THE U(1)-ANOMALY, THE NON-COMPACT INDEX THEOREM, AND THE (SUPERSYMMETRIC) BA-EFFECT

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The fractional discrepancy between the global U(1) chiral anomaly (described by a flux-integral of gauge-fields and not necessarily an integer on non-compact, euclidean space–times) and the index of the Dirac operator $\not{D}$ is shown to be just $(\delta_+(0)-\delta_-(0))/\pi$ where $\delta_\pm(0)$ are the left- and right-handed zero energy phase shifts.

It is generally accepted that once the quantity $\Gamma = \ln\det(\mathrm{i}\not{D})$, where $\not{D}$ is the Dirac operator, is regularized so as to take care of both its infra-red (IR) and ulta-violet (UV) divergences then in even $(d=2n)$ dimensions, its U(1) chiral variation $\delta\Gamma$, or chiral anomaly [1], is given by the formula

$$\delta\Gamma = \delta \ln \det{}_{\mathrm{reg}}(\mathrm{i}\not{D}) = K\int \alpha(x)\phi(x)\,\mathrm{d}^{2n}x,$$
$$K=(2\mathrm{i}/n!)(1/4\pi)^n, \tag{1}$$

where reg denotes both regularizations, $\alpha(x)$ is the infinitesimal parameter of U(1) chiral transformations, and $\phi(x)$ is a pseudo-scalar which is a divergence of a local function of the gauge-potentials i.e.

$$\phi(x) = \epsilon^{\alpha_1\cdots\alpha_{2n}}F_{\alpha_1\alpha_2}\ldots F_{\alpha_{2n-1}\alpha_{2n}} = \partial_\alpha\phi_\alpha(x). \tag{2}$$

Furthermore, this formula holds not only if $\alpha(x)$ is purely local ($\alpha(x)\to 0$ as $|x|\to\infty$), which is the case usually considered in perturbative field theory [2], and if $\alpha(x)$ is a constant ($\alpha(x)=c$ for all x) which is the case usually considered in geometric discussions of the global anomaly, but also in the more general case that $\alpha(x)$ is local in the sense that it may vary with x, but $\alpha(x)\to c\neq 0$ as $|x|\to\infty$ [3].

In this paper we wish to establish three results concerning the chiral anomaly (1). First, we wish to show that the formula (1) has a natural decomposition into a local and a global (infra-red) part i.e. that

$$\frac{\partial\Gamma}{\partial\alpha_{,\mu}(x)} = \frac{\partial}{\partial\alpha_{,\mu}(x)}\ln\det{}_{\mathrm{U}}(\mathrm{i}\not{D}) = \phi_\alpha(x),$$
$$\frac{\partial\Gamma}{\partial c} = \oint \phi_\alpha(x)\,\mathrm{d}s^\alpha, \tag{3}$$

where $\partial\Gamma/\partial\alpha_{,\mu}$ contains no IR-divergent, and $\partial\Gamma/\partial c$ no UV-divergent part. This decomposition is, of course, obvious on a compact space, where it amounts to no more than the extraction of the zero modes of $\mathrm{i}\not{D}$, but our point is that there exists a natural infra-red regularization for which it holds even on non-compact spaces, for which the continuous spectrum of $\mathrm{i}\not{D}$ is not, in general, bounded away from zero.

We then concentrate on the IR or, global, chiral variation and our second result is to show that this part is given by the formula

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$$\frac{\partial}{\partial c}\ln\det(\mathrm{i}\not{D}) = 2\mathrm{i}\ \mathrm{tr}\ \Delta P\gamma_5$$
$$= 2\mathrm{i}(n_+ - n_-) + \frac{2\mathrm{i}}{\pi}\sum_k\left(\delta_+^k(0) - \delta_-^k(0)\right), \quad (4)$$

where $\mathrm{tr}(\Delta P\gamma_5)$ denotes the jump at zero-energy in the trace measure associated with the spectral projections $P(\lambda)$ of $-\not{D}^2$, $n_\pm$ denote the left- and right-handed zero modes of $\mathrm{i}\not{D}$, and $\delta_\pm^k(0)$ means the limit of the scattering phase shifts for $-\not{D}^2$ as the energy tends to zero (labelled in a suitable angular momentum basis k). This formula shows that in the non-compact case the usual zero-mode contribution to the global anomaly is supplemented by a contribution from the continuous spectrum of $\mathrm{i}\not{D}$, and that this contribution has a simple physical interpretation in terms of phase-shifts.

Finally, by eliminating the anomaly $\partial\Gamma/\partial c$ from eqs. (4) and (3) one obtains the independent relationship

$$\oint \phi_\alpha(x)\,\mathrm{d}s^\alpha = 2\mathrm{i}(n_+ - n_-) + \frac{2\mathrm{i}}{\pi}\sum_k\left(\delta_+^k(0) - \delta_-^k(0)\right), \quad (5)$$

between the surface integral of the gauge-fields and the zero-modes and phase-shifts of the differential operator $-\not{D}^2$. This formula is evidently a generalization to non-compact spaces of the Atiyah–Singer index theorem [4], and since it is independent of the anomaly, our third result is to derive it directly i.e. using only ordinary quantum mechanical scattering theory for the simplest case of a two-dimensional space ($d = 2$). The case $d = 2$ has the added interest that the phase-shifts $\delta_\pm(0)$ reduce to the (supersymmetric) Bohm–Aharonov phase-shifts [5]. It may be amusing to note that eq. (5) actually incorporates three well-known but apparently uncorrelated results, namely, the Levinson theorem, the Atiyah–Singer index theorem and the Bohm–Aharonov theorem, as can be seen by putting the flux term, the phase-shift term, and the bound state term respectively equal to zero!

To establish the above results we first note that the IR and UV divergences of $\Gamma = \mathrm{tr}\ln(\mathrm{i}\not{D})$ have two very different origins. The IR-divergence comes from the fact the *operator* $\ln(\mathrm{i}\not{D})$ does not exist when the spectrum of $\mathrm{i}\not{D}$ is not bounded away from zero. This means that it can be removed by modifying the operator $\mathrm{i}\not{D}$, and the usual modification is to introduce a small imaginary mass-term i.e. by letting $\mathrm{i}\not{D} \to \mathrm{i}(\not{D} + m)$. However, because m is not chirally invariant ($m \to m\exp(2\mathrm{i}\gamma_5\alpha)$ it is actually more convenient to replace m by a small chiral doublet $M = m + \mathrm{i}n\gamma_5$, i.e. to let $\mathrm{i}\not{D} \to \mathrm{i}(\not{D} + M)$ where the doublet (m, n) rotates under chiral transformations (if desired M may be thought of as the (small) vacuum expectation value of a chiral field doublet [6] after a spontaneous breakdown of chiral symmetry). The UV-divergence, on the other hand, comes from the fact that the *trace* of the operator $\ln\mathrm{i}(\not{D} + M)$ does not exist, and thus it is removed by using one of the conventional UV-regularization schemes. Having removed it in this way one may write

$$\Gamma = \mathrm{tr}_\mathrm{U}\ln\mathrm{i}(\not{D} + M), \quad (6)$$

where U denotes UV-regularization, and it makes sense to talk of chiral variations of Γ. It is then easy to see that the chiral variation of Γ decomposes naturally into two parts corresponding to the variation $\mathrm{i}\not{D}$ and M;

$$\delta\Gamma = \delta\Gamma_\mathrm{D} + \delta\Gamma_M$$
$$= \mathrm{tr}_\mathrm{U}(\not{D} + M)^{-1}\delta\not{D} + \mathrm{tr}(\not{D} + M)^{-1}\delta M \quad (7)$$

and the advantage of the decomposition is that it is simultaneously a decomposition into parts which are proportional to $\alpha(x)_{,\mu}$ and $\alpha(x)$, and into parts which are IR- and UV-convergent respectively. That is to say,

$$\delta\Gamma_\mathrm{D} = \mathrm{tr}_\mathrm{U}\left[(\not{D} + M)^{-1}\mathrm{i}\gamma^\mu\gamma_5\alpha(x)_{,\mu}\right], \quad (8)$$

which is IR-convergent in the sense that the limit $M \to 0$ may be taken inside the trace (and is actually only mildly UV-divergent) and

$$\delta\Gamma_M = \mathrm{tr}\left[(\not{D} + M)^{-1}2\mathrm{i}\gamma_5 M\alpha(x)\right]$$
$$= 2\mathrm{i}\ \mathrm{tr}\left\{\left[\frac{\rho^2}{\rho^2 - \not{D}_+^2} - \frac{\rho^2}{\rho^2 - \not{D}_-^2}\right]\alpha(x)\right\}, \quad (9)$$

where ρ^2 denotes the chiral invariant combination $\rho^2 = m^2 + n^2$ and $\not{D}_\pm = \frac{1}{2}(1 \pm \gamma_5)\not{D}$, which is UV-convergent on account of the minus sign (due to

γ_5). From now on we shall consider only the UV-convergent variation $\delta\Gamma_M$ which contains all the infra-red information.

When $\alpha(x)$ is constant and the zero-eigen values of $(-\not{D}^2)$ are isolated (as happens typically in the compact case) one sees at once from (9) that

$$\lim_{\rho\to 0} \frac{\partial \Gamma_M}{\partial \alpha} = 2i(n_+ - n_-), \tag{10}$$

where $n_\pm$ are the multiplicities of the left- and right-handed zero modes of $i\not{D}$. In general, however, the zero-eigenvalues of $-\not{D}^2$ are not isolated (the continuous part of the spectrum stretches down to zero) and eq. (10) must be modified. By using the spectral representations of $-\not{D}^2_\pm$ it is evident that in this more general case the formal modification is of the form

$$\lim_{\rho\to 0} \frac{\partial \Gamma_M}{\partial \alpha} = 2i \lim_{\rho\to 0} \rho^2 \int_0^\infty \frac{d\sigma(\lambda)}{\rho^2 + \lambda}$$
$$= 2i\sigma(0) = 2i\sigma_+(0) - 2i\sigma_-(0), \tag{11}$$

where $\sigma(\lambda)$ is the trace of the difference of the spectral measures $P_+(\lambda) - P_-(\lambda)$ where $-\not{D}^2_\pm = \int_0^\infty \lambda \, dP_\pm(\lambda)$ and $\sigma(0)$ means the limit of $\sigma(\lambda)$ as λ tends to zero from the + direction. (Note that in contrast to the ordinary measures $(f, P(\lambda) f)$ the measure $\sigma(\lambda)$ may have a discontinuity not associated with a bound state. Note also that the total trace $\sigma(\infty)$ is not necessarily unity, or even finite.)

What we now wish to show is that the formula (11) leads to eq. (4) i.e. $\sigma(\lambda)$ is just the sum of the phase shifts. This result is actually a consequence of a more general statement, namely, that for any Schrödinger hamiltonian H the projection valued spectral measure $P(\lambda)$ is just the logarithm of the S-matrix (on the mass-shell). That is to say, there is a general result

$$P_0(E) P(\lambda) P_0(E)$$
$$= k(E, \lambda) + P_0(E) \ln S(\lambda) P_0(E), \tag{12}$$

where $P_0(E)$ are the projections onto scattering states of energy E of the free hamiltonian and $k(E, \lambda)$ is a universal function (independent of H), and the result for $\delta(\lambda)$ then follows by noting that $\ln S = 2i\delta$ and taking the trace of (12) for $P = (P_+ - P_-)$. Since the general result (12) does not appear to be well-known (at least in this direct form) we now sketch the derivation. First by using the representation $(1/2\pi i) \lim_{\epsilon\downarrow 0} \ln[(s - i\epsilon)/(s + i\epsilon)]$ for the characteristic function $\theta(s)$ of the positive real axis, we see that for any positive hamiltonian H,

$$P(\lambda) = \int_0^\infty dP(x)\, \theta(\lambda - x)$$
$$= \frac{1}{2\pi i} \lim_{\epsilon\downarrow 0} \int_0^\infty dP(x) \ln \frac{\lambda - x - i\epsilon}{\lambda - x + i\epsilon}$$
$$= \frac{1}{2\pi i} \lim_{\epsilon\downarrow 0} \ln \frac{\lambda - H - i\epsilon}{\lambda - H + i\epsilon}, \tag{13}$$

an equation which expresses the projections $P(\lambda)$ as explicit functions of the operator. Eq. (12) then follows by sandwiching the logarithm of the identity

$$\frac{\lambda - H - i\epsilon}{\lambda - H + i\epsilon} = \frac{1}{\lambda - H_0 + i\epsilon} \Sigma(\epsilon, \lambda)(\lambda - H_0 - i\epsilon), \tag{14}$$

where

$$\Sigma(\epsilon, \lambda) = \left[1 - V\left(1 + \frac{1}{\lambda - H + i\epsilon} V\right) \times \frac{2i\epsilon}{(\lambda - H)^2 + \epsilon^2}\right],$$
$$V = H - H_0, \tag{15}$$

between the free-projections $P_0(E)$, and noting that the limit as $\epsilon \to 0$ of $\Sigma(\epsilon, \lambda)$ is just the S-matrix $S(\lambda)$ as conventionally defined [7]. Thus finally

$$P_0(E)[P_+(\lambda) - P_-(\lambda)] P_0(E)$$
$$= \frac{1}{\pi} P_0(E)[\delta_+(\lambda) - \delta_-(\lambda)] P_0(E), \tag{16}$$

where $\delta_\pm(\lambda)$ are the generalized phase shifts. On taking the trace in an angular momentum basis (for fixed E) one then sees that $\pi\delta(\lambda)$ is just the sum of the conventional phase shifts at energy λ.

Our final task is to give a direct derivation of the generalized index theorem (5) using only the theory of differential equations. Such derivations are enormously simplified by exploiting the fact that the operator $-\not{D}^2$ is supersymmetric [8], i.e.

$$-\not{D}^2 = \{Q^+, Q^-\},$$

$$\text{where } Q^\pm = \tfrac{1}{2}(1 \pm \gamma_5)\mathrm{i}\not{D}\tfrac{1}{2}(1 \mp \gamma_5) \tag{17}$$

and in the two-dimensional case that we shall consider this reduces to the statement that

$$\mathrm{i}\not{D} = \begin{pmatrix} 0 & \mathrm{i}D_1 + D_2 \\ \mathrm{i}D_1 - D_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}. \tag{18}$$

For simplicity we shall treat the case where the two-dimensional gauge field is spherically symmetric i.e. $F_{ij}(x) = \epsilon_{ij}B(x)$ where $B(x) = B(r)$, and is of compact support i.e. there is a radius $r = a$ such that $B(r) = 0$ for $r \geqslant a$, and only sketch the generalization to arbitrary smooth B-fields of finite flux. A great advantage of the supersymmetric formulation in the two-dimensional case is that the zero modes of $-\not{D}^2$ are just the zero modes of the first-order differential operators $\not{D}$ i.e.

$$D_+\varphi_+ = 0, \quad D - \varphi_- = 0, \tag{19}$$

respectively, and this circumstance allows us to obtain an expression for the flux through any circle of radius r (not necessarily $r \geqslant a$) in terms of the radial derivative and orbital angular momentum m of the fields, even without solving the equations explicitly. That is to say, by expressing (19) in polar coordinates (r, θ) and integrating over θ one has

$$\begin{aligned} 2\pi\phi(r) &= \oint A_\theta r\,\mathrm{d}\theta \\ &= -\oint \frac{\partial_\theta \varphi_\epsilon}{\varphi_\epsilon}\mathrm{d}\theta - \epsilon\oint\frac{D_r\varphi_\epsilon}{\varphi_3} r\,\mathrm{d}\theta \\ &= -2\pi m - \epsilon\oint\frac{D_r\varphi_\epsilon}{\varphi_\epsilon} r\,\mathrm{d}\theta, \end{aligned} \tag{20}$$

where m is the angular momentum of the field φ_ϵ and $\epsilon = \pm$ according to whether $\varphi_\epsilon = \varphi_+$ or φ_-. Note that each integral in (20) is seperately gauge-invariant and that for $r \geqslant a$, $\phi(r)$, and hence the $(\not{D}_r\varphi)$-integral become independent of r. From eq. (20) one can already find the bound states, since they must be smaller than r^{-1} both as $r \to 0$ and as $r \to \infty$. Hence they are just those for which

$$m\epsilon < 1, \quad (\phi + m)\epsilon > 1, \tag{21}$$

where ϕ denotes $\phi(r)$ for $r \geqslant a$ and we have used the fact that $\phi(0) = 0$. It follows from (21) that $\phi\epsilon > 0$. Hence for each sign of ϕ there can be bound states for only one choice of ϵ, and then only if m has the same sign as $-\phi$ and $0 \leqslant |m| < |\phi| - 1$. In particular, for $|\phi| \leqslant 1$ there are no bound states of any kind [‡1]. In this way one sees that (20) is already sufficient to establish the integer part of the index theorem i.e.

$$[\phi] = n_+ - n_-, \tag{22}$$

where $[\phi]$ denotes the integer part of ϕ for the generic case when ϕ is not an integer. This is the case in which we are most interested, but for the record it should be mentioned that if ϕ is integer (22) becomes [‡1] $\Delta n = |\phi| - 1$ for $|\phi| \geqslant 2$.

Our main interest, however, is the fractional part. For that we have to investigate the continuous part of the spectrum of $-\not{D}^2$, especially at low energy and that is described by the *second-order* differential equations

$$-\not{D}^2\psi = k^2\psi \Rightarrow \left(-D^2 + \tfrac{1}{2}\epsilon B\right)\psi_\epsilon = k^2\psi_\epsilon,$$

$$k^2 = E > 0, \quad \epsilon = \pm 1. \tag{23}$$

Although we cannot completely solve these second-order differential equations explicitly, we are saved by the fact that we can solve them explicitly in the exterior region $r \geqslant a$, and we can approximate them by the zero-energy solutions in the interior region (for $k \to 0$) and have good control on the error because the inside region $r \leqslant a$ is compact. So the problem reduces essentially to matching the inside and outside solutions at $r = a$, and since the overall normalization does not matter for phase-shift analysis, and the system is spherically symmetric, the matching problem reduces to matching the log-derivatives $\gamma = (r\partial_r\psi)/\psi$ at $r =$

‡1 There is no eigenvalue for $|\phi| = 1$, and only $|\phi| - 1$ eigenvalues for $|\phi|$ integer and $|\phi| > 1$ because in two dimensions functions which fall off like r^{-1} are not square-integrable. The discrepancy between this formula and the more conventional result $|\Delta n| = |\phi|$ may be removed by changing the range of the phase shift from the conventional range $0 \leqslant \delta < \pi$ to $0 < \delta \leqslant \pi$.

a. We now consider the outside and inside solutions in turn.

Outside. In the radial gauge $A_r = 0$, the gauge-potential reduces to the usual BA-potential $A_\theta = \phi/r$ outside, and since $B = 0$, eq. (23) reduces to the Bessel equation of order W where $W = |m + \phi|$, for each angular momentum m. Thus the general outside solution (for non-integer W) is

$$\psi = \sum \psi_m(r)\, e^{im\theta},$$
$$\psi_m(r) = \alpha J_W(kr) + \beta J_{-W}(kr), \tag{24}$$

where $J_{\pm W}$ are the conventional Bessel functions, and α and β are constants, whose ratio determines the phase-shift. In fact, from the asymptotic ($kr \to \infty$) form of the Bessel functions

$$J_W(kt) \to (\tfrac{1}{2}kr)^{-1/2}\cos(kr - \tfrac{1}{2}W\pi - \tfrac{1}{4}\pi), \tag{25}$$

one sees at once that the scattering phase-shift (relative to $\phi = 0$) is given by

$$\tan\delta = \left(\frac{\beta-\alpha}{\beta+\alpha}\right)\tan(\tfrac{1}{2}\pi W). \tag{26}$$

In particular $\tan\delta = \pm\tan\tfrac{1}{2}\pi W$ for $\alpha = 0$ and $\beta = 0$ respectively.

Inside. As we have said, the inside solutions ψ may be approximated by the inside part of the zero-energy solutions φ in (19) (whether or not the φ themselves correspond to bound states), and since all that we shall need for matching are the log-derivatives ($r\partial_r \ln\psi$) theerror can be controlled by controlling the quantity $(r\partial_r \ln(\psi/\varphi))_a$ which from (23) is easily seen to be

$$(r\,\partial_r \ln(\psi/\varphi))_a = -k^2\int_0^a \varphi\psi r\,\mathrm{d}r \Big/ \varphi(a)\psi(a)$$
$$= -k^2\int_0^a \varphi^2 r\,\mathrm{d}r \Big/ \varphi^2(a). \tag{27}$$

Thus, to first order in k^2

$$\gamma(\text{inside}) = \left(\frac{r\,\partial_r\psi}{\psi}\right)_a = \left(\frac{r\,\partial_r\varphi}{\varphi}\right)_a - (ka)^2\Delta^2,$$
$$\Delta^2 = \int_0^a \varphi^2 r\,\mathrm{d}r \Big/ a^2\varphi^2(a) > 0, \tag{28}$$

where Δ^2 is strictly positive and independent of k^2. Hence, for matching, we need only know the zero-energy quantity $(r\,\partial_r\varphi/\varphi)_a$, and, as already emphasized in the bound-state discussion, this can be obtained from the flux-equation (20) without actually solving for $\varphi(r)$, $r \leqslant a$ explicitly. In fact from (20) one has

$$(r\partial_r\varphi/\varphi)_a = -\epsilon(\phi + m) \tag{29}$$

and thus finally

$$\gamma(\text{inside}) = -\epsilon(\phi + m) - (ka)^2\Delta^2, \quad \Delta^2 > 0. \tag{30}$$

Matching. To match with γ(inside) we must now compute γ(outside), and for this we use the fact that since a is fixed and $k \to 0$, the Bessel functions $J_W(ka)$ may be approximated for $k \to 0$ by their values in the neighbourhood of the origin. Thus for small k and fixed a one has

$$\psi_{\text{outside}}(ka) \simeq \frac{\alpha}{\Gamma(1+W)}(\tfrac{1}{2}ka)^W + \frac{\beta}{\Gamma(1-W)}(\tfrac{1}{2}ka)^{-W} \tag{31}$$

From this equation it is easy to see that

$$\frac{\beta}{\alpha} = \frac{\Gamma(1-W)}{\Gamma(1+W)}(\tfrac{1}{2}ka)^{2W}\left(\frac{W-\gamma}{W+\gamma}\right),$$
$$\text{where } \gamma = \left(\frac{r\,\partial_r\psi_{\text{outside}}}{\psi_{\text{outside}}}\right)_a \tag{32}$$

is the relationship between the ratio β/α and the matching log-derivative γ. On making the match $\gamma(\text{inside}) = \gamma$ from (30) one then obtains

$$\frac{\beta}{\alpha} = \frac{\Gamma(1-W)}{\Gamma(1+W)}(\tfrac{1}{2}ka)^{2W} \times \frac{W + \epsilon(\phi+m) + (ka)^2\Delta^2}{W - \epsilon(\phi+m) - (ka)^2\Delta^2}, \tag{33}$$

where $W = |\phi + m|$, as the equation to determine β/α.

From eq. (33) one sees that the ratio $\beta/\alpha \to 0$ as $k \to 0$ and hence the phase shift is given by $\tan\delta = -\tan(\pi W/2)$ in all cases except when $\epsilon(\phi + m) = W < 1$, or, equivalently, $0 < \epsilon(m + \phi) < 1$, in which cases the $\alpha/\beta \to 0$ as $k \to 0$ and $\tan\delta = +\tan(\pi W/2)$. Thus for each flux $2\pi\phi$ and chirality ϵ there is only one special value m_s of m for which the sign of the phase-shift is the reverse of the normal (BA) sign, and, furthermore, since the inside solutions have to be regular at the origin, there is the further condition $m\epsilon < 1$ from eq. (21), which show that the sign-reversal takes place for only one sign of ϵ for each given ϕ. In other words, the anomalous phase-shift $\tan \delta = \tan(\pi W/2)$ occurs for only one angular momentum sector $m = m_s$ and only one chirality, and its value is just π-times the fractional part f of ϕ, i.e.

$$\delta^m(0) = \delta_+^m(0) - \delta_-^m(0) = \pi f \delta_{m m_s}. \tag{34}$$

Combining this result with the result (22) for the bound states we see finally that

$$\phi = [\phi] + f = (n_+ - n_-) + f$$
$$= (n_+ - n_-) + \frac{1}{\pi} \sum_m (\delta_+^m(0) - \delta_-^m(0)), \tag{35}$$

which, since $2\pi\phi$ is just the flux-integral (1) for the two-dimensional case, establishes the result for that case. In fact we have established a little more, namely, that the integer contribution actually comes from the angular momentum sectors $|m| < |\phi| - 1$ $(m\phi \leqslant 0,\ \epsilon\phi \geqslant 0)$, with unit multiplicity for each m, and that the fractional contribution comes from only the "missing" angular momentum sector $|\phi| - 1 < |m| < |\phi|$ $(m\phi \leqslant 0,\ \epsilon\phi \geqslant 0)$. Note that the formula (35) holds also for the case of integer ϕ if one lets $f \to 1$ and $\delta_+ - \delta_- \to \pi$ (not $f \to 0$ and $\delta_+ - \delta_- \to 0$). Those familiar with supersymmetric quantum mechanics will also note that the existence of contributions only for the angular momenta $|m| < |\phi| - 1$ $(m\phi \leqslant 0,\ \epsilon\phi \geqslant 0)$ implies that these are the only angular momentum sectors for which the supersymmetry is not spontaneously broken, and this can be verified directly. Finally it might be mentioned that, although we have restricted ourselves to the spherically symmetric, compact support case for clarity, there is no real difficulty in extending the results to the general two-dimensional case. This is because the finite-flux condition $\int B(x)\, d^2x < \infty$ implies that B falls off as $r \to \infty$, and that the system is asymptotically spherically symmetric. (In the non-asymptotic region the angular momentum generalizes to a winding number.)

After this work was completed we became aware of two other recent derivations [9,10] of the central formula (5). However the approach in these two derivations is quite different, in fact complementary, to ours. In particular in both derivations the relationship between the high- and low-energy parts of the anomaly is used to obtain information about the low-energy part (using the eikonal approximation in the case of ref. [10]) whereas we go immediately to the low-energy (global) anomaly and derive the information directly using the relation (9). Furthermore, in ref. [9] the index theorem (5) is verified for two examples, using conicutional methods, whereas we have verified for only one of these examples, but our explicit exploitation of the supersymmetry of the Dirac operator explicitly leads to a considerable simplification of the proof.

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