the gauge fields:

$$(P_1 A_\pm)(x^0, x^1) \equiv A^t_\mp(x^0, -x^1) \ . \tag{2.45}$$

The P_1-invariant reduction procedure does not preserve the parity symmetry P_2, but it is possible to consider reductions preserving just P_2 instead of P_1. In fact, such reductions can be obtained by taking $\tilde{\Gamma} = \Gamma$ and $\tilde{M} = M$.

Finally, it is obvious that to construct parity invariant WZNW reductions in general, for some arbitrary but non-compact real form $\mathcal{G}$ of the complex simple Lie algebras, one can use $-\sigma$ instead of the transpose, where σ is a Cartan involution of $\mathcal{G}$.

3. Polynomiality in KM reductions and the $\mathcal{W}_S^\mathcal{G}$-algebras

In the previous chapter we described the conditions for (2.2) defining first class constraints and for $L_H(J)$ in (2.10) being a gauge invariant quantity on this constraint surface. It is clear that the KM Poisson brackets of the gauge invariant differential polynomials of the current always close on such polynomials and δ-distributions. The algebra of the gauge invariant differential polynomials is of special interest in the conformally invariant case when it is a polynomial extension of the Virasoro algebra. In Section 3.1 we shall give an additional condition on the triple (Γ, M, H) which allows one to construct out of the current in (2.2) a complete set of gauge invariant differential polynomials by means of a differential polynomial gauge fixing algorithm. We call the KM reduction polynomial if such a polynomial gauge fixing algorithm is available, and also call the corresponding gauges Drinfeld-Sokolov (DS) gauges, since our construction is a generalization of the one given in [5]. The KM Poisson bracket algebra of the gauge invariant differential polynomials becomes the Dirac bracket algebra of the current components in the DS gauges, which we consider in Section 3.2. The extended conformal algebra $\mathcal{W}_S^\mathcal{G}$ mentioned in the Introduction is especially interesting in that its primary field basis is manifest and given by the $sl(2)$ structure, as we shall see in Section 3.3. One of our main results is that we shall find here first class KM constraints underlying this algebra, such that they satisfy our sufficient condition for polynomiality. Thus we can represent $\mathcal{W}_S^\mathcal{G}$ as a KM Poisson bracket algebra of gauge invariant differential polynomials, which in principle allows for its quantization through the KM representation theory. The importance of the $\mathcal{W}_S^\mathcal{G}$-algebras is clearly demonstrated by the result of Section 3.4, where we show that the W_n^l-algebras of [26] can be interpreted as further reductions of particular $\mathcal{W}_S^\mathcal{G}$-algebras. This makes it possible to exhibit primary fields for the W_n^l-algebras and to describe their structure in detail in terms of the corresponding $\mathcal{W}_S^\mathcal{G}$-algebras, which is the subject of [37].

3.1. A sufficient condition for polynomiality

Let us suppose that (Γ, M, H) satisfy the previously given conditions, (2.6) and (2.13), for

$$J(x) = M + j(x) , \qquad j(x) \in \Gamma^\perp \tag{3.1}$$

describing the constraint surface of conformally invariant first class constraints, where H is a *grading operator* and M is subject to

$$[H, M] = -M\,, \qquad\qquad M \notin \Gamma^{\perp}\ . \tag{3.2}$$

Then, as we shall show, the following two additional conditions:

$$\Gamma \cap \mathcal{K}_M = \{0\}\ , \qquad \text{where} \qquad \mathcal{K}_M = \mathrm{Ker}(\mathrm{ad}_M)\ , \tag{3.3}$$

and

$$\Gamma^{\perp} \subset \mathcal{G}_{>-1}\ , \tag{3.4a}$$

allow for establishing *a differential polynomial gauge fixing algorithm* whereby one can construct out of $J(x)$ in (3.1) a complete set of gauge invariant differential polynomials.

Before proving this result, we discuss some consequences of the conditions, which we shall need later. In the present situation Γ, $\Gamma^{\perp}$ and $\mathcal{G}$ are graded by the eigenvalues of ad_H, and first we note that (3.4a) is *equivalent* to

$$\mathcal{G}_{\geq 1} \subset \Gamma\ . \tag{3.4b}$$

Indeed, this follows from the fact that the spaces $\mathcal{G}_h$ and $\mathcal{G}_{-h}$ are dual to each other with respect to the Cartan-Killing form, which is a consequence of its non-degeneracy and invariance under ad_H. Of course, here and below the grading is the one defined by H, and we note that $\mathcal{G}_{\pm 1}$ are non-trivial because of (3.2). The condition given by (3.4a) plays a technical role in our considerations, but perhaps it can be argued for also physically, on the basis that it ensures that the conformal weights of the primary field components of $j(x)$ in (3.1) are *non-negative* with respect to L_H (2.10). Second, let us observe that in our situation M satisfying (3.2) is *uniquely determined*, that is, there is no possibility of shifting it by elements from $\Gamma^{\perp}$, simply because there are no grade -1 elements in $\Gamma^{\perp}$, on account of (3.4a). Equation (3.3) means that the operator ad_M maps Γ into $\Gamma^{\perp}$ in an *injective* manner, and for this reason we refer to (3.3) as the *non-degeneracy condition.* Combining the non-degeneracy condition with (3.2), (3.4a) and (2.7) we see that our gauge algebra Γ can contain only *positive* grades:

$$\Gamma \subset \mathcal{G}_{>0}\ . \tag{3.5}$$

This implies that every $\gamma \in \Gamma$ is represented by a nilpotent operator in any finite dimensional representation of $\mathcal{G}$, and that

$$\mathcal{G}_{\geq 0} \subset \Gamma^{\perp}\ . \tag{3.6}$$

It follows from (3.2) that $[H, \mathcal{K}_M] \subset \mathcal{K}_M$, which is telling us that $\mathcal{K}_M$ is also graded, and we see from (3.3) and (3.4b) that

$$\mathcal{K}_M \subset \mathcal{G}_{<1} \ . \tag{3.7}$$

Finally, we wish to establish a certain relationship between the dimensions of $\mathcal{G}$ and $\mathcal{K}_M$. For this purpose we consider an arbitrary complementary space $\mathcal{T}_M$ to $\mathcal{K}_M$, defining a linear direct sum decomposition

$$\mathcal{G} = \mathcal{K}_M + \mathcal{T}_M \ . \tag{3.8}$$

It is easy to see that for the 2-form ω_M we have $\omega_M(\mathcal{K}_M, \mathcal{G}) = 0$, and the restriction of ω_M to $\mathcal{T}_M$ is a *symplectic* form, in other words:

$$\omega_M(\mathcal{T}_M, \mathcal{T}_M) \qquad \text{is} \quad \text{non}-\text{degenerate} \ . \tag{3.9}$$

(We note in passing that $\mathcal{T}_M$ can be identified with the tangent space at M to the coadjoint orbit of G through M, and in this picture ω_M becomes the Kirillov-Kostant symplectic form of the orbit [34].) The non-degeneracy condition (3.3) says that one can choose the space $\mathcal{T}_M$ in (3.8) in such a way that $\Gamma \subset \mathcal{T}_M$. One then obtains the inequality

$$\dim(\Gamma) \leq \frac{1}{2}\dim(\mathcal{T}_M) = \frac{1}{2}\big(\dim(\mathcal{G}) - \dim(\mathcal{K}_M)\big) \ , \tag{3.10}$$

where the factor $\frac{1}{2}$ arises since ω_M is a symplectic form on $\mathcal{T}_M$, which vanishes, by (2.6), on the subspace $\Gamma \subset \mathcal{T}_M$.

After the above clarification of the meaning of conditions (3.3) and (3.4), we now wish to show that they indeed allow for exhibiting a complete set of gauge invariant differential polynomials among the gauge invariant functions. Generalizing the arguments of [5,13,15], this will be achieved by demonstrating that an arbitrary current $J(x)$ subject to (3.1) *can be brought to a certain normal form by a unique gauge transformation which depends on $J(x)$ in a differential polynomial way.*

A normal form suitable for this purpose can be associated to any *graded* subspace $\Theta \subset \mathcal{G}$ which is *dual to* Γ *with respect to the* 2*-form* ω_M. Given such a space Θ, it is possible to choose bases γ_h^i and θ_k^j in Γ and Θ respectively such that

$$\omega_M(\gamma_h^l, \theta_k^i) = \delta_{il}\delta_{hk}, \tag{3.11}$$

where the subscript h on γ_h^l denotes the grade, and the indices i and l denote the additional labels which are necessary to specify the base vectors at fixed grade. It is to be noted that, by definition, the subsript k on elements $\theta_k^j \in \Theta$ does not denote the grade, which is $(1-k)$. The normal (or *reduced*) form corresponding to Θ is given by the following equation:

$$J_{\rm red}(x) = M + j_{\rm red}(x) \quad \text{where} \quad j_{\rm red}(x) \in \Gamma^\perp \cap \Theta^\perp \ . \tag{3.12}$$

In other words, the set of reduced currents is obtained by supplementing the first class constraints of equation (2.3) by the gauge fixing condition

$$\chi_\theta(x) = \langle J(x), \theta\rangle - \langle M, \theta\rangle = 0 \ , \qquad \forall \theta \in \Theta \ . \tag{3.13}$$

We call a gauge which can be obtained in the above manner a Drinfeld-Sokolov (DS) gauge. It is not hard to see that the space $\mathcal{V} = \Gamma^\perp \cap \Theta^\perp$ is a graded subspace of $\Gamma^\perp$ which is disjoint from the image of Γ under the operator ${\rm ad}_M$ and is in fact complementary to the image, i.e., one has

$$\Gamma^\perp = [M, \Gamma] + \mathcal{V} \ . \tag{3.14}$$

It also follows from the non-degeneracy condition (3.3) that any graded complement $\mathcal{V}$ in (3.14) can be obtained in the above manner, by means of using some Θ. Thus it is possible to define the DS normal form of the current directly in terms of a complementary space $\mathcal{V}$ as well, as has been done in special cases in [5,13,18].

As the first step in proving that any current in (3.1) is gauge equivalent to one in the DS gauge, let us consider the gauge transformation by $g_h(x^+) = \exp[\sum_l a_h^l(x^+)\gamma_h^l]$ for some fixed grade h. Suppressing the summation over l, it can be written as*

$$j(x) \to j^{g_h}(x) = e^{a_h\cdot\gamma_h}(j(x)+M)e^{-a_h\cdot\gamma_h} + (e^{a_h\cdot\gamma_h})' e^{-a_h\cdot\gamma_h} - M \ . \tag{3.15}$$

Taking the inner product of this equation with the basis vectors θ_k^i in (3.11) for all $k \leq h$, we see that there is no contribution from the derivative term. We also see that the only contribution from

$$e^{a_h\cdot\gamma_h} j(x) e^{-a_h\cdot\gamma_h} = j(x) + [a_h(x^+)\cdot\gamma_h,\, j(x)] + \dots \tag{3.16}$$

---

* Throughout the chapter, all equations involving gauge transformations, Poisson brackets, etc., are to be evaluated by using a fixed time, since they are all consequences of equation (2.1). By this convention, they are valid both on the canonical phase space and on the chiral KM phase space belonging to space of solutions of the theory.

is the one coming from the first term, since all commutators containing the elements γ_h^l drop out from the inner product in question as a consequence of the following crucial relation:

$$[\gamma_h^l, \theta_k^i] \in \Gamma, \quad \text{for} \quad k \leq h, \tag{3.17}$$

which follows from (3.4b) by noting that the grade of this commutator, $(1+h-k)$, is at least 1 for $k \leq h$. Taking these into account, and computing the contribution from those two terms in $j^{g_h}(x)$ which contain M by using (3.11), we obtain

$$\langle \theta_k^i, j^{g_h}(x) \rangle = \langle \theta_k^i, j(x) \rangle - a_h^i(x^+)\delta_{hk}, \quad \text{for all} \quad k \leq h. \tag{3.18}$$

We see from this equation that

$$\langle \theta_k^i, j(x) \rangle = 0 \quad \Longleftrightarrow \quad \langle \theta_k^i, j^{g_h}(x) \rangle = 0 \ , \quad \text{for} \quad k < h \ , \tag{3.19}$$

and

$$a_h^i(x^+) = \langle \theta_h^i, j(x) \rangle \quad \Rightarrow \quad \langle \theta_h^i, j^{g_h}(x) \rangle = 0 \ , \quad \text{for} \quad k = h. \tag{3.20}$$

These last two equations tell us that if the gauge-fixing condition $\langle \theta_k^i, j(x) \rangle = 0$ is satisfied for all $k < h$ then we can ensure that the same condition holds for $j^{g_h}(x)$ for the *extended range of indices* $k \leq h$, by choosing $a_h^i(x^+)$ to be $\langle \theta_h^i, j(x) \rangle$. From this it is easy to see that the DS gauge (3.13) can be reached by an iterative process of gauge transformations, and the gauge-parameters $a_h^i(x^+)$ are unique polynomials in the current at each stage of the iteration.

In more detail, let us write the general element $g(a(x^+)) \in e^\Gamma$ of the gauge group as a product in order of descending grades, i.e., as

$$g(a(x^+)) = g_{h_n} \cdot g_{h_{n-1}} \cdots g_{h_1}, \quad \text{with} \quad g_{h_i}(x^+) = e^{a_{h_i}(x^+) \cdot \gamma_{h_i}} \ , \tag{3.21a}$$

where

$$h_n > h_{n-1} > \ldots > h_1 \tag{3.21b}$$

is the list of grades occurring in Γ. Let us then insert this expression into

$$j \to j^g = g(j+M)g^{-1} + g'g^{-1} - M \ , \tag{3.22a}$$

and consider the condition

$$j^g(x) = j_{\rm red}(x) \ , \tag{3.22b}$$

with $j_{\rm red}(x)$ in (3.12), as an equation for the gauge-parameters $a_h(x^+)$. One sees from the above considerations that this equation is uniquely soluble for the components of the $a_h(x^+)$ and the solution is a differential polynomial in $j(x)$. This implies that the components of $j_{\rm red}(x)$ can also be uniquely computed from (3.22), and *the solution yields a complete set of gauge invariant differential polynomials of* $j(x)$, which establishes the required result. The above iterative procedure is in fact a convenient tool for computing the gauge invariant differential polynomials in practice [15]. We remark that, of course, any unique gauge fixing can be used to define gauge invariant quantities, but they are in general not polynomial, not even local in $j(x)$.

We also wish to note that an arbitrary linear subspace of $\mathcal{G}$ which is dual to $\mathcal{V}$ in (3.14) with respect to the Cartan-Killing form can be used in a natural way as the space of parameters for describing those current dependent KM transformations which preserve the DS gauge. In fact, it is possible to give an algorithm which computes the $\mathcal{W}$-algebra and its action on the other fields of the corresponding constrained WZNW theory by finding the gauge preserving KM transformations implementing the $\mathcal{W}$-transformations. This algorithm presupposes the existence of such gauge invariant differential polynomials which reduce to the current components in the DS gauge, which is ensured by the above gauge fixing algorithm, but it works without actually computing them. This issue is treated in detail in [13,18] in special cases, but the results given there apply also to the general situation investigated in the above.

3.2. The polynomiality of the Dirac bracket

It follows from the polynomiality of the gauge fixing that the components of the gauge fixed current $j_{\rm red}$ in (3.12) generate a differential polynomial algebra *under Dirac bracket*. In our proof of the polynomiality we actually only used that the graded subspace Θ of $\mathcal{G}$ is dual to the graded gauge algebra Γ with respect to ω_M and satisfies the condition

$$([\Theta\,,\,\Gamma])_{\geq 1}\subset\Gamma\ , \tag{3.23}$$

which is equivalent to the existence of the bases γ_h^l and θ_k^i satisfying (3.11) and (3.17). We have seen that this condition follows from (3.3) and (3.4), but it should be noted that it is a *more general condition*, since the converse is not true, as is shown by an example at the end of this section.

Below we wish to give a direct proof for the polynomiality of the Dirac bracket algebra belonging to the *second class* constraints:

$$c_\tau(x) = \langle \tau\,,\, J(x) - M\rangle = 0 \quad \text{where} \quad \tau \in \{\gamma_h^l\} \cup \{\theta_k^i\}\ . \tag{3.24}$$

The proof will shed a new light on the polynomiality condition. We note that for certain purposes second class constraints might be more natural to use than first class ones since in the second class formalism one directly deals with the physical fields. For example, the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra mentioned in the Introduction is very natural from the second class point of view and can be realized by starting with a number of different first class systems of constraints, as we shall see in the next section.

We first recall that, by definition, the Dirac bracket algebra of the reduced currents is

$$\begin{aligned}\{j^u_{\rm red}(x), j^v_{\rm red}(y)\}^* &= \{j^u_{\rm red}(x), j^v_{\rm red}(y)\}\\ &-\sum_{\mu\nu}\int dz^1dw^1\{j^u_{\rm red}(x), c_\mu(z)\}\Delta_{\mu\nu}(z,w)\{c_\nu(w), j^v_{\rm red}(y)\}\,,\end{aligned} \tag{3.25}$$

where, for any $u\in\mathcal{G}$, $j^u_{\rm red}(x) = \langle u, j_{\rm red}(x)\rangle$ is to be substituted by $\langle u, J(x)-M\rangle$ under the KM Poisson bracket, and $\Delta_{\mu\nu}(z,w)$ is the inverse of the kernel

$$D_{\mu\nu}(z,w) = \{c_\mu(z), c_\nu(w)\}\ , \tag{3.26}$$

in the sense that (on the constraint surface)

$$\sum_\nu \int dx^1\Delta_{\mu\nu}(z,x)D_{\nu\sigma}(x,w) = \delta_{\mu\sigma}\delta(z^1-w^1). \tag{3.27}$$

To establish the polynomiality of the Dirac bracket, it is useful to consider the *matrix differential operator* $D_{\mu\nu}(z)$ defined by the kernel $D_{\mu\nu}(z,w)$ in the usual way, i.e.,

$$\sum_\nu D_{\mu\nu}(z)f_\nu(z) = \sum_\nu\int dw^1 D_{\mu\nu}(z,w)f_\nu(w)\ , \tag{3.28}$$

for a vector of smooth functions $f_\nu(z)$, which are periodic in z^1. From the structure of the constraints in (3.24), $c_\tau = (\phi_\gamma, \chi_\theta)$, one sees that $D_{\mu\nu}(z)$ is a first order differential operator possessing the following block structure

$$D_{\mu\nu} = \begin{pmatrix} D_{\gamma\tilde\gamma} & D_{\gamma\theta}\\ D_{\tilde\theta\tilde\gamma} & D_{\tilde\theta\theta}\end{pmatrix} = \begin{pmatrix} 0 & E\\ -E^\dagger & F\end{pmatrix}, \tag{3.29}$$

where $E^\dagger$ is the formal Hermitian conjugate of the matrix E, $(E^\dagger)_{\theta\gamma} = (E_{\gamma\theta})^\dagger$. It is clear that the Dirac bracket in (3.25) is a differential polynomial in $j_{\rm red}(x)$ and $\delta(x^1-y^1)$

whenever the inverse operator $D^{-1}(z)$, whose kernel is $\Delta_{\mu\nu}(z,w)$, is a differential operator whose coefficients are differential polynomials in $j_{\rm red}(z)$. On the other hand, we see from (3.29) that the operator D is invertible if and only if its block E is invertible, and in that case the inverse takes the form

$$(D^{-1})_{\mu\nu} = \begin{pmatrix} (E^\dagger)^{-1}FE^{-1} & -(E^\dagger)^{-1} \\ E^{-1} & 0 \end{pmatrix} . \tag{3.30}$$

Since $E(z)$ and $F(z)$ are polynomial (even linear) in $j_{\rm red}(z)$ and in ∂_z and the inverse of $F(z)$ does not occur in $D^{-1}(z)$, it follows that $D^{-1}(z)$ is a polynomial differential operator if and only if $E^{-1}(z)$ is a polynomial differential operator.

To show that E^{-1} exists and is a polynomial differential operator we note that in terms of the basis of $(\Gamma+\Theta)$ in (3.24) the matrix E is given explicitly by the following formula:

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{mn} + \langle [\gamma_h^m,\theta_k^n], j_{\rm red}(z)\rangle + \langle \gamma_h^m,\theta_k^n\rangle \partial_z \ . \tag{3.31}$$

The crucial point is that, by the grading and the property in (3.17), we have

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{nm}\ , \qquad \text{for} \qquad k\leq h\ . \tag{3.32}$$

The matrix E has a block structure labelled by the (block) row and (block) column indices h and k, respectively, and (3.32) means that the blocks in the diagonal of E are unit matrices and the blocks below the diagonal vanish. In other words, E is of the form $E=1+\varepsilon$, where ε is a strictly upper triangular matrix. It is clear that such a matrix differential operator is polynomially invertible, namely by a *finite series* of the form

$$E^{-1} = 1-\varepsilon+\varepsilon^2+\ldots+(-1)^N\varepsilon^N\ , \qquad (\varepsilon^{N+1}=0), \tag{3.33}$$

which finishes our proof of the polynomiality of the Dirac bracket in (3.25). One can use the arguments in the above proof to set up an algorithm for actually computing the Dirac bracket. The proof also shows that the polynomiality of the Dirac bracket is guaranteed whenever E is of the form $(1+\varepsilon)$ with ε being *nilpotent as a matrix.* In our case this was ensured by a special grading assumption, and it appears an interesting question whether polynomial reductions can be obtained at all without using some grading structure.

The zero block occurs in D^{-1} in (3.30) because the second class constraints originate from the gauge fixing of first class ones. We note that the presence of this zero block implies that the Dirac brackets of the gauge invariant quantities coincide with their original Poisson brackets, namely one sees this from the formula of the Dirac bracket by keeping in mind that the gauge invariant quantities weakly commute with the first class constraints.

Finally, we want to show that condition (3.23) is weaker than (3.3-4). This is best seen by considering an example. To this let now $\mathcal{G}$ be the maximally non-compact real form of a complex simple Lie algebra. If $\{M_-, M_0, M_+\}$ is the principal $sl(2)$ embedding in $\mathcal{G}$, with commutation rules as in (3.34) below, we simply choose the one-dimensional gauge algebra $\Gamma \equiv \{M_+\}$ and take $M \equiv M_-$. The ω_M-dual to M_+ can be taken to be $\theta = M_0$, and then (3.23) holds. To show that conditions (3.3-4) cannot be satisfied, we prove that a grading operator H for which $[H, M_-] = -M_-$ and $\mathcal{G}^H_{\geq 1} \subset \Gamma$, does not exist. First of all, $[H, M_-] = -M_-$ and $\langle M_-, M_+\rangle \neq 0$ imply $[H, M_+] = M_+$, and thus $\Gamma^H_{\geq 1} = \{M_+\}$. Furthermore, writing $H = (M_0 + \Delta)$, we find from $[H, M_\pm] = \pm M_\pm$ that Δ must b e an $sl(2)$ singlet in the adjoint of $\mathcal{G}$. However, in the case of the principal $sl(2)$ embedding, there is no such singlet in the adjoint, and hence $H = M_0$. But then the condition $\mathcal{G}^{M_0}_{\geq 1} \subset \Gamma$ is not fulfilled.

3.3. First class constraints for the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras

Let $\mathcal{S} = \{M_-\,,\, M_0\,,\, M_+\}$ be an $sl(2)$ subalgebra of the simple Lie algebra $\mathcal{G}$:

$$[M_0, M_\pm] = \pm M_\pm\ , \qquad [M_+, M_-] = 2M_0\ . \tag{3.34}$$

We argued in the Introduction that it is natural to associate an extended conformal algebra, denoted as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, to any such $sl(2)$ embedding [16,18]. Namely, we defined the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra to be the Dirac bracket algebra generated by the components of the constrained KM current of the the following special form:

$$J_{\mathrm{red}}(x) = M_- + j_{\mathrm{red}}(x)\ , \qquad \text{with} \qquad j_{\mathrm{red}}(x) \in \mathrm{Ker}(\mathrm{ad}_{M_+})\ , \tag{3.35}$$

which means that $j_{\mathrm{red}}(x)$ is a linear combination of the $sl(2)$ highest weight states in the adjoint of $\mathcal{G}$. This definition is indeed natural in the sense that the conformal properties are manifest, since, as we shall see below, with the exception of the M_+-component the spin s component of $j_{\mathrm{red}}(x)$ turns out to be a primary field of conformal weight $(s+1)$ with respect to L_{M_0}. Before showing this, we shall construct here *first class* KM constraints underlying the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra, which will be used in Chapter 4 to construct generalized Toda theories which realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras as their chiral algebras. We expect the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras to play an important organizing role in describing the (primary field content of) conformally invariant KM reductions in general, and shall give arguments in favour of this idea later.

We wish to find a gauge algebra Γ for which the triple $(\Gamma, H = M_0, M = M_-)$ satisfies our sufficient conditions for polynomiality and (3.35) represents a DS gauge for the corresponding conformally invariant first class constraints. We start by noticing that the dimension of such a Γ has to satisfy the relation

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_+}) = \dim \mathcal{W}_{\mathcal{S}}^{\mathcal{G}} = \dim \mathcal{G} - 2\dim \Gamma \ . \tag{3.36}$$

From this, since the kernels of $\mathrm{ad}_{M_\pm}$ are of equal dimension, we obtain that

$$\dim \Gamma = \frac{1}{2}\dim \mathcal{G} - \frac{1}{2}\dim \mathrm{Ker}(\mathrm{ad}_{M_-}) \ , \tag{3.37}$$

which means by (3.10) that we are looking for a Γ of *maximal* dimension. By the representation theory of $sl(2)$, the above equality is equivalent to

$$\dim \Gamma = \dim \mathcal{G}_{\geq 1} + \frac{1}{2}\dim \mathcal{G}_{\frac{1}{2}} \ , \tag{3.38}$$

where the grading is by the, in general half-integral, eigenvalues of ad_{M_0}. We also know, (3.4b) and (3.5), that for our purpose we have to choose the graded Lie subalgebra Γ of $\mathcal{G}$ in such a way that $\mathcal{G}_{\geq 1} \subset \Gamma \subset \mathcal{G}_{>0}$. Observe that the non-degeneracy condition (3.3) is automatically satisfied for any such Γ since in the present case $\mathrm{Ker}(\mathrm{ad}_{M_-}) \subset \mathcal{G}_{\leq 0}$, and $M_0 \in \Gamma^\perp$ is also ensured, which guarantees the conformal invariance, see (2.13).

It is obvious from the above that in the special case of an *integral* $sl(2)$ subalgebra, for which $\mathcal{G}_{\frac{1}{2}}$ is empty, one can simply take

$$\Gamma = \mathcal{G}_{\geq 1} \ . \tag{3.39}$$

For grading reasons,

$$\omega_{M_-}(\mathcal{G}_{\geq 1}, \mathcal{G}_{\geq 1}) = 0 \tag{3.40}$$

holds, and thus one indeed obtains first class constraints in this way.

One sees from (3.38) that for finding the gauge algebra in the non-trivial case of a *half-integral* $sl(2)$ subalgebra, one should somehow add half of $\mathcal{G}_{\frac{1}{2}}$ to $\mathcal{G}_{\geq 1}$, in order to have the correct dimension. The key observation for defining the required *halving* of $\mathcal{G}_{\frac{1}{2}}$ consists in noticing that the restriction of the 2-form ω_{M_-} to $\mathcal{G}_{\frac{1}{2}}$ is non-degenerate. This can be seen as a consequence of (3.9), but is also easy to verify directly. By the well known Darboux normal form of symplectic forms [34], there exists a (non-unique) direct sum decomposition

$$\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}} \tag{3.41}$$

such that ω_{M_-} vanishes on the subspaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$ separately. The spaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, which are the analogues of the usual momentum and coordinate subspaces of the phase space in analytic mechanics, are of equal dimension and dual to each other with respect to ω_{M_-}. The point is that the first-classness conditions in (2.6) are satisfied if we define the gauge algebra to be

$$\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} \ , \tag{3.42}$$

by using *any symplectic halving* of the above kind. It is obvious from the construction that the first class constraints,

$$J(x) = M_- + j(x) \qquad \text{with} \qquad j(x) \in \Gamma^{\perp} \ , \tag{3.43}$$

obtained by using Γ in (3.42) satisfy the sufficient conditions for polynomiality given in Section 3.1. With this Γ we have

$$\Gamma^{\perp} = \mathcal{G}_{\geq 0} + \mathcal{Q}_{-\frac{1}{2}} \ , \tag{3.44a}$$

where $\mathcal{Q}_{-\frac{1}{2}}$ is the subspace of $\mathcal{G}_{-\frac{1}{2}}$ given by

$$\mathcal{Q}_{-\frac{1}{2}} = [M_-, \mathcal{P}_{\frac{1}{2}}] \ . \tag{3.44b}$$

By combining (3.42) and (3.44) one also easily verifies the following direct sum decomposition:

$$\Gamma^{\perp} = [M_-, \Gamma] + \mathrm{Ker}(\mathrm{ad}_{M_+}) \ , \tag{3.45}$$

which is just (3.14) with $\mathcal{V} = \mathrm{Ker}(\mathrm{ad}_{M_+})$. This means that (3.35) is indeed nothing but the equation of a particular DS gauge for the first class constraints in (3.43), as required. This special DS gauge is called the *highest weight gauge* [13]. Similarly as for any DS gauge, there exists therefore a basis of gauge invariant differential polynomials of the current in (3.43) such that the base elements reduce to the components of $j_{\mathrm{red}}(x)$ in (3.35) by the gauge fixing. The KM Poisson bracket algebra of these gauge invariant differential polynomials is clearly identical to the Dirac bracket algebra of the corresponding current components, and we can thus realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra as a KM Poisson bracket algebra of gauge invariant differential polynomials.

The second class constraints defining the highest weight gauge (3.35) are natural in the sense that in this case τ in (3.24) runs over the basis of the space $\mathcal{T}_{M_-} = [M_+\,,\,\mathcal{G}]$ which is a natural complement of $\mathcal{K}_{M_-} = \mathrm{Ker}(\mathrm{ad}_{M_-})$ in $\mathcal{G}$, eq. (3.8).

In the second class formalism, the conformal action generated by L_{M_0} on the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra is given by the following formula:

$$\delta^*_{f,M_0}\, j_{\mathrm{red}}(x) \equiv -\int dy^1\, f(y^+)\, \{L_{M_0}(y)\,,\, j_{\mathrm{red}}(x)\}^*\ , \tag{3.46}$$

where the parameter function $f(x^+)$ refers to the conformal coordinate transformation $\delta_f\, x^+ = -f(x^+)$, cf. (2.11), and $j_{\mathrm{red}}(x)$ is to be substituted by $J(x)-M_-$ when evaluating the KM Poisson brackets entering into (3.46), like in (3.25). To actually evaluate (3.46), we first replace L_{M_0} by the object

$$L_{\mathrm{mod}}(x) = L_{M_0}(x) - \frac{1}{2}\langle M_+\,,\, J''(x)\rangle\ , \tag{3.47}$$

which is allowed under the Dirac bracket since the difference (the second term) vanishes upon imposing the constraints. The crucial point to notice is that L_{mod} weakly commutes with *all* the constraints defining (3.35) (not only with the first class ones) under the KM Poisson bracket. This implies that with L_{mod} the Dirac bracket in (3.46) is in fact identical to the original KM Poisson bracket and by this observation we easily obtain

$$\delta^*_{f,M_0}\, j_{\mathrm{red}}(x) = f(x^+)\, j'_{\mathrm{red}}(x) + f'(x^+)\big(j_{\mathrm{red}}(x) + [M_0, j_{\mathrm{red}}(x)]\big) - \frac{1}{2} f'''(x^+) M_+. \tag{3.48}$$

This proves that, with the exception of the M_+-component, the $sl(2)$ highest weight components of $j_{\mathrm{red}}(x)$ in (3.35) transform as conformal primary fields, whereby the conformal content of $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ is determined by the decomposition of the adjoint of $\mathcal{G}$ under $\mathcal{S}$ in the aforementioned manner. We end this discussion by noting that in the highest weight gauge $L_{M_0}(x)$ becomes a linear combination of the M_+-component of $j_{\mathrm{red}}(x)$ and a quadratic expression in the components corresponding to the singlets of $\mathcal{S}$ in $\mathcal{G}$. From this we see that $L_{M_0}(x)$ and the primary fields corresponding to the $sl(2)$ highest weight states give a basis for the differential polynomials contained in $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, which is thus indeed a (classical) $\mathcal{W}$-algebra in the sense of the general idea in [20].

In the above we proposed a 'halving procedure' for finding *purely first class* constraints for which $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ appears as the algebra of the corresponding gauge invariant differential polynomials. We now wish to clarify the relationship between our method and the construction in a recent paper by Bais *et al* [16], where the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra has been described, in the special case of $\mathcal{G} = sl(n)$, by using a different method. We recall that the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra has been constructed in [16] by adding to the first class constraints defined by the pair $(\mathcal{G}_{\geq 1}, M_-)$ the *second class* constraints

$$\langle u\,,\, J(x)\rangle = 0\ , \qquad \text{for} \quad \forall\ u \in \mathcal{G}_{\frac{1}{2}}\ . \tag{3.49}$$

Clearly, we recover these constraints by first imposing our complete set of first class constraint belonging to (Γ, M_-) with Γ in (3.42), and then partially fixing the gauge by imposing the condition

$$\langle u\,,\, J(x)\rangle = 0\ , \qquad \text{for} \quad \forall\ u \in \mathcal{Q}_{\frac{1}{2}}\ . \tag{3.50}$$

One of the advantages of our construction is that by using only first class KM constraints it is easy to construct generalized Toda theories which possess $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ as their chiral algebra, for any $sl(2)$ subalgebra, namely by using our general method of WZNW reductions. This will be elaborated in the next chapter. We note that in [16] the authors were actually also led to replacing the original constraints by a first class system of constraints, in order to be able to consider the BRST quantization of the theory. For this purpose they introduced unphysical 'auxiliary fields' and thus constructed first class constraints in an extended phase space. However, in that construction one has to check that the auxiliary fields finally disappear from the physical quantities. Another important advantage of our halving procedure is that it renders the use of any such auxiliary fields completely unnecessary, since one can start by imposing a complete system of first class constraints on the KM phase space from the very beginning. We study some aspects of the BRST quantization in Chapter 5, and we shall see that the Virasoro central charge given in [16] agrees with the one computed by taking our first class constraints as the starting point.

The first class constraints leading to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ are not unique, for example one can consider an arbitrary halving in (3.41) to define Γ. We conjecture that these $\mathcal{W}$-algebras always occur under certain natural assumptions on the constraints. To be more exact, let us suppose that we have conformally invariant first class constraints determined by the pair (Γ, M_-) where M_- is a *nilpotent* matrix and the *non-degeneracy* condition (3.3) holds together with equation (3.37). By the Jacobson-Morozov theorem, it is possible to extend the nilpotent generator M_- to an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$. It is also worth noting that the conjugacy class of $\mathcal{S}$ under the automorphism group of $\mathcal{G}$ is uniquely determined by the conjugacy class of the nilpotent element M_-. For this and other questions concerning the theory of $sl(2)$ embeddings into semi-simple Lie algebras the reader may consult refs. [32,33,38,39]. We expect that the above assumptions on (Γ, M_-) are sufficient for the existence of a complete set of gauge invariant differential polynomials and their algebra is isomorphic to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, where $M_- \in \mathcal{S}$. We are not yet able to prove this conjecture in general, but below we wish to sketch the proof in an important special case which illustrates the idea.

Let us assume that we have conformally invariant first class constraints described by (Γ, M_-, H) subject to the sufficient conditions for polynomiality given in Section 3.1, such that H is an *integral grading operator* of $\mathcal{G}$. We note that these are exactly the assumptions satisfied by the constraints in the non-degenerate case of the generalized Toda theories associated to integral gradings [18]. In this case equation (3.37) is actually automatically satisfied as a consequence of the non-degeneracy condition (3.3). One can also show that it is possible to find an $sl(2)$ algebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which in addition to $[H, M_-] = -M_-$ one has

$$[H, M_0] = 0 \qquad \text{and} \qquad [H, M_+] = M_+ \ , \tag{3.51}$$

and that for this $sl(2)$ algebra the relation

$$\mathrm{Ker}(\mathrm{ad}_{M_+}) \subset \mathcal{G}_{\geq 0}^{H} \tag{3.52}$$

holds, where the superscript indicates that the grading is defined by H. For the $sl(2)$ subject to (3.51) the latter property is in fact equivalent to $\mathrm{Ker}(\mathrm{ad}_{M_-}) \subset \mathcal{G}_{\leq 0}^{H}$, which is just the non-degeneracy condition as in our case $\Gamma = \mathcal{G}_{>0}^{H}$. The proof of these statements is given in Appendix B.

We introduce a definition at this point, which will be used in the rest of the paper. Namely, we call an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ an *H-compatible $sl(2)$* from now on if there exists an integral grading operator H such that $[H, M_\pm] = \pm M_\pm$ is satisfied together with the non-degeneracy condition. The non-degeneracy condition can be expressed in various equivalent forms, it can be given for example as the relation in (3.52), and its (equivalent) analogue for M_-.

Turning back to the problem at hand, we now point out that by using the H-compatible $sl(2)$ we have the following direct sum decomposition of $\Gamma^\perp = \mathcal{G}_{\geq 0}^{H}$:

$$\mathcal{G}_{\geq 0}^{H} = [M_-, \mathcal{G}_{>0}^{H}] + \mathrm{Ker}(\mathrm{ad}_{M_+}). \tag{3.53}$$

This means that the set of currents of the form (3.35) represents a DS gauge for the present first class constraints. This implies the required result, that is that the $\mathcal{W}$-algebra belonging to the constraints defined by $\Gamma = \mathcal{G}_{>0}^{H}$ together with a non-degenerate

M_- is isomorphic to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ with $M_- \in \mathcal{S}$. In this example both $L_H(x)$ and $L_{M_0}(x)$ are gauge invariant differential polynomials. Although the spectrum of ad_H is *integral* by assumption, in some cases the H-compatible $sl(2)$ is embedded into $\mathcal{G}$ in a *half-integral* manner, i.e., the spectrum of ad_{M_0} can be half-integral in certain cases. We shall return to this point later. We further note that in general it is clearly impossible to build such an $sl(2)$ out of M_- for which H would play the role of M_0. It is possible to prove that in those cases there is no full set of primary fields with repect to L_H which would complete this Virasoro density to a generating set of the corresponding differential polynomial $\mathcal{W}$-algebra. We have seen that such a conformal basis is manifest for $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, which seems to indicate that in the present situation the conformal structure defined by the $sl(2)$, L_{M_0}, is preferred in comparison to the one defined by L_H.

We also would like to mention an interesting general fact about the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras, which will be used in the next section. Let us consider the decomposition of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$. In general, we shall find singlet states and they span a Lie subalgebra in the Lie subalgebra $\mathrm{Ker}(\mathrm{ad}_{M_+})$ of $\mathcal{G}$. Let us denote this zero spin subalgebra as $\mathcal{Z}$. It is easy to see that we have the semi-direct sum decomposition

$$\mathrm{Ker}(\mathrm{ad}_{M_+}) = \mathcal{Z} + \mathcal{R}, \qquad [\mathcal{Z},\mathcal{R}] \subset \mathcal{R}, \qquad [\mathcal{Z},\mathcal{Z}] \subset \mathcal{Z}, \tag{3.54}$$

where $\mathcal{R}$ is the linear space spanned by the rest of the highest weight states, which have non-zero spin. It is not hard to prove that the subalgebra of the original KM algebra which belongs to $\mathcal{Z}$, survives the reduction to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$. In other words, the Dirac brackets of the $\mathcal{Z}$-components of the highest weight gauge current, j_{red} in (3.35), coincide with their original KM Poisson brackets, given by (2.1). Furthermore, this $\mathcal{Z}$ KM subalgebra acts on the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra by the corresponding original KM transformations, which preserve the highest weight gauge:

$$J_{\mathrm{red}}(x) \rightarrow e^{a^i(x^+)\zeta_i}\, J_{\mathrm{red}}(x)\, e^{-a^i(x^+)\zeta_i} + (e^{a^i(x^+)\zeta_i})'\, e^{-a^i(x^+)\zeta_i}, \tag{3.55}$$

where the ζ_i form a basis of $\mathcal{Z}$. In particular, one sees that the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra inherites the semi-direct sum structure given by (3.54) [16]. The point we wish to make is that it is possible to *further reduce* the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra by applying the general method of conformally invariant KM reductions to the present $\mathcal{Z}$ KM symmetry. In principle, one can generate a huge number of new conformally invariant systems out of the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras in this way, i.e., by applying conformally invariant constraints to their singlet KM subalgebras. For example, if one can find a subalgebra of $\mathcal{Z}$ on which the Cartan-Killing form of $\mathcal{G}$ vanishes, then one can consider the obviously conformally invariant reduction obtained by constraining the corresponding components of $j_{\rm red}$ in (3.35) to zero. We do not explore these 'secondary' reductions of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras in this paper. However, their potential importance will be highlighted by the example of the next section.

Finally, we note that, for a half-integral $sl(2)$, one can consider (instead of using Γ in (3.42)) also those conformally invariant first class constraints which are defined by the triple (Γ, M_0, M_-) with any graded Γ for which $\mathcal{G}_{\geq 1} \subset \Gamma \subset (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}})$. The polynomiality conditions of Section 3.1 are clearly satisfied with any such non-maximal Γ, and the corresponding extended conformal algebras are in a sense between the KM and $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras.

3.4. The $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ interpretation of the W_n^l-algebras

The W_n^l-algebras are certain conformally invariant reductions of the $sl(n,R)$ KM algebra introduced by Bershadsky [26] using a mixed set of *first class and second class* constraints. It is known [16] that the simplest non-trivial case W_3^2, originally proposed by Polyakov [27], coincides with the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra belonging to the highest root $sl(2)$ of $sl(3,R)$. The purpose of this section is to understand whether or not these reduced KM systems fit into our framework, which is based on using *purely first class* constraints, and to uncover their possible connection with the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras in the general case. (In this section, $\mathcal{G} = sl(n,R)$.) In fact, we shall construct here purely first class KM constraints leading to the W_n^l-algebras. The construction will demonstrate that the W_n^l-algebras can in general be identified as *further reductions of particular* $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$*-algebras*. The secondary reduction process is obtained by means of the singlet KM subalgebras of the relevant $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras, in the manner mentioned in the previous section.

By definition [26], the KM reduction yielding the W_n^l-algebra is obtained by constraining the current to take the following form:

$$J_{\rm B}(x) = M_- + j_{\rm B}(x), \qquad j_{\rm B}(x) \in \Delta^\perp, \tag{3.56}$$

where Δ denotes the set of all strictly upper triangular $n \times n$ matrices and

$$M_- = e_{l+1,1} + e_{l+2,2} + \ldots + e_{n,n-l}, \tag{3.57}$$

the e's being the standard $sl(n,R)$ generators ($l \leq n-1$), i.e., M_- has 1's all along the l-th slanted line below the diagonal. The current in (3.56) corresponds to imposing the constraints $\phi_\delta(x) = 0$ for all $\delta \in \Delta$, like in (2.3). Generally, these constraints comprise first and second class parts, where the first class part is the one belonging to the subalgebra $\mathcal{D}$ of Δ defined by the relation $\omega_{M_-}(\mathcal{D}, \Delta) = 0$, (see (2.4)). The second class part belongs to the complementary space, $\mathcal{C}$, of $\mathcal{D}$ in Δ. In fact, for $l = 1$ the constraints are the usual first class ones which yield the standard $\mathcal{W}$-algebras, but the second class part is non-empty for $l > 1$. The above KM reduction is so constructed that it is conformally invariant, since the constraints weakly commute with the Virasoro density $L_{H_l}(x)$, see (2.10), where $H_l = \frac{1}{l} H_1$ and H_1 is the standard grading operator of $sl(n, R)$, for which $[H_1\, ,\, e_{ik}] = (k - i) e_{ik}$.

We start our construction by extending the nilpotent generator M_- in (3.57) to an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$. In fact, parametrizing $n = ml + r$ with $m = \left[\frac{n}{l}\right]$ and $0 \leq r < l$, we can take

$$M_0 = \mathrm{diag}\Big(\overbrace{\frac{m}{2}, \cdots,}^{r \text{ times}} \overbrace{\frac{m-1}{2}, \cdots,}^{(l-r) \text{ times}} \cdots, \overbrace{-\frac{m}{2}, \cdots}^{r \text{ times}}\Big), \tag{3.58}$$

where the mutiplicities, r and $(l - r)$, occur alternately and end with r. The meaning of this formula is that the fundamental of $sl(n, R)$ branches into l irreducible representations under $\mathcal{S}$, r of spin $\frac{m}{2}$ and $l - r$ of spin $\frac{m-1}{2}$. The explicit form of M_+ is a certain linear combination of the e_{ik}'s with $(k - i) = l$, which is straightforward to compute.

We describe next the first and the second class parts of the constraints in (3.56) in more detail by using the grading defined by M_0. We observe first that in terms of this grading the space Δ admits the decomposition

$$\Delta = \Delta_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{G}_1 + \mathcal{G}_{>1}\ . \tag{3.59}$$

From this and the definition of ω_{M_-}, the subalgebra $\mathcal{D}$ comprising the first class part can also be decomposed into

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{G}_{>1}\ , \tag{3.60}$$

where

$$\mathcal{D}_0 = \mathrm{Ker}\,(\mathrm{ad}_{M_-}) \cap \Delta_0 \tag{3.61}$$

is the set of the $sl(2)$ singlets in Δ, and $\mathcal{D}_1$ is a subspace of $\mathcal{G}_1$ which we do not need to specify. By combining (3.59) and (3.60), we see that the complementary space $\mathcal{C}$, to which the second class part belongs, has the structure

$$\mathcal{C} = \mathcal{Q}_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{P}_1\ , \tag{3.62}$$

where the subspace $\mathcal{Q}_0$ is complementary to $\mathcal{D}_0$ in Δ_0, and $\mathcal{P}_1$ is complementary to $\mathcal{D}_1$ in $\mathcal{G}_1$. The 2-form ω_{M_-} is non-degenerate on $\mathcal{C}$ by construction, and this implies by the grading that the spaces $\mathcal{Q}_0$ and $\mathcal{P}_1$ are symplectically conjugate to each other, which is reflected by the notation.

We shall construct a gauge algebra, Γ, so that Bershadsky's constraints will be recovered by a partial gauge fixing from the first class ones belonging to Γ. As a generalization of the halving procedure of the previous section, we take the following ansatz:

$$\Gamma = \mathcal{D} + \mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1 \ , \tag{3.63}$$

where $\mathcal{P}_{\frac{1}{2}}$ is defined by means of some symplectic halving $\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}}$, like in (3.41). It is important to notice that this equation can be recasted into

$$\Gamma = \mathcal{D}_0 + \mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} \ , \tag{3.64}$$

which would be just the familiar formula (3.42) if $\mathcal{D}_0$ was not here. By using (3.57) and (3.58), $\mathcal{D}_0$ can be identified as the set of $n \times n$ block-diagonal matrices, σ, of the following form:

$$\sigma = \text{block-diag}\{\Sigma_0, \sigma_0, \Sigma_0, \ldots.., \Sigma_0, \sigma_0, \Sigma_0\}, \tag{3.65}$$

where the Σ_0's and the σ_0's are identical copies of strictly upper triangular $r \times r$ and $(l-r) \times (l-r)$ matrices respectively. This implies that

$$\dim \mathcal{D}_0 = \frac{1}{4}[l(l-2) + (l-2r)^2] \ , \tag{3.66}$$

which shows that $\mathcal{D}_0$ is non-empty except when $l = 2$, $r = 1$, which is the case of W_n^2 with $n = $ odd. The fact that $\mathcal{D}_0$ is in general non-empty gives us a trouble at this stage, namely, we have now no guarantee that the above Γ is actually a *subalgebra* of $\mathcal{G}$. By using the grading and the fact that $\mathcal{D}_0$ is a subalgebra, we see that Γ in (3.64) becomes a subalgebra if and only if

$$[\mathcal{D}_0 \, , \, \mathcal{P}_{\frac{1}{2}}] \subset \mathcal{P}_{\frac{1}{2}} . \tag{3.67}$$

We next show that it is possible to find such a 'good halving' of $\mathcal{G}_{\frac{1}{2}}$ for which $\mathcal{P}_{\frac{1}{2}}$ satisfies (3.67).

For this purpose, we use yet another grading here. This grading is provided by using the particular diagonal matrix, $H \in \mathcal{G}$, which we construct out of M_0 in (3.58) by first adding $\frac{1}{2}$ to its half-integral eigenvalues, and then substracting a multiple of the unit matrix so as to make the result traceless. In the adjoint representation, we then have

$\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on the tensors, and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm 1/2$ on the spinors. We notice from this that the H-grading is an integral grading. In fact, the relationship between the two gradings allows us to define a good halving of $\mathcal{G}_{\frac{1}{2}}$ as follows:

$$\mathcal{P}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}^H_1 , \qquad \text{and} \qquad \mathcal{Q}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}^H_0 . \tag{3.68}$$

Since M_- is of grade -1 with respect to both gradings, the spaces given by (3.68) clearly yield a sympectic halving of $\mathcal{G}_{\frac{1}{2}}$ with respect to ω_{M_-}. That this is a good halving, i.e., it ensures the condition (3.67), can also be seen easily by observing that $\mathcal{D}_0$ has grade 0 in the H-grading, too. Thus we obtain the required subalgebra Γ of $\mathcal{G}$ by using this particular $\mathcal{P}_{\frac{1}{2}}$ in (3.64).

Let us consider now the first class constraints corresponding to the above constructed gauge algebra Γ, $\phi_\gamma(x) = 0$ for $\gamma \in \Gamma$, which bring the current into the form

$$J_\Gamma(x) = M_- + j_\Gamma(x) , \qquad j_\Gamma(x) \in \Gamma^\perp . \tag{3.69}$$

It is easy to verify that the original constraint surface (3.56) can be recovered from (3.69) by a partial gauge fixing in such a way that the residual gauge transformations are exactly the ones belonging to the space $\mathcal{D}$. In fact, this is achieved by fixing the gauge freedom corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1)$ of Γ, (3.63), by imposing the partial gauge fixing condition

$$\phi_{q_i}(x) = 0 , \qquad q_i \in (\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}}), \tag{3.70}$$

where the q_i form a basis of the space $(\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}})$ and the ϕ_q's are defined like in (2.3). This implies that the reduced phase space defined by the constraints in (3.69) is the same as the one determined by the original constraints (3.56). In conclusion, our purely first class constraints, (3.69), have the same physical content as Bershadsky's original mixed set of constraints, (3.56).

Finally, we give the relationship between Bershadsky's W^l_n-algebras and the $sl(2)$ systems. Having seen that the reduced KM phase spaces carrying the W^l_n-algebras can be realized by starting from the first class constraints in (3.69), it follows from (3.64) that the W^l_n-algebras coincide with particular $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras if and only if the space $\mathcal{D}_0$ is empty, i.e., for W^2_n with $n =$ odd. In order to establish the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ interpretation of W^l_n in the general case, we point out that the reduced phase space can be reached from (3.69) by means of the following two step process based on the $sl(2)$ structure. Namely, one can proceed by first fixing the gauge freedom corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1})$ of Γ, and then fixing the rest of the gauge freedom. Clearly, the constraint surface resulting in the first step is the same as the one obtained by putting to zero those components of the highest weight gauge current representing $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ which correspond to $\mathcal{D}_0$. The final reduced phase space is obtained in the second step by fixing the gauge freedom generated by the constraints belonging to $\mathcal{D}_0$, which we have seen to be the space of the upper triangular singlets of $\mathcal{S}$. Thus we can conclude that W^l_n can be regarded as a further reduction of the corresponding $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, where the 'secondary reduction' is of the type mentioned at the end of Section 3.3. One can exhibit primary field bases for the W^l_n-algebras and describe their structure in detail in terms of the underlying $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras by further analysing the secondary reduction, but this is outside the scope of the present paper, see [37].

4. Generalized Toda theories

Let us remind ourselves that, as has been detailed in the Introduction, the standard conformal Toda field theories can be naturally regarded as reduced WZNW theories, and as a consequence these theories possess the chiral algebras $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$ as their canonical symmetries, where $\mathcal{S}$ is the principal $sl(2)$ subalgebra of the maximally non-compact real Lie algebra $\mathcal{G}$. It is natural to seek for WZNW reductions leading to effective field theories which would realize $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$ as their chiral algebras *for any $sl(2)$ subalgebra $\mathcal{S}$ of any simple real Lie algebra*. The main purpose of this chapter is to obtain, by combining the results of sections 2.3 and 3.3, *generalized Toda theories* meeting the above requirement in the non-trivial case of the *half-integral* $sl(2)$ subalgebras of the simple Lie algebras. Before turning to describing these new theories, next we briefly recall the main features of those generalized Toda theories, associated to the *integral gradings* of the simple Lie algebras, which have been studied before [3,4,14-18]. The simplicity of the latter theories will motivate some subsequent developments.

4.1. Generalized Toda theories associated with integral gradings

The WZNW reduction leading to the generalized Toda theories in question is set up by considering an integral grading operator H of $\mathcal{G}$, and taking the special case

$$\Gamma = \mathcal{G}_{\geq 1}^{H} \qquad \text{and} \qquad \tilde{\Gamma} = \mathcal{G}_{\leq -1}^{H} \ , \tag{4.1}$$

and any non-zero

$$M \in \mathcal{G}_{-1}^{H} \qquad \text{and} \qquad \tilde{M} \in \mathcal{G}_{1}^{H} \ , \tag{4.2}$$

in the general construction given in Section 2.3. We note that by an integral grading operator $H \in \mathcal{G}$ we mean a diagonalizable element whose spectrum in the adjoint of $\mathcal{G}$ consists of integers and contains ± 1, and that $\mathcal{G}_n^H$ denotes the grade n subspace defined by H. In the present case $\mathcal{B}$ in (2.25b) is the subalgebra $\mathcal{G}_0^H$ of $\mathcal{G}$, and, because of the grading structure, the properties expressed by equation (2.34) hold. Thus the effective field equation reads as (2.37) and the corresponding action is given by the simple formula

$$I_{\text{eff}}^{H}(b) = S_{\text{WZ}}(b) - \int d^2x \, \langle b\tilde{M}b^{-1}, M \rangle \ , \tag{4.3}$$

where the field b varies in the little group G_0^H of H in G.

Generalized, or non-Abelian, Toda theories of this type have been first investigated by Leznov and Saveliev [1,3], who defined these theories by postulating their Lax potential,

$$\mathcal{A}_+^H = \partial_+ b \cdot b^{-1} + M \;, \qquad \mathcal{A}_-^H = -b\tilde{M}b^{-1} \;, \tag{4.4}$$

which they obtained by considering the problem that if one requires a $\mathcal{G}$-valued pure-gauge Lax potential to take some special form, then the consistency of the system of equations coming from the zero curvature condition becomes a non-trivial problem. In comparison, we have seen in Section 2.3 that in the WZNW framework the Lax potential originates from the chiral zero curvature equation (1.9), and the consistency and the integrability of the effective theory arising from the reduction is automatic.

It was shown in [3,4,16] in the special case when H, M and $\tilde{M}$ are taken to be the standard generators of an integral $sl(2)$ subalgebra of $\mathcal{G}$, that the non-Abelian Toda equation allows for conserved chiral currents underlying its exact integrability. These currents then generate chiral $\mathcal{W}$-algebras of the type $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, for integrally embedded $sl(2)$'s.

By means of the argument given in Section 3.3, we can establish the structure of the chiral algebras of a wider class of non-Abelian Toda systems [18]. Namely, we see that if M and $\tilde{M}$ in (4.2) satisfy the non-degeneracy conditions

$$\mathrm{Ker}(\mathrm{ad}_M) \cap \mathcal{G}_{\geq 1}^H = \{0\} \qquad \text{and} \qquad \mathrm{Ker}(\mathrm{ad}_{\tilde{M}}) \cap \mathcal{G}_{\leq -1}^H = \{0\} \;, \tag{4.5}$$

then the left$\times$right chiral algebra of the corresponding generalized Toda theory is isomorphic to $\mathcal{W}_{\mathcal{S}_-}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}_+}^{\mathcal{G}}$, where $\mathcal{S}_-$ ($\mathcal{S}_+$) is an $sl(2)$ subalgebra of $\mathcal{G}$ containing the nilpotent generator M ($\tilde{M}$), respectively. The H-compatible $sl(2)$ algebras $\mathcal{S}_\pm$ occurring here are *not always integrally embedded* ones. Thus for certain *half-integral* $sl(2)$ algebras $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ can be realized in a generalized Toda theory of the type (4.3). As we would like to have generalized Toda theories which possess $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ as their symmetry algebra for an arbitrary $sl(2)$ subalgebra, we have to ask whether the theories given above are already enough for this purpose or not. This leads to the technical question as to whether for every half-integral $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ of $\mathcal{G}$ there exists an integral grading operator H such that $\mathcal{S}$ is an H-compatible $sl(2)$, in the sense introduced in Section 3.3. The answer to this question is negative, as proven in Appendix C, where the relationship between integral gradings and $sl(2)$ subalgebras is studied in detail. Thus we have to find new integrable conformal field theories for our purpose.

4.2. Generalized Toda theories for half-integral $sl(2)$ embeddings

In the following we exhibit a generalized Toda theory possessing the left $\times$ right chiral algebra $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ for an arbitrarily chosen half-integral $sl(2)$ subalgebra $\mathcal{S} = \{M_-\,,\, M_0\,,\, M_+\}$ of the arbitrary but non-compact simple real Lie algebra $\mathcal{G}$. Clearly, if one imposes first class constraints of the type described in Section 3.3 on the currents of the WZNW theory then the resulting effective field theory will have the required chiral algebra. We shall choose the left and right gauge algebras in such a way to be dual to each other with respect to the Cartan-Killing form.

Turning to the details, first we choose a direct sum decomposition of $\mathcal{G}_{\frac{1}{2}}$ of the type in (3.41), and then define the *induced decomposition* $\mathcal{G}_{-\frac{1}{2}} = \mathcal{P}_{-\frac{1}{2}} + \mathcal{Q}_{-\frac{1}{2}}$ to be given by the subspaces

$$\mathcal{Q}_{-\frac{1}{2}} \equiv \mathcal{P}_{\frac{1}{2}}^{\perp} \cap \mathcal{G}_{-\frac{1}{2}} = [M_-\,,\, \mathcal{P}_{\frac{1}{2}}] \quad \text{and} \quad \mathcal{P}_{-\frac{1}{2}} \equiv \mathcal{Q}_{\frac{1}{2}}^{\perp} \cap \mathcal{G}_{-\frac{1}{2}} = [M_-\,,\, \mathcal{Q}_{\frac{1}{2}}]\,. \tag{4.6}$$

It is easy to see that the 2-form ω_{M_+} vanishes on the above subspaces of $\mathcal{G}_{-\frac{1}{2}}$ as a consequence of the vanishing of ω_{M_-} on the corresponding subspaces of $\mathcal{G}_{\frac{1}{2}}$. Thus we can take the left and right gauge algebras to be

$$\Gamma = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}}) \quad \text{and} \quad \tilde{\Gamma} = (\mathcal{G}_{\leq -1} + \mathcal{P}_{-\frac{1}{2}})\ , \tag{4.7}$$

with the constant matrices M and $\tilde{M}$ entering the constraints given by M_- and M_+, respectively. The duality hypothesis of Section 2.3 is obviously satisfied by this construction.

In principle, the action and the Lax potential of the effective theory can be obtained by specializing the general formulas of Section 2.3 to the present particular case. In our case

$$\mathcal{B} = \mathcal{Q}_{\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{-\frac{1}{2}}\ , \tag{4.8}$$

and the physical modes, which are given by the entries of b in the generalized Gauss decomposition $g = abc$ with $a \in e^{\Gamma}$ and $c \in e^{\tilde{\Gamma}}$, are now conveniently parametrized as

$$b(x) = \exp[q_{\frac{1}{2}}(x)] \cdot g_0(x) \cdot \exp[q_{-\frac{1}{2}}(x)]\ , \tag{4.9}$$

where $q_{\pm\frac{1}{2}}(x) \in \mathcal{Q}_{\pm\frac{1}{2}}$ and $g_0(x) \in G_0$, the little group of M_0 in G. Next we introduce some notation which will be useful for describing the effective theory.

The operator Ad_{g_0} maps $\mathcal{G}_{-\frac{1}{2}}$ to itself and, by writing the general element u of $\mathcal{G}_{-\frac{1}{2}}$ as a two-component column vector $u = (u_1\ u_2)^t$ with $u_1 \in \mathcal{P}_{-\frac{1}{2}}$ and $u_2 \in \mathcal{Q}_{-\frac{1}{2}}$, we can designate this operator as a 2×2 matrix:

$$\mathrm{Ad}_{g_0 \,|\mathcal{G}_{-\frac{1}{2}}} = \begin{pmatrix} X_{11}(g_0) & X_{12}(g_0) \\ X_{21}(g_0) & X_{22}(g_0) \end{pmatrix} , \tag{4.10}$$

where, for example, $X_{11}(g_0)$ and $X_{12}(g_0)$ are linear operators mapping $\mathcal{P}_{-\frac{1}{2}}$ and $\mathcal{Q}_{-\frac{1}{2}}$ to $\mathcal{P}_{-\frac{1}{2}}$, respectively. Analogously, we introduce the notation

$$\mathrm{Ad}_{g_0^{-1} \,|\mathcal{G}_{\frac{1}{2}}} = \begin{pmatrix} Y_{11}(g_0) & Y_{12}(g_0) \\ Y_{21}(g_0) & Y_{22}(g_0) \end{pmatrix} , \tag{4.11}$$

which corresponds to writing the general element of $\mathcal{G}_{\frac{1}{2}}$ as a column vector, whose upper and lower components belong to $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, respectively.

The action functional of the effective field theory resulting from the WZNW reduction at hand reads as follows:

$$\begin{aligned} I^{\mathcal{S}}_{\text{eff}}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) = S_{\mathrm{WZ}}(g_0) - \int d^2x\, \langle g_0 M_+ g_0^{-1}\, ,\, M_- \rangle \\ + \int d^2x\, \left(\langle \partial_- q_{\frac{1}{2}}\, ,\, g_0 \partial_+ q_{-\frac{1}{2}} g_0^{-1} \rangle + \langle \eta_{\frac{1}{2}}\, ,\, X_{11}^{-1} \cdot \eta_{-\frac{1}{2}} \rangle \right) , \end{aligned} \tag{4.12a}$$

where the objects $\eta_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$ are given by the formulae

$$\eta_{\frac{1}{2}} = [M_+, q_{-\frac{1}{2}}] + Y_{12} \cdot \partial_- q_{\frac{1}{2}} \quad \text{and} \quad \eta_{-\frac{1}{2}} = [M_-, q_{\frac{1}{2}}] - X_{12} \cdot \partial_+ q_{-\frac{1}{2}} . \tag{4.12b}$$

The Euler-Lagrange equation of this action is the zero curvature condition of the following Lax potential:

$$\begin{aligned} \mathcal{A}^{\mathcal{S}}_+ =& M_- + \partial_+ g_0 \cdot g_0^{-1} + g_0 (\partial_+ q_{-\frac{1}{2}} + X_{11}^{-1} \cdot \eta_{-\frac{1}{2}}) g_0^{-1} , \\ \mathcal{A}^{\mathcal{S}}_- =& - g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1} \cdot \eta_{\frac{1}{2}} . \end{aligned} \tag{4.13}$$

The above new (conformally invariant) effective action and Lax potential are among the main results of the present paper. Clearly, for an integrally embedded $sl(2)$ this action and Lax potential simplify to the ones given by equation (4.3) and (4.4).

The derivation of the above formulae is not completely straightforward, and next we wish to sketch the main steps. First, let us remember that, by (2.29a), to specialize the general effective action given by (2.40) and the Lax potential given by (2.32) to our situation, we should express the objects $\partial_+ c c^{-1}$ and $a^{-1}\partial_- a$ in terms of b by using the constraints on J and $\tilde{J}$, respectively. (In the present case it would be tedious to compute the inverse matrix of V_{ij} in (2.27), which would be needed for using directly (2.29b).) For this purpose it turns out to be convenient to parametrize the WZNW field g by using the grading defined by the $sl(2)$, i.e., as

$$g = g_+ \cdot g_0 \cdot g_- \quad \text{where} \quad g_+ = a \cdot \exp[q_{\frac{1}{2}}], \quad g_- = \exp[q_{-\frac{1}{2}}] \cdot c\,. \tag{4.14}$$

We recall that the fields a, c, g_0 and $q_{\pm\frac{1}{2}}$ have been introduced previously by means of the parametrization $g = abc$, with b in (4.9). Also for later convenience, we write $g_\pm$ as

$$g_+ = \exp[r_{\geq 1} + p_{\frac{1}{2}} + q_{\frac{1}{2}}] \quad \text{and} \quad g_- = \exp[r_{\leq -1} + p_{-\frac{1}{2}} + q_{-\frac{1}{2}}]\,. \tag{4.15}$$

Note that here and below the subscript denotes the grade of the variables, and $p_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$. In our case this parametrization of g is advantageous, since, as shown below, the use of the grading structure facilitates solving the constraints.

For example, the left constraint are restrictions on $J_{<0}$, for which we have

$$J_{<0} = (g_+ g_0 N g_0^{-1} g_+^{-1})_{<0} \quad \text{with} \quad N = \partial_+ g_- \cdot g_-^{-1}\,. \tag{4.16}$$

By considering this equation grade by grade, starting from the lowest grade, it is easy to see that the constraints corresponding to $\mathcal{G}_{\geq 1} \subset \Gamma$ are equivalent to the relation

$$N_{\leq -1} = g_0^{-1} M_- g_0\ . \tag{4.17}$$

The remaining left constraints set the $\mathcal{P}_{-\frac{1}{2}}$ part of $J_{-\frac{1}{2}}$ to zero, and to unfold these constraints first we note that

$$J_{-\frac{1}{2}} = [p_{\frac{1}{2}} + q_{\frac{1}{2}}\ , M_-] + g_0 \cdot N_{-\frac{1}{2}} \cdot g_0^{-1}\,, \quad \text{with} \quad N_{-\frac{1}{2}} = \partial_+ p_{-\frac{1}{2}} + \partial_+ q_{-\frac{1}{2}}\,. \tag{4.18}$$

By using the notation introduced in (4.10), the vanishing of the projection of J to $\mathcal{P}_{-\frac{1}{2}}$ is written as

$$[q_{\frac{1}{2}}\ , M_-] + X_{11} \cdot \partial_+ p_{-\frac{1}{2}} + X_{12} \cdot \partial_+ q_{-\frac{1}{2}} = 0\ , \tag{4.19}$$

and from this we obtain

$$\partial_+ p_{-\frac{1}{2}} = X_{11}^{-1} \cdot \left\{ [M_-\ , q_{\frac{1}{2}}] - X_{12} \cdot \partial_+ q_{-\frac{1}{2}} \right\}\ . \tag{4.20}$$

Combining our previous formulae, finally we obtain that on the constraint surface of the WZNW theory

$$N = g_0^{-1} M_- g_0 + \partial_+ q_{-\frac{1}{2}} + X_{11}^{-1}(g_0) \cdot \left\{ [M_-\ , q_{\frac{1}{2}}] - X_{12}(g_0) \cdot \partial_+ q_{-\frac{1}{2}} \right\}\ . \tag{4.21}$$

A similar analysis applied to the right constraints yields that they are equivalent to the following equation:

$$-g_+^{-1} \cdot \partial_- g_+ = -g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1}(g_0) \cdot \left\{ [M_+ \, , \, q_{-\frac{1}{2}}] + Y_{12}(g_0) \cdot \partial_- q_{\frac{1}{2}} \right\} . \quad (4.22)$$

By using the relations established above, we can at this stage easily compute $b^{-1} T b = \partial_+ c c^{-1}$ and $b \tilde{T} b^{-1} = a^{-1} \partial_- a$ as well, and substituting these into (2.40), and using the Polyakov-Wiegmann identity to rewrite $S_{\rm WZ}(b)$ for b in (4.9), results in the action in (4.12) indeed. The Lax potential in (4.13) is obtained from the general expression in (2.32) by an additional 'gauge transformation' by the field $\exp[-q_{\frac{1}{2}}]$, which made the final result simpler. Of course, for the above analysis we have to restrict ourselves to a neighbourhood of the identity where the operators $X_{11}(g_0)$ and $Y_{11}(g_0)$ are invertible.

The choice of the constraints leading to the effective theory (4.12) guarantees that the chiral algebra of this theory is the required one, $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$, and thus one should be able to express the $\mathcal{W}$-currents in terms of the local fields in the action. To this first we recall that in Section 3.1 we have given an algorithm for constructing the gauge invariant differential polynomials $W(J)$. The point we wish to make is that the expression of the gauge invariant object $W(J)$ in terms of the local fields in (4.12) is simply $W(\partial_+ b \, b^{-1} + T(b))$, where b is given by (4.9). Applying the reasoning of [40,18] to the present case, this follows since the function W is form-invariant under any gauge transformation of its argument, and the quantity $(\partial_+ b \, b^{-1} + T(b))$ is obtained by a (non-chiral) gauge transformation from J, namely by the gauge transformation defined by the field $a^{-1} \in e^{\Gamma}$, see equations (2.31-2). (In analogy, when considering a right moving $\mathcal{W}$-current one gauge transforms the argument $\tilde{J}$ by the field $c \in e^{\tilde{\Gamma}}$.) We can in principle compute the object $T(b)$, as explained in the above, and thus we have an algorithm for finding the formulae of the W's in terms of the local fields g_0 and $q_{\pm\frac{1}{2}}$.

The conformal symmetry of the effective theory (4.12) is determined by the left and right Virasoro densities $L_{M_0}(J)$ and $L_{-M_0}(\tilde{J})$, which survive the reduction. To see this conformal symmetry explicitly, it is useful to extract the *Liouville field* ϕ by means of the decomposition $g_0 = e^{\phi M_0} \cdot \hat{g}_0$, where $\hat{g}_0$ contains the generators from $\mathcal{G}_0$ orthogonal to M_0. One can easily rewrite the action in terms of the new variables and then its conformal symmetry becomes manifest since e^{ϕ} is of conformal weight $(1,1)$, $\hat{g}_0$ is conformal scalar, and the fields $q_{\pm\frac{1}{2}}$ have conformal weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. This assignment of the conformal weights can be established in a number of ways, one can for example derive it from the corresponding conformal symmetry transformation of the WZNW field g in the gauged WZNW theory, see eq. (5.30). We also note that the action (4.12) can be made generally covariant and thereby our generalized Toda theory can be re-interpreted as a theory of two-dimensional gravity since ϕ becomes the gravitational Liouville mode [14].

We would like to point out the relationship between the generalized Toda theory given by (4.12) and certain non-linear integrable equations which have been associated to the half-integral $sl(2)$ subalgebras of the simple Lie algebras by Leznov and Saveliev, by using a different method. (See, e.g., equation (1.24) in the review paper in *J. Sov. Math.* referred to in [3].) To this we note that, in the half-integral case, one can also consider that WZNW reduction which is defined by imposing the left and right constraints corresponding to the subalgebras $\mathcal{G}_{\geq 1}$ and $\mathcal{G}_{\leq -1}$ of Γ and $\tilde{\Gamma}$ in (4.7). In fact, the Lax potential of the effective field theory corresponding to this WZNW reduction coincides with the Lax potential postulated by Leznov and Saveliev to set up their theory. Thus, in a sense, their theory lies between the WZNW theory and our generalized Toda theory which has been obtained by imposing a larger set of first class KM constraints. This means that the theory given by (4.12) can also be regarded as a reduction of their theory.

There is a certain freedom in constructing a field theory possessing the required chiral algebra $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, for example, one has a freedom of choice in the halving procedure used here to set up the gauge algebra. The theories in (4.12) obtained by using different halvings in equation (3.41) have their chiral algebras in common, but it is not quite obvious if these theories are always completely equivalent local Lagrangean field theories or not. We have not investigated this 'equivalence problem' in general.

A special case of this problem arises from the fact that one can expect that in some cases the theory in (4.12) is equivalent to one of the form (4.3). This is certainly so in those cases when for the half-integral $sl(2)$ of M_0 and $M_\pm$ one can find an integral grading operator H such that: (i) $[H\,,\,M_\pm]=\pm M_\pm$, (ii) $\mathcal{P}_{\frac{1}{2}}+\mathcal{G}_{\geq 1}=\mathcal{G}^H_{\geq 1}$, (iii) $\mathcal{P}_{-\frac{1}{2}}+\mathcal{G}_{\leq -1}=\mathcal{G}^H_{\leq -1}$, (iv) $\mathcal{Q}_{-\frac{1}{2}}+\mathcal{G}_0+\mathcal{Q}_{\frac{1}{2}}=\mathcal{G}^H_0$, where one uses the M_0 grading and the H-grading on the left- and on the right hand sides of these conditions, respectively. By definition, we call the halving $\mathcal{G}_{\frac{1}{2}}=\mathcal{P}_{\frac{1}{2}}+\mathcal{Q}_{\frac{1}{2}}$ an *H-compatible halving* if these conditions are met. (We note in passing that an $sl(2)$ which allows for an H-compatible halving is automatically an H-compatible $sl(2)$ in the sense defined in Section 3.3, but, as shown in Appendix C, not every H-compatible $sl(2)$ allows for an H-compatible halving.) Those generalized Toda theories in (4.12) which have been obtained by using H-compatible halvings in the WZNW reduction can be rewritten in the simpler form (4.3) by means of a renaming of the variables, since in this case the relevant first class constraints are in the overlap of the ones which have been considered for the integral gradings and for the half-integral $sl(2)$'s to derive the respective theories. Since the form of the action in (4.3) is much simpler than the one in (4.12), it appears important to know the list of those $sl(2)$ embeddings which allow for an H-compatible halving, i.e., for which conditions (i) ... (iv) can be satisfied with some integral grading operator H and halving. We study this group theoretic question for the $sl(2)$ subalgebras of the maximally non-compact real forms of the classical Lie algebras in Appendix C. We show that the existence of an H-compatible halving is a very restrictive condition on the half-integral $sl(2)$ subalgebras of the symplectic and orthogonal Lie algebras, where such a halving exists only for the special $sl(2)$ embeddings listed at the end of Appendix C. In contrast, it turns out that for $\mathcal{G} = sl(n, R)$ an H-compatible halving can be found for every $sl(2)$ subalgebra, since in this case one can construct such a halving by proceeding similarly as we did in Section 3.4 (see (3.68)). This means that in the case of $\mathcal{G} = sl(n, R)$ any chiral algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ can be realized in a generalized Toda theory associated to an integral grading.

It is interesting to observe that those theories which can be alternatively written in both forms (4.3) and (4.12) allow for several conformal structures. This is so since in this case at least two different Virasoro densities, namely L_H and L_{M_0}, survive the WZNW reduction.

4.3. Two examples of generalized Toda theories

We wish to illustrate here the general construction of the previous section by working out two examples. First we shall describe a generalized Toda theory associated to the highest root $sl(2)$ of $sl(n+2, R)$. This is a half-integral $sl(2)$ embedding, but, as we shall see explicitly, the theory (4.12) can in this case be recasted in the form (4.3), since the corresponding halving is H-compatible. We note that the $\mathcal{W}$-algebras defined by these $sl(2)$ embeddings have been investigated before by using auxiliary fields in [29]. It is perhaps worth stressing that our method does not require the use of auxiliary fields when reducing the WZNW theory to the generalized Toda theories which possess these $\mathcal{W}$-algebras as their symmetry algebras, see also Section 5.3. According to the group theoretic analysis in Appendix C, the simplest case when a $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra defined by a half-integral $sl(2)$ embedding cannot be realized in a theory of the type (4.3) is the case of $\mathcal{G} = sp(4,R)$. As our second example, we shall elaborate on the generalized Toda theory in (4.12) which realizes the $\mathcal{W}$-algebra belonging to the highest root $sl(2)$ of $sp(4,R)$.

i) Highest root $sl(2)$ of $sl(n+2,R)$

In the usual basis where the Cartan subalgebra consists of diagonal matrices, the $sl(2)$ subalgebra $\mathcal{S}$ is generated by the elements

$$M_0 = \frac{1}{2}\begin{pmatrix} 1 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & -1 \end{pmatrix} \qquad \text{and} \qquad M_+ = M_-^t = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}. \tag{4.23}$$

Note that here and below dots mean 0's in the entries of the various matrices. The adjoint of $sl(n+2)$ decomposes into one triplet, $2n$ doublets and n^2 singlets under this $\mathcal{S}$. It is convenient to parametrize the general element, g_0, of the little group of M_0 as

$$g_0 = e^{\phi M_0} \cdot e^{\psi T} \cdot \begin{pmatrix} 1 & \dots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2+n}\begin{pmatrix} n & \cdots & 0 \\ 0 & -2I_n & 0 \\ 0 & \cdots & n \end{pmatrix} \tag{4.24}$$

is trace orthogonal to M_0 and $\tilde{g}_0$ is from $sl(n)$. We note that T and M_0 generate the centre of the corresponding subalgebra, $\mathcal{G}_0$. We consider the halving of $\mathcal{G}_{\pm\frac{1}{2}}$ which is defined by the subspaces $\mathcal{P}_{\pm\frac{1}{2}}$ and $\mathcal{Q}_{\pm\frac{1}{2}}$ consisting of matrices of the following form:

$$\begin{aligned} p_{\frac{1}{2}} &= \begin{pmatrix} 0 & p^t & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad q_{\frac{1}{2}} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & q \\ 0 & \cdots & 0 \end{pmatrix}, \\ p_{-\frac{1}{2}} &= \begin{pmatrix} 0 & \cdots & 0 \\ \tilde{p} & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad q_{-\frac{1}{2}} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \tilde{q}^{\,t} & 0 \end{pmatrix}, \end{aligned} \tag{4.25}$$

where q and $\tilde{p}$ are n-dimensional column vectors and p^t and $\tilde{q}^{\,t}$ are n-dimensional row vectors, respectively. One sees that the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{\pm\frac{1}{2}}$ are invariant under the adjoint action of g_0, which means that the block-matrices in (4.10) and (4.11) are diagonal, and thus $\eta_{\pm\frac{1}{2}} = [M_\pm, q_{\mp\frac{1}{2}}]$. One can also verify that $X_{11} = e^{-\frac{1}{2}\phi-\psi}\tilde{g}_0$, and that using this the effective action (4.12) can be written as follows:

$$\begin{aligned} I_{\rm eff}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) = S_{\rm WZ}(g_0) - \int d^2x \Big[& e^{\phi} - e^{-\frac{1}{2}\phi+\psi}\ (\partial_+\tilde{q})^t \cdot \tilde{g}_0^{-1} \cdot (\partial_- q) \\ & + e^{\frac{1}{2}\phi+\psi}\ \tilde{q}^t \cdot \tilde{g}_0^{-1} \cdot q \Big], \end{aligned} \tag{4.26}$$

where dot means usual matrix multiplication. With respect to the conformal structure defined by M_0, e^ϕ has weights $(1,1)$, the fields q and $\tilde{q}$ have half-integer weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively, ψ and $\tilde{g}_0$ are conformal scalars. In particular, we see that ϕ is the Liouville mode with respect to this conformal structure.

In fact, the halving considered in (4.25) can be written like the one in (3.68), by using the integral grading operator H given explicitly as

$$H = M_0 + \frac{1}{2}T = \frac{1}{n+2}\begin{pmatrix} n+1 & 0 \\ 0 & -I_{n+1} \end{pmatrix}. \tag{4.27}$$

It is an *H-compatible halving* as one can verify that it satisfies the conditions (i)...(iv) mentioned at the end of Section 4.2, see also Appendix C. It follows that our reduced WZNW theory can also be regarded as a generalized Toda theory associated with the integral grading H. In other words, it is possible to identify the effective action (4.26) as a special case of the one in (4.3). To see this in concrete terms, it is convenient to parametrize the little group of H as

$$b = \exp(q_{\frac{1}{2}}) \cdot g_0 \cdot \exp(q_{-\frac{1}{2}}), \qquad \text{where} \qquad g_0 = e^{\Phi H} \cdot e^{\xi S} \cdot \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \tag{4.28}$$

and $S = M_0 - (\frac{n+2}{2n})T$ is trace orthogonal to H. It is easy to check that by inserting this decomposition into the effective action (4.3) and using the Polyakov-Wiegmann identity one recovers indeed the effective action (4.26), with

$$\phi = \Phi + \xi \qquad \text{and} \qquad \psi = \frac{1}{2}\Phi - \frac{2+n}{2n}\xi. \tag{4.29}$$

The conformal structure defined by H is different from the one defined by M_0. In fact, with respect to the former conformal structure Φ is the Liouville mode and all other fields, including q and $\tilde{q}$, are conformal scalars.

ii) Highest root $sl(2)$ of $sp(4, R)$

We use the convention when the symplectic matrices have the form

$$g = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \qquad \text{where} \qquad B = B^t\,,\ C = C^t\,, \tag{4.30}$$

and the Cartan subalgebra is diagonal. The $sl(2)$ subalgebra $\mathcal{S}$ corresponding to the highest root of $sp(4,R)$ is generated by the matrices

$$M_0 = \frac{1}{2}(e_{11} - e_{33}), \quad M_+ = e_{13}\,, \quad \text{and} \quad M_- = e_{31}\,, \tag{4.31}$$

where e_{ij} denotes the elementary 4×4 matrix containing a single 1 in the ij-position. The adjoint of $sp(4)$ branches into $\underline{3} + 2\cdot\underline{2} + 3\cdot\underline{1}$ under $\mathcal{S}$. The three singlets generate an $sl(2)$ subalgebra different from $\mathcal{S}$, so that the little group of M_0 is $GL(1)\times SL(2)$. $GL(1)$ is generated by M_0 itself and the corresponding field is the Liouville mode. Using usual Gauss-parameters for the $SL(2)$, we can parametrize the little group of M_0 as

$$g_0 = e^{\phi M_0}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\psi}+\alpha\beta e^{-\psi} & 0 & \alpha e^{-\psi} \\ 0 & 0 & 1 & 0 \\ 0 & \beta e^{-\psi} & 0 & e^{-\psi} \end{pmatrix}. \tag{4.32}$$

We decompose the $\mathcal{G}_{\pm\frac{1}{2}}$ subspaces (spanned by the two doublets) into their $\mathcal{P}$ and $\mathcal{Q}$ parts as follows

$$p_{\frac{1}{2}} + q_{\frac{1}{2}} = \begin{pmatrix} 0 & p & 0 & q \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -p & 0 \end{pmatrix}, \qquad p_{-\frac{1}{2}} + q_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tilde{p} & 0 & 0 & 0 \\ 0 & \tilde{q} & 0 & -\tilde{p} \\ \tilde{q} & 0 & 0 & 0 \end{pmatrix}. \tag{4.33}$$

Now the little group, or more precisely the $SL(2)$ generated by the three singlets, mixes the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{-\frac{1}{2}}$ so that the matrices X_{ij} and Y_{ij} in (4.10) and (4.11) possess off-diagonal elements:

$$X_{ij} = e^{-\frac{1}{2}\phi}\begin{pmatrix} e^{\psi}+\alpha\beta e^{-\psi} & \alpha e^{-\psi} \\ \beta e^{-\psi} & e^{-\psi} \end{pmatrix}, \qquad Y_{ij} = X_{ji}. \tag{4.34}$$

Inserting this into (4.12) yields the following effective action:

$$\begin{aligned} I^{\mathcal{S}}_{\text{eff}}(g_0, q, \tilde{q}) =& S_{\text{WZ}}(g_0) - \int d^2x \Bigg[e^{\phi} - 2e^{-\frac{1}{2}\phi-\psi}(\partial_- q)\cdot(\partial_+\tilde{q}) \\ &+ 2e^{\frac{1}{2}\phi}\,\frac{\left(\tilde{q} + e^{-\frac{1}{2}\phi-\psi}\beta\partial_- q\right)\cdot\left(q + e^{-\frac{1}{2}\phi-\psi}\alpha\partial_+\tilde{q}\right)}{e^{\psi}+\alpha\beta e^{-\psi}} \Bigg], \end{aligned} \tag{4.35}$$

for the Liouville mode ϕ, the conformal scalars ψ, α, β and the fields q, $\tilde{q}$ with weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively.

It is easy to see directly from its formula that it is impossible to obtain the above action as a special case of (4.3). Indeed, if the expression in (4.35) was obtained from (4.3) then the non-derivative term $\sim \tilde{q}\,q(e^{\psi}+\alpha\beta e^{-\psi})^{-1}$ could only be gotten from the second term in (4.3), but, since g_0 and b are matrices of unit determinant, this term could never produce the denominator in the non-derivative term in (4.35).

5. Quantum framework for WZNW reductions

In this chapter we study the quantum version of the WZNW reduction by using the path-integral formalism and also re-examine some of the classical aspects discussed in the previous chapters. We first show that the configuration space path-integral of the constrained WZNW theory can be realized by the gauged WZNW theory of Section 2.2. We then point out that the effective action of the reduced theory, (2.40), can be derived by integrating out the gauge fields in a convenient gauge, the physical gauge, in which the gauge degrees of freedom are frozen. A nontrivial feature of the quantum theory may appear in the path-integral measure. We shall find that for the generalized Toda theories associated with integral gradings the effective measure takes the form determined from the symplectic structure of the reduced theory. This means that in this case the quantum Hamiltonian reduction results in the quantization of the reduced classical theory; in other words, the two procedures, the reduction and the quantization, commute. We shall also exhibit the $\mathcal{W}$-symmetry of the effective action for this example. By using the gauged WZNW theory, we can construct the BRST formalism for the WZNW reduction in the general case. For conformally invariant reductions, this allows for computing the corresponding Virasoro centre explicitly. In particular, we derive here a nice formula for the Virasoro centre of $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ for an arbitrary $sl(2)$ embedding. We shall verify that our result agrees with the one obtained in [16], in spite of the apparent difference in the structure of the constraints.

5.1. Path-integral for constrained WZNW theory

In this section we wish to set up the path-integral formalism for the constrained WZNW theory. For this, we recall that classically the reduced theory has been obtained by imposing a set of first-class constraints in the *Hamiltonian formalism*. Thus what we should do is to write down the path-integral of the WZNW theory first in phase space with the constraints implemented and then find the corresponding configuration space expression. The phase space path-integral can formally be defined once the canonical variables of the theory are specified. A practical way to find the canonical variables is the following [41]. Let us start from the WZNW action $S_{\mathrm{WZ}}(g)$ in (1.2) and parametrize the group element $g \in G$ in some arbitrary way, $g = g(\xi)$. We shall regard the parameters

ξ^a, $a = 1, ..., \dim G$, as the canonical coordinates in the theory. To find the canonical momenta, we introduce the 2-form $\mathcal{A} = \frac{1}{2}\mathcal{A}_{ab}(\xi)\, d\xi^a d\xi^b$ to rewrite the Wess-Zumino term as

$$\frac{1}{3}\mathrm{Tr}\,(dg\, g^{-1})^3 = d\mathcal{A}. \tag{5.1}$$

The 2-form $\mathcal{A}$ is well-defined only locally on G, since the Wess-Zumino 3-form is closed but not exact. Fortunately we do not need to specify $\mathcal{A}$ explicitly below. We next define $N_{ab}(\xi)$ by

$$\Big(\frac{\partial g}{\partial \xi^a}\Big)\, g^{-1} = N_{ab}(\xi) T^b, \tag{5.2}$$

where T^b are the generators of $\mathcal{G}$. The matrix N is easily shown to be non-singular, $\det N \neq 0$. Upon writing $S_{\mathrm{WZ}}(g) = \int d^2x\, \mathcal{L}(g)$, the canonical momentum conjugate to ξ^a is found to be

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \partial_0 \xi^a} = \kappa\Big[N_{ab}(\xi)(\partial_0 g\, g^{-1})^b - \mathcal{A}_{ab}(\xi)\partial_1 \xi^b\Big]. \tag{5.3}$$

The Hamiltonian of the WZNW theory is then given by $H = \int dx^1 \mathcal{H}$ with

$$\mathcal{H} = \Pi_a \partial_0 \xi^a - \mathcal{L} = \frac{1}{2\kappa}\mathrm{Tr}\,\big[P^2 + (\kappa \partial_1 g\, g^{-1})^2\big], \tag{5.4}$$

where

$$P^a = (N^{-1})^{ab}(\Pi_b + \kappa \mathcal{A}_{bc}\partial_1 \xi^c). \tag{5.5}$$

Since $P = \kappa \partial_0 g\, g^{-1}$ in the original variables, the Hamiltonian density takes the usual Sugawara form as expected.

Classically, the constrained WZNW theory has been defined as the usual WZNW theory with its KM phase space reduced by the set of constraints given by (2.16), which in the canonical variables read

$$\begin{aligned} \phi_i &= \langle \gamma_i, P + \kappa(\partial_1 g\, g^{-1} - M)\rangle = 0, \\ \tilde{\phi}_i &= \langle \tilde{\gamma}_i, g^{-1} P g - \kappa(g^{-1}\partial_1 g + \tilde{M})\rangle = 0, \end{aligned} \tag{5.6}$$

with the bases $\gamma_i \in \Gamma$, $\tilde{\gamma}_i \in \tilde{\Gamma}$. As in Section 2.2, no relationship is assumed here between the two subalgebras, Γ and $\tilde{\Gamma}$. Now we write down the phase space path-integral for the constrained WZNW theory. According to Faddeev's prescription [42] it is defined as

$$\begin{aligned} Z = \int d\Pi d\xi\, \delta(\phi)\delta(\tilde{\phi})\delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}| \\ \times \exp\Big(i\int d^2x\, (\Pi_a \partial_0 \xi^a - \mathcal{H})\Big), \end{aligned} \tag{5.7}$$

where we implement the first class constraints by inserting $\delta(\phi)$ and $\delta(\tilde{\phi})$ in the path-integral. The δ-functions of χ and $\tilde{\chi}$ refer to gauge fixing conditions corresponding to the constraints, ϕ and $\tilde{\phi}$, which act as generators of gauge symmetries. By introducing Lagrange-multiplier fields, $A_- = A_-^i \gamma_i$ and $A_+ = A_+^i \tilde{\gamma}_i$, (5.7) can be written as

$$
\begin{aligned}
Z = \int & d\Pi d\xi dA_+ dA_- \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi, \chi\}| \det|\{\tilde{\phi}, \tilde{\chi}\}| \\
& \times \exp\Big(i \int d^2x \left[\mathrm{Tr}\,(\Pi \partial_0 \xi + A_- \phi + A_+ \tilde{\phi}) - \mathcal{H} \right] \Big).
\end{aligned}
\tag{5.8}
$$

By changing the momentum variable from Π_a to P^a in (5.5), the measure acquires a determinant factor, $d\Pi = dP \det N$, and the integrand of the exponent in (5.8) becomes

$$
\begin{aligned}
& \mathrm{Tr}\,(\Pi \partial_0 \xi + A_- \phi + A_+ \tilde{\phi}) - \mathcal{H} \\
& \quad = \kappa \mathrm{Tr} \Big[-\frac{1}{2} \big(\frac{1}{\kappa} P\big)^2 + \frac{1}{\kappa} P (A_- + g A_+ g^{-1} + \partial_0 g\, g^{-1}) - N^{-1} \mathcal{A}\, \partial_1 \xi (\partial_0 g\, g^{-1}) \\
& \qquad - \frac{1}{2} (\partial_1 g\, g^{-1})^2 + A_- (\partial_1 g\, g^{-1} - M) - A_+ (g^{-1} \partial_1 g + \tilde{M}) \Big].
\end{aligned}
\tag{5.9}
$$

Since the matrix $N(\xi)$ is independent of P, we can easily perform the integration over P provided that the remaining δ-functions and the determinant factors are also P-independent. We can choose the gauge fixing conditions, χ and $\tilde{\chi}$, so that this is true. (For example, the physical gauge which we will choose in the next section fulfills this demand.) Then we end up with the following formula of the configuration space path-integral:

$$
Z = \int d\xi \det N \, dA_+ dA_- \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi, \chi\}| \det|\{\tilde{\phi}, \tilde{\chi}\}| \, e^{iI(g, A_-, A_+)},
\tag{5.10}
$$

where $I(g, A_-, A_+)$ is the gauged WZNW action (2.18). We note that the measure for the coordinates in this path-integral is the invariant Haar measure,

$$
d\mu(g) = \prod_a d\xi^a \det N = \prod_a (dg\, g^{-1})^a.
\tag{5.11}
$$

This is a consequence of the fact that the phase space measure in (5.7) is invariant under canonical transformations to which the group transformations belong.

The above formula for the configuration space path-integral means that the gauged WZNW theory provides the Lagrangian realization of the Hamiltonian reduction, which we have already seen on the basis of a classical argument in Section 2.2.

5.2. Effective theory in the physical gauge

Having seen how the constrained WZNW theory is realized as the gauged WZNW theory, we next discuss the effective theory which arises when we eliminate all the unphysical degrees of freedom in a particularly convenient gauge, the physical gauge. We shall rederive, in the path-integral formalism, the effective action which appeared in the classical context earlier in this paper. For this purpose, within this section we restrict our attention to the left-right dual reductions considered in Section 2.3. It, however, should be noted that this restriction is not absolutely necessary to get an effective action by the method given below. In this respect, it is also worth noting that Polyakov's 2-dimensional gravity action in the light-cone gauge can be regarded as an effective action in a non-dual reduction, which is obtained by imposing a constraint only on the left-current for $G = SL(2)$ [43,12]. We will not pursue the non-dual cases here.

To eliminate all the unphysical gauge degrees of freedom, we simply gauge them away from g, i.e., we gauge fix the Gauss decomposed g in (2.25) into the form

$$g = abc \to b. \tag{5.12}$$

More specifically, with the parametrization $a(x) = \exp\left[\sigma_i(x)\gamma_i\right]$, $c(x) = \exp\left[\tilde{\sigma}_i(x)\tilde{\gamma}_i\right]$ we define the *physical gauge* by

$$\chi_i = \sigma_i = 0, \qquad \tilde{\chi}_i = \tilde{\sigma}_i = 0. \tag{5.13}$$

We here note that for this gauge the determinant factors in (5.8) are actually constants. Now the effective action is obtained by performing the $A_\pm$ integrations in (5.10). The integration of A_- gives rise to the delta-function,

$$\prod_i \delta\Big(\langle \gamma_i, bA_+b^{-1} + \partial_+ b\, b^{-1} - M\rangle\Big), \tag{5.14}$$

with $\gamma_i \in \Gamma$ normalized by the duality condition (2.22). One then notices that the delta-function (5.14) implies exactly condition (2.29) with $\partial_+ c\, c^{-1}$ replaced by A_+. Hence, with the help of the matrix $V_{ij}(b)$ in (2.27) and $T(b)$ in (2.29), it can be rewritten as

$$(\det V)^{-1}\, \delta\Big(A_+ - b^{-1}T(b)b\Big). \tag{5.15}$$

Finally, the integration of A_+ yields

$$Z = \int d\mu_{\text{eff}}(b)\, e^{I_{\text{eff}}(b)}, \tag{5.16}$$

where $I_{\rm eff}(b)$ is the effective action (2.40)*, and $d\mu_{\rm eff}(b)$ is the effective measure given by

$$d\mu_{\rm eff}(b) = (\det V)^{-1}\, d\mu(g)\delta(\sigma)\delta(\tilde\sigma) = (\det V)^{-1}\, \frac{d\mu(g)}{d\sigma d\tilde\sigma}\bigg|_{\sigma=\tilde\sigma=0}. \tag{5.17}$$

Of course, as far as the effective action is concerned, the path-integral approach should give the same result as the classical one, because the integration of the gauge fields is Gaussian and hence equivalent to the classical elimination of the gauge fields by their field equations. However, a non-trivial feature may arise at the quantum level when the effective path-integral measure (5.17) is taken into account. Let us examine the effective measure in the simple case where the space $\mathcal{B} = (\Gamma + \tilde\Gamma)^\perp$, with which $b \in e^{\mathcal{B}}$, forms a subalgebra of $\mathcal{G}$ satisfying (2.34), and thus the effective action in (5.16) simplifies to

$$I_{\rm eff}(b) = S_{\rm WZ}(b) - \kappa \int d^2x\, \langle b\tilde M b^{-1}, M\rangle. \tag{5.18}$$

In this case, the 1-form appearing in the measure $d\mu(g)$ of (5.11),

$$dg\, g^{-1} = da\, a^{-1} + a(db\, b^{-1})a^{-1} + ab(dc\, c^{-1})b^{-1}a^{-1}, \tag{5.19}$$

turns out, in the physical gauge, to be

$$dg\, g^{-1}\big|_{\sigma=\tilde\sigma=0} = \gamma_i d\sigma_i + db\, b^{-1} + V_{ij}(b)\tilde\gamma_i d\tilde\sigma_j. \tag{5.20}$$

As a result, the determinant factor in (5.17) is cancelled by the one coming from (5.20), and the effective measure admits a simple form:

$$d\mu_{\rm eff}(b) = db\, b^{-1}. \tag{5.21}$$

The point is that this is exactly the measure which is determined from the symplectic structure of the effective theory (5.18) obtained by the *classical* Hamiltonian reduction. This tells us that in this case the *quantum* Hamiltonian reduction results in the quantization of the reduced classical theory. In particular, since the above assumption for $\mathcal{B}$ is satisfied for the generalized Toda theories associated with integral gradings, we conclude that these generalized Toda theories are equivalent to the corresponding constrained

* Actually, the effective action always takes the form (2.40) if one restricts the WZNW field to be of the form $g = abc$ with $a \in e^\Gamma$, $c \in e^{\tilde\Gamma}$ and b such that $V_{ij}(b)$ is invertible. The duality between Γ and $\tilde\Gamma$ is not necessary but can be used to ensure this technical assumption.

(gauged) WZNW theories even at the quantum level, i.e., including the measure. This result has been established before in the special case of the standard Toda theory (1.1) in [44], where the measure $d\mu_{\rm eff}(b)$ is simply given by $\prod_i d\varphi^i$.

We end this section by noting that it is not clear whether the measure determined from the symplectic structure of the reduced classical theory is identical to the effective measure (5.17) in general. In the general case both measures in question could become quite involved and thus one would need some geometric argument to see if they are identical or not.

5.3. The $\mathcal{W}$-symmetry of the generalized Toda action $I^H_{\rm eff}(b)$

In the previous section we have seen the quantum equivalence of the generalized Toda theories given by (4.3) and the corresponding constrained WZNW theories. It follows from their WZNW origin that the generalized Toda theories possess conserved $\mathcal{W}$-currents. It is thus natural to expect that their effective actions, $I^H_{\rm eff}$ in (4.3) and $I^{\mathcal{S}}_{\rm eff}$ in (4.12), allow for *symmetry transformations yielding the $\mathcal{W}$-currents as the corresponding Noether currents*. We demonstrate below that this is indeed the case on the example of the theories associated with integral gradings, when the action takes a simple form. We however believe that there are symmetries of the effective action corresponding to the conserved chiral currents inherited from the KM algebra for any reduced WZNW theory.

Let us consider a gauge invariant differential polynomial $W(J)$ in the constrained WZNW theory giving rise to the effective theory described by the action in (4.3). In terms of the generalized Toda field $b(x)$, this conserved $\mathcal{W}$-current is given by the differential polynomial

$$W_{\rm eff}(\beta)=W(M+\beta), \qquad \text{where} \qquad \beta\equiv\partial_+ b\, b^{-1}. \tag{5.22}$$

This equality [34,15] holds because the constrained current J and $(M+\beta)$ (which is, incidentally, just the Lax potential $\mathcal{A}^H_+$ in (4.4)) are related by a gauge transformation, as we have seen. By choosing some test function $f(x^+)$, we now associate to $W_{\rm eff}(\beta)$ the following transformation of the field $b(x)$:

$$\delta_W b(y)=\Big[\int d^2x\, f(x^+)\frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}\Big]\cdot b(y)\ , \tag{5.23}$$

and we wish to show that $\delta_W b$ is a symmetry of the action $I^H_{\rm eff}(b)$. Before proving this, we notice, by combining the definition in (5.23) with (5.22), that $(\delta_W b)b^{-1}$ is a polynomial expression in f, β and their ∂_+-derivatives up to some finite order.

We start the proof by noting that the change of the action under an arbitrary variation δb is given by the formula

$$\begin{aligned} \delta I^H_{\rm eff}(b) &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\, ,\, b(y)\frac{\delta I^H_{\rm eff}}{\delta b(y)}\rangle \\ &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\, ,\, \partial_-\beta(y) + [b(y)\tilde{M}b^{-1}(y), M]\rangle\, . \end{aligned} \tag{5.24}$$

In the next step, we use the field equation to replace $\partial_-\beta$ by $-[b\tilde{M}b^{-1}, M]$ in the obvious equality

$$\partial_- W_{\rm eff}(x) = \int d^2y\, \langle \frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}, \partial_-\beta(y)\rangle, \tag{5.25}$$

and then, from the fact that $\partial_- W_{\rm eff} = 0$ on-shell, we obtain the following identity:

$$\int d^2y\, \langle \frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}, [b(y)\tilde{M}b^{-1}(y), M]\rangle = 0\, , \tag{5.26}$$

Of course, the previous argument only implies that (5.26) holds on-shell. However, we now make the crucial observation that (5.26) is an *off-shell identity*, i.e., it is valid for any field $b(x)$ not only for the solutions of the field equation. This follows by noticing that the object in (5.26) is a local expression in $b(x)$ containing only x^+-derivatives. In fact, any such object which vanishes on-shell has to vanish also off-shell, because one can find solutions of the field equation for which the x^+-dependence of the field b is prescribed in an arbitrary way at an arbitrarily chosen fixed value of x^-.

By using the above observation, it is easy to show that $\delta_W b$ in (5.23) is indeed a symmetry of the action. First, simply inserting (5.23) into (5.24), we have

$$\delta_W I^H_{\rm eff}(b) = -\int d^2x\, f(x^+) \int d^2y\, \langle \frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}\, ,\, \partial_-\beta(y) + [b(y)\tilde{M}b^{-1}(y), M]\rangle. \tag{5.27}$$

We then rewrite this equation as

$$\delta_W I^H_{\rm eff}(b) = -\int d^2x\, f(x^+)\partial_- W_{\rm eff}(x), \tag{5.28}$$

with the aid of the identities (5.26) and (5.25). This then proves that

$$\delta_W I^H_{\rm eff}(b) = 0\, , \tag{5.29}$$

since the integrand in (5.28) is a total derivative, thanks to $\partial_- f = 0$. One can also see, from equation (5.23), that $W_{\rm eff}$ is the Noether charge density corresponding to the symmetry transformation $\delta_W b$ of $I^H_{\rm eff}(b)$.

5.4. BRST formalism for WZNW reductions

Since the constrained WZNW theory can be regarded as the gauged WZNW theory (2.18), one is naturally led to construct the BRST formalism for the theory as a basis for quantization. Below we discuss the BRST formalism based on the gauge symmetry (2.19) and thus return to the general situation of Section 5.1 where no relationship between the two subalgebras, Γ and $\tilde{\Gamma}$, is supposed.

Prior to the construction we here note how the conformal symmetry is realized in the gauged WZNW theory when there is an operator H satisfying the condition (2.13). (For simplicity, in what follows we discuss the symmetry associated to the left-moving sector.) In fact, with such H and a chiral test function $f^+(x^+)$ one can define the following transformation,

$$\begin{aligned} \delta g &= f^+\partial_+ g + \partial_+ f^+ H g, \\ \delta A_- &= f^+\partial_+ A_- + \partial_+ f^+ [H, A_-], \\ \delta A_+ &= f^+\partial_+ A_+ + \partial_+ f^+ A_+, \end{aligned} \tag{5.30}$$

which leaves the gauged WZNW action $I(g, A_-, A_+)$ invariant. This corresponds exactly to the conformal transformation in the constrained WZNW theory generated by the Virasoro density L_H in (2.10), as can be confirmed by observing that (5.30) implies the conformal action (2.11) for the current with $f(x^+) = f^+(x^+)$. We shall derive later the Virasoro density as the Noether charge density in the BRST system.

Turning to the construction of the BRST formalism, we first choose the space $\Gamma^* \subset \mathcal{G}$ which is dual to Γ with respect to the Cartan-Killing form (and similarly $\tilde{\Gamma}^*$ dual to $\tilde{\Gamma}$). Following the standard procedure [45] we introduce two sets of ghost, anti-ghost and Nakanishi-Lautrup fields, $\{c \in \Gamma,\ \bar{c}_+, B_+ \in \Gamma^*\}$ and $\{b \in \tilde{\Gamma},\ \bar{b}_-, B_- \in \tilde{\Gamma}^*\}$. The BRST transformation corresponding to the (left-sector of the) local gauge transformation (2.19) is given by

$$\begin{aligned} \delta_{\rm B} g &= -cg, & \delta_{\rm B} \bar{c}_+ &= iB_+, \\ \delta_{\rm B} A_- &= D_- c, & \delta_{\rm B} B_+ &= 0, \\ \delta_{\rm B} c &= -c^2, & \delta_{\rm B}(\text{others}) &= 0, \end{aligned} \tag{5.31}$$

with $D_\pm = \partial_\pm \mp [A_\pm, \quad]$. After defining the BRST transformation $\bar{\delta}_{\rm B}$ for the right-sector in an analogous way, we write the BRST action by adding a gauge fixing term and a ghost term to the gauged action,

$$I_{\rm BRST} = I(g, A_-, A_+) + I_{\rm gf} + I_{\rm ghost}. \tag{5.32}$$

The additional terms can be constructed by the manifestly BRST invariant expression,

$$\begin{aligned} I_{\rm gf} + I_{\rm ghost} &= -i\kappa(\delta_{\rm B} + \bar{\delta}_{\rm B}) \int d^2x \, \big(\langle \bar{c}_+, A_- \rangle + \langle \bar{b}_-, A_+ \rangle \big) \\ &= \kappa \int d^2x \, \big(\langle B_+, A_- \rangle + \langle B_-, A_+ \rangle + i\langle \bar{c}_+, D_- c \rangle + i\langle \bar{b}_-, D_+ b \rangle \big), \end{aligned} \tag{5.33}$$

where we have chosen the gauge fixing conditions as $A_\pm = 0$. Then the path-integral for the BRST system is given by

$$Z = \int d\mu(g) \, dA_+ dA_- dc \, d\bar{c}_+ db \, d\bar{b}_- dB_+ dB_- \, e^{iI_{\rm BRST}}, \tag{5.34}$$

which, upon integration of the ghosts and the Nakanishi-Lautrup fields, reduces to (5.10). (Strictly speaking, for this we have to generalize the gauge fixing conditions in (5.10) to be dependent on the gauge fields.) By this construction the nilpotency, $\delta_{\rm B}^2 = 0$, and the BRST invariance of the action, $\delta_{\rm B} I_{\rm BRST} = 0$, are easily checked.

It is, however, convenient to deal with the simplified BRST theory obtained by performing the trivial integrations of $A_\pm$ and $B_\pm$ in (5.34),

$$I_{\rm BRST}(g, c, \bar{c}_+, b, \bar{b}_-) = S_{\rm WZ}(g) + i\kappa \int d^2x \, \big(\langle \bar{c}_+, \partial_- c \rangle + \langle \bar{b}_-, \partial_+ b \rangle \big). \tag{5.35}$$

We note that this effective BRST theory is not merely a sum of a free WZNW sector and free ghost sector as it appears, but rather it consists of the two interrelated sectors in the physical space specified by the BRST charge defined below. At this stage the BRST transformation which leaves the simplified BRST action (5.35) invariant reads

$$\begin{aligned} &\delta_{\rm B} g = -cg, \qquad \delta_{\rm B} \bar{c}_+ = -\pi_{\Gamma^*} \Big[i(\partial_+ g \, g^{-1} - M_-) + (c\bar{c}_+ + \bar{c}_+ c) \Big], \\ &\delta_{\rm B} c = -c^2, \qquad \delta_{\rm B}(\text{others}) = 0, \end{aligned} \tag{5.36}$$

where $\pi_{\Gamma^*} = \sum_i |\gamma_i^*\rangle\langle\gamma_i|$ is the projection operator onto the dual space Γ^* with the normalized bases, $\langle\gamma_i, \gamma_j^*\rangle = \delta_{ij}$. From the associated conserved Noether current, $\partial_- j_+^{\mathrm{B}} = 0$, the BRST charge Q_{B} is defined to be

$$Q_{\mathrm{B}} = \int dx^+ j_+^{\mathrm{B}}(x) = \int dx^+ \langle c, \partial_+ g\, g^{-1} - M - c\bar{c}_+\rangle. \tag{5.37}$$

The physical space is then specified by the condition,

$$Q_{\mathrm{B}}|\mathrm{phys}\rangle = 0. \tag{5.38}$$

In the simple case of the WZNW reduction which leads to the standard Toda theory, the BRST charge (5.37) agrees with the one discussed earlier [46].

In the case where there is an H operator which guarantees the conformal invariance, the BRST system also has the corresponding conformal symmetry,

$$\begin{aligned} \delta g &= f^+\partial_+ g + \partial_+ f^+ H g, & \delta b &= f^+\partial_+ b, \\ \delta c &= f^+\partial_+ c + \partial_+ f^+ [H, c], & \delta\bar{b}_- &= f^+\partial_+\bar{b}_-, \\ \delta\bar{c}_+ &= f^+\partial_+\bar{c}_+ + \partial_+ f^+(\bar{c}_+ + [H, \bar{c}_+]), & & \end{aligned} \tag{5.39}$$

inherited from the one (5.30) in the gauged WZNW theory. If the H operator further provides a grading, one finds from (5.39) that the currents of grade $-h$ have the (left-) conformal weight $1-h$, except the H-component, which is not a primary field. Similarly, the ghosts c, $\bar{c}_+$ of grade h, $-h$ have the conformal weight h, $1-h$, respectively, whereas the ghosts b, $\bar{b}$ are conformal scalars. Now we define the total Virasoro density operator L_{tot} from the associated Noether current, $\partial_- j_+^{\mathrm{C}} = 0$, by

$$\int dx^+ j_+^{\mathrm{C}}(x) = \frac{1}{\kappa}\int dx^+ f^+(x^+) L_{\mathrm{tot}}(x). \tag{5.40}$$

The (on-shell) expression is found to be the sum of the two parts, $L_{\mathrm{tot}} = L_H + L_{\mathrm{ghost}}$, where L_H is indeed the Virasoro operator (2.10) for the WZNW part, and

$$L_{\mathrm{ghost}} = i\kappa\big(\langle\bar{c}_+, \partial_+ c\rangle + \partial_+\langle H, c\bar{c}_+ + \bar{c}_+ c\rangle\big), \tag{5.41}$$

is the part for the ghosts. The conformal invariance of the BRST charge, $\delta Q_{\mathrm{B}} = 0$, or equivalently, the BRST invariance of the total conformal charge, $\delta_{\mathrm{B}} L_{\mathrm{tot}} = 0$, are readily confirmed.

Let us find the Virasoro centre of our BRST system. The total Virasoro centre c_{tot} is given by the sum of the two contributions, c from the WZNW part and c_{ghost} from the ghost one. The Viraso centre from L_H is given by

$$c = \frac{k \dim \mathcal{G}}{k+g} - 12k\langle H, H\rangle, \tag{5.42}$$

where k is the level of the KM algebra and g is the dual Coxeter number. On the other hand, the ghosts contribute to the Virasoro centre by the usual formula,

$$c_{\mathrm{ghost}} = -2\sum_{\Gamma}\big[1 + 6h(h-1)\big], \tag{5.43}$$

where the summation is performed over the eigenvectors of ad_H in the subalgebra Γ. (One can confirm (5.43) by performing the operator product expansion with L_{ghost} in (5.41).)

5.5. The Virasoro centre in two examples

By elaborating on the general result of the previous section, we here derive explicit formulas for the total Virasoro centre in two important special cases of the WZNW reduction.

i) The generalized Toda theory $I^H_{\mathrm{eff}}(b)$

In this case the summation in (5.43) is over the eigenstates of ad_H with eigenvalues $h > 0$, since $\Gamma = \mathcal{G}^H_{>0}$. We can establish a concise formula for c_{tot}, (5.46) below, by using the following group theoretic facts.

First, we can assume that the grading operator $H \in \mathcal{G}$ is from the Cartan subalgebra of the complex simple Lie algebra $\mathcal{G}_c$ containing $\mathcal{G}$. Second, the scalar product $\langle\ ,\ \rangle$ defines a natural isomorphism between the Cartan subalgebra and the space of roots, and we introduce the notation $\vec{\delta}$ for the vector in root space corresponding to H under this isomorphism. More concretely, this means that we set $H = \sum_i \delta_i H_i$ by using an orthonormal Cartan basis, $\langle H_i, H_j\rangle = \delta_{ij}$. Third, we recall the *strange formula* of Freudenthal-de Vries [47], which (by taking into account the normalization of $\langle\ ,\ \rangle$ and the duality between the root space and the Cartan subalgebra) reads

$$\dim \mathcal{G} = \frac{12}{g}|\vec{\rho}|^2\ , \tag{5.44}$$

where $\vec{\rho}$ is the Weyl vector, given by half the sum of the positive roots. Fourth, we choose the simple positive roots in such a way that the corresponding step operators, which are in general in $\mathcal{G}_c$ and not in $\mathcal{G}$, have non-negative grades with respect to H.

By using the above conventions, it is straightforward to obtain the following expressions

$$\begin{aligned} &\sum_{h>0} 1 = \dim\Gamma = \frac{1}{2}(\dim\mathcal{G} - \dim\mathcal{G}_0^H), \qquad \sum_{h>0} h = 2(\vec{\rho}\cdot\vec{\delta}), \\ &\sum_{h>0} h^2 = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_H)^2 = g\langle H, H\rangle = g|\vec{\delta}|^2, \end{aligned} \tag{5.45}$$

for the corresponding terms in (5.43). Substituting these into (5.43) and also (5.44) into (5.42),

one can finally establish the following nice formula of the total Virasoro centre [14]:

$$c_{\mathrm{tot}} = c + c_{\mathrm{ghost}} = \dim\mathcal{G}_0^H - 12\Big|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\Big|^2. \tag{5.46}$$

In particular, in the case of the reduction leading to the standard Toda theory (1.1) the result (5.46) is consistent with the one directly obtained in the reduced theory [8,10].

ii) The $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra for half-integral $sl(2)$ embeddings

For $sl(2)$ embeddings the role of the H is played by M_0 and in the half-integral case we have $\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} = \mathcal{G}_{>0} - \mathcal{Q}_{\frac{1}{2}}$. It follows that the value of the total Virasoro centre can now be obtained by substracting the contribution of the 'missing ghosts' corresponding to $\mathcal{Q}_{\frac{1}{2}}$, which is $\frac{1}{2}\dim\mathcal{G}_{\frac{1}{2}}$, from the expression in (5.46). We thus obtain that in this case

$$c_{\mathrm{tot}} = N_t - \frac{1}{2}N_s - 12\Big|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\Big|^2, \tag{5.47a}$$

where

$$N_t = \dim\mathcal{G}_0\ , \qquad \text{and} \qquad N_s = \dim\mathcal{G}_{\frac{1}{2}}\ , \tag{5.47b}$$

are the number of tensor and spinor multiplets in the decomposition of the adjoint of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$, respectively. We note that, as proven by Dynkin [39], it is possible to choose a system of positive simple roots so that the grade of the corresponding step operators is from the set $\{0, \frac{1}{2}, 1\}$, and that $\vec{\delta}$ is ($\frac{1}{2}\times$) the so called *defining vector* of the $sl(2)$ embedding in Dynkin's terminology.

As has been mentioned in Section 3.3, Bais *et al* [16] (see also [29]) studied a similar reduction of the KM algebra for half-integral $sl(2)$ embeddings where all the current components corresponding to $\mathcal{G}_{>0}$ are constrained from the very beginning. In their system, the constraints (3.49) of $\mathcal{G}_{\frac{1}{2}}$, being inevitably second-class, are modified into first-class by introducing an auxiliary field to each constraint of $\mathcal{G}_{\frac{1}{2}}$. Accordingly, the auxiliary fields give rise to the extra contribution $-\frac{1}{2}\dim \mathcal{G}_{\frac{1}{2}}$ in the total Virasoro centre. It is clear that adding this to the sum of the WZNW and ghost parts (which is of the form (5.46) with M_0 substituted for H), renders the total Virasoro centre of their system identical to that of our system, given by (5.47). This result is natural if we recall the fact that their reduced phase space (after complete gauge fixing) is actually identical to ours. It is obvious that our method, which is based on purely first-class KM constraints and does not require auxiliary fields, provides a simpler way to reach the identical reduced theory.

6. Discussion

The main purpose of this paper has been to study the general structure of the Hamiltonian reductions of the WZNW theory. Considering the number of interesting examples resulting from the reduction, this problem appears important for the theory of two-dimensional integrable systems and in particular for conformal field theory.

Our most important result perhaps is that we established the gauged WZNW setting of the Hamiltonian reduction by first class constraints in full generality. It was then used here to set up the BRST formalism in the general case, and for obtaining the effective actions for the left-right dual reductions. We hope that the general framework we set up will be useful for further studies of this very rich problem.

The other major concern of the paper has been to investigate the $\mathcal{W}$-algebras and their field theoretic realizations arising from the WZNW reduction. We found first class KM constraints leading to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras which allowed us to construct generalized Toda theories realizing these interesting extended conformal algebras. We believe that the $sl(2)$-embeddings underlying the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras are to play an important organizing role in general for understanding the structure, especially the primary field content, of the conformally invariant reduced KM systems. We illustrated this idea by showing that the W_n^l-algebras are nothing but further reductions of $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras belonging to particular $sl(2)$-embeddings (see also [37]). In our study of $\mathcal{W}$-algebras we employed two (apparently) new methods, which are likely to have a wider range of applicability than what we exploited here. The first is *the method of symplectic halving* whereby we constructed purely first class KM constraint for the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ as well as for the W_n^l-algebras. The second is what we call *the $sl(2)$-method*, which can be summarized by saying that if one has conformally invariant first class constraints given by some (Γ, M_-) with M_- nilpotent, then one should build the $sl(2)$ containing M_- and try to analyse the system in terms of this $sl(2)$. We used this method to investigate, in the non-degenerate case, the generalized Toda sytems belonging to integral gradings, and also to provide the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-interpretation of the W_n^l-algebras.

We wish to remark here that, as far as we know, the technical problem concerning the inequivalence of those $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras which belong to group theoretically inequivalent $sl(2)$ embeddings has not been tackled yet.

It is well known [22] that the standard $\mathcal{W}$-algebras can be identified as the second

Poisson bracket structure of the generalized KdV hierarchies of Drinfeld-Sokolov [5]. A similar relationship between $\mathcal{W}$-algebras and KdV type hierarchies has been established very recently in more general cases [28,48,49]. In particular, the W_n^l-algebras have been related to the so called fractional KdV hierarchies. It would be clearly worthwhile to study in general the relationship between the generalized Drinfeld-Sokolov hierachies of [48] and the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras together with their further reductions, see also [16,17].

We gave a general local analysis of the effective theories arising in the left-right dual case of the reduction, and investigated in particular the generalized Toda theories obtained by the reduction in some detail. In the case of the generalized Toda theories associated with the integral gradings we exhibited the way in which the $\mathcal{W}$-symmetry operates as an ordinary symmety of the action, and demonstrated that the quantum Hamiltonian reduction is consistent with the canonical quantization of the reduced classical theory. It would be nice to have the analogous problems under control also in more general cases. In our analysis we restricted the considerations to Gauss-decomposable fields. The fact that the Gauss decomposition may break down can introduce apparent singularities in the local description of the effective theories, but the WZNW description is inherently global and remains valid for non Gauss-decomposable fields as well [12,13]. It is hence an interesting problem to further analyze the global (topological) aspects of the phase space of the reduced WZNW theories.

We should also note that it is possible to remove the technical assumption of left-right duality. In particular, the study of purely chiral WZNW reductions could be of importance, as they are likely to give natural generalizations of Polyakov's 2d gravity action [43,12].

In this paper we assumed the existence of a gauge invariant Virasoro density L_H, of the form given by (2.10), for obtaining conformally invariant reductions. Based on this assumption, we came to realize that, when H provides a grading of Γ and M, the $sl(2)$ built out of $M = M_-$ plays an important role. However, the example of Appendix A indicates that there is another class of conformally invariant reductions where the form of the surviving Virasoro density is different from that of an L_H. The study of this novel way of preserving the conformal invariance may open up a new perspective on conformal reductions of the WZNW theory as well as on $\mathcal{W}$-algebras.

There are many further interesting questions related to the Hamiltonian reductions of the WZNW theory, which we could not mention in this paper. We hope to be able to present those in future publications.

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Note added. After finishing this paper, there appeared a preprint [50] also advocating the importance of $sl(2)$ structures in classifying $\mathcal{W}$-algebras.

Appendix A: A solvable but not nilpotent gauge algebra

In all the cases of the reduction we considered in Chapters 3 and 4, the gauge algebra Γ was a graded nilpotent subalgebra of $\mathcal{G}$. On the other hand, we have seen in Section 2.1 that the first-classness of the constraints imply that Γ is solvable. We want here to discuss a constrained WZNW model for which the gauge algebra is *solvable but not nilpotent*. Interestingly enough, it turns out that in this example no H satisfying (2.13) exists which would render the constraints conformally invariant. However, conformal invariance can still be maintained, showing clearly that the existence of such an H is only a sufficient but not a necessary condition.

We choose the Lie algebra $\mathcal{G}$ to be $sl(3,R)$ and the gauge algebra Γ as generated by the following three generators

$$\gamma_1 = E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.1a}$$

$$\gamma_3 = \frac{1}{\sqrt{3}}(2H_1 + H_2) + \frac{1}{2}(E_{\alpha_2} - E_{-\alpha_2}) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{pmatrix}, \tag{A.1b}$$

where the Cartan-Weyl generators are normalized by $[H_i, E_{\pm\alpha_i}] = \pm E_{\pm\alpha_i}$ and $[E_{\alpha_i}, E_{-\alpha_i}] = 2H_i$, for the simple positive roots α_i. Note that, being diagonalizable over the complex numbers, γ_3 is not a nilpotent operator. The algebra of Γ is

$$[\gamma_1, \gamma_2] = 0, \quad [\gamma_1, \gamma_3] = -\frac{\sqrt{3}}{2}\gamma_1 + \frac{1}{2}\gamma_2, \quad [\gamma_2, \gamma_3] = -\frac{1}{2}\gamma_1 - \frac{\sqrt{3}}{2}\gamma_2. \tag{A.2}$$

It is easy to verify that Γ is a solvable, not-nilpotent Lie algebra. It qualifies as a gauge algebra since $\mathrm{Tr}\,(\gamma_i\,\gamma_j) = 0$.

It is readily checked that the spaces $\Gamma^\perp$ and $[\Gamma,\Gamma]^\perp$ are given by

$$\begin{aligned} \Gamma^\perp &= \mathrm{span}\{H_2, E_{\alpha_1}, E_{\alpha_1+\alpha_2}, 2H_1 + \sqrt{3}E_{\alpha_2}, 2H_1 - \sqrt{3}E_{-\alpha_2}\}, \\ [\Gamma,\Gamma]^\perp &= \mathrm{span}\{H_1, H_2, E_{\alpha_1}, E_{\alpha_1+\alpha_2}, E_{\alpha_2}, E_{-\alpha_2}\}. \end{aligned} \tag{A.3}$$

Thus $[\Gamma,\Gamma]^\perp/\Gamma^\perp$, which is the space of the M's leading to first class constraints, is one-dimensional, and we can take

$$M = \mu Y \equiv \frac{\mu}{\sqrt{3}}(4H_1 + 2H_2) = \frac{\mu}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.4}$$

without loss of generality.

The next question is the conformal invariance. As discussed in Section 2.1, a sufficient condition for conformal invariance is provided by the existence of a (modified) Virasoro density $L_H = L_{KM} - \partial_x \langle H, J(x) \rangle$ weakly commuting with the constraints. For this to work, the generator H must satisfy the three conditions in (2.13). However, it is an easy matter to show that those conditions are contradictory in the present case, and therefore no such H exists.

The above analysis can also be carried out for the simpler gauge algebra spanned by γ_3 only. This gauge algebra is obviously nilpotent, since it is Abelian. Nevertheless, the previous conclusions remain: There exists no H which would render the first class constraints conformally invariant, for any $M \neq 0$ from $[\Gamma, \Gamma]^\perp / \Gamma^\perp$. This shows the importance of the gauge generators being nilpotent operators, rather than the gauge algebra being nilpotent. It would be interesting to know whether there is always an H satisfying (2.13) for gauge algebras consisting of nilpotent operators.

Although there is no H such that the constraints are preserved by L_H, we can nevertheless construct another Virasoro density Λ which does preserve the constraints. It is given by

$$\Lambda(x) = L_{KM}(x) - \mu \langle \gamma_3^t, J(x) \rangle. \tag{A.5}$$

For M given in (A.4), the constraints read

$$\langle \gamma_1, J(x) \rangle = \langle \gamma_2, J(x) \rangle = 0 \,, \qquad \langle \gamma_3, J(x) \rangle = \mu \,, \tag{A.6}$$

and are checked to weakly commute with Λ: $\{\Lambda(x), \langle \gamma_i, J(y) \rangle\} \approx 0$ on the constraint surface (A.6). (Note that, when going from L_{KM} to Λ, we have not changed the conformal central charge, which is classically zero.) Therefore we expect the reduced theory to be invariant under the conformal transformation generated by Λ being its Noether charge density. We now proceed to show that it is indeed the case. Before doing this, we display the form of Λ on the constraint surface:

$$\Lambda(x) = T_1^2(x) + T_2^2(x) \,, \tag{A.7a}$$

$$T_1 = \frac{1}{2} \langle E_{\alpha_2} + E_{-\alpha_2}, J \rangle \,, \qquad T_2 = \langle H_2, J \rangle. \tag{A.7b}$$

Following the analysis of Section 2.3, we take the left and right gauge algebras to be dual to each other ($\langle \gamma_i, \tilde{\gamma}_j \rangle = \delta_{ij}$)

$$\Gamma = \mathrm{span}\{\gamma_1, \gamma_2, \gamma_3\}, \quad \tilde{\Gamma} = \mathrm{span}\{\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3\} = \mathrm{span}\{\gamma_1^t, \gamma_2^t, \gamma_3^t\}, \tag{A.8}$$

and consider $M = \mu Y$ and $\tilde{M} = \nu Y^t = \nu Y$. We write the $SL(3,R)$ group elements as $g = a \cdot b \cdot c$, with $a \in \exp\Gamma$, $b \in \exp\mathcal{H}$ and $c \in \exp\tilde{\Gamma}$, with $\mathcal{H} = \mathrm{span}\{Y, H_2\}$ the Cartan subalgebra. We did not conform to the general prescription given in Section 2.3, which required to write $g = abc$ with $b \in \exp\mathcal{B}$ for a space $\mathcal{B}$ complementary to $\Gamma + \tilde{\Gamma}$ in $\mathcal{G}$, eqs.(2.25-26). Had we done that, the resulting effective action would have looked much more complicated. Here, we simply take a set of coordinates in which the action looks simple.

The reduction yields an effective theory for the group-valued field b, of which the effective action is given by (2.40) with (2.29b). Using the parametrization $b = \exp(\alpha Y) \cdot \exp(2\beta H_2)$, the explicit form of the effective action is

$$I_{\text{eff}}(\alpha,\beta) = \int d^2x \left\{ \partial_+\alpha\partial_-\alpha + \partial_+\beta\partial_-\beta - \frac{(\partial_+\alpha - \mu)(\partial_-\alpha - \nu)}{\cosh^2\beta} \right\}. \tag{A.9}$$

By inspection, we see that this effective action is going to be conformally invariant if the field β is a scalar, and if the transformation of α is such that $\mu - \partial_+\alpha$ and $\nu - \partial_-\alpha$ are (1,0) and (0,1) vectors respectively. It implies that, under a conformal transformation $x^\pm \longrightarrow x^\pm - f^\pm(x^\pm)$, the fields α and β transform as

$$\begin{aligned} \delta\alpha &= f^+\,(\partial_+\alpha - \mu) + f^-\,(\partial_-\alpha - \nu), \\ \delta\beta &= f^+\,\partial_+\beta + f^-\,\partial_-\beta. \end{aligned} \tag{A.10}$$

We now want to show our previous claim: the action (A.9) is conformally invariant under the conserved Virasoro density $\Lambda(x)$, which reproduces the f^+-transformations (A.10) by Poisson brackets. (The f^--transformations could also be realized by constructing the corresponding Virasoro density $\tilde{\Lambda}$ in the right-handed sector in a similar way.) For this, we first note that in terms of the reduced variables α and β the two current components T_1 and T_2 of (A.7b) read

$$T_1 = -(\mu - \partial_+\alpha)\,\tanh\beta\,, \quad \text{and} \quad T_2 = \partial_+\beta. \tag{A.11}$$

These expressions can be obtained as follows. Writing $g = a \cdot b \cdot c$ and using the constraints (2.29b), the constrained current reads

$$J = a[T(b) + \partial_+ b \cdot b^{-1}]a^{-1} + \partial_+ a \cdot a^{-1}, \tag{A.12}$$

with $T(b)$ given by (2.29). Although neither T_1 nor T_2 is gauge invariant, the quantity we want to compute, $\Lambda(x)$, is gauge invariant. As a result, it cannot depend on the gauge variables contained in a. Hence we can just as well put $a = 1$ in (A.12). Doing that, the definitions (A.7b) yield (A.11). We thus find the following expression for Λ:

$$\Lambda = (\mu - \partial_+\alpha)^2 \tanh^2\beta + (\partial_+\beta)^2. \tag{A.13}$$

It is an easy matter to show, by using the field equations obtained from the action (A.9),

$$\begin{aligned}
&\sinh^2\beta\,\partial_+\partial_-\alpha + \tanh\beta\left[\partial_+\beta(\partial_-\alpha - \nu) + \partial_-\beta(\partial_+\alpha - \mu)\right] = 0\,,\\
&\cosh^2\beta\,\partial_+\partial_-\beta - \tanh\beta\,(\partial_-\alpha - \nu)(\partial_+\alpha - \mu) = 0\,,
\end{aligned} \tag{A.14}$$

that Λ is indeed chiral, satisfying

$$\partial_-\Lambda = 0\,. \tag{A.15}$$

Moreover one also checks the following Poisson brackets

$$\begin{aligned}
\{\Lambda(x), \alpha(y)\} &= -(\partial_+\alpha - \mu)\,\delta(x^1 - y^1)\,,\\
\{\Lambda(x), \beta(y)\} &= -(\partial_+\beta)\,\delta(x^1 - y^1),
\end{aligned} \tag{A.16}$$

which reproduce the transformations (A.10). Thus the density Λ features all what is expected from the Noether charge density associated with the conformal symmetry.

Finally, we present here for completeness the general solution of the equations of motion (A.14). Along the lines of Section 2.3, it can be obtained as follows:

$$\begin{aligned}
\alpha &= (\eta_L + \eta_R) + \tan^{-1}\left[\frac{\sinh(\theta_L - \theta_R)}{\sinh(\theta_L + \theta_R)}\tan(\lambda_L - \rho_R)\right] + \mu x^+ + \nu x^-,\\
\cosh(2\beta) &= \cosh(2\theta_L)\cosh(2\theta_R) + \sinh(2\theta_L)\sinh(2\theta_R)\cos(2(\lambda_L - \rho_R)),
\end{aligned} \tag{A.17}$$

where $\{\eta_L, \lambda_L, \theta_L\}$ and $\{\eta_R, \rho_R, \theta_R\}$ are arbitrary functions of x^+ and x^- only, respectively, and the three functions of each chirality are related by the equations,

$$\partial_+\eta_L + \partial_+\lambda_L\,\cosh(2\theta_L) = 0\,, \qquad \partial_-\eta_R + \partial_-\rho_R\,\cosh(2\theta_R) = 0\,. \tag{A.18}$$

Appendix B: H-compatible $sl(2)$ and the non-degeneracy condition

Our purpose in this technical appendix is to analyse the notion of the H-compatible $sl(2)$ subalgebra, which has been introduced in Section 3.3. We recall that the $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ of the simple Lie algebra $\mathcal{G}$ is called H-compatible if H is an integral grading operator, $[H\,,\, M_\pm] = \pm M_\pm$, and $M_\pm$ satisfy the non-degeneracy conditions

$$\mathrm{Ker}(\mathrm{ad}_{M_\pm}) \cap \mathcal{G}^H_\mp = \{0\}. \tag{$B.1$}$$

Note that the second property in this definition is equivalent to the fact that $\mathcal{S}$ commutes with $(H - M_0)$. We prove here the results stated in Section 3.3, and also establish an alternative form of the non-degeneracy condition, which will be used in Appendix C.

Let us first consider an arbitrary (not necessarily integral) grading operator H of $\mathcal{G}$ and some non-zero element M_- from $\mathcal{G}^H_{-1}$. We wish to show that to each such pair (H, M_-) there exists an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which $M_+ \in \mathcal{G}^H_{+1}$. ommutes To exhibit the $\mathcal{S}$-triple in question, we need the Jacobson-Morozov theorem, which has already been mentioned in Section 3.3. In addition, we shall also use the following lemma, which can be found in [33] (Lemma 7 on page 98, attributed to Morozov).

Lemma: Let $\mathcal{L}$ be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose $\mathcal{L}$ contains elements h and e such that $[h\,,\, e] = -e$ and $h \in [\mathcal{L}\,,\, e]$. Then there exists an element $f \in \mathcal{L}$ such that

$$[h\,,\, f] = f \quad \text{and} \quad [f\,,\, e] = 2h\ . \tag{$B.2$}$$

Turning to the proof, we first use the Jacobson-Morozov theorem to find generators (m_-, m_0, m_+) in $\mathcal{G}$ completing $m_- \equiv M_-$ to an $sl(2)$ subalgebra. We then decompose the elements m_0 and m_+ into their components of definite grade, i.e., we write

$$m_0 = \sum_n m_0^n \quad \text{and} \quad m_+ = \sum_n m_+^n\ , \tag{$B.3$}$$

where n runs over the spectrum of the grading operator H. Since M_- is of grade -1, it follows from the $sl(2)$ commutation relations that

$$[m_0^0\,,\, M_-] = -M_- \quad \text{and} \quad [m_+^1\,,\, M_-] = 2m_0^0\ , \tag{$B.4$}$$

and these relations tell us that $h = m_0^0$ and $e = M_-$ satisfy the conditions of the above lemma. Thus there exists an element f satisfying (B.2), which we can write as $f = \sum_n f^n$ by using the H-grading again. The proof is finished by verifying that $M_+ \equiv f^1$ and $M_0 \equiv m_0^0$ together with M_- span the required $sl(2)$ subalgebra of $\mathcal{G}$.

From now on, let H be an *integral* grading operator. For an element $M_\pm$ of grade ± 1, respectively, the pair $(H, M_\pm)$ is called *non-degenerate* if it satisfies the corresponding condition in (B.1).

We claim that if $\mathcal{S} = \{M_-, M_0, M_+\}$ is an $sl(2)$ for which the generators $M_\pm$ are from $\mathcal{G}^H_{\pm 1}$, then the non-degeracy of the pairs (H, M_-) and (H, M_+) are *equivalent statements*. This will follow immediately from the $sl(2)$ structure if we prove that the *non-degeneracy of the pair* $(H, M_\pm)$ *is equivalent to the following equality*:

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_\pm}) = \dim \mathcal{G}^H_0 \ . \qquad (B.5)$$

It is enough to prove this latter statement for a pair (H, M_-), since then for a pair (H, M_+) it can be obtained by changing H to $-H$. To prove this let us first rearrange the identity

$$\dim \mathcal{G} = \dim \mathrm{Ker}(\mathrm{ad}_{M_-}) + \dim [M_-, \mathcal{G}] \qquad (B.6)$$

by using the grading as

$$\begin{aligned} \dim \mathrm{Ker}(\mathrm{ad}_{M_-}) - \dim \mathcal{G}^H_0 =& \{\dim \mathcal{G}^H_+ - \dim [M_-, \mathcal{G}^H_+]\} \\ &+ \{\dim \mathcal{G}^H_- - \dim [M_-, \mathcal{G}^H_0 + \mathcal{G}^H_-]\} \end{aligned} . \qquad (B.7)$$

Since both terms on the right hand side of this equation are non-negative, we see that

$$\dim \mathrm{Ker}(\mathrm{ad}_{M_-}) \geq \dim \mathcal{G}^H_0 \ , \qquad (B.8)$$

and equality is achieved here if and only if

$$\dim \mathcal{G}^H_+ = \dim [M_-, \mathcal{G}^H_+] \quad \text{and} \quad [M_-, \mathcal{G}^H_0 + \mathcal{G}^H_-] = \mathcal{G}^H_- \ . \qquad (B.9)$$

On the other hand, we can show that the two equalities in (B.9) are actually equivalent to each other. To see this, let us assume that the second equality in (B.9) is not true. This is clearly equivalent to the existence of some non-zero $u \in \mathcal{G}^H_+$ such that $\langle u\,, [M_-, \mathcal{G}^H_0 + \mathcal{G}^H_-]\rangle = \{0\}$. By the invariance and the non-degeneracy of the Cartan-Killing form, this is in turn equivalent to $[M_-, u] = 0$, which means that the first equality in (B.9) is not true. By noticing that the first equality in (B.9) is just the non-degeneracy condition for the pair (H, M_-), we can conclude that the non-degeneracy condition is indeed equivalent to the equality in (B.5).

We wish to mention a consequence of the results proven in the above. To this let us consider a non-degenerate pair (H, M_-). By our more general result, we know that there exists such an $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ for which M_+ is from $\mathcal{G}^H_{+1}$. The point to mention is that this $\mathcal{S}$ is an H-compatible $sl(2)$ subalgebra, as has already been sated in Section 3.3. In fact, it is now easy to see that this follows from the equivalence of (B.1) with (B.5) by taking into account that the kernels of $\mathrm{ad}_{M_\pm}$ are of equal dimension by the $sl(2)$ structure.

Appendix C: H-compatible $sl(2)$ embeddings and halvings

In Section 3.3, we showed that, given a triple (Γ, M, H) satisfying the conditions for first-classness, conformal invariance and polynomiality (eqs. (2.6), (2.13) and (3.2-4)), the corresponding $\mathcal{W}$-algebra is isomorphic to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, provided that H is an integral grading operator. Here $\mathcal{S} = \{M_-, M_0, M_+\}$ is some $sl(2)$ subalgebra containing $M_- = M$. A natural question is what $sl(2)$ subalgebras arise in this way, or equivalently, given an arbitrary $sl(2)$ subalgebra, can the resulting $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra be obtained as the $\mathcal{W}$-algebra corresponding to the triple (Γ, M, H), for some integral grading operator H? Whether this occurs or not depends only on how the $sl(2)$ is embedded, and it is therefore a pure group-theoretic question. According to Section 3.3, the $sl(2)$ subalgebras having this property are the H-compatible ones. This appendix is devoted to establishing when a given $sl(2)$ embedding is H-compatible, and if so, what the corresponding H is.

The question of an $sl(2)$ being H-compatible is very much related to another one, which was mentioned at the end of Section 4.2. We noted that in some instances, a generalized Toda theory associated to an $sl(2)$ embedding could as well be regarded as a Toda theory associated to an integral grading operator H. This means that the effective action of the theory is a special case of both (4.12) and (4.3) at the same time. We have seen that this is the case when the corresponding halving is H-compatible, i.e., when the Lie algebra decomposition $\mathcal{G} = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}}) + (\mathcal{Q}_{\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{-\frac{1}{2}}) + (\mathcal{P}_{-\frac{1}{2}} + \mathcal{G}_{\leq -1})$ (subscripts are M_0-grades) can be nicely recasted into $\mathcal{G} = \mathcal{G}_{\geq 1}^H + \mathcal{G}_0^H + \mathcal{G}_{\leq -1}^H$. Our second problem, addressed at the end of the appendix, is to find the list of those $sl(2)$ subalgebras which allow for an H-compatible halving. Clearly, an $sl(2)$ subalgebra which possesses an H-compatible halving is also H-compatible in the above sense, but it will turn out that the converse is not true.

Let $\mathcal{S} = \{M_-, M_0, M_+\}$ be an $sl(2)$ subalgebra embedded in a maximally non-compact real simple Lie algebra $\mathcal{G}$. For the classical algebras A_l, B_l, C_l and D_l, these real forms are respectively $sl(l+1, R)$, $so(l, l+1, R)$, $sp(2l, R)$ and $so(l, l, R)$. (We do not consider the exceptional Lie algebras.) For $\mathcal{S}$ to be an H-compatible $sl(2)$, one should find an H in $\mathcal{G}$ with the following properties:

1. ad_H is diagonalizable with eigenvalues being integers,
2. $H - M_0$ must commute with the $\mathcal{S}$-triple,

3. $\dim \mathrm{Ker}(\mathrm{ad}_H) = \dim \mathrm{Ker}(\mathrm{ad}_{M_\pm})$.

We remark that here the equivalence of relations (B.1) and (B.5), proven in the previous appendix, has been taken into account. Under conditions 1-3, the decomposition

$$\Gamma^\perp = [M_-, \Gamma] + \mathrm{Ker}(\mathrm{ad}_{M_+}) \tag{C.1}$$

holds, where $\Gamma = \mathcal{G}^H_{\geq 1}$ in the (Γ, M_-, H) setting, or $\Gamma = \mathcal{P}_{\frac{1}{2}} + \mathcal{G}^{M_0}_{\geq 1}$ in the $sl(2)$ setting, respectively. (For clarity, note that these two gauge algebras are in general not equal.) As a consequence, $J_{\mathrm{red}}(x) = M_- + j_{\mathrm{red}}(x)$ with $j_{\mathrm{red}}(x) \in \mathrm{Ker}(\mathrm{ad}_{M_+})$ is a DS gauge in both settings, and thus the $\mathcal{W}$-algebras are the same.

In order to answer the question of whether an $sl(2)$ embedding is H-compatible, it is useful to know what these embeddings actually are. For a classical *complex* Lie algebra $\mathcal{G}_c$, this question has been completely answered by Malcev (and Dynkin for the exceptional complex Lie algebras) [39]. The result can be nicely stated in terms of the way the fundamental vector representation reduces into irreducible representations of the $sl(2)$:

A_l : the $sl(2)$ reduction of the (l+1)-dimensional representation can be arbitrary,

B_l : the $(2l+1)$-dimensional representation of B_l reduces in such a way that the multiplicity of each $sl(2)$ *spinor* appearing in the reduction is even,

C_l : the $2l$-dimensional representation of C_l reduces in such a way that the multiplicity of each $sl(2)$ *tensor* appearing in the reduction is even,

D_l : same restriction as the B_l series: the spinors come in pairs.

The above conditions are necessary and sufficient, i.e., every possible $sl(2)$ content satisfying the above requirements actually occurs for some $sl(2)$ embedding. Moreover, for the classical complex Lie algebras, the way the fundamental reduces completely specifies the $sl(2)$ subalgebra, up to automorphisms of the embedding $\mathcal{G}_c$ [39].

The above description of the $sl(2)$ embeddings remains valid for the maximally non-compact classical real Lie algebras, except the last statement. First of all, this means that the above restrictions apply to the possible decompositions of the fundamental under the $sl(2)$ subalgebras in the real case as well. It is also obvious that those $sl(2)$ embeddings for which the content of the fundemantal is different are inequivalent. The converse however ceases to be true in the real case in general: inequivalent $sl(2)$ subalgebras can have the same multiplet content in the fundamental of $\mathcal{G}$. The answer to the problem of H-compatibility will in fact be provided by looking more closely at the decomposition of the fundamental of $\mathcal{G}$ under the $sl(2)$ subalgebra in question, as will be clear below.

As an immediate consequence of condition 2, $H - M_0$ is an $sl(2)$ invariant and can only depend on the value of the Casimir. If, in the reduction of the fundamental of $\mathcal{G}$, a spin j representation occurs with multiplicity m_j, the $sl(2)$ generators $\vec{M}$ and H can be written

$$\vec{M} = \sum_j \vec{M}^{(j)} \times I_{m_j}, \tag{C.2a}$$

$$H = M_0 + \sum_j I_{2j+1} \times D(j), \tag{C.2b}$$

where I_n denotes the unit $n \times n$ matrix, and the $D(j)$'s are $m_j \times m_j$ diagonal matrices. Hence, within each irreducible representation of $sl(2)$, H is equal to M_0 shifted by a constant. Obviously, this is also true in the adjoint representation and, in turn, this implies that ad_H takes the value zero at most once in each $sl(2)$ multiplet in the adjoint of $\mathcal{G}$. From condition 3, ad_H must take the value zero exactly once, i.e., each $sl(2)$ representation must intersect $\mathrm{Ker}(\mathrm{ad}_H)$ exactly once. In particular, the $sl(2)$ singlets must be ad_H-eigenvectors with zero eigenvalue.

The trivial solution $H = M_0$ exists whenever ad_{M_0} is diagonalizable on the integers, i.e., when the reduction of the fundamental of $\mathcal{G}$ is either purely tensorial or purely spinorial. From now on, we suppose that the reduction involves both kinds of $sl(2)$ representations.

1) A_l algebras.

The problem for the A_l series is simple to solve since, in this case, an H *always* exists. As a proof, we explicitly give an H which fulfills all the requirements. In (C.2b), we set

$$D(j) = \begin{cases} \lambda \cdot I_{m_j} & \text{if } j \in N, \\ (\lambda + \frac{1}{2}) \cdot I_{m_j} & \text{if } j \in N + \frac{1}{2}, \end{cases} \tag{C.3}$$

where λ is a constant that makes H traceless. In order to show that the H so defined has the required properties, we recall that for the A_l algebras, the adjoint representation is obtained by tensoring the fundamental with its contragredient. As a result, the roots are the differences of the weights of the fundamental (up to a singlet) and we have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) - D(j_2)], \tag{C.4}$$

where j_1 and j_2 are the spins of the states in the fundamental representation from which a given state in the adjoint representation is formed. That the conditions 1-3 are satisfied is obvious from the fact that $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on tensors and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm \frac{1}{2}$ on spinors, with $+\frac{1}{2}$ occurring as many times as $-\frac{1}{2}$.

It should be pointed out that (C.3) is by no means the only solution. Since in the product $j_1 \times j_2$, the highest weights have an M_0-eigenvalue at least equal to $|j_1 - j_2|$, another solution is given by $D(j) = (\lambda + j) \cdot I_{m_j}$.

2) C_l algebras.

For the symplectic algebras, the adjoint representation is obtained from the *symmetric* product of the fundamental with itself and we therefore have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) + D(j_2)]. \tag{C.5}$$

Since the symmetric product of a tensor with itself produces a singlet, which must belong to $\mathrm{Ker}(\mathrm{ad}_H)$, we have $2D(t) = 0$ for every integer $j = t$. Hence in the fundamental representation, $H = M_0$ on tensors. Similarly, the symmetric product of a spinor with itself always produces a triplet, one member of which must belong to $\mathrm{Ker}(\mathrm{ad}_H)$. This implies that the diagonal entries of $2D(s)$ are either 0 or ± 1, for every half-integer $j = s$. However $D(s)$ cannot have a zero on the diagonal, because ad_H would not be integral on the representations contained in $s \times t$. Therefore, in the fundamental, $H = M_0 \pm \frac{1}{2}$ on spinors.

Let us now look at the m_s spinor representations of spin s, say $s^1, s^2, \ldots, s^{m_s}$. The product $s^i \times s^j$ of any two of those contains a singlet, and that implies $D(s^i) + D(s^j) = 0$. This equality must hold for any pair of spin s representations, which is impossible unless $m_s \leq 2$.

Let us consider the restriction g_s of the symplectic form to the spin s representations. The restricted form is non-degenerate, because the original non-degenerate metric is block-diagonal with respect to the eigenvalues of the $sl(2)$ Casimir.

If $m_s = 1$, then the H given by $M_0 \pm \frac{1}{2} \cdot I$ on the unique spin s representation, should be in the symplectic algebra: $g_s H + H^t g_s = 0$. Since M_0 is already symplectic, we require that the identity be symplectic, which is impossible for a non-degenerate form. Hence m_s must be 2.

If $m_s = 2$, $H - M_0$ and g_s look like (in the basis where M_0 and H are diagonal)

$$H - M_0 = \pm \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad g_s = \begin{pmatrix} a & b \\ -b^t & c \end{pmatrix}, \tag{C.6}$$

where the blocks a and c are antisymmetric. $H - M_0$ being symplectic leads to $a = c = 0$.

To summarize, for an integral H to exist, the $sl(2)$ embedding must be such that: (i) the multiplicity of any spinor representation in the fundamental of $\mathcal{G}$ is 2, (ii) if (s, s') is such a pair of spinors, they must be the dual of each other with respect to the symplectic form. If these two conditions are met, then H is given in the fundamental by

$$H = \begin{cases} M_0 & \text{on tensors,} \\ M_0 +/- \frac{1}{2} & \text{on a pair of spinors } s/s'. \end{cases} \tag{C.7}$$

Conditions 1-3 are satisfied since (C.7) implies $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on singlets, $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm (1$ or $0)$ on tensors and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm \frac{1}{2}$ on spinors.

3) B_l and D_l algebras.

The analysis here is similar to what has been done in 2), and we can therefore go through the proof quickly.

For the orthogonal algebras, the adjoint is got from the *antisymmetric* product of the fundamental with itself and we still have

$$\mathrm{ad}_H = \mathrm{ad}_{M_0} + [D(j_1) + D(j_2)]. \tag{C.8}$$

The antisymmetric product of a tensor (spinor) with itself produces a triplet (singlet), so that with respect to the symplectic algebras, the situation is reversed in the sense that the tensors and the spinors have their roles interchanged: $H = M_0 \pm \frac{1}{2}$ on tensors, $H = M_0$ on spinors and $m_t \leq 2$ for any tensor representation of spin t.

If as in 2), we look at the restriction g_t of the orthogonal metric to the spin t tensors, we have $m_t = 2$ on account of the non-degeneracy of g_t. From this, we get at once that there can be no solution for the B_l algebras. Indeed, the fundamental being odd-dimensional, at least one tensor representation must come on its own.

On the $2(2t+1)$-dimensional subspace made up by the two spin t tensors, $H - M_0$ and g_t take the form

$$H - M_0 = \pm \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad g_s = \begin{pmatrix} a & b \\ b^t & c \end{pmatrix}, \tag{C.9}$$

where a and c are now symmetric. Requiring that $H - M_0$ be orthogonal, we again obtain $a = c = 0$.

Therefore, for the orthogonal algebras, we get the following conclusions. There is no solution for the B_l series if the $sl(2)$ embedding is not integral. As to the D_l series, the $sl(2)$ embedding must be such that: (i) every tensor in the fundamental of $\mathcal{G}$ has a multiplicity equal to 2, (ii) if (t, t') is such a pair of tensors, they must be the dual of each other with respect to the orthogonal metric. In this case, H is given in the fundamental by

$$H = \begin{cases} M_0 +/- \frac{1}{2} & \text{on a pair of tensors } t/t', \\ M_0 & \text{on spinors.} \end{cases} \tag{C.10}$$

Summarizing the analysis, the H-compatible $sl(2)$ embeddings are the following ones:

A_l : any $sl(2)$ subalgebra,

B_l : only the integral $sl(2)$'s,

C_l : those for which each *spinor* occurs in the fundamental of C_l with a multiplicity 0 or 2, the pairs of spinors being symplectically dual,

D_l : those for which each *tensor* occurs in the fundamental of D_l with a multiplicity 0 or 2, the pairs of tensors being orthogonally dual.

The reader may wish to check that the above results are consistent with the isomorphisms $B_2 \sim C_2$ and $A_3 \sim D_3$.

We now come to the second question alluded to at the beginning of this appendix, namely the problem of H-compatible halvings. From the definition, an $sl(2)$ subalgebra allows for an H-compatible halving if in addition to conditions 1-3 one also has

4. $\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} = \mathcal{G}^H_{\geq 1}$, and $\mathcal{P}_{-\frac{1}{2}} + \mathcal{G}_{\leq -1} = \mathcal{G}^H_{\leq -1}$.

In particular, this fourth condition implies $\mathcal{G}_0^{M_0} \subset \mathcal{G}_0^H$. So we readily obtain that H and M_0 must satisfy

$$\mathrm{ad}_H = \mathrm{ad}_{M_0}, \qquad \text{on} \quad \text{tensors}, \tag{C.11}$$

since we know, from the previous analysis, that $\mathrm{ad}_H - \mathrm{ad}_{M_0}$ is a constant in every representation (condition 2). Therefore, we can simply look at those solutions of the first problem which satisfy (C.11) and check if condition 4 is fully satisfied or not. We get that the $sl(2)$ embeddings allowing for an H-compatible halving are as follows:

A_l : any $sl(2)$ subalgebra. There are only two solutions for H given by setting in (C.2b): $D(j) = (\lambda \pm \epsilon(j)) \cdot I_{m_j}$ with $\epsilon(j) = 0/\frac{1}{2}$ for a tensor/spinor,

B_l : only the integral $sl(2)$'s with $H = M_0$,

C_l : only the integral $sl(2)$'s,

D_l : the integral $sl(2)$'s, and those for which the fundamental of D_l reduces into spinors and two singlets, with H given by (C.10).

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