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**Kac-Moody Realization of $\mathcal{W}$-Algebras**

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Abstract:

By realizing the $\mathcal{W}$-algebras of Toda field-theories as the algebras of gauge-invariant polynomials of constrained Kac-Moody systems we obtain a simple algorithm for constructing $\mathcal{W}$-algebras without computing the $\mathcal{W}$-generators themselves. In particular this realization yields an identification of a primary field basis for all the $\mathcal{W}$-algebras, quadratic bases for the $A, B, C$-algebras, and the relation of $\mathcal{W}$-algebras to Casimir algebras. At the quantum level it yields the general formula for the Virasoro centre in terms of the KM- level.

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The so-called $\mathcal{W}$-algebras are defined as *polynomial* extensions of the Virasoro algebra by higher ($\geq 3$) conformal spin fields. The study of such polynomial extensions of the Virasoro algebra has been initiated by Zamolodchikov [1]. It has been found [2] that a large class of polynomial extensions of the Virasoro algebra is realized at the Poisson-bracket (PB) level by the Gelfand-Dickey (GD) Hamiltonian structure of Lax operators. More recently Toda theories have also been shown to provide a realization of $\mathcal{W}$-algebras [3]. Both of these theories may be regarded as constrained Wess-Zumino-Novikov-Witten (WZNW) models. The Gelfand-Dickey Poisson bracket structure can be obtained by a Hamiltonian reduction from the phase space of a Kac-Moody (KM) algebra [4] and as shown in ref. [5], Toda theories may be regarded as WZNW models reduced by certain conformally invariant constraints. Therefore it is natural to expect that a unified treatment of the $\mathcal{W}$-algebras could be obtained by exploiting the structure of the constrained KM (CKM) structure. The aim of this paper is to demonstrate that indeed the $\mathcal{W}$-algebras in the Toda theory and the GD algebras are different manifestations of the same CKM algebra. Furthermore the study of $\mathcal{W}$-algebra is much simpler at the CKM level. The simplicity is due to the fact that the key eqs. of the CKM theory can be solved by a linear, iterative, algebraic process, which is due to the nilpotent nature of the constraint algebra. Here we shall only present the main results and for a more complete mathematical elaboration refer to [6].

All our results are based on the fact that the CKM theory is a gauge theory (with the gauge transformations generated by the first class constraints) and that the gauge group is nilpotent.

Our first set of results rests on the observation that the differential eqs. used to define the $\mathcal{W}$-algebra in the Toda and GD theories [2,3] are nothing but the gauge-invariant form of the standard WZNW equation $\partial g = J \cdot g$ which relates the group-valued WZNW field $g(z)$ to the KM current $J(z)$. This observation leads to the identification of the $\mathcal{W}$-algebras of Toda theory with the PB algebra of gauge invariant polynomials of the constrained currents and their derivatives (differential polynomials). Since the original constraints together with any complete gauge fixing form a system of second class constraints we get an alternative identification of the $\mathcal{W}$-algebra as the Dirac-bracket algebra of polynomials of gauge fixed currents (and their derivatives). Chosing a special gauge, the diagonal gauge, we immediately obtain the free-field representation of the $\mathcal{W}$-algebra and find that it is nothing but the gauge invariant extension of the corresponding Casimir algebra. As a by-product we also find a simple criterion to decide on the differential or pseudo-differential nature of the relevant operator, which yields that for the groups $A$, $B$, $C$, $G_2$ one has a differential operator while for the $D$, $E$ and $F_4$ groups one has a pseudo-differential one.

Our second set of results is based on the observation that the Dirac bracket version of the $\mathcal{W}$-algebra may be implemented by the action of the KM algebra only. By using the fact that the gauge fixing constraints can be chosen to be linear in the currents (DS-gauges) the KM-implementation becomes particularly simple and can be solved in a linear, algebraic and iterative manner. By using DS-gauges we identify a primary field basis for all the $\mathcal{W}$-algebras. We also show that quadratic bases exist only for the $A$, $B$, $C$ groups and identify them in these cases.

Our final result is a first step in quantizing the CKM theory. We compute the quantum corrections to the classical center, $c$, of the Virasoro subalgebra of the corresponding $\mathcal{W}$-algebra for highest weight representations. There are three contributions to $c$, namely, those from the Sugawara energy-momentum tensor, the improvement term (which emerges automatically in this framework) and the BRST ghosts. The sum of the three contributions produces the simple and elegant formula

$$c = l - 12\Big(\sqrt{k+g}\ \hat{\rho} - \frac{1}{\sqrt{k+g}}\ \rho\Big)^2 , \tag{1}$$

where $l$ is the rank of $G$, $k$ is the level of the KM algebra, $g$ is the dual Coxeter number of $G$, and $\rho$ (resp. $\hat{\rho}$) is half the sum of the positive roots (resp. coroots) of $G$. As usual the norm of the highest root is taken to be $\sqrt{2}$. Of course, until other considerations, such as the unitarity of the representations are investigated, the range of $k$ (in particular its sign) is not determined.

We begin by defining the constrained WZNW or KM theory. Let us recall that in the WZNW model the group valued field, $g(z,\bar{z})$ and the corresponding KM currents $J(z)$, $\tilde{J}(\bar{z})$, subject to the equations of motion, satisfy

$$g(z,\bar{z}) = g(z)\cdot g(\bar{z}) \qquad \partial g(z) = J(z)\cdot g(z)\,, \quad \bar{\partial} g(\bar{z}) = -g(\bar{z})\tilde{J}(\bar{z})\,. \tag{2}$$

Let us now consider WZNW fields with values in a maximally non-compact Lie group $G$ (i.e. a Lie group generated by the *real* linear span of the conventional Cartan – Weyl operators $(H_i, E_{\pm\varphi_k})$, $i = 1\ldots l$, $k = 1\ldots,(D-l)/2$, $(D = \dim G)$. We define the constrained KM theory by imposing the following constraints on the currents

$$\mathrm{tr}\, JE_\varphi = J^{-\varphi} = \mu_\varphi\ , \qquad \varphi \in \Phi^+, \tag{3.a}$$

where $\Phi^+$ denotes the set of positive roots and the constants $\mu_\varphi$ are zero for all but the simple roots. As shown previously [5] these constraints reduce the WZNW theories to Toda theories. By a suitable choice of the constants $\mu_\varphi$ the currents fulfilling (3.a) are of the form

$$\begin{aligned} J(z) =& I_- + j(z) \quad , \quad I_- = \sum_{i=1}^{l} E_{-\alpha_i}, \\ j(z) =& \sum_{i=1}^{l} j^i(z) H_i + \sum_{\varphi\in\Phi^+} j^\varphi(z) E_\varphi, \end{aligned} \tag{3.b}$$

where $\{E_{\alpha_i}\}$ are the $l$ simple root vectors. The constraints for $\tilde{J}$ are very similar to (3.a) with $\varphi_k$ replaced by $-\varphi_k$. The maximal subgroup of the KM group $G$ which leaves the form (3.b) of the currents invariant is the maximally nilpotent subgroup, $N$, generated by the $E_\varphi$, $(\varphi \in \Phi^+)$ and implemented by the $(D-l)/2$ constrained KM-currents $J^{-\varphi}(z)$. Thus the CKM theory is actually a gauge theory, in which all but $l$ of the $(D+l)/2$ components of $J$ are gauge components. This situation is reflected at the CWZNW-level by the fact that there are only $l$ independent gauge-invariant functions of $g(z,\bar{z})$ (e.g. the $l$ Toda fields). In particular there exists a unique Virasoro density $L(J)$ that commutes weakly with the constraints (3.a) and is therefore gauge-invariant, namely

$$L(J) = \frac{1}{2}\mathrm{tr}\, J^2 - \mathrm{tr}\, HJ', \tag{4}$$

where $H = \hat{\rho}^i H_i$ and $\hat{\rho}$ is as in (1) [5]. $L(J)$ is actually the $T_{++}$-component of the improved energy momentum tensor of the Toda theory. Under conformal transformations generated by this improved stress tensor the current transforms as

$$\{L(x), J(y)\} = \Big([H, J(x)] + J(x)\Big)\delta' + [H, J'(x)]\delta - H\delta'', \tag{5}$$

where $\delta = \delta(x–y)$, i.e. with exception of the $H$-component of $J$ all entries are primary with conformal weights equal to the $H$-weights plus one.

The currents $J(z)$ in (2), and the gauge-transformations corresponding to the $E_\varphi$, act on each column of $g(z)$ separately, and since each column is a copy of the defining representation of $G$, it contains only one component $\overset{\circ}{e}$ say (namely the lowest weight component) satisfying $E_\varphi \overset{\circ}{e} = 0$, and hence only one component that is gauge-invariant. It is natural to eliminate the gauge-covariant elements of each column in favour of $\overset{\circ}{e}$, and if one does so, one finds that because of the form of $J(z)$ in (3) the elimination procedure is iterative and leads to a linear differential (or pseudo-differential) equation for $\overset{\circ}{e}$, i.e. an equation of the form

$$\Big(\partial^n + \sum_{r=1}^{n-1} P^r(J)\partial^r\Big)\, \overset{\circ}{e} = 0 \tag{6}$$

(or a similar pseudo-differential equation in which some $\partial^{-1}$'s appear). In this equation the coefficients $P^r(J)$ of the $\partial^r$ are polynomials in the currents and their derivatives, and are gauge-invariant. The operator on the l.h.s of (6) is used in the Toda and GD theories to define the $\mathcal{W}$-algebra (the independent coefficients of the $\partial^r$ being the base elements) and thus the reduction (2) $\to$ (6) identifies the $\mathcal{W}$-algebras as the algebras of gauge-invariant polynomials of the constrained currents and their derivatives.

We now introduce a special $SL(2,R)$ subgroup of $G$, which plays a central role. This $SL(2,R)$ subgroup, which we shall denote by $S$, is defined as that generated by $\{I_{\pm}, H\}$, where $I_-$ is as in (3.b), $H$ is as in (4), and $I_+$ is then a unique linear combination of the $E_{\alpha_i}$ which we do not need to exhibit explicitly. The importance of the group $S$ is that

(a) the simple roots-vectors $E_{\alpha_i}$ are all eigenvectors of $\mathrm{ad}\, H$ with eigenvalue unity,

(b) the adjoint representation of $G$ decomposes into $l$ irreducible (tensor) representations of $S$ whose highest weights $j$ are just the exponents (orders of the Casimirs minus one) of $G$,

(c) the condition for the linear constraints imposed on the current components to be a gauge fixing is that $J_{gf}$ must have one non-zero component in each of the $l$ irreducible representations of $S$ appearing in the decomposition of the adjoint of $G$.

Two natural gauges which satisfy this condition are the diagonal and Drinfeld-Sokolov (DS) gauges, i.e.

$$j(z) = \sum_{i=1}^{l} j^i(z) H_i \qquad \text{and} \qquad j_{DS}(z) = \sum_{p \geq 2} j^p_{DS}(z) F_p, \tag{7}$$

where the $p$ are the orders of the $l$ independent Casimirs* of $G$ and the $F_p$ are generators with $H$-weights $(p-1)$ satisfying condition (c) above. Note that $j^2_{DS}$ is just $L(J)$.

In the diagonal gauge the $\mathcal{W}$-algebra generators, $P^r(J)$, reduce to $P^r(j^i H_i)$, and since the Cartan KM currents $j^i$ satisfy free-field KM commutation relations, this gauge provides us immediately with the free-field representation of the $\mathcal{W}$-algebra. Furthermore, if in the algebra of the $P^r(j^i H_i)$ derivatives higher than the first are dropped, the resulting 'truncated' $\mathcal{W}$-algebra is easily seen to be just KM-Casimir algebra studied by Bais et. al [7], restricted to its Cartan subalgebra. But at the Poisson-bracket level the restriction to the Cartan subalgebra is actually isomorphic to the full KM-Casimir algebra [6]. Hence, at the PB-level the truncated $\mathcal{W}$-algebra may be identified with the Casimir-algebra. Conversely, one may say that the full $\mathcal{W}$-algebra is a deformation of the Casimir algebra, where the Casimir operators are restricted to the space of constrained currents (3.b) and higher derivatives are included in order to ensure gauge-invariance. From this correspondence and the orders of the Casimirs it is easy to see that a quadratic basis for the $\mathcal{W}$-algebras can exist only for the $A, B, C$ groups [6].

The $S$-group may also be used to obtain a simple criterion for the pseudo-differential nature of the Toda-GD equation (6) as follows: from the form of $J(z)$ in

* Note that $D_{2n}$ possesses two independent Casimirs of order $2n$

(2) one sees that the elimination procedure (2) $\to$ (6) requires no $\partial^{-1}$'s if, and only if, the only element in the kernel of $I_+$ is $\overset{\circ}{e}$ itself. But this is just the condition for the defining representation, $F$, of $G$ to be irreducible with respect to the group $S$. Thus the criterion for eq. (6) to be pseudo-differential is just the $S$-reducibility of $F$. It is easy to verify that $F$ is $S$-irreducible for the $A$, $B$, $C$, $G_2$-groups and $S$-reducible for the $D$, $E$, $F_4$ groups, which leads to the result stated earlier.

The importance of the DS-gauges is that the gauge-transformation $U(J)$ that transforms any current $J(z)$ to its DS-form

$$UJU^{-1} + \partial UU^{-1} = J_{DS}(z) \tag{8}$$

is unique and is a polynomial in $J(z)$ and its derivatives. The uniqueness of $U(J)$ means that the DS-gauges define a complete gauge-fixing (with no Gribov ambiguities for example). Thus the ring of all gauge-invariant differential polynomials $\{W(J)\}$ is finitely generated and in particular the $l$ components $j^p_{DS}$ of $J_{DS}$ can be used as the representatives (in this gauge) of $l$ base elements $W^p(J)$ of $\{W(J)\}$.

For example taking $G = SL(2,R)$:

$$U \begin{pmatrix} \theta(z) & j(z) \\ 1 & -\theta(z) \end{pmatrix} U^{-1} + \partial UU^{-1} = \begin{pmatrix} 0 & j_{DS}(z) \\ 1 & 0 \end{pmatrix} \tag{9}$$

we get

$$U(J) = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad j_{DS}(J) = \theta^2 + j - \theta'\,. \tag{10}$$

In fact $j_{DS} = L(J)$, where $L(J)$ is the improved Virasoro operator defined in (4) and is the simplest example of a gauge invariant polynomial. Note, that in the identification $W^p(J) = j^p_{DS}$ each DS-gauge corresponds to a different choice of basis of the $\mathcal{W}$-algebra.

The straightforward way to compute the $\mathcal{W}$-algebra relations explicitly would be to choose a basis $\{W^p\}$ where the $l$ elements $W^p$ are differential polynomials of the currents and simply calculate their Poisson brackets. In practice this is not tractable, since first one has to find the $W^p$'s as explicit functions of the KM currents which is quite cumbersome and second the calculation of the Poisson brackets of complicated polynomials is rather tedious. We now show that there is a way to compute the $\mathcal{W}$-algebra relations without computing the $W^p$'s themselves. As the $W^p$'s are gauge invariant it suffices to know them as functions of the $l$ independent gauge fixed currents $J^p_{gf}$. In fact there is a class of gauges where the $W^p$'s are just the $J^p_{gf}$'s themselves. The price one has to pay is that the computations to be done involve Dirac brackets, rather than Poisson brackets, because the Toda and the gauge fixing constraints together form a second class system. Though in general the Dirac procedure is quite hard to implement it becomes tractable in our case due to the special nilpotent form of the Toda constraints and the linear gauge choices.

The $\mathcal{W}$-algebra relations may be summarized by the standard formula:

$$\delta W^p(x) = \int dy\; a_q(y)\,\{W^q(y), W^p(x)\}, \tag{11.a}$$

where the $p, q$ are the orders of the independent Casimirs and the $a_q$'s arbitrary parameter functions. After gauge fixing eq. (11.a) becomes:

$$\delta w^p(x) = \int dy\; a_q(y)\,\{w^q(y), w^p(x)\}^\star\,, \qquad \text{where} \quad w^p = W^p(J^{gf}). \tag{11.b}$$

In eq. (11.b) $\{,\}^\star$ denotes the Dirac bracket. To compute the r.h.s. of (11.b) we write it in the form:

$$\int dy\; \Big(a_q(y)\{w^q(y), w^p(x)\} + a_\beta(y)\{c^\beta(y), w^p(x)\}\Big), \tag{12.a}$$

where the $a_\beta$ are defined as solution of

$$a_q\,\{w^q, c^\alpha\} + a_\beta\,\{c^\beta, c^\alpha\} = 0, \tag{12.b}$$

and $\{c^\alpha\}$ is the set of $D-l$ constraints. Although eqs. (12.b) need only be solved on the 'constraint surface' (defined by $c^\alpha = 0$) for $a_\beta(a_q, J_{gf})$, in general it is difficult to find the solution even then, since it is equivalent to inverting the 'constraint matrix', $\{c^\alpha, c^\beta\}$. However the Toda constraints are such that (12.b) is tractable, and when the result is substituted into (12.a) $\delta w^p$ is quite simple. In fact in the DS-gauges (12.b) can be solved in an iterative, algebraic manner.

The crucial property of the DS-gauges is that all gauge fixing conditions are *linear* in the currents and that the parameters of the gauge transformations are *differential polynomials* of the currents. These properties allow us to identify the $w^p$'s with the currents in the DS gauge $j^p_{DS}$. Thus in the DS gauges the $w^p$'s and the constraints $c^\alpha$ in (12) are just the current components, up to constants, therefore (12.a) can be rewritten as:

$$\delta j^p_{DS}(x) = \int dy \Big(a_q(y)\,\{j^q_{DS}(y), j^p_{DS}(x)\} + a_\beta(y)\,\{J^\beta(y), j^p_{DS}(x)\}\Big)\big|_{c^\alpha=0} \tag{13.a}$$

where the $a_\beta$'s satisfy

$$\Big(a_q\,\{j^q_{DS}, J^\alpha\} + a_\beta\,\{J^\beta, J^\alpha\}\Big)\big|_{c^\gamma=0} = 0. \tag{13.b}$$

Exploiting the linearity of eq. (13.a) in $j_{DS}$ we may rewrite it as:

$$\delta_R J_{DS} = [R, J_{DS}] + R', \qquad R(a) = a_q F^q + a_\beta T^\beta, \tag{14}$$

where the $F^q$ are conjugate to the $F_p$ in (7), $\mathrm{Tr}\, F^q F_p = \delta^q_p$, and $T^\beta$ are the remaining (matrix) generators of the Lie algebra of $G$. Now eq. (13.b) can be interpreted in the following way. Given the $a_q$'s as free parameter functions, find $R(a_q, j_{DS})$ such that the KM-transformation (14) keeps the DS-current form-invariant. In other words determine the $a_\beta(a_q, j_{DS})$ such that $J_{DS} + \delta_R J_{DS}$ is still in the same gauge as $J_{DS}$ and use these $a_\beta$ to compute $\delta_R J_{DS}$. It turns out that this is a very efficient way to calculate the $\delta_R J_{DS}$ and hence the $\mathcal{W}$-algebra relations, as we shall illustrate on the example of $SL(3,R)$. It may seem surprising that the problem of inverting the 'constraint matrix' can be solved in such a relatively simple way, but this may perhaps be understood as follows. In the DS gauges all of the constraints are linear and hence the problem would be completely trivial if the symplectic form were constant. Thus the only complication in our case (apart from the problem being infinite dimensional in a rather harmless way) is that the KM symplectic form is not constant but linear in the variables $J$. This latter fact and the nature of the Toda constraints makes the problem non-trivial but still tractable.

As the simplest non-trivial example let us consider $SL(3,R)$. In this case the special $sl(2,R)$ subalgebra $S$ is generated by

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad I_+ = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{15}$$

and the gauge fixed currents $J_{DS}$ in (7) are

$$J_{DS}(x) = I_- + j^2(x) F_2 + j^3(x) F_3 \ , \tag{16.a}$$

where

$$F_2 = F_2(t) = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 1-t \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{16.b}$$

Here $t$ is a parameter which distinguishes the different DS-gauges. The 'Wronskian gauge' $t=1$ is the one usually used in the literature [2,3] and in this gauge the $F^q$, $q=2,3$, in (14) are just the transpose of $F_2$ and $F_3$. Applying our algorithm we find that

$$R(a, j_{DS}) = a_2 F^2 + a_3 F^3 + \Delta R(a, j_{DS}), \tag{17}$$

where

$$\Delta R(a, j_{DS}) = a_2\, j_{DS} + a_2'\, H + \begin{pmatrix} u & v-u' & v'-2u'' \\ 0 & u & v-2u' \\ 0 & a_2 - a_3' & -2u \end{pmatrix} \tag{18.a}$$

and

$$u = \frac{1}{3}(a_3 j^2_{DS} - a_3'') \quad , \quad v = a_3 j^3_{DS} - a_2''. \tag{18.b}$$

The variation of $J_{DS}$ under the KM transformation (14) using this $R$ is then found to be :

$$\begin{aligned}
\delta_2 j^2 &= a_2 (j^2)' + 2a_2' j^2 - 2a_2''' \\
\delta_2 j^3 &= a_2 (j^3)' + 3a_2' j^3 + a_2'' j^2 - a_2^{(\mathrm{IV})} \\
\delta_3 j^2 &= 2a_3 (j^3)' + 3a_3' j^3 - (a_3 j^2)'' + a_3^{(\mathrm{IV})} \\
\delta_3 j^3 &= a_3 [(j^3)'' + \tfrac{2}{3} j^2 (j^2)' - \tfrac{2}{3}(j^2)'''] \\
&\quad + a_3' [\tfrac{2}{3}(j^2)^2 + 2(j^3)' - 2(j^2)''] - 2a_3''(j^2)' - \tfrac{4}{3} a_3''' j^2 + \tfrac{2}{3} a_3^{(\mathrm{V})} \; .
\end{aligned} \tag{19}$$

Note that the components of $R(a, j_{DS})$ in (17) are *linear* functions of the current components, and consequently $\delta_R J_{DS}$ is at most quadratic in $J_{DS}$, which implies that the Poisson brackets of the $\mathcal{W}$-generators are also (at most) quadratic polynomials. This is not always the case, as can be checked on the example of $B_2$.
From (19) one can read off the $SL(3,R)$ $\mathcal{W}$-algebra relations:

$$\begin{aligned}
\{W^2(x), W^2(y)\} &= (W^2)'(x)\delta + 2W^2(x)\delta' - 2\delta''' \\
\{W^2(x), W^3(y)\} &= 2(W^3)'(x)\delta + 3W^3(x)\delta' - [W^2(x)\delta]'' + \delta^{(\mathrm{IV})} \\
\{W^3(x), W^3(y)\} &= [\tfrac{2}{3}(W^2)'W^2 + (W^3)'' - \tfrac{2}{3}(W^2)'''](x)\delta \\
&+ [\tfrac{2}{3}(W^2)^2 + 2(W^3)' - 2(W^2)''](x)\delta' - 2(W^2)'(x)\delta'' - \tfrac{4}{3}W^2(x)\delta''' + \tfrac{2}{3}\delta^{(\mathrm{V})} \; ,
\end{aligned} \tag{20}$$

where $\delta = \delta(x-y)$. Note that for the CKM group-valued field we have $\delta g = Rg$, from which the variations of any field, such as the Toda field can be found. Of course the $SL(3,R)$ $\mathcal{W}$-algebra is well known and has been used here only for illustration. The true power of the algorithm becomes apparent only when it is applied to more complicated groups. In Ref. [6], for example, it was used to obtain the $\mathcal{W}$-algebras of the $B_2$ and $G_2$ groups.

Let us now consider the question of primary field bases for $\mathcal{W}$-algebras. Observe that the $W^3$ generator in (20) is not a primary field with respect to $W^2$, however

$$\tilde{W}^3 = W^3 - \frac{1}{2}(W^2)'$$

is a primary field of weight 3. We now show that a primary field basis can be identified for any group $G$. For this we chose the special DS gauge (highest weight gauge) in which the generators $F_p$ in (7) are the highest weight vectors, $E_p$ of the $S$ group occuring in the adjoint of $G$ (for the $SL(3,R)$ example above this is the $t = 1/2$ gauge). That is we now chose the DS gauge

$$j_{HW}(z) = \sum_{p \geq 2} j^p_{HW}(z) E_p = L(J) I_+ + \sum_{p \geq 3} j^p_{HW}(z) E_p \tag{21.a}$$

where

$$[I_+, E_p] = 0. \qquad (21.b)$$

The conformal transformations (5) do not respect this (or any other DS) gauge because the inhomogeneous term in (5) is diagonal and the DS-gauges have no diagonal component. However, we can find a closely related gauge which has a diagonal component to absorb the inhomogeneous term by making the gauge transformation $\exp(\theta I_+)$, where $\theta^2 - \theta' = j^2_{HW}$. In this way we obtain the gauge

$$j_\theta = 2\theta \cdot H + \sum_{p \geq 3} j^p_{HW} E_p \,, \qquad (22)$$

where the $j^p_{HW}$, $p \geq 3$ in (22) and (21.a) are the *same*, because $I_+$ commutes with the $E_p$'s and $\theta' I_+$ has no component along the $E_p$ for $p \geq 3$. The conformal transformation (5) leaves (22) form-invariant and gives rise to the following variation of the currents:

$$\delta\theta = (a\theta)' + \frac{a''}{2} \qquad \delta j^p_{HW} = a(j^p_{HW})' + p\, a' j^p_{HW} \,, \quad p \geq 3. \qquad (23)$$

Eq. (23) shows that $\theta$ is not a primary field, but the $j^p_{HW}$ $(p \geq 3)$ are *primary fields* of conformal spin $p$. On the other hand, because the highest weight gauge is a DS gauge the current components $j^p_{HW}$, $p \geq 3$, together with $L(J)$, define a basis, $W^p$ of the $\mathcal{W}$-algebra. Combining the two results we see that this is the required primary field basis of the $\mathcal{W}$-algebra and the weights of a base element $W^p$, $p \geq 3$ is just $p$ itself.

It is easy to see from (14) that the degree of the polynomial $\delta j_{DS}(z)$ in $j_{DS}(z)$ cannot exceed the nilpotency index of $\mathrm{ad}\, j_{DS}(z)$, and this means that in the corresponding basis $W^p$, the degree of the $\mathcal{W}$-algebra cannot exceed the nilpotency index. In particular, the $\mathcal{W}$-algebra will be quadratic if the nilpotency index of $j_{DS}(z)$ is 2. We already know that the $\mathcal{W}$-algebra cannot be quadratic for the $D$, $E$, $F_4$, $G_2$ groups, but we can now show that for the remaining ($A$,$B$,$C$) groups there exist bases in which it is quadratic. The point is that, for these groups, the $j_{DS}(z)$ can be chosen to lie within any rectangular block in the upper triangle that intersects $I_+$ (for example the current (16) is not of block-form unless $t = 1$) and then the nilpotency index of $j_{DS}(z)$ is automatically 2. Note that in general the highest weight gauge does not lie within such a rectangular block and does not have nilpotency index 2.

Finally we turn to the derivation of the result (1) for the central charge of the quantized CKM theory. The contribution of the CKM-fields to the central charge is just the central charge of the improved energy-momentum tensor $L(J)$ in (4), that is the central charge of the Sugawara energy-momentum tensor plus the contribution of the improvement term, i.e.

$$c_{CKM} = \frac{k \cdot D}{k+g} - 12k\,\mathrm{Tr}H^2. \tag{24}$$

To determine the contribution of the BRST ghosts, which are necessary to implement the constraints (3.a), we first observe that, according to (5), the currents to be constrained are primary fields with weights $\Delta(J^{-\varphi}) = 1 - h_\varphi$, where $h_\varphi$ is the $H$-weight of $E_\varphi$. Hence we must introduce a ghost pair $(b_\varphi,\ c_\varphi)$ with weight $(1{+}h_\varphi, -h_\varphi)$ for each of the $(D-l)/2$ constraints (3.a). Each ghost pair contributes $12(h_\varphi(1-h_\varphi) - \frac{1}{6})$ to the central charge of the CWZW model [8]. Making use of

$$\sum_{\Phi+} h_\varphi = 2\hat{\rho}\cdot\rho \qquad \text{and} \qquad \sum_{\Phi+} h_\varphi^2 = \frac{1}{2}\mathrm{tr}_{\mathrm{adj}} H^2\,,$$

the total contribution of the ghost system can be written as

$$c_{\mathrm{gh}} = l - D - 12g\ \mathrm{Tr}H^2 + 24\,\hat{\rho}\cdot\rho\,, \tag{25}$$

where $g\mathrm{Tr}H^2 = \frac{1}{2}\mathrm{tr}_{\mathrm{adj}}H^2$.

By using the Freudenthal-deVries formula $gD = 12\rho^2$, one arrives at eq. (1) for the total central charge of the CWZW model. For simply laced algebras $\hat{\rho} = \rho$ and (1) reduces to

$$c = l - \frac{gD}{k+g}(k+g-1)^2\,, \tag{26.$a$}$$

which agrees with the $sl(n,R)$ result in [9]. By writing $k{+}g$ as $r/s$ it takes the form

$$c = l\left[1 - g(g+1)\frac{(r-s)^2}{rs}\right], \tag{26.$b$}$$

which was obtained directly from Toda theory (without using the KM imbedding) by Bilal and Gervais [3].

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