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Instantons and Gribov Copies in the Maximally Abelian Gauge

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Abstract

We calculate the Faddeev-Popov operator corresponding to the maximally Abelian gauge for gauge group $SU(N)$. Specializing to $SU(2)$ we look for explicit zero modes of this operator. Within an illuminating toy model (Yang-Mills mechanics) the problem can be completely solved and understood. In the field theory case we are able to find an analytic expression for a normalizable zero mode in the background of a single 't Hooft instanton. Accordingly, such an instanton corresponds to a horizon configuration in the maximally Abelian gauge. Possible physical implications are discussed.

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1 Introduction

The configuration space $\mathfrak{A}$ of gauge theories is a "bigger-than-real-life-space" [1]. This is due to the fact that the action of the gauge group $\mathfrak{G}$ relates physically equivalent configurations along the gauge orbits. Therefore, this action has to be divided out. In principle, this division leads to the *physical* configuration space, $\mathfrak{A}_{\rm phys} = \mathfrak{A}/\mathfrak{G}$. In practice, however, this division is not easily performed. The most efficient method to do so is gauge fixing, where a subset of $\mathfrak{A}$ is identified with $\mathfrak{A}_{\rm phys}$. This subset is characterized by choosing some condition on the gauge potentials A of the form $\chi[A] = 0$. Prominent examples are the covariant gauge, $\chi_{\rm cov} = \partial_\mu A_\mu$, or the axial gauge, $\chi_{\rm ax} = n \cdot A$. One hopes that this condition satisfies both the requirements of existence and uniqueness. Existence means that the hypersurface $\Gamma : \chi = 0$ intersects every orbit, while uniqueness requires that it does so once and only once. It has first been shown by Gribov that the latter requirement cannot be satisfied for non-Abelian gauge theories in the covariant and Coulomb gauge [2]. Shortly afterwards, Gribov's observation has been proven for a large class of continuous gauge fixings [3]. In the physics community, the lack of uniqueness has become known as the Gribov problem. This just paraphrases the difficulty in constructing the physical configuration space which, by definition, is void of any (residual) gauge (or Gribov) copies.

In order to analyse this issue it has turned out useful to describe the gauge fixing not simply by a condition $\chi = 0$. Instead, in order to study the global aspects of the problem, one formulates the gauge fixing procedure in terms of a variational principle [4, 5, 6, 7, 8]. To this end one tries to define an 'action' functional F in such a way that the associated 'classical trajectories' are nowhere parallel to the orbits so that their union defines a gauge fixing hypersurface. By this construction one completely suppresses fluctuations in gauge directions which in the unfixed formulation do not cost energy (or action) and thus make the path integral ill–defined. Of course, by conservation of difficulties, one cannot avoid the Gribov problem this way.

The variational approach to gauge fixing has mainly been studied for background type gauges like the Coulomb gauge, where one can indeed construct a functional $F[A; U]$ with the following generic properties: the critical points of F along the orbits generated by U are the potentials A satisfying the Coulomb gauge condition, $\partial_i A_i = 0$. The Hessian of F at these points is the Faddeev-Popov operator FP. The Gribov region Ω^0 is defined as the set of transverse gauge fields for which det FP is positive. This is the set of relative minima of F. Its boundary $\partial\Omega^0$ is the Gribov horizon, where, accordingly, det FP $= 0$ because the lowest eigenvalue of FP changes sign. It has been shown that $\Omega^0 \subset \mathfrak{A}$ is convex [7, 9, 10]. This is basically due to the linearity of FP in A [11]. Contrary to early expectations the Gribov region still contains Gribov copies [7, 12, 13]. Only if one restricts to the set Λ of *absolute* minima and performs certain boundary identifications, one ends up with the physical configuration space (also called the fundamental modular domain) [11].

As stated above, the appearance of horizon configurations $A \in \partial\Omega^0$ implies that the gauge is not uniquely fixed; in other words, there are gauge fixing degeneracies. Somewhat symbolically, this can be shown as follows. Let $A \in \Gamma$ have the infinitesimal gauge variation $\delta A = D[A]\,\delta\phi$, D denoting the covariant derivative. To check whether the gauge transform $A + \delta A$ also satisfies the gauge condition, one calculates

$$\chi[A + \delta A] = \chi[A] + \frac{\delta\chi}{\delta A}\frac{\delta A}{\delta\phi}\,\delta\phi \equiv N[A] \cdot D[A]\,\delta\phi\ . \tag{1.1}$$

Here we have used that $\chi[A] = 0$ and defined the normal N to the gauge fixing hypersurface Γ. Now, if $A + \delta A$ is also in Γ we see that the FP operator,

$$\mathrm{FP}[A] \equiv N[A] \cdot D[A]\ , \tag{1.2}$$

must have a zero mode given by the infinitesimal gauge transformation $\delta\phi$. In this case, there are two gauge equivalent fields A and $A + \delta A$ on Γ and A is a horizon configuration. From (1.2) one infers that there are two generic reasons for this to happen. First, $\delta\phi$ can be a zero mode already of $D[A]$. As the latter can be viewed as the 'velocity' of a fictitious motion along the orbits, its vanishing (on $\delta\phi$) corresponds to a fixed point under the action of the gauge group. In this case, the configuration A is called reducible [3, 14]. Obviously, these are always horizon configurations. One might speculate whether there is a gauge fixing such that reducible configurations are the *only* horizon configurations [15]. The second possibility for det FP to vanish is that 'orbit velocity' D and normal N are orthogonal, which means that a particular orbit is tangent to Γ. This is what usually happens for background type gauges like the Coulomb gauge where N is constant, i.e. independent of A.

In general, it is very hard to explicitly find horizon configurations. For this reason, one has to concentrate on rather simple and/or symmetric gauge potentials. Again, the case best studied is the Coulomb gauge. It is known that there are Gribov copies of the classical vacuum $A = 0$ [11, 16, 17]. An even simpler example of Gribov copies is provided by constant Abelian gauge fields on the torus (the torons) [11, 18]. Configurations with a radial symmetry have been discussed in the original work of Gribov [2]. An explicit example with axial symmetry has been given by Henyey [11, 19].

On the lattice, the detection of Gribov copies has been reported for the first time in [20]. It turns out that some of these copies are lattice artifacts while others survive in the continuum limit [21]. In a sense, therefore, the Gribov problem becomes even more pronounced upon gauge fixing on the lattice. This is of particular relevance for the lattice studies of the dual superconductor hypothesis of confinement [22, 23, 24], where one mainly uses (a lattice version [25]) of 't Hooft's maximally Abelian gauge (MAG) [6]. In order to extract physical results within this approach one clearly has to control the influence of Gribov copies. Finding the critical points of the lattice gauge fixing functional is similar to a spin glass problem due to the high degree of degeneracy. The difficulties in numerically determining the absolute maximum[1] of the lattice functional lead to an inaccuracy in observables of the order of 10 % [26, 27].

The Gribov problem for the MAG so far has not been discussed in the continuum. The purpose of this paper is to (at least partly) fill this gap. The MAG and its defining functional will be reviewed in Section 2. The Hessian of this functional is the FP operator which is calculated for gauge group $SU(N)$. In Section 3 we specialize to $SU(2)$ and give general arguments showing the existence of a Gribov horizon. To provide some intuition, Section 4 introduces a simple toy model for which the FP operator and determinant can be calculated exactly. The presence of Gribov copies is shown explicitly. Finally, in Section 5, we return to field theory and calculate the FP operator in the background of a single instanton (in the singular gauge). Again, we find an analytic expression for a normalizable zero mode which shows that the single instanton is a horizon configuration in the MAG. Some technical issues are discussed in Appendices A to D.

2 The Maximally Abelian Gauge

As explained in Appendix A, we decompose the gauge potential A into diagonal ($A^{\|} \in \mathcal{H}^{\|}$) and off-diagonal ($A^{\perp} \in \mathcal{H}^{\perp}$) components, $A = A^{\|} + A^{\perp}$. The MAG is then defined by minimizing the following functional

$$F[A;U] \equiv \|({}^{U}\!A)^{\perp}\|^2 \; . \tag{2.1}$$

F is thus a functional of both the gauge field A and the gauge transformation $U \in SU(N)$. Via the parametrization

$$U(\phi) = \exp(-i\phi) = \exp(-i\phi^a T^a) \; , \quad \phi \in su(N) \; , \tag{2.2}$$

F equivalently can be viewed as depending on the argument ϕ of U. The action of U on A is

$${}^{U}\!A_\mu = U^{-1} A_\mu U + i U^{-1} \partial_\mu U \; . \tag{2.3}$$

With F of (2.1) we are thus minimizing the 'charged' component $A^{\perp}$ along its orbit, which, roughly speaking, amounts to maximizing the Abelian or 'neutral' component $A^{\|}$. Hence the name 'maximally Abelian gauge'.

The Yang-Mills norm in (2.1) is the same as in the Yang-Mills action and induced by the scalar product (A.6),

$$\|A\|^2 \equiv \langle A, A \rangle \equiv \int d^d x \, \mathrm{tr} A^2 \; . \tag{2.4}$$

[1]The maximization of the lattice functional corresponds to a minimization of the continuum functional.

Note that our conventions are such that this norm is positive for *hermitian* gauge fields A with values in $su(N)$. The norm (2.4) can be viewed as the distance (squared) between A and the zero configuration $A = 0$. As the space $\mathfrak{A}$ of gauge potentials is affine, the norm is gauge invariant in the following sense,

$$\|A - B\| = \|{}^U\!A - {}^U\!B\| \tag{2.5}$$

If the configuration B is kept fixed, however, the norm ceases to be gauge invariant and explicitly depends on U or ϕ. The same is thus true for F which accordingly changes along the orbit of A unless there is some (residual) invariance. For the functional (2.1) such an invariance can indeed be found. Let $V = \exp(-i\theta^{\|})$ be an Abelian gauge transformation and consider

$$F[A;V] = \|({}^V\!A)^{\perp}\|^2 = \|(V^{-1}A^{\|}V + V^{-1}A^{\perp}V + iV^{-1}dV)^{\perp}\|^2 \ . \tag{2.6}$$

As V is Abelian, the first and last terms on the r.h.s. of (2.6) vanish due to the projection on $\mathcal{H}^{\perp}$, and we are left with

$$F[A;V] = \|(V^{-1}A^{\perp}V)^{\perp}\|^2 \ . \tag{2.7}$$

At this point it is crucial to note that $V^{-1}A^{\perp}V$ is in $\mathcal{H}^{\perp}$,

$$\mathrm{tr}(H_i V^{-1}A^{\perp}V) = \mathrm{tr}(VH_iV^{-1}A^{\perp}) = \mathrm{tr}(H_iA^{\perp}) = 0 \ . \tag{2.8}$$

Therefore, we can write for (2.6),

$$F[A;V] = \|V^{-1}A^{\perp}V\|^2 = \|A^{\perp}\|^2 = F[A;\mathbb{1}] \ . \tag{2.9}$$

This immediately leads to the following Abelian invariance of F,

$$F[A;VU] = F[{}^U\!A;V] = F[{}^U\!A;\mathbb{1}] = F[A;U] \ . \tag{2.10}$$

Note that our notation is such that U acts prior to V, i.e.

$${}^{VU}\!A = (UV)^{-1}A\,UV + i(UV)^{-1}d(UV) \ . \tag{2.11}$$

Roughly speaking, the Abelian invariance implies that F can be thought of as some kind of 'mexican hat' with the residual symmetry corresponding to (Abelian gauge) rotations around its symmetry axis. Accordingly, the Hessian of F will have trivial zero modes asssociated with the constant directions of F.

We are interested in the behaviour of $F[A;U]$ around the point $U = \mathbb{1}$, i.e. $\phi = 0$, on the orbit of A. To this end we Taylor expand F as

$$F[A;U] \equiv F[A;\phi] = F[A;0] + \langle F'[A;0], \phi\rangle + \frac{1}{2}\langle\phi, F''[A;0]\,\phi\rangle + O(\phi^3) \tag{2.12}$$

In order to do so we need the gauge transform ${}^U\!A$ as a power series in ϕ. The former can easily be found from (B.1) and (B.2) with the result

$$
\begin{aligned}
{}^U\!A_\mu &= A_\mu + \frac{\exp(i\,\mathrm{ad}\phi) - \mathbb{1}}{i\,\mathrm{ad}\phi}\,(D_\mu\phi) \\
&= A_\mu + D_\mu\phi + \frac{i}{2}[\phi, D_\mu\phi] + \frac{i^2}{3!}\Big[\phi, [\phi, D_\mu\phi]\Big] + \ldots \,. \qquad (2.13)
\end{aligned}
$$

Not surprisingly, the covariant derivative $D_\mu = \partial_\mu - i\,\mathrm{ad}(A_\mu)$ with $\mathrm{ad}(A)\,B \equiv [A,B]$ appears at this stage. Inserting (2.13) into (2.1) we obtain

$$
F[A;\phi] = \|A_\mu^\perp\|^2 + 2\,\langle A_\mu^\perp, (D_\mu\phi)^\perp\rangle + \langle (D_\mu\phi)^\perp, (D_\mu\phi)^\perp\rangle + i\,\langle A_\mu^\perp, [\phi, D_\mu\phi]^\perp\rangle + \ldots \,. \qquad (2.14)
$$

In the following we are going to evaluate this expression term by term. This requires some preparations. We will need the commutator identity,

$$
\langle A, [B,C]\rangle = \langle B, [C,A]\rangle = \langle C, [A,B]\rangle \,, \qquad (2.15)
$$

which follows straightforwardly from the definition of the scalar product. The latter equation shows that both the operator $\mathrm{ad}(A)$ and the covariant derivative $D[A]$ are anti-hermitean,

$$
\begin{aligned}
\langle \phi,\ \mathrm{ad}(A)\psi\rangle &= -\langle\ \mathrm{ad}(A)\phi, \psi\rangle \,, \qquad (2.16)\\
\langle \phi, D[A]\psi\rangle &= -\langle D[A]\phi, \psi\rangle \,. \qquad (2.17)
\end{aligned}
$$

The last two identities allow for an evaluation of the first derivative F',

$$
\langle A_\mu^\perp, (D_\mu\phi)^\perp\rangle = \langle A_\mu^\perp, D_\mu\phi\rangle = -\langle D_\mu^{||} A_\mu^\perp, \phi\rangle = -\langle D_\mu^{||} A_\mu^\perp, \phi^\perp\rangle \,, \qquad (2.18)
$$

with $D_\mu^{||} \equiv \partial_\mu - i\,\mathrm{ad}A_\mu^{||}$. We thus have, to first order in ϕ,

$$
F[A;\phi] = \|A_\mu^\perp\|^2 - 2\,\langle D_\mu^{||} A_\mu^\perp, \phi^\perp\rangle + O(\phi^2) \,. \qquad (2.19)
$$

Note that to this order, F does not depend on the Cartan component $\phi^{||}$. We immediately read off the critical points defining the MAG,

$$
D_\mu^{||} A_\mu^\perp \equiv D_\mu A_\mu^\perp = 0 \,. \qquad (2.20)
$$

The second derivative requires considerably more efforts. We relegate the explicit calculations to Appendix C, where we obtain for the Taylor expansion of $F[A;\phi]$,

$$
\begin{aligned}
F[A;\phi] &= \|A_\mu^\perp\|^2 - 2\langle D_\mu^{||} A_\mu^\perp, \phi^\perp\rangle + i\langle \phi^\perp,\ \mathrm{ad}(D_\mu A_\mu^\perp)\phi^{||}\rangle \\
&- \langle \phi^\perp, \Big[D_\mu^{||} D_\mu^{||} +\ \mathrm{ad}^2 A_\mu^\perp - i(\ \mathrm{ad}A_\mu^\perp)\mathbb{Q} D_\mu - i\,\mathrm{ad}(D_\mu^{||} A_\mu^\perp)\Big]\phi^\perp\rangle \\
&+ O(\phi^3) \,. \qquad (2.21)
\end{aligned}
$$

Here we have defined a projection $\mathbb{Q}$ onto the complement $\mathcal{H}^\perp$ of the Cartan subalgebra such that $\mathbb{Q}\phi = \phi^\perp$. The term in (2.21) depending on $\phi^{\|}$ may seem somewhat strange but is actually necessary to guarantee the Abelian invariance (2.10). It vanishes on the gauge fixing hypersurface Γ defined by (2.20).

From (2.21) we can easily read off the Faddeev-Popov operator which is the Hessian of F evaluated on Γ (i.e. at the critical points),

$$\mathrm{FP} = -\mathbb{Q}\left(D_\mu^{\|}D_\mu^{\|} + \ \mathrm{ad}^2 A_\mu^\perp - i(\ \mathrm{ad}A_\mu^\perp)\mathbb{Q}D_\mu\right)\mathbb{Q}\ . \tag{2.22}$$

In effect we have performed a saddle point approximation to the functional $F[A;\phi]$. The 'equation of motion' is the gauge fixing condition, and the fluctuation operator is the FP operator. In this approximation the functional on Γ reads

$$F[A;\phi] = \|A_\mu^\perp\|^2 + \langle\phi, \mathrm{FP}\phi\rangle + O(\phi^3)\ . \tag{2.23}$$

As stated in the introduction, it is in general rather difficult (in a continuum formulation) to find explicit examples of Gribov copies. The MAG is no exception from this rule. The nontrivial task is to find normalizable zero modes of FP given by (2.22) which is a complicated partial differential operator. We are, however, encouraged by lattice calculations, in which such copies have been detected numerically, for the first time in [28] and with refined techniques in [26, 27]. One should keep in mind, though, that some (if not all) of these copies can be lattice artifacts which do not survive in the continuum limit. To study the possible appearance of Gribov copies in the continuum we have to perform several simplifications. The first one will be to consider the case of gauge group $SU(2)$.

3 The FP Operator for $SU(2)$ — General Considerations

For $SU(2)$, the gauge fixing condition (2.20) of the MAG can be rewritten in terms of the gauge field components $A_\mu^3 \in \mathcal{H}^{\|}$ and $A_\mu^\pm \in \mathcal{H}^\perp$,

$$(\partial_\mu \pm iA_\mu^3)A_\mu^\pm = 0\ , \quad A_\mu^\pm \equiv A_\mu^1 \pm iA_\mu^2\ . \tag{3.1}$$

The fact that these are only *two* requirements already implies (by counting of degrees of freedom) that there remains a residual gauge freedom corresponding to a one-dimensional subgroup which can only be $U(1)$. Superficially, the gauge fixing looks like a background gauge which would actually be true if the neutral component A_μ^3 were independent of the charged one, $A_\mu^\perp$. As these, however, are two components of one and the same configuration they are not independent, and the gauge fixing condition is quadratic, i.e. *nonlinear* in A_μ. This makes life somewhat complicated (although it does not spoil the renormalizability of the gauge [29]). A BRST approach, for example, necessitates the introduction of four-ghost terms. In a path integral formulation, these ghost interactions 'regularize' the usual bilinear FP ghost term in the presence of zero modes [30].

The FP operator for $SU(2)$ simplifies considerably as the last term in (2.22) vanishes. One is thus left with the following sum of two operators,

$$\mathrm{FP} = -\mathbb{Q}\Big(D_\mu^{\|}D_\mu^{\|} + \,\mathrm{ad}^2(A_\mu^\perp)\Big)\mathbb{Q}\;. \tag{3.2}$$

Using the notation (A.8), FP can be viewed as a 3×3 matrix in color space. The operator $\mathbb{Q}$ projects onto the two directions perpendicular to the z-axis so that the third row and column of FP vanish identically. The associated trivial zero mode corresponds to the residual $U(1)$ gauge freedom which remains unfixed by the MAG. Explicitly, one has for the nonvanishing entries of FP,

$$(D_\mu^{\|}D_\mu^{\|})^{\bar a\bar b} = \delta^{\bar a\bar b}(\Box - A_\mu^3 A_\mu^3) - \epsilon^{\bar a\bar b}(\partial_\mu A_\mu^3 + 2A_\mu^3\partial_\mu)\;, \tag{3.3}$$

$$(\,\mathrm{ad}^2(A_\mu^\perp))^{\bar a\bar b} = \delta^{\bar a\bar b}A_\mu^{\bar c}A_\mu^{\bar c} - A_\mu^{\bar a}A_\mu^{\bar b}\;. \tag{3.4}$$

Summing these two terms leads to the representation of FP given in equation (12) of [31][2].

Being (the negative of) a Laplacian, the operator $-D_\mu^{\|}D_\mu^{\|}$ is nonnegative. The same is true for $\,\mathrm{ad}(A_\mu^\perp)\,\mathrm{ad}(A_\mu^\perp)$ as will be shown in what follows. We define the *hermitean* matrix C via

$$[A_\mu^\perp, \phi^\perp] \equiv iC\;, \tag{3.5}$$

and calculate, using (2.16),

$$\langle\phi, \mathbb{Q}\,\mathrm{ad}(A_\mu^\perp)\,\mathrm{ad}(A_\mu^\perp)\,\mathbb{Q}\,\phi\rangle = -\langle\,\mathrm{ad}A_\mu^\perp\phi^\perp,\;\mathrm{ad}A_\mu^\perp\phi^\perp\rangle \tag{3.6}$$

$$= -\langle iC, iC\rangle = \langle C, C\rangle \geq 0\;. \tag{3.7}$$

One can as well use the representations (3.3), (3.4) and the Cauchy–Schwarz inequality to end up with the same result. The $SU(2)$ FP operator from (3.2) is thus the *difference of two positive semidefinite operators* which we abbreviate for the time being as

$$\mathrm{FP} = A - B\;, \quad A, B \geq 0\;. \tag{3.8}$$

The inequality denotes the fact that A and B have nonnegative spectrum. The identity (3.8) already suggests that if B is 'sufficiently large', FP will develop a vanishing eigenvalue. Let us make this statement slightly more rigorous. To this end we modify an argument used in [14, 32] for background type gauges.

First of all we note that together with the configuration $(A^{\|}, A^\perp)$ also the scaled configuration $(A^{\|}, \lambda A^\perp)$, with λ some (positive real) parameter, will be in the MAG. The associated FP operator is

$$\mathrm{FP}[A^{\|}, \lambda A^\perp] \equiv \mathrm{FP}(\lambda) = A - \lambda^2 B\;. \tag{3.9}$$

[2]Note, however, that in this reference the gauge potentials are defined as being *anti-hermitean.*

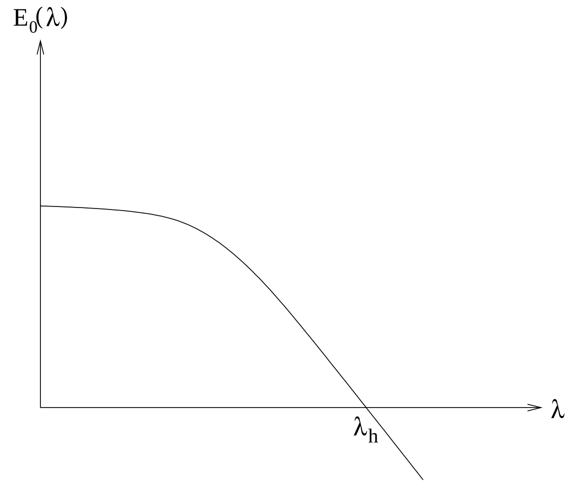

Figure 1: Qualitative behavior of the lowest eigenvalue of FP as a function of the 'flow parameter' λ. The parameter value λ_h corresponds to a horizon configuration.

Let us denote the lowest eigenvalue and the associated eigenfunction of $\mathrm{FP}(\lambda)$ by $E_0(\lambda)$ and $\phi_0(\lambda)$, respectively,

$$\mathrm{FP}(\lambda)\,\phi_0(\lambda) = E_0(\lambda)\,\phi_0(\lambda)\ . \tag{3.10}$$

From (3.8) one must have $E_0(0) \geq 0$. If we turn on λ, a straightforward application of the Hellmann–Feynman theorem leads to

$$\frac{\partial}{\partial\lambda} E_0(\lambda) = -2\lambda\,\langle \phi_0(\lambda), B\,\phi_0(\lambda)\rangle \leq 0\ , \tag{3.11}$$

whence the function $E_0(\lambda)$ has negative slope. In addition, it has to be concave [33][3] so that, for λ sufficiently large, there will be a zero-mode at some value, say λ_h (see Fig. 1). In a way we have thus determined a 'path' within the MAG fixing hypersurface that leads us from the interior of the Gribov region ($\lambda = 0$) to its boundary ($\lambda = \lambda_h$).

As a result we can state that generically there have to be Gribov copies within the MAG if the non-diagonal components $A^{\perp}$ of the gauge fields become sufficiently large.

4 A Toy Model

In order to have an illustration of the somewhat abstract notions of the preceding sections we will analyse an example with a finite number of degrees of

[3]It is exactly for this reason that the second order perturbation theory correction to any groundstate is always negative.

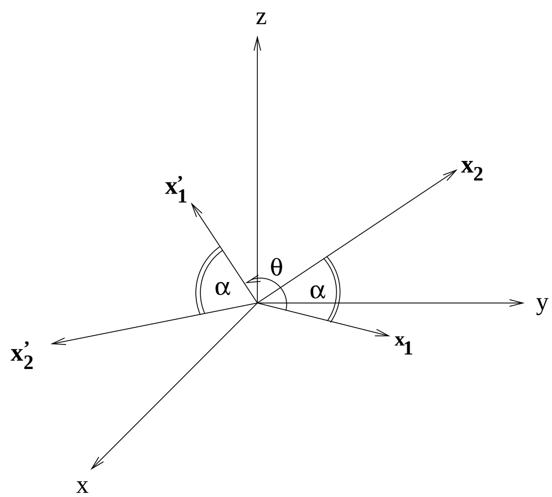

Figure 2: An isospace (gauge) rotation (by an angle θ) in the toy model, transforming the configuration $(x_1, x_2) \to (x_1', x_2')$. The lengths of the vectors and the angle α inbetween them are invariant.

freedom [34]. To this end we employ a Hamiltonian formulation in $d = 2 + 1$ and consider only gauge potentials A_μ which are spatially constant. Renaming $A_i^a = x_i^a$, $i = 1, 2$, $a = 1, 2, 3$, the Lagrangian becomes

$$L = \frac{1}{2}(D_0^{ab} x_i^b)^2 \equiv \frac{1}{2}(\dot{x}_i^a - \epsilon^{abc} A_0^c x_i^b)^2 \; . \tag{4.1}$$

One way of arriving at this Lagrangian is by gauging a free particle Lagrangian $L_0 = \dot{x}_i^a \dot{x}_i^a / 2$ via minimal substitution, i.e. by replacing the ordinary time derivative ∂_0 with the covariant derivative D_0^{ab}. To keep things as simple as possible, we have not introduced any (Yang-Mills type) interaction; we are anyhow only interested in the kinematics of the problem.

Defining the canonical momenta $p_i^a = D_0^{ab} x_i^b$, the Lagrangian (4.1) can be recast in first order form

$$L = p_i^a \dot{x}_i^a - \frac{1}{2} p_i^a p_i^a + A_0^a G^a \; , \tag{4.2}$$

where we have introduced the operator G^a leading to Gauss's law

$$G^a \equiv \epsilon^{abc} x_i^b p_i^c \equiv D_i^{ab} p_i^b = 0 \; . \tag{4.3}$$

Obviously, G^a is the total angular momentum of two point particles in $\mathbb{R}^3$ (= color isospace) with position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Gauge transformations are thus $SO(3)$ rotations of these vectors which do not change their relative orientation (i.e. the angle α inbetween them). This is illustrated in Fig. 2.

As usual we will work in the Weyl gauge, $A_0 = 0$, so that Gauss's law has to be imposed 'by hand', and, after quantization, holds upon acting on physical states. Once the Weyl gauge has been chosen, there still is the freedom of performing time independent gauge transformations. This will be (partially) fixed using the MAG. For the case at hand, there are several equivalent ways of formulating the latter.

To avoid writing too many indices we denote $\boldsymbol{x}_1 \equiv \boldsymbol{x} = (x, y, z)$, $\boldsymbol{x}_2 \equiv \boldsymbol{X} = (X, Y, Z)$. An arbitrary vector $\boldsymbol{A}$ will be decomposed according to

$$\boldsymbol{A}_{||} \equiv A_z \boldsymbol{e}_z \, , \tag{4.4}$$

$$\boldsymbol{A}_{\perp} \equiv A_x \boldsymbol{e}_x + A_y \boldsymbol{e}_y \, , \tag{4.5}$$

which represents the decomposition into Cartan ($= z$) component and its complement. The MAG condition then reads explicitly

$$\chi^a \equiv D_i^{ab}(x_{i||}) x_{i\perp} = \epsilon^{abc} x_{i||}^b x_{i\perp}^c = 0 \, , \tag{4.6}$$

or, in components,

$$\begin{aligned} \chi^1 &= -yz - YZ = 0 \, , \\ \chi^2 &= xz + XZ = 0 \, , \\ \chi^3 &= 0 \, . \end{aligned} \tag{4.7}$$

The last condition is just an empty tautology so that there are in fact only *two* gauge conditions[4]. Of course, this just corresponds to the fact that the gauge rotations generated by G^3 (the rotations around the z–axis) remain unfixed (cf. the remark after (3.1)).

The MAG conditions (4.7) can be easily visualized. The projections $\boldsymbol{x}_\perp$ and $\boldsymbol{X}_\perp$ have to be collinear with their magnitudes being related through

$$|z| \, x_\perp = |Z| \, X_\perp \, . \tag{4.8}$$

The MAG is thus obtained by rotating the configuration $(\boldsymbol{x}, \boldsymbol{X})$ in such a way that both vectors are as close to the z-axis as possible. This is achieved as shown in Fig. 3. $\boldsymbol{x}$ and $\boldsymbol{X}$ are the diagonals of two rectangles with sides $|z|$, $x_\perp$ and $|Z|$, $X_\perp$, respectively. If the areas a and A of the rectangles coincide, $a = A$, the configuration is in the MAG. Algebraically, the notion of being 'close to the z–axis' is measured by the function

$$F(\boldsymbol{x}, \boldsymbol{X}) \equiv x_\perp^2 + X_\perp^2 \, . \tag{4.9}$$

One can easily show that the conditions (4.6) or (4.7) minimize F and thus make the 'nondiagonal' components of $\boldsymbol{x}$ and $\boldsymbol{X}$ as small as possible. We mention

[4]In Dirac's terminology [35], χ^3 is *strongly* zero and thus does not contribute in the calculation of any Poisson bracket.

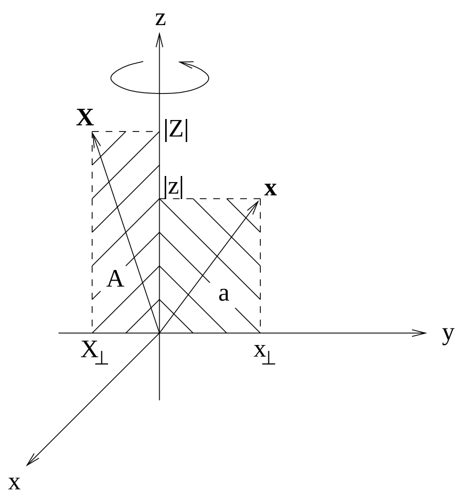

Figure 3: The MAG condition in the toy model. The areas A and a have to be the same. We have arbitrarily chosen $\boldsymbol{x}$ and $\boldsymbol{X}$ to lie in the yz–plane. The residual $U(1)$ gauge freedom corrresponds to rotations around the z–axis.

in passing that the trivial solution of (4.7) given by $z = Z = 0$ corrresponds to a maximum of F so that we can always assume z or $Z \neq 0$ (except for the zero–configuration representing the origin).

It is obvious from Fig. 3 that rotations around the z–axis leave both F and the MAG condition invariant and thus correspond to a residual $U(1)$ gauge freedom. As expected, this situation is reflected in the FP operator,

$$\mathrm{FP} = \left\{ \chi^a \, , \, G^b \right\} \Big|_{\boldsymbol{\chi}=0} \, , \tag{4.10}$$

which, in matrix notation, can be written as

$$\mathrm{FP} = \begin{pmatrix} z^2 + Z^2 - y^2 - Y^2 & xy + XY & 0 \\ xy + XY & z^2 + Z^2 - x^2 - X^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.11}$$

The zero entries in the third row and column are a trivial consequence of the residual $U(1)$ and correspond to the action of the $\mathbb{Q}$–projection in (2.22). The eigenvalues of FP are found to be

$$\begin{aligned} E_3 &= 0 \, , & (4.12)\\ E_+ &= z^2 + Z^2 \, , & (4.13)\\ E_- &= z^2 + Z^2 - x_\perp^2 - X_\perp^2 \, . & (4.14) \end{aligned}$$

Let us concentrate on the eigenvalues $E_\pm$ which are not related to the residual Abelian gauge freedom. Configurations where one of these vanishes are located on the Gribov horizon and reflect some non-trivial residual gauge freedom different from the $U(1)$ above. A particular (in some sense trivial) class of horizon configurations consists in the reducible configurations as discussed in the introduction. These have a higher symmetry than generic configurations (a nontrivial stabilizer or isotropy group). In other words, they are fixed points under the action of (a subgroup of) the gauge group. Technically, they show up by inducing zero modes of the Laplacian $\Delta^{ab} = D_i^{ac} D_i^{cb}$ (see Appendix D). Within our example, the reducible configurations are readily identified [34, 36] by simple symmetry considerations. The origin is invariant under the whole action of $SO(3)$ while configurations with $\boldsymbol{x}$ and $\boldsymbol{X}$ collinear are invariant under rotations around their common direction which clearly corresponds to a $U(1)$. This is nicely reflected in the spectrum of FP. At the origin, both $E_\pm$ vanish, while a collinear configuration can always be rotated in the z–axis so that its stabilizer coincides with the standard residual $U(1)$ corresponding to $E_3 = 0$. This U(1) stabilizer is thus 'hidden' in the residual $U(1)$. Fixing the latter by demanding e.g. $x = X = 0$, does, however, not affect configurations collinear along the z–axis so that these will induce zero modes of FP even after residual gauge fixing [34].

There is a remaining possibility for a vanishing eigenvalue. While E_+ is always positive, E_- vanishes if $z^2 + Z^2 = x_\perp^2 + X_\perp^2$. This happens for configurations where $\boldsymbol{x}$ and $\boldsymbol{X}$ are of the same length and orthogonal to each other. Elementary trigonometry implies that in this case the two areas a and A are always the same, irrespective of the location of the configuration relative to the z–axis. Thus, there is an additional residual $U(1)$ gauge freedom for such exceptional configurations. This can be nicely illustrated in terms of a 'spectral flow' as a function of $x_\perp^2 + X_\perp^2$ (see Fig. 4). We thus have found an explicit realization of the general results of Section 3, in particular of Fig. 1.

5 The FP Operator in an Instanton Background

The natural question arising at this point is the following: is there a way of extending the results of the toy model to the realistic field theory case? The answer given in this section will be affirmative.

Our motivation stems from the observation made by Brower et al. [37] that the single 't Hooft instanton *both* in the singular *and* regular gauge satisfies the MAG condition (2.20). For the instanton in the singular gauge[5] (or 'singular instanton', for short) given by

$$A_\mu^{\mathrm{sing}}(x) = 2\,\bar{\eta}^a_{\mu\nu}\,\frac{\rho^2}{x^2}\,\frac{x_\nu}{x^2+\rho^2}\,\sigma^a/2\ , \tag{5.1}$$

with ρ denoting the instanton size, the MAG fixing functional F is finite, while

[5] We use the conventions of [38].

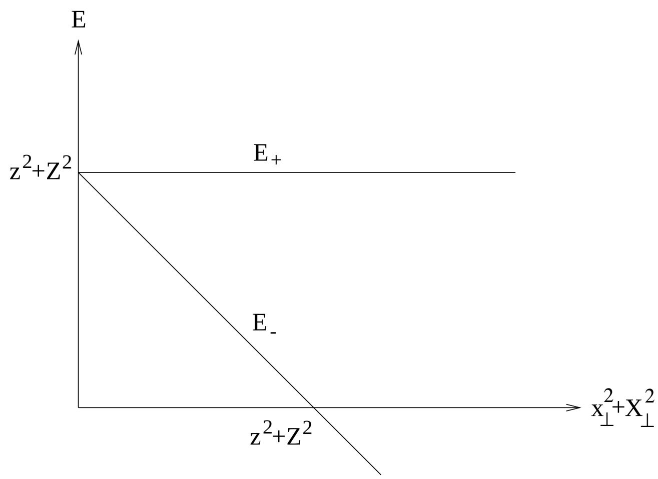

Figure 4: Behavior of the eigenvalues of FP in the toy model as a function of the magnitude $x_\perp^2 + X_\perp^2$ of the 'nondiagonal' components. A zero mode arises when $x_\perp^2 + X_\perp^2 = z^2 + Z^2$.

for the instanton in the regular gauge,

$$A_\mu^{\rm reg}(x) = 2\,\eta^a_{\mu\nu}\,\frac{x_\nu}{x^2+\rho^2}\,\sigma^a/2\;, \tag{5.2}$$

it diverges. The two configurations $A_\mu^{\rm sing}$ and $A_\mu^{\rm reg}$ are related through the gauge transformation

$$g(x) = \hat{x}_4 + i\,\hat{x}^a\sigma^a\;, \tag{5.3}$$

where $\hat{x}_\mu = x_\mu/r$, $r = (x^2)^{1/2}$ denoting the modulus of the Euclidean position x. If we adopt the point of view that we have to take the minima of F to define the Gribov region Ω^0 of the MAG then $A_\mu^{\rm sing}$ is located in Ω^0 while $A_\mu^{\rm reg}$ is not. This is corroborated by the quoted work of Brower et al. [37] which, when translated into our language, amounts to the following. One numerically constructs a 'path' $\gamma(R) \in \Gamma$ connecting $A_\mu^{\rm sing}$ with $A_\mu^{\rm reg}$. Along this path[6] (beginning at the singular instanton) the MAG functional F is monotonically rising. The configurations $A_\mu(R)$ along the path are determined by applying a (singular) gauge transformation Ω which takes the singular instanton to $A_\mu(R)$, i.e. $A_\mu(R) = {}^\Omega A_\mu^{\rm sing}$. Hence $\gamma(R)$ is a path *both* within Γ *and* the single instanton orbit. Accordingly, there must be an infinitesimal gauge transformation of the singular instanton that does not leave Γ and thus must be a zero mode of FP $[A^{\rm sing}]$. In what follows we will try to explicitly determine this zero mode.

The first step of this program consists in the calculation of the FP operator in the background of a singular instanton. Plugging (5.1) into (3.3) and (3.4) one

[6] In [37] the parameter R is the radius of a monopole loop associated with the configuration $A_\mu(R)$ located on γ somewhere inbetween $A_\mu^{\rm sing} = A_\mu(R=0)$ and $A_\mu^{\rm reg} = A_\mu(R=\infty)$.

obtains the result

$$\mathrm{FP}^{\bar{a}\bar{b}} = -\delta^{\bar{a}\bar{b}}\Box + 2\,\epsilon^{\bar{a}\bar{b}}\,a(r)(x_2\partial_1 - x_1\partial_2 + x_3\partial_4 - x_4\partial_3)\ . \tag{5.4}$$

We have discarded the vanishing third row and column (resulting from the action of $\mathbb{Q}$) and introduced the (singular) instanton profile function,

$$a(r) = 2\,\frac{\rho^2/r^2}{r^2+\rho^2} = 2\left(\frac{1}{r^2} - \frac{1}{r^2+\rho^2}\right)\ . \tag{5.5}$$

We are looking for normalizable zero modes ϕ of the FP operator,

$$\mathrm{FP}\,\phi = 0\ , \quad \langle\phi,\phi\rangle < \infty\ , \tag{5.6}$$

where $\phi(x)$ now is a two-component vector (field) living in the complement of the Cartan subalgebra. Solving the equation (5.6) for the zero mode is basically an exercise in group theory as will become clear in a moment. If we define the generators of four-dimensional Euclidean rotations as

$$L_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\ , \quad \mu,\nu = 1,\ldots,4\ , \tag{5.7}$$

the FP operator can be written in 2×2 matrix notation as

$$\mathrm{FP} = \begin{pmatrix} -\Box & -2i\,a(r)(L_{12}-L_{34}) \\ 2i\,a(r)(L_{12}-L_{34}) & -\Box \end{pmatrix} \tag{5.8}$$

It is straightforward to check that the $L_{\mu\nu}$ indeed satisfy the Lie algebra of $SO(4)$. In analogy with the Lorentz group one introduces the angular momentum and 'boost' generators

$$L_i \equiv \frac{1}{2}\epsilon_{ijk}L_{jk} \tag{5.9}$$

$$K_i \equiv L_{i4}\ , \tag{5.10}$$

and their linear combinations,

$$M_i \equiv \frac{1}{2}(L_i - K_i) = -\frac{i}{2}\,\bar{\eta}^i_{\mu\nu}\,x_\mu\partial_\nu\ , \tag{5.11}$$

$$N_i \equiv \frac{1}{2}(L_i + K_i) = -\frac{i}{2}\,\eta^i_{\mu\nu}\,x_\mu\partial_\nu\ . \tag{5.12}$$

These can be viewed as the self-dual and anti-self-dual parts of $L_{\mu\nu}$, if 'duality' is understood as the exchange of $\mathbf{L}$ and $\mathbf{K}$. The operators M_i and N_i generate two independent $SU(2)$ subgroups with Casimirs M^2 and N^2 having eigenvalues $M(M+1)$ and $N(N+1)$, respectively [39]. It is important to note that M and N will in general be half-integer,

$$M, N \in \{0, 1/2, 1, \ldots\}\ . \tag{5.13}$$

This fact is well known from the algebraic treatment of the hydrogen atom which has a hidden dynamical $O(4)$ symmetry (see e.g. [40]). In addition, as FP is a $2{\times}2$ matrix, it can be expanded in terms of Pauli matrices, so that altogether we find the rather compact result,

$$\mathrm{FP} = -\Box \mathbb{1} + 4a(r) M_3\, \sigma_2 \ . \tag{5.14}$$

Plugging this into (5.6) results in a four-dimensional Schrödinger equation with spin having a high degree of symmetry. A complete set of commuting observables is given by the Casimirs M^2 and N^2, their projections M_3 and N_3 (with eigenvalues m and n) and the Pauli matrix σ_2 (eigenvalues $s = \pm 1$). Replacing σ_2 by its eigenvalue and rewriting the Laplacian in terms of the radial coordinate r we are left with

$$\mathrm{FP}(s) \equiv -\partial_r^2 - \frac{3}{r}\partial_r + \frac{2}{r^2}(\mathbf{M}^2 + \mathbf{N}^2) + 4a(r) M_3\, s \tag{5.15}$$

This is indeed a $4d$ radially symmetric Hamiltonian. Upon closer inspection, the Casimir term turns out to become even simpler. Using the representations (5.7), (5.9) and (5.10) one finds that

$$\mathbf{N}^2 - \mathbf{M}^2 = \mathbf{L} \cdot \mathbf{K} = 0 \ , \tag{5.16}$$

so that FP finally becomes

$$\mathrm{FP}(s) = -\partial_r^2 - \frac{3}{r}\partial_r + \frac{4}{r^2}\mathbf{M}^2 + 4a(r) M_3\, s \ . \tag{5.17}$$

The eigenfunctions of FP will therefore depend on the quantum numbers $M \in \{0, 1/2, 1, \ldots\}$, $m, n \in \{-M, -M+1, \ldots, M\}$ and $s = \pm 1$. Chosing the coordinates

$$x = r\,(\cos\theta\cos\varphi_{12}, \cos\theta\sin\varphi_{12}, \sin\theta\cos\varphi_{34}, \sin\theta\sin\varphi_{34}) \ , \tag{5.18}$$

with $0 \le \theta \le \pi/2$, $0 \le \varphi_{12}, \varphi_{34} \le 2\pi$, the eigenfunctions can be written as follows,

$$\phi = f_{Mm}(r)\, h_{Mmn}(\theta)\, y_{mn}(\varphi_{12})\, z_{mn}(\varphi_{34})\, \chi_s \ . \tag{5.19}$$

The χ_s are the eigenspinors of σ_2,

$$\chi_\pm = \frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ \pm i \end{array} \right] . \tag{5.20}$$

The Schrödinger equation factorizes accordingly. Introducing the dimensionless variable $R = r/\rho$ and defining a function $g(R)$ via

$$f(R) \equiv g(R)/R^{3/2} \ , \tag{5.21}$$

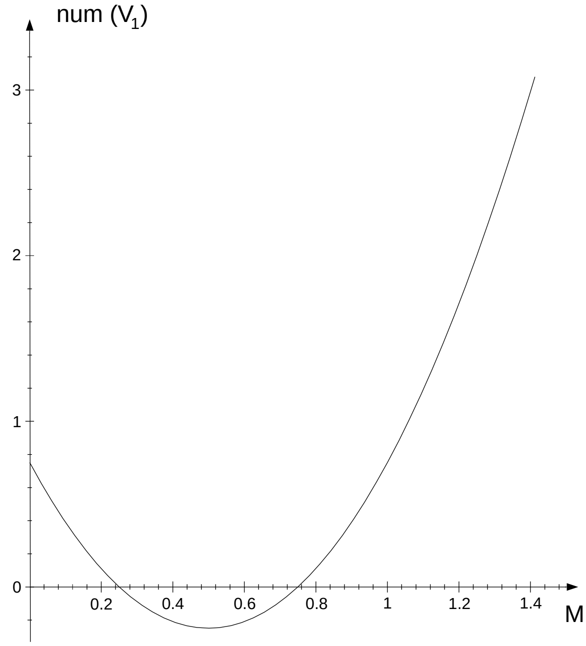

Figure 5: The numerator of the potential term V_1 as a function of the quantum number M. The only (half-)integer leading to attraction (negative V_1) is $M = 1/2$.

(we omit the subscripts of f) the radial equation for the zero mode becomes

$$\left[-\partial_R^2 + \frac{4M(M+1)+3/4}{R^2} + \frac{8ms}{R^2(1+R^2)}\right] g(R) = 0 \ . \tag{5.22}$$

We are looking for a normalizable zero mode, or, in other words, a bound state with vanishing energy. For this we need an attractive potential. We thus must have $ms < 0$, and we choose $s = -1$, $m > 0$ in what follows. The bound state equation (5.22) thus becomes

$$\left[-\partial_R^2 + \frac{4M(M+1)-8m+3/4}{R^2} + \frac{8m}{1+R^2}\right] g(R) = 0 \ . \tag{5.23}$$

This equation has already been obtained by Brower et al. [37] in the stability analysis of their monopole solutions. These authors, however, have overlooked the fact that M is half-integer which is crucial for obtaining the correct solution (see below). In addition they approximated the profile function $a(r)$ by $1/r^2$ (in the limit of small monopole loops). We will instead solve (5.23) exactly. The latter is an effective one–dimensional Schrödinger equation with a Hamiltonian

$$H_R \equiv -\partial_R^2 + V_1(R) + V_2(R) \ . \tag{5.24}$$

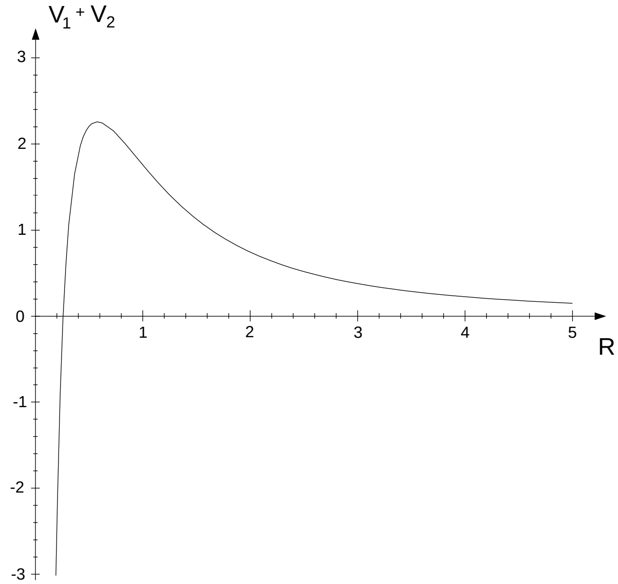

Figure 6: The bound-state potential $V_1 + V_2$ as a function of $R = r/\rho$ for the attractive case (quantum numbers $M = m = 1/2$).

The second potential term, V_2, is always positive (for $m > 0$). Only the first term, V_1 has a chance of becoming negative leading to attraction. Technically, this is due to the relative minus sign in the profile function (5.5) of the singular instanton. In the regular gauge, this is absent so that both V_1 and V_2 are positive and there are no normalizable zero modes. As m is bounded by M, the Casimir term $M(M+1)$ in (5.23) will always win for large M. We thus should make M as small and m as large as possible. We thus take $m = M$ and plot the numerator of V_1 as a function of M (see Fig. 5). Obviously, there is exactly one solution for M which makes V_1 negative, namely $M = 1/2 = m$. We have explicitly checked that for $M > 1/2$ there is no bound state solution[7]. The associated potential $V_1 + V_2$ is plotted in Fig. 6. For $M = 1/2$, the normalizable solution of (5.23) is given by

$$g(R) = \sqrt{R}\left[1 - (1 + R^2)\ln\left(1 + \frac{1}{R^2}\right)\right] \ . \tag{5.25}$$

Close to the origin, $f(R) = g(R)/R^{3/2}$ behaves as

$$f(R) = \frac{1}{R}(1 + 2\ln R) - R(1 - 2\ln R) + O(R^2) \ , \tag{5.26}$$

while asymptotically it drops as $1/R^3$. Both types of behavior are sufficient to make f (or ϕ) normalizable. The radial wave function $f(R)$ and the associated

[7]The claim of Brower et al. [37], that attraction occurs for $m = 1$ with the ground state having $M = 1$, thus cannot be substantiated.

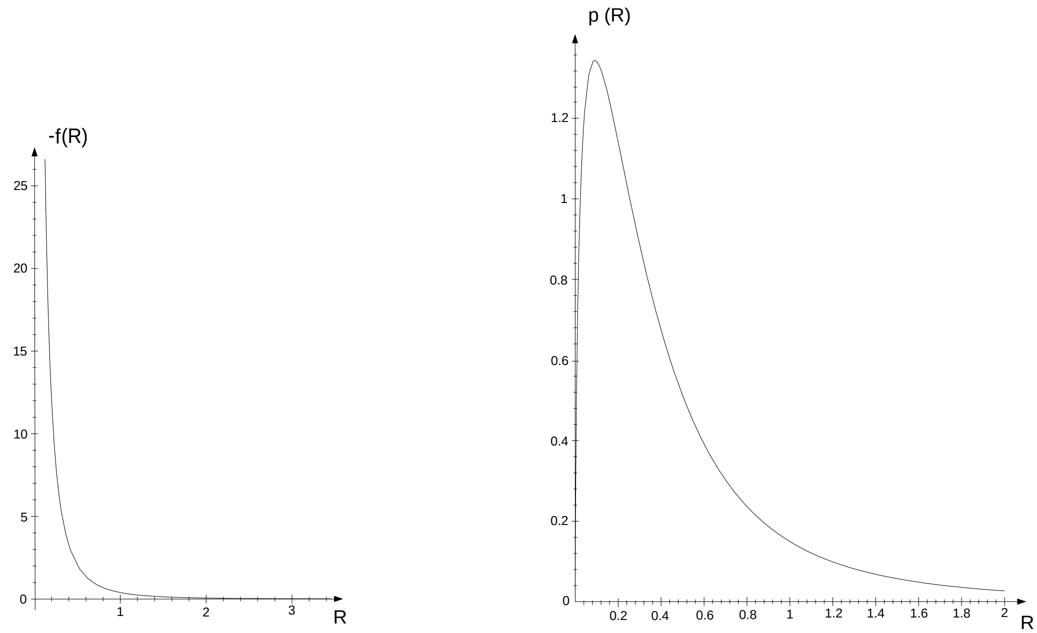

Figure 7: The radial wave function $-f(R)$ of the zero mode and the associated probability distribution $p(R)$. While f diverges at the origin, p vanishes due to the radial measure factor R^3.

probability distribution $p(R) = R^3 f^2(R)$ are shown in Fig. 7. From this figure it is obvious that f has no nodes and therefore corresponds to the ground state in the sector with $M = 1/2$ (cf. the analogous reasoning in [11]).

The degeneracy of the solution is found as follows. FP does not depend on N_3, therefore n can arbitrarily be chosen as an half–integer from $\{-M, -M + 1, \ldots, M\}$, i.e. for $M = 1/2$, one has $n = \pm\, 1/2$. Furthermore, FP is invariant under $(m, s) \to (-m, -s)$, so that, altogether, there is a four–fold degeneracy. In terms of abstract states $|M, m, n, s\rangle$ the zero modes are linear combinations of the four degenerate basis states $|1/2, 1/2, \pm 1/2, -\rangle$ and $|1/2, -1/2, \pm 1/2, +\rangle$.

To explicitly determine the zero mode, we still have to find the functions h_{Mmn}, y_{mn} and z_{mn} for $M = 1/2$. y_{mn} and z_{mn} are eigenfunctions of the operators L_3 and K_3 so that their product becomes an eigenfunction of the two operators

$$M_3 = \frac{1}{2}(L_3 - K_3) = -\frac{i}{2}\left(\frac{\partial}{\partial\varphi_{12}} - \frac{\partial}{\partial\varphi_{34}}\right) , \tag{5.27}$$

$$N_3 = \frac{1}{2}(L_3 + K_3) = -\frac{i}{2}\left(\frac{\partial}{\partial\varphi_{12}} + \frac{\partial}{\partial\varphi_{34}}\right) , \tag{5.28}$$

according to

$$M_3\, y_{mn}\, z_{mn} = m\, y_{mn}\, z_{mn} \ , \tag{5.29}$$

$$N_3\, y_{mn}\, z_{mn} = n\, y_{mn}\, z_{mn} \ . \tag{5.30}$$

Explicitly, one finds

$$\begin{aligned} y_{mn}(\varphi_{12}) &= \mathrm{e}^{i(m+n)\varphi_{12}} , && (5.31) \\ z_{mn}(\varphi_{34}) &= \mathrm{e}^{-i(m-n)\varphi_{34}} . && (5.32) \end{aligned}$$

The function $h_{Mmn}(\theta)$ satisfies the differential equation

$$\left[\frac{1}{\sin 2\theta}\frac{\partial}{\partial\theta}\sin 2\theta\frac{\partial}{\partial\theta} + 4M(M+1) - \frac{(m+n)^2}{\cos^2\theta} - \frac{(m-n)^2}{\sin^2\theta}\right] h_{Mmn}(\theta) = 0 . \tag{5.33}$$

For $M = 1/2$, we can circumvent solving this equation by considering only the two extremal states in a multiplet with $m = \pm M$, which obey

$$M_\pm|M, \pm M, n\rangle = 0 . \tag{5.34}$$

The associated differential equation is much simpler than (5.33) and straightforwardly solved in terms of the functions

$$\begin{aligned} h_{M,-M,n}(\theta) &= \cos^{M-n}\theta \, \sin^{M+n}\theta , \\ h_{M,M,n}(\theta) &= \sin^{M-n}\theta \, \cos^{M+n}\theta . \end{aligned} \tag{5.35}$$

Direct application to $m = \pm 1/2$ finally yields the four degenerate zero modes for $M = 1/2$ (using the notation ϕ_{mns}),

$$\begin{aligned} \phi_{-1/2,-1/2,+}(x) &= c f(r) \cos\theta \, \mathrm{e}^{-i\varphi_{12}} \chi_+ &\equiv \Phi_1 , \\ \phi_{-1/2,1/2,+}(x) &= c f(r) \sin\theta \, \mathrm{e}^{i\varphi_{34}} \chi_+ &\equiv \Phi_4 , \\ \phi_{1/2,1/2,-}(x) &= c f(r) \cos\theta \, \mathrm{e}^{i\varphi_{12}} \chi_- &\equiv \Phi_2 , \\ \phi_{1/2,-1/2,-}(x) &= c f(r) \sin\theta \, \mathrm{e}^{-i\varphi_{34}} \chi_- &\equiv \Phi_3 , \end{aligned} \tag{5.36}$$

where c denotes a normalization constant which will be determined in a moment. To this end we rewrite the measure

$$d^4x = r^3 dr \cos\theta \sin\theta \, d\theta \, d\varphi_{12} \, d\varphi_{34} , \tag{5.37}$$

and calculate the integral (Φ denoting any of the basic zero modes)

$$\int d^4x \; \Phi^*(x) \cdot \Phi(x) = c^2 \rho^4 \, \frac{\pi^2}{6} \left(1 + \frac{\pi^2}{3}\right) \stackrel{!}{=} 1 . \tag{5.38}$$

This determines the normalization c. Any zero mode ϕ of FP satisfying (5.6) must be a linear combination of the four basis modes (5.36). For the following considerations it is convenient to introduce the real basis,

$$\begin{aligned} \Psi_1 &\equiv \tfrac{1}{2i}(\Phi_3 - \Phi_4) &= \tfrac{c}{\sqrt{2}} \tfrac{f(r)}{r} \begin{bmatrix} -x_4 \\ -x_3 \end{bmatrix} , \\ \Psi_2 &\equiv \tfrac{1}{2}(\Phi_3 + \Phi_4) &= \tfrac{c}{\sqrt{2}} \tfrac{f(r)}{r} \begin{bmatrix} x_3 \\ -x_4 \end{bmatrix} , \\ \Psi_3 &\equiv \tfrac{1}{2i}(\Phi_1 - \Phi_2) &= \tfrac{c}{\sqrt{2}} \tfrac{f(r)}{r} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} , \\ \Psi_4 &\equiv \tfrac{1}{2}(\Phi_1 + \Phi_2) &= \tfrac{c}{\sqrt{2}} \tfrac{f(r)}{r} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \end{aligned} \tag{5.39}$$

which, upon using the properties of ‘t Hooft’s η symbols [38] can be compactly written as

$$\Psi_\mu^{\bar{a}}(x) = \frac{c}{\sqrt{2}} f(r)\, \bar{\eta}^{\bar{a}}_{\mu\nu}\, \hat{x}_\nu \; . \tag{5.40}$$

A general linear combination thus assumes the form

$$\phi^{\bar{a}}(x) \equiv \frac{\sqrt{2}}{c} n_\mu \Psi_\mu^{\bar{a}} = n^\mu\, \bar{\eta}^{\bar{a}}_{\mu\nu}\, \hat{x}^\nu\, f(r) \equiv m^{\bar{a}} f(r)/r \; , \tag{5.41}$$

where n_μ is a constant four vector. The latter is particularly suited for obtaining the finite transformation,

$$\Omega = \exp i\phi^{\bar{a}}\sigma^{\bar{a}}/2 = \mathbb{1}\cos\varphi/2 + iN^{\bar{a}}\sigma^{\bar{a}}\sin\varphi/2 \; , \tag{5.42}$$

with $\varphi = (\phi^{\bar{a}}\phi^{\bar{a}})^{1/2}$ and $N^{\bar{a}} = \phi^{\bar{a}}/\varphi$. Using (5.41) one finds the explicit representation

$$\varphi = \frac{f(r)}{r}\sqrt{m^{\bar{a}}m^{\bar{a}}} \; , \tag{5.43}$$

$$N^{\bar{a}} = \frac{m^{\bar{a}}}{\sqrt{m^{\bar{a}}m^{\bar{a}}}} \; . \tag{5.44}$$

Applying the gauge transformation Ω from (5.42) to the singular instanton leads to a configuration that is no longer in the MAG. This is at variance with the solution Ω_R found by Brower et al. [37] which yields a monopole configuration *within* the MAG. To illustrate this difference we plot the modulus φ (denoted β in [37]) for the choice $\phi = \Psi_4$ or, correspondingly, $n = (0,0,0,1)$, $\mathbf{m} = (x_1, x_2)^T$. The result is shown in Fig. 8 which clearly differs from the analogous Fig. 2 in [37]. The presence of a zero mode as given by (5.41) shows that the instanton in the singular gauge is located on the Gribov horizon of the MAG. For (covariant) background type gauges, an analogous result has been obtained in [41].

6 Discussion

Among the different Abelian gauges used for the lattice study of the dual superconductor hypothesis, the MAG is the one that has been analysed in greatest detail. In this paper we have tried to supplement these achievements by analytic investigations. As the gauge fixing is nonlinear, this requires some effort. We have calculated the FP operator for general gauge group $SU(N)$. The result is fairly complicated; considerable simplifications only seem to arise for gauge group $SU(2)$. For this particular case we were able to show by quite general reasoning that there must be Gribov copies. This finding was confirmed both for a simple toy model and the full field theory. In the latter case it turns out that the singular instanton is a horizon configuration in the MAG. The associated zero modes of the FP operator have explicitly been constructed.

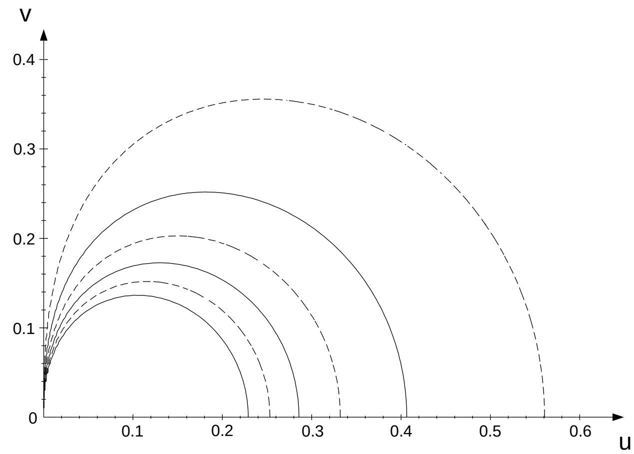

Figure 8: Lines of constant modulus φ of the zero mode $\phi \equiv \Psi_4$ as a function of $u \equiv R\cos\theta$ and $v \equiv R\sin\theta$. The dashed lines correspond to $\varphi = \pi/2$, $3\pi/2$, $5\pi/2$, ..., the full ones to $\varphi = \pi$, 2π, 3π, ..., with φ increasing from the outermost curve towards the origin.

Let us finally discuss some possible physical consequences of our findings. The two pronounced manifestations of the QCD vacuum are confinement and spontaneous breakdown of chiral symmetry. As stated above, the MAG is well suited for studying the former by checking dual superconductivity which is believed to be due to monopole condensation [22, 23, 24]. On the lattice, condensation of monopoles has been confirmed for various Abelian gauges [42, 43, 44, 45]. The monopole vacuum, however, does not provide a straightforward explanation of chiral symmetry breaking which is due to instantons rather than monopoles [46, 47, 38]. It is thus of conceptual importance to relate these two complementary pictures of the vacuum to each other. In computer 'experiments' correlations between instantons and monopoles have indeed been detected [48, 49, 50, 51, 37, 52, 53, 54, 55]. The dynamical origin of these correlations, however, remains unclear, despite considerable efforts to investigate this problem analytically, in the MAG [56, 37, 57, 55], the Polyakov gauge [58, 59, 60, 61, 62] and other Abelian gauges [63, 64]. For the MAG, the situation is as follows. There are basically three known solutions which represent *finite* transformations from the singular instanton A^{sing} into another MAG configuration. These are (i) the transformation (5.3) to the regular gauge instanton A^{reg}, (ii) the 'hedgehog' transformation of Chernodub and Gubarev [56], and (iii) the family of solutions $\{A(R)\}$ given by Brower et al. [37], interpolating between A^{sing} and A^{reg}. Of these solutions only (ii) and (iii) induce magnetic monopoles. Solution (ii) leads to an infinite Dirac string, solution (iii) to a monopole loop of radius R. The associated MAG functional F diverges in cases (i) and (ii). In case (iii) it is finite, however such that $F[A(R)] > F[A(0)] \equiv F[A^{\text{sing}}]$. As a result one concludes that the instanton in the singular gauge defines the global minimum of F along the single instanton orbit. In other words, the MAG functional does not support monopoles associated with single instantons as these configurations give rise to a larger value of F. This is actually consistent with lattice results. In [28, 26, 27] it was observed that the number of monopoles decreases the better the MAG is fixed, i.e. the closer one approaches the absolute maximum of the lattice MAG functional. Due to monopole dominance, the string tension also becomes smaller. This effect might well be due to the suppression of monopole loops associated with single instantons.

In favor of the instanton–monopole correlation, Brower et al. argue that a possible zero mode of FP can be interpreted as a kinematical instability of the singular instanton against monopole formation. In the limit of small monopole loops, $R \ll \rho$, their solution (eq. (31) in [37]) indeed is a zero mode of FP. It goes like $\sin 2\theta \sim \sin\theta\cos\theta$, and thus, upon comparing with (5.35), is seen to correspond to $M = 1$. Therefore, from our general analysis in the preceding section, it is *not normalizable* and thus should be discarded from the stability analysis. It is probably not too surprising that singular gauge transformations like the ones found in [37] lead to zero modes with diverging norm.

The physical interpretation of the normalizable $M = 1/2$ zero mode given in (5.41) is not completely clear. We have checked that it is not due to any of the known space-time symmetries of the instanton. Contrary to our expectations, it also has nothing to do with the solution of Brower et al. In particular, it does *not* induce monopole singularities. Furthermore, as stated in the last section, the finite transformation (5.42) even leads out of the MAG. All this confirms the result that in the MAG *single* instantons are not correlated with monopoles. One is thus left with a possible correlation between *multi*-instanton configurations and monopoles. Numerically, this has been observed [48, 49, 50, 51, 37, 52]. In particular, the instanton-anti-instanton (IA) system seems to be physically interesting. In this case one finds that both I and A are surrounded by a single monopole loop if the IA distance is large. Below a critical distance, however, the two loops merge into a single one [37] which can be viewed as a 'kinematical precursor' to monopole percolation. Of course, an analytic treatment of multi-instanton systems is quite involved, but maybe not hopeless. In this respect let us just mention Rossi's old construction of the BPS monopole in terms of an infinite number of instantons aligned along the time axis [65]. We have performed some preliminary investigations of the IA system which show that the simple sum ansatz, $A^{IA} = A^I + A^A$ is not in the MAG. The ansatz suggested by Yung [66], however, does fulfill the differential MAG conditions (3.1), though the MAG functional probably diverges. Further work in this direction is surely necessary.

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A Notations and Conventions

The generators of $SU(N)$ are *hermitean* matrices denoted by T^a with normalization $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$. Any gauge field $A = A^aT^a$ is decomposed into a component $A^\parallel$ in the Cartan subalgebra $\mathcal{H}^\parallel \subset su(N)$ and a component $A^\perp$ in the complement, $\mathcal{H}^\perp$, such that $su(N) = \mathcal{H}^\parallel \oplus \mathcal{H}^\perp$ and

$$A = A^\parallel + A^\perp = A^i H_i + A^\alpha E_\alpha \ . \tag{A.1}$$

The different generators obey the commutation relations [67]

$$\begin{aligned}
[H_i, H_j] &= 0 \ , \quad i = 1, \ldots, r \ , && \text{(A.2)}\\
[H_i, E_\alpha] &= \alpha_i E_\alpha \ , && \text{(A.3)}\\
[E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta} \ , \quad \alpha + \beta \neq 0 \ , && \text{(A.4)}\\
[E_\alpha, E_{-\alpha}] &= \alpha_i H_i \ . && \text{(A.5)}
\end{aligned}$$

The rank of the Lie algebra is denoted by r, the α_i are the roots, and $N_{\alpha\beta}$ is a normalization the value of which is not important for us. For $SU(2)$, which has only two roots $\pm\alpha$, the third commutator (A.4) becomes obsolete, and the situation simplifies considerably.

The decomposition (A.1) is orthogonal with respect to the scalar product

$$\langle A, B \rangle \equiv \int d^dx \, \mathrm{tr} AB \ , \tag{A.6}$$

where A and B denote some arbitrary Lie algebra valued $\mathbb{L}_2$ functions. Thus we have

$$\langle A^\perp, B^\parallel \rangle = 0 \ . \tag{A.7}$$

We will also use an alternative notation [31] where we simply divide the $N^2 - 1$ generators T^a into 'neutral' and 'charged' ones by means of their superscripts, namely $T^a = (T^{a_0}, T^{\bar{a}})$ with $T^{a_0} \in \mathcal{H}^\parallel$ and $T^{\bar{a}} \in \mathcal{H}^\perp$. A gauge field thus is decomposed as

$$A_\mu = A_\mu^a T^a = A_\mu^{a_0} T^{a_0} + A_\mu^{\bar{a}} T^{\bar{a}} \ . \tag{A.8}$$

The superscripts a_0 and $\bar{a}$ take on r and $N^2 - 1 - r$ values, respectively. For $SU(2)$, for example, we have $a_0 = 3$ and $\bar{a} \in \{1, 2\}$, while for $SU(3)$, $a_0 \in \{3, 8\}$ etc.

B Group-Theoretical Identities

In this appendix we prove the two useful identities,

$$U^{-1}A_\mu U = \exp(i\,\mathrm{ad}\phi)A_\mu \;, \tag{B.1}$$

$$iU^{-1}\partial_\mu U = \frac{\exp(i\,\mathrm{ad}\phi)-\mathbb{1}}{i\,\mathrm{ad}\phi}\,\partial_\mu\phi \;, \tag{B.2}$$

which hold for an arbitrary gauge transformation $U = \exp(-i\phi)$. In the above, we have denoted $\mathrm{ad}(A)B \equiv [A,B]$.

(B.1) is simply the definition of the adjoint representation of a Lie group expressed in terms of the adjoint representation of the Lie algebra [68],

$$\exp(i\phi)X\exp(-i\phi) \equiv \mathrm{Ad}\Big(\exp(i\phi)\Big)X = \exp(i\,\mathrm{ad}\phi)X \;, \tag{B.3}$$

where X is an arbitrary Lie algebra element. Equation (B.2) is obtained from the identity

$$i\exp(is\phi)\partial_\mu\exp(-is\phi) = \frac{\exp(is\,\mathrm{ad}\phi)-\mathbb{1}}{i\,\mathrm{ad}\phi}\partial_\mu\phi \;, \tag{B.4}$$

for $s=1$. To show (B.4) we first note that it is obviously true for $s=0$. Differentiating with respect to s, we find

$$\begin{aligned} \frac{\partial}{\partial s}\text{l.h.s.} &= \exp(is\phi)(\partial_\mu\phi)\exp(-is\phi) \;, \\ \frac{\partial}{\partial s}\text{r.h.s.} &= \exp(is\,\mathrm{ad}\phi)\partial_\mu\phi = \exp(is\phi)(\partial_\mu\phi)\exp(-is\phi) \;, \end{aligned} \tag{B.5}$$

where in the last step we have used (B.3). Upon inspection we note that both sides of (B.4) obey the same first order differential equation in s and initial condition at $s=0$. Thus (B.4) is true for all s.

C The Second Derivative of the MAG Functional

In this appendix we calculate the second derivative of the MAG functional given by the last two terms in (2.14). First we evaluate $(D_\mu\phi)^\perp$,

$$\begin{aligned} (D_\mu\phi)^\perp &= \partial_\mu\phi^\perp - i[A_\mu^{||},\phi^\perp] - i[A_\mu^\perp,\phi^\perp]^\perp - i[A_\mu^\perp,\phi^{||}] \\ &= (D_\mu\phi^\perp)^\perp - i[A_\mu^\perp,\phi^{||}] \;. \end{aligned} \tag{C.1}$$

This yields for the square term in (2.14),

$$\begin{aligned} \|(D_\mu\phi)^\perp\|^2 &= \|(D_\mu\phi^\perp)^\perp - i\,[A_\mu^\perp,\phi^{||}]\|^2 \\ &= \langle D_\mu\phi^\perp,(D_\mu\phi^\perp)^\perp\rangle - 2i\,\langle D_\mu\phi^\perp,[A_\mu^\perp,\phi^{||}]\rangle - \langle[A_\mu^\perp,\phi^{||}],[A_\mu^\perp,\phi^{||}]\rangle \\ &= -\langle\phi^\perp, D_\mu\mathbb{Q}D_\mu\phi^\perp\rangle + 2i\,\langle\phi^\perp,[A_\mu^\perp,D_\mu\phi^{||}]\rangle + \langle\phi^{||},[A_\mu^\perp,[A_\mu^\perp,\phi^{||}]]\rangle \\ &+ 2i\,\langle\phi^\perp,[D_\mu A_\mu^\perp,\phi^{||}]\rangle \;. \end{aligned} \tag{C.2}$$

In the last equality, we have made use of the 'Leibniz rule',

$$D_\mu[B,C] = [D_\mu B, C] + [B, D_\mu C] \;, \tag{C.3}$$

and defined a projection $\mathbb{Q}$ onto the Cartan complement, $\mathbb{Q}A = A^\perp$. Note that the last term in (C.2) vanishes at the critical points (2.20). The second term of order ϕ^2 in (2.14) is

$$\begin{aligned} i\langle A_\mu^\perp, [\phi, D_\mu\phi]^\perp\rangle &= -i\langle\phi, [A_\mu^\perp, D_\mu\phi]\rangle \\ &= -i\langle\phi^\perp, [A_\mu^\perp, D_\mu\phi^\perp]\rangle - i\langle\phi^\perp, [A_\mu^\perp, D_\mu\phi^{||}]\rangle \\ &\quad -i\langle\phi^{||}, [A_\mu^\perp, D_\mu\phi^\perp]\rangle - i\langle\phi^{||}, [A_\mu^\perp, D_\mu\phi^{||}]\rangle \;. \end{aligned} \tag{C.4}$$

The third term can be reshuffled and evaluated with the rule (C.3) yielding

$$-i\langle\phi^{||}, [A_\mu^\perp, D_\mu\phi^\perp]\rangle = -i\langle\phi^\perp, [D_\mu A_\mu^\perp, \phi^{||}]\rangle - i\langle\phi^\perp, [A_\mu^\perp, D_\mu\phi^{||}]\rangle \;. \tag{C.5}$$

Plugging this into (C.4) and adding (C.2) we see that the terms which mix $\phi^\perp$ and $D_\mu\phi^{||}$ cancel. The $O(\phi^2)$ term in F thus becomes

$$\begin{aligned} F^{(2)}[A;\phi] &\equiv -\langle\phi^\perp, D_\mu\mathbb{Q}D_\mu\phi^\perp\rangle - i\,\langle\phi^\perp, [A_\mu^\perp, D_\mu\phi^\perp]\rangle + i\langle\phi^\perp, [D_\mu A_\mu^\perp, \phi^{||}]\rangle \\ &\quad +\langle\phi^{||}, [A_\mu^\perp, [A_\mu^\perp, \phi^{||}]]\rangle - i\,\langle\phi^{||}, [A_\mu^\perp, D_\mu\phi^{||}]\rangle \;. \end{aligned} \tag{C.6}$$

The two terms bilinear in $\phi^{||}$ add up to zero according to

$$-i\langle\phi^{||}, [A_\mu^\perp, (D_\mu + i\,\mathrm{ad}A_\mu^\perp)(\phi^{||})]\rangle = -i\langle\phi^{||}, [A_\mu^\perp, D_\mu^{||}\phi^{||}]\rangle = -i\langle\phi^{||}, [A_\mu^\perp, \partial_\mu\phi^{||}]\rangle = 0 \;, \tag{C.7}$$

where the last identity holds because the commutator is in the Cartan complement $\mathcal{H}^\perp$. Expression (C.6) thus simplifies to

$$\begin{aligned} F^{(2)}[A;\phi] &= -\langle\phi^\perp, D_\mu\mathbb{Q}D_\mu\phi^\perp - i[A_\mu^\perp, D_\mu\phi^\perp]\rangle + i\langle\phi^\perp, [D_\mu A_\mu^\perp, \phi^{||}]\rangle \\ &\equiv F^{(2)}[A;\phi^\perp] + i\langle\phi^\perp, [D_\mu A_\mu^\perp, \phi^{||}]\rangle \;. \end{aligned} \tag{C.8}$$

Introducing $\mathbb{P} = \mathbb{1} - \mathbb{Q}$, the terms quadratic in $\phi^\perp$ assume the following form,

$$\begin{aligned} F^{(2)}[A;\phi^\perp] &\equiv -\langle\phi^\perp, (D_\mu\mathbb{Q} + i\,\mathrm{ad}A_\mu^\perp)(D_\mu\phi^\perp)\rangle \\ &= -\langle\phi^\perp, (D_\mu\mathbb{Q} + i\,\mathrm{ad}A_\mu^\perp)(\mathbb{P} + \mathbb{Q})(D_\mu\phi^\perp)\rangle \\ &= -\langle\phi^\perp, D_\mu^{||}\mathbb{Q}\,D_\mu\phi^\perp + i\,\mathrm{ad}A_\mu^\perp\,\mathbb{P}D_\mu\phi^\perp\rangle \;. \end{aligned} \tag{C.9}$$

We thus need the projections

$$\mathbb{P}D_\mu\phi^\perp = -i\mathbb{P}[A_\mu^\perp, \phi^\perp] \;, \tag{C.10}$$

$$\mathbb{Q}D_\mu\phi^\perp = D_\mu^{||}\phi^\perp - i\mathbb{Q}[A_\mu^\perp, \phi^\perp] \;. \tag{C.11}$$

Using this and the identity $\langle \phi^\perp, \mathbb{Q} D^{||}_\mu A \rangle = \langle \phi^\perp, D^{||}_\mu A \rangle$, (C.9) becomes

$$
\begin{aligned}
&F^{(2)}[A;\phi^\perp] = \\
&= -\langle \phi^\perp, D^{||}_\mu D^{||}_\mu \phi^\perp - i D^{||}_\mu \mathbb{Q}[A^\perp_\mu, \phi^\perp] + [A^\perp_\mu, \mathbb{P}[A^\perp_\mu, \phi^\perp]] \rangle \\
&= -\langle \phi^\perp, D^{||}_\mu D^{||}_\mu \phi^\perp - i[D^{||}_\mu A^\perp_\mu, \phi^\perp] - i[A^\perp_\mu, D^{||}_\mu \phi^\perp + i\mathbb{P}[A^\perp_\mu, \phi^\perp]] \rangle \\
&= -\langle \phi^\perp, D^{||}_\mu D^{||}_\mu \phi^\perp - i[D^{||}_\mu A^\perp_\mu, \phi^\perp] - i[A^\perp_\mu, D^{||}_\mu \phi^\perp - i\mathbb{Q}[A^\perp_\mu, \phi^\perp] + i[A^\perp_\mu, \phi^\perp]] \rangle \\
&= -\langle \phi^\perp, D^{||}_\mu D^{||}_\mu \phi^\perp - i[D^{||}_\mu A^\perp_\mu, \phi^\perp] - i[A^\perp_\mu, \mathbb{Q} D_\mu \phi^\perp] + [A^\perp_\mu, [A^\perp_\mu, \phi^\perp]] \rangle \; .
\end{aligned} \tag{C.12}
$$

This is the result used in (2.21).

D The Laplacian of the Toy Model

Using matrix notation, the Laplacian $\Delta^{ab} = D^{ac}_i D^{cb}_i$ of the toy model is given by

$$
\Delta = \begin{pmatrix} -y^2 - z^2 - Y^2 - Z^2 & xy + XY & xz + XZ \\ yx + YX & -x^2 - z^2 - X^2 - Z^2 & yz + YZ \\ zx + ZX & zy + ZY & -x^2 - y^2 - X^2 - Y^2 \end{pmatrix} . \tag{D.1}
$$

Denoting $r \equiv |\boldsymbol{x}|$ and $R \equiv |\boldsymbol{X}|$, the determinant of the Laplacian becomes

$$
\det \Delta = -(r^2 + R^2)\,(\boldsymbol{x} \times \boldsymbol{X})^2 \equiv -(r^2 + R^2)\, d^2 \leq 0 \; . \tag{D.2}
$$

As is appropriate for a Laplacian, Δ is a negative-semidefinite operator. It has zero modes for reducible configurations only [14], which for the case at hand are given by the zero configuration, $\boldsymbol{x} = \boldsymbol{X} = 0$ (with the full group $SO(3)$ as its stabilizer), and the collinear configurations, $\boldsymbol{x} = \alpha \boldsymbol{X}$, with $U(1)$ stabilizer.

The eigenvalues of $-\Delta$ are given by

$$
E_0 = r^2 + R^2 \, , \tag{D.3}
$$

$$
E_\pm = \frac{1}{2}(r^2 + R^2) \pm \frac{1}{2}\sqrt{(r^2 - R^2)^2 + 4(\boldsymbol{x} \cdot \boldsymbol{X})^2} \; . \tag{D.4}
$$

It is reassuring to note that the eigenvalues and, accordingly, the determinant only depend on the gauge invariant scalar products r^2, R^2 and $\boldsymbol{x} \cdot \boldsymbol{X}$. At the origin, which has the largest stabilizer, all eigenvalues vanish. For collinear configurations with $\boldsymbol{x} \cdot \boldsymbol{X} = \pm rR$, the eigenvalues are $E_- = 0$ and $E_0 = E_+ = r^2 + R^2$, so that there is a zero mode and the first 'excited' state is degenerate. For the horizon configurations of the MAG, having $\boldsymbol{x} \cdot \boldsymbol{X} = 0$, $r = R \equiv \rho$, one finds $E_\pm = \rho^2$ and $E_0 = 2\rho^2$. Thus, the groundstate becomes degenerate. The latter fact corresponds to the gauge fixing degeneracies of the *Laplacian gauge* [69, 70, 71] as particularly discussed in [70]. As the MAG and the Laplacian gauge coincide for constant gauge fields, the degeneracies have to be the same, and, indeed, they are.

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