On 4D-Hawking Radiation from Effective Action

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Abstract

We determine the s-waves contribution of a scalar field to the four dimensional effective action for arbitrary spherically symmetric external gravitational fields. The result is applied to $4d$-black holes and it is shown that the energy momentum tensor derived from the (nonlocal) effective action contains the Hawking radiation. The luminosity is close to the expected one in the s-channel. The energy momentum tensor may be used as starting point to study the backreaction problem.

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1 Introduction

One of the interesting problems in quantum field theory in curved spacetimes is to derive the induced energy momentum tensor and in particular the Hawking radiation [1] from the effective action approach [2, 3, 4]. The solution of this problem is important for studying the backreaction problem for black holes [5-8]. The effective action for quantized matter fields in a black hole metric is strongly nonlocal and should describe both the asymptotic Hawking radiation and the vacuum polarization effects [9].

Most of the recent works on the Hawking radiation and backreaction problem (see, for instance [2,5-8]) concerned $2d$ black holes. In particular, it has been shown that the $2d$ Hawking radiation can be derived from the $2d$ effective action [2,9]. It is a priori not clear whether these results are of relevance for real $4d$ black holes. On the other hand, the covariant perturbation theory for the $4d$ effective action Γ as developed in [3,10] seems to be very involved for concrete calculations. The results obtained so far are far from being complete.

In this paper we shall simplify the problem by considering s-modes of minimally coupled massless scalar fields propagating in an arbitrary spherically symmetric $4d$ spacetime. We compute the contribution of these modes to the $4d$ effective action Γ. The part of Γ which is not invariant under Weyl-rescalings of the (r,t)-part of the metric is exactly calculated. For the invariant part an appropriate perturbation expansion is developed. As an application the s-wave contribution to the Hawking flux is obtained from the s-channel effective action for $4d$ black holes. We demonstrate why and how the $2d$-calculations [2,5-8] are relevant for realistic $4d$ black holes.

The calculations are performed in the Euclidean formalism. The sign conventions, e.g. for the Riemann tensor and the signature when we return to Lorentzian space-time after the calculations have been done, are the same as in [11]. We set $c=\hbar=G=1$.

2 Setup

The Euclidean action for the coupled gravitational and scalar fields is

$$S_E = S_E^{grav} + S_E^{\phi} = -\frac{1}{16\pi}\int \mathcal{R}\sqrt{g}\, d^4x + \frac{1}{2}\int \sqrt{g} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\, d^4x. \tag{1}$$

For spherically symmetric space-times it is convenient to choose adapted coordinates for which the metric takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ab}(x^a)dx^a dx^b + \Omega^2(x^a)\,\omega_{ij}dx^i dx^j, \tag{2}$$

where

$$\omega_{ij}dx^i dx^j = (d\theta^2 + \sin^2\theta d\varphi^2) \tag{3}$$

is the metric of S^2. The function Ω depends only on the coordinates $(x^a) = (x^0, x^1) = (t, r)$ and $\gamma_{ab}(x^a)$ is the metric in the $t-r$ sector. Note that $g^{ab} = \gamma^{ab}$.

In a spherically symmetric space-time we can expand a matter field into spherical harmonics. In particular scalar fields in the s-channel depend only on t and r, $\phi = \phi(x^a)$. For s-waves the action (1) reduces to the following $2d$ action

$$\begin{aligned} S_E = S_E^{grav} + S_E^{\phi} &= -\frac{1}{4}\int \Big(\Omega^2\,{}^\gamma\mathcal{R} + {}^\omega\mathcal{R} + 2(\nabla\Omega)^2\Big)\sqrt{\gamma}\,d^2x \\ &+ 2\pi\int \Omega^2(\nabla\phi)^2\sqrt{\gamma}\,d^2x, \end{aligned} \tag{4}$$

where we took into account that the volume of S^2 is equal to 4π. Here ${}^\gamma\mathcal{R}$ is the scalar curvature of the $2d$-space with metric γ_{ab}, ${}^\omega\mathcal{R} = 2$ is the scalar curvature of S^2 and $(\nabla\Omega)^2 = \gamma^{ab}\partial_a\Omega\partial_b\Omega$.

The purely gravitational part of the action (4) is almost the action belonging to $2d$ dilatonic gravity with two exceptions: first, the numerical coefficient in front of $(\nabla\Omega)^2$ is different and second, the action (4) is not invariant under Weyl transformations due to the ${}^\omega\mathcal{R}$ term which is the $2d$ analog of the cosmological constant in 4 dimensions. The action for the scalar field $\phi(t, r)$ is quite different from the actions usually considered in $2d$-field theories [5-8] because of the unusual coupling of ϕ to the dilaton field Ω. The action (4) is the $4d$-action for spherically symmetric gravitational and scalar fields and as such should not be regarded as just another $2d$-toy-model for gravity.

The independent field equations which follow from (4) are

$$\triangle_\gamma\Omega - \Big(\frac{{}^\gamma\mathcal{R}}{2} - 4\pi(\nabla\phi)^2\Big)\Omega = 0 \tag{5}$$

from the variation of Ω,

$$\triangle_\gamma \Omega^2 - \ {}^{\omega}\mathcal{R} + 8\pi\Omega^2(\nabla\phi)^2 = 0, \qquad (6)$$

which is the trace of the variation with respect to γ_{ab}, and the equation for the scalar field

$$\nabla^a(\Omega^2\nabla_a\phi) = 0. \qquad (7)$$

Here $\triangle_\gamma$ is the Laplace-Beltrami operator in 2-dimensional space-time with the metric γ_{ab}. Note that the matter part of the action (4) is invariant under 2-dimensional Weyl transformations, $\gamma_{ab} \to e^{2\sigma}\gamma_{ab}$, and hence the partial trace $T^a_{\ a}$ of the energy momentum tensor vanishes for spherically symmetric scalar fields.

Without matter ($\phi = 0$) the $2d$-Euclidean black holes

$$^{(2)}ds^2 = \Big(1 - \frac{r_g}{r}\Big)dt^2 + \frac{dr^2}{1 - r_g/r}, \qquad \Omega = r \qquad (8)$$

are solutions of (5,6) as it should be.

3 Effective Action

In this section we determine the s-wave contribution of the quantized scalar field to the effective action. In particular we shall show how this problem can be reduced to a $2d$ problem. Then we shall calculate the non Weyl-invariant part of the s-channel effective action exactly and develop a perturbation theory for the Weyl-invariant part.

The Euclidean $4d$-effective action Γ is defined as

$$e^{-\Gamma} = \int \mathcal{D}\phi \, e^{-S^\phi_E}, \qquad (9)$$

where

$$S^\phi_E = -\frac{1}{2}\int \phi\triangle\phi \sqrt{g}\, d^4x \qquad (10)$$

is the Euclidean action for the minimally coupled scalar field and $\triangle = \triangle_g$ is the $4d$-Laplace-Beltrami operator. To define the (formal) diffeomorphism invariant measure in the path integral (9) we expand the field $\phi(x^\alpha)$ in terms of the eigenfunctions of $-\triangle$. For a spherically symmetric space-time and adapted coordinates (2) this expansion reads

$$\phi(x^\alpha) = \sum_{nlm} \phi_{nlm}, \tag{11}$$

where

$$\phi_{nlm} = \phi_n^{lm}(t,r) Y_{lm}(\theta,\phi), \qquad -\triangle\phi_{nlm} = \lambda_{nl}\phi_{nlm}. \tag{12}$$

Here the Y_{lm} are the spherical harmonics and the eigenmodes are normalized with respect to the 4-metric:

$$\langle\phi_{nlm}|\phi_{n'l'm'}\rangle = \int \phi_{nlm}\phi_{n'l'm'}\sqrt{g}\, d^4x = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{13}$$

Then the path integral becomes

$$e^{-\Gamma} = \int \prod_{nlm} dc_{nlm} \exp\Big(-\frac{1}{2}\sum_{nlm}\lambda_{nl}\, c_{nlm}^2\Big) = \exp\Big(-\sum_l (2l+1)\Gamma_l\Big), \tag{14}$$

where

$$e^{-\Gamma_l} = \int \prod_n dc_{nlm} \exp\Big(-\frac{1}{2}\sum_n \lambda_{nl}\, c_{nlm}^2\Big) \tag{15}$$

is the contribution of the modes with quantum numbers (l,m) and we took into account that the eigenvalues do not depend on the magnetic quantum number m.

The integrals (14,15) are of course ultraviolet divergent and must be regularized. We shall use the *zeta*-function regularization [12]. Any other covariant regularization of (14) would yield the same result up to integrals of local terms of the form $\sqrt{g}$, $\sqrt{g}\mathcal{R}$ and $\sqrt{g}\mathcal{R}^2$, $\sqrt{g}\partial\partial\mathcal{R}$ [13]. The coefficients of these ambigues terms should be determined by experiments or observations in any case. One can regularize every Γ_l separately and then sum over all angular momenta to recover the total effective action $\Gamma = \sum_l (2l+1)\Gamma_l$. In general the sum of the regularized Γ_l's is still quadratically divergent. However, the remaining quadratic and logarithmic divergences can be absorbed by redefining coefficients of the local counterterms. For the finite non-local terms of interest the regularization commutes with taking the sum over the angular momentum sectors. Hence, to obtain the nonlocal contribution of the different sectors to Γ one can apply the ζ-function regularization for every sector separately.

In this paper we shall calculate only the contribution $\Gamma_s \equiv \Gamma_0$ of the s-wave scalar fields to the total $4d$ effective action:

$$e^{-\Gamma_s} = \int \prod_n dc_{n00} \exp\Big(-\frac{1}{2} \sum_n \lambda_{n0}\, c_{n00}^2 \Big). \tag{16}$$

Now we shall show how (16) relates to the effective action of a 2-dimensional theory. For that we introduce the complete set of rescaled s-modes

$$\varphi_n(t,r) = \sqrt{4\pi}\, \Omega(t,r) \phi_{n00}(t,r) \tag{17}$$

which are orthonormal with respect to the 2-metric γ_{ab}

$$\langle \varphi_n | \varphi_{n'} \rangle = \int \ \varphi_n \varphi_{n'} \sqrt{\gamma}\, d^2x = \delta_{nn'}, \tag{18}$$

contrary to the ϕ_{n00}, which are orthonormal with respect to the 4-metric $g_{\alpha\beta}$. Then any field $\varphi(t,r) = \sqrt{4\pi}\Omega\phi_{l=0}(t,r)$ can be expanded as

$$\varphi = \sum_n \ c_n \varphi_n. \tag{19}$$

Also note that since Ω in (2) had the dimension of a length (if we keep the dimension of the gravitational constant G), φ becomes dimensionless as required for a 2-dimensional scalar field.

It is easy to see that the φ_n are the eigenmodes of the $2d$-operator

$$\hat{O} = \Big(-\triangle_\gamma + \frac{\triangle_\gamma \Omega}{\Omega} \Big) \tag{20}$$

with the same eigenvalues $\lambda_n = \lambda_{n0}$ an in (12). Then (16) can be rewritten as the functional integral of a field theory in 2-dimensional space time with the metric γ_{ab}:

$$e^{-\Gamma_s} = \int \mathcal{D}\varphi \exp\Big(-\frac{1}{2} \int \varphi \hat{O} \varphi \sqrt{\gamma}\, d^2x \Big), \tag{21}$$

where the measure $\mathcal{D}\varphi$ is the usual (formal) Lebeques measure. The classical action in (21) is of course just the action S_E^ϕ in (4) rewritten in terms of the $2d$-scalar field φ. From (21) it follows at once that

$$\Gamma_s = \frac{1}{2}\log\det\hat{O}. \tag{22}$$

Thus, calculating the s-waves contribution to the effective action reduces to the problem of calculating the determinant of the operator $\hat{O}$ defined in $2d$-space with metric γ_{ab}. This operator has the nice and important property that it transforms homogeneously under Weyl-rescalings

$$\gamma_{ab} \to e^{2\sigma}\gamma_{ab} \Longrightarrow \hat{O} \to e^{-2\sigma}\hat{O}. \tag{23}$$

This immediately implies that the classical $2d$-action for φ is Weyl invariant. As it is well-known [14], the $2d$-effective action ceases to be Weyl invariant and the breaking is determined by the trace anomaly which is proportional to the first Seeley-deWitt coefficient a_1, which in our case is

$$a_1 = \frac{{}^\gamma\mathcal{R}}{6} - \frac{\triangle_\gamma\Omega}{\Omega}. \tag{24}$$

Thus we have

$$\gamma^{ab}\frac{2}{\sqrt{\gamma}}\frac{\delta\Gamma_s}{\delta\gamma^{ab}} = \frac{a_1}{4\pi}. \tag{25}$$

This equation can easily be integrated if we choose isothermal coordinates (the conformal gauge)

$$\gamma_{ab} = e^{2\sigma}\gamma^f_{ab}, \tag{26}$$

where γ^f_{ab} is the metric of the flat $2d$ space. In this gauge

$${}^\gamma\mathcal{R} = -2\triangle_\gamma\sigma = -2e^{-2\sigma}\triangle_f\sigma \tag{27}$$

and (25) simplifies to

$$\frac{\delta\Gamma_s}{\delta\sigma} = \frac{1}{4\pi}\Big(\frac{1}{3}\triangle_f\sigma + \frac{\triangle_f\Omega}{\Omega}\Big), \tag{28}$$

where $\triangle_f$ is the Laplace operator on flat space which does not depend on σ.

Integrating (28) and expressing σ in terms of ${}^{\gamma}\mathcal{R}$ by means of eq. (27) one ends up with

$$\begin{aligned} {}^{(n)}\Gamma_s[\sigma,\Omega] &\equiv \Gamma_s[\sigma,\Omega]-\Gamma_s[\sigma=0,\Omega] \\ &= \frac{1}{8\pi}\int\Big[\frac{1}{12}\,{}^{\gamma}\mathcal{R}\frac{1}{\triangle_\gamma}\,{}^{\gamma}\mathcal{R}-\frac{\triangle_\gamma\Omega}{\Omega}\frac{1}{\triangle_\gamma}\,{}^{\gamma}\mathcal{R}\Big]\sqrt{\gamma}\,d^2x \end{aligned} \tag{29}$$

which is manifestly invariant under $2d$-coordinate transformations. The first term on the r.h.s in (29) has been obtained previously in [2,9].In [2] it has been shown that it leads to the Hawking radiation for $2d$ black holes. We shall see in the following section how it is related with the s-wave radiation of $4d$ black holes. However, there is the second term which, as can easily be checked, yields a infalling radiation flux. The amplitude of this radiation exceeds the outgoing flux coming from the first term by a factor 6. However, we must not forget that in (29) we calculated only that part of Γ_s which is non-invariant under $2d$-Weyl transformations. The total s-wave effective action is

$$\Gamma_s = {}^{(n)}\Gamma_s + {}^{(i)}\Gamma_s, \tag{30}$$

where

$${}^{(i)}\Gamma_s=\Gamma_s[\sigma=0,\Omega]=\frac{1}{2}\log\det\Big(-\triangle_f+\frac{\triangle_f\Omega}{\Omega}\Big)\equiv\frac{1}{2}\log\det\hat{O}_f \tag{31}$$

is the part of the total effective action which is invariant under $2d$-Weyl transformations. To get the complete result we should also calculate the determinant of $\hat{O}_f$ on flat space and then restore the metric γ_{ab} in the obtained expression such as to recover general covariance.

Unfortunately $\log\det\hat{O}_f$ cannot be calculated exactly and we must resort to some perturbation expansion. The covariant perturbation theory developed in [3,10] seems to be of no help here because of severe infrared divergences. These are related with the non-analyticity of the effective action in the potential

$$V^f=\frac{\triangle_f\Omega}{\Omega}. \tag{32}$$

Instead we write the heat kernel for $\hat{O}_f$ in the form

$$K(x,x;\tau)=\frac{\mu^2}{4\pi\tau}\exp\Big(-\frac{\tau}{\mu^2}W^f(x;\tau)\Big) \tag{33}$$

and develop the perturbation theory for W in powers of the potential V (see appendix B). The arbitrary mass μ has been introduced such that τ becomes dimensionless.

Using the ζ-function regularization (see appendix A) we immediately arrive at the following finite expression for the effective action in terms of W and V:

$$\begin{aligned}
{}^{(i)}\Gamma_s &= \Gamma_s^{CW}+\Gamma_s^{BS}, \quad \text{where} \\
\Gamma_s^{CW} &= \frac{1}{8\pi}\int\Big\{V^f-V^f\log\frac{V^f}{\mu^2}\Big\}\sqrt{\gamma^f}d^2x \\
\Gamma_s^{BS} &= \int_0^\infty d\tau\Big\{\frac{W^f-V^f}{\tau W^f}(\tau W^f)'\exp(-\frac{\tau}{\mu^2}W^f)\Big\}\sqrt{\gamma^f}d^2x
\end{aligned} \tag{34}$$

and the prime means differentiation with respect to τ. Γ_s^{CW} correspond to the $2d$ Coleman-Weinberg potential [16]. For constant V^f we have $W^f=V^f$ so that Γ_s^{BS} vanishes and the Coleman-Weinberg potential is the exact result. In this section we will neglect Γ_s^{BS} in (34) which is proportional to the derivatives of the potential V. This approximation corresponds to the simple classical approximation to the heat kernel (33) and yields the $4d$ s-channel Hawking radiation without backscattering effects. The more involved problem of including the backscattering will be discussed in section 5.

Next we need to covariantize the Weyl-invariant Coleman-Weinberg contribution to (34), that is restore the original metric γ_{ab}. Taking into account that

$$V^f\equiv\frac{\triangle_f\Omega}{\Omega}=e^{2\sigma}\frac{\triangle_\gamma\Omega}{\Omega} \tag{35}$$

and expressing σ in terms of of $\mathcal{R}$ via (27) we obtain the following $2d$-covariant result

$$\Gamma_s^{CW}=\frac{1}{8\pi}\int\frac{\triangle_\gamma\Omega}{\Omega}\Big(1-\log\frac{1}{\mu^2}\frac{\triangle_\gamma\Omega}{\Omega}+\frac{1}{\triangle_\gamma}{}^\gamma\mathcal{R}\Big)\sqrt{\gamma}\,d^2x. \tag{36}$$

Note that only the last term is nonlocal and contributes to the Hawking flux.

The action Γ_s^{CW} is invariant under Weyl-transformations (26) as required and hence does not contribute to the trace of the $2d$ energy momentum tensor (EMT). In the next section we shall see that it also does not contribute to the partial trace T_a^a of the $4d$ EMT. However, it contributes to the total trace T_α^α. The free mass-parameter μ corresponds to the renormalization arbitrariness.

Combining (29) and (36) one finally obtains for the s-channel effective action in the no-backscattering approximation

$$^{(n)}\Gamma_s + \Gamma_s^{CW} = \frac{1}{8\pi}\int\Big(\frac{1}{12}\,{}^\gamma\mathcal{R}\frac{1}{\triangle_\gamma}\,{}^\gamma\mathcal{R} - \frac{\triangle_\gamma\Omega}{\Omega}(1+\log\frac{\triangle_\gamma\Omega}{\mu^2\Omega})\Big)\sqrt{\gamma}\,d^2x. \quad (37)$$

We see that the nonlocal term in (36) cancels against the second term in (29) which yields a negative contribution to the Hawking flux. However, in the section 5 we shall see that the region near a black hole contributes significantly to the (so far neglected) Γ_s^{BS} in (34) and effectively reduces the coefficient in front of the nonlocal term in (37). Physically this corresponds to a decreasing of the Hawking flux due to backscattering effects.

4 $4d$-Energy Momentum Tensor and Hawking Radiation

The $4d$-EMT can be derived from the $4d$ effective action (9) according to

$$T_{\alpha\beta} = \frac{2}{\sqrt{g}}\frac{\delta\Gamma}{\delta g^{\alpha\beta}}. \quad (38)$$

The s-waves contribution to $T_{\alpha\beta}$ is then gotten by inserting Γ_s for Γ in (38). For that we rewrite the s-channel effective action (22) in terms of the $4d$ metric as

$$\Gamma_s = \frac{1}{4\pi}\int\frac{1}{\Omega^2}\,\mathcal{L}_s\sqrt{g}\,d^4x. \quad (39)$$

Then using the symmetry properties and taking into account that $g_{\alpha\beta} = (\gamma_{ab}, \Omega^2\omega_{ij})$ one obtains the following formulae for the non-vanishing components of the $4d$ EMT:

$$T^a_b = \frac{1}{2\pi\Omega^2}\frac{1}{\sqrt{\gamma}}\gamma^{ac}\frac{\delta\Gamma_s}{\delta\gamma^{cb}} \quad , \quad T^i_j = -\frac{1}{8\pi\Omega}\frac{1}{\sqrt{\gamma}}\frac{\delta\Gamma_s}{\delta\Omega}\delta^i{}_j. \tag{40}$$

Without backscattering effects Γ_s is given by (37) and the functional derivative is to be calculated in $2d$-space. Straightforward calculations lead to the following explicit expressions for the EMT ($\triangle = \triangle_\gamma, \mathcal{R} = \mathcal{R}^\gamma$):

$$\begin{aligned} T^a_b \;&=\; \frac{1}{4\pi\Omega^2}\frac{1}{48\pi}\Big[-2\nabla^a\nabla_b(\frac{1}{\triangle}\mathcal{R}) + \nabla^a(\frac{1}{\triangle}\mathcal{R})\nabla_b\frac{1}{\triangle}\mathcal{R} \\ &\qquad + \delta^a_b\Big(2\mathcal{R} - \frac{1}{2}\nabla^c(\frac{1}{\triangle}\mathcal{R})\nabla_c(\frac{1}{\triangle}\mathcal{R})\Big)\Big] + \text{local terms} \end{aligned} \tag{41}$$

The components T^i_j contain only local terms which give rise to vacuum polarization effect. Here we are mainly interested in particle creation and for that reason skipped all local terms in (41). We stress that up to this point our results apply to arbitrary spherically symmetric backgrounds. Thus (41) describes the s-channel particle creation (and vacuum polarization) for minimally coupled scalars propagating in an *arbitrary spherically symmetric* spacetime, e.g. of a collapsing star. Here we are mainly concerned with the Hawking radiation and leave other interesting applications to a forthcoming publication.

To get the flux of the Hawking radiation we need to go back to Lorentzian space-time by changing the signs in the appropriate places. According to the results in [2,17] we arrive at the in-vacuum EMT by replacing $-1/\triangle$ by the retarded Greens function G^-. Only the second term in (41) contributes to the Hawking radiation. The calculations leading to the corresponding flux are analogous to the ones which have been done by Frolov and Vilkovisky [2]. Thus we may skip them here by referring the reader to that paper.

The luminosity of the black hole, which is obtained from the Lorentzian (in-vacuum) version of the EMT (41) is then found to be

$$L = -\frac{\pi}{12}\frac{1}{(8\pi M)^2}, \tag{42}$$

where M is the mass of the black hole. This exactly coincides with the total s-waves flux of the Hawking radiation obtained by standard methods [14] without taking backscattering effects into account.

5 Backscattering effect

To take into account the backscattering of Hawking radiation we must calculate Γ_s^{BS} in (34),

$$\Gamma_s^{BS} = \frac{1}{8\pi} \int I \sqrt{\gamma^f} d^2x \quad , \quad I = \int d\tau \frac{W^f - V^f}{\tau W^f} (W^f \tau)' \, e^{-W^f \tau}, \tag{43}$$

to the effective action. For that we develop the perturbation expansion for W^f in powers of the potential V^f. This expansion in presented in appendix B. Up to linear order in V^f we find

$$W^f(x;\tau) = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} (\frac{\tau \triangle}{\mu^2})^n \, V^f(x) + O(\nabla V^f \cdot \nabla V^f). \tag{44}$$

To simplify the analysis we choose the natural radial variable introduced by Regge and Wheeler

$$r^* = r + 2M \log |\frac{r}{2M} - 1| \tag{45}$$

so that the (r,t)-part of the (Euclidean) black hole metric takes the form

$$ds^2 = (1 - \frac{2M}{r})(dt^2 + dr^{*2}). \tag{46}$$

Note that $r^* \to \infty$ as $r \to \infty$, but also $r^* \to -\infty$ as $r \to 2M$. In this coordinate system the potential V^f reads

$$V^f(r^*) = e^{2\sigma} \frac{\triangle_\gamma \Omega}{\Omega} = (1 - \frac{2M}{r}) \frac{2M}{r^3}, \tag{47}$$

where r should be expressed in terms of r^* via (45). Since the potential V^f depends only on one coordinate, namely r^*, the asymptotic series (44) can be converted into an integral

$$W^f(r^*;\tau) = \sqrt{\frac{\pi \mu^2}{4\tau}} \int_{-\infty}^{\infty} V(\tilde{r}^*) \left(1 - \Phi(\frac{\mu |\tilde{r}^* - r^*|}{\sqrt{\tau}})\right) d\tilde{r}^*, \tag{48}$$

where

$$\Phi(x) = \int_0^x e^{-t^2} dt \tag{49}$$

is the error function. For $r^* \gg 2M$, where $r^* \approx r$, we can find a good approximation to the integral (48) for different values of τ/μ^2. The τ/μ^2-dependence of W^f for $r \gg 2M$ is depicted in figure 1. We see that for $\tau/\mu^2 \ll V^f/\triangle_f V^f$ the function W^f slowly increases as a function of τ, starting with $W^f(r;\tau=0) = V^f(r)$. Then, in a very short interval in the vicinity of $\tau/\mu^2 \sim V^f/\triangle_f V^f$ it increases dramatically from M/r^3 to $1/Mr$. When τ is much bigger then W^f decreases as $1/\sqrt{\tau}$. Since $W^f\tau \sim \sqrt{\tau}$ when $\tau \to \infty$ the expression for the effective action is infrared-convergent.

Clearly, the small interval in the vicinity of $V^f/\triangle_f V^f$, where W^f changes a lot, gives the main contribution to the integral in (34). In this interval we have $W^f \gg V^f$, $(W^f)'\tau \gg W^f$ and we can estimate the integral (43) as

$$I \sim \int_0^\infty d\tau\, (W^f)' \exp\,(- \frac{W^f \tau}{\mu^2}) \sim W^f(\tau_0), \tag{50}$$

where τ_0 is the value of τ for which $\tau_0 W^f(\tau_0) \sim \mu^2$. For potentials for which W^f has the qualitative shape depicted in figure 1, we have $\tau_0/\mu^2 \sim V^f/\triangle_f V^f$. Thus one obtains

$$I \sim \xi \frac{\triangle_f V^f}{V^f}, \tag{51}$$

where ξ is some fudge coefficient. In our approximate treatment of the integral in (43) we cannot get the exact value for this coefficient. Note that at $\tau \sim \tau_0$ the main contribution to the integral (48) which defines W^f comes from the region near the black hole horizon. This confirms that (51) actually takes into account the backscattering of the Hawking radiation in the potential of the black hole, which is most effective near a black hole.

Now we need to restore the metric γ_{ab} in (51). Taking into account eqs. (27) and (36) we find that (51) leads to the following contribution to the total effective action

$$\Gamma_s^{BS} = -\frac{\xi}{8\pi} \int \Big({}^\gamma\mathcal{R} \frac{1}{\triangle_\gamma}\, {}^\gamma\mathcal{R} + \text{local terms}\Big) \sqrt{\gamma} d^2x. \tag{52}$$

This must be added to (37) to get the s-channel effective action. Notice that the nonlocal term in (52) cancels part of the nonlocal term in (37) and diminishes the total Hawking flux. Comparing our result with that obtained by other means [] we conclude that ξ should be about 10 percent less than 1/12.

6 Conclusions

We have calculated the contribution of the s-waves of massless minimally coupled scalars to the $4d$-effective action in an arbitrary spherically symmetric external gravitational field. The problem was to a large extend simplified by reducing the s-waves sector to an effective 2-dimensional, classically Weyl-invariant theory. Of course, it is obvious that the s-wave channel reduces to a 2-dimensional theory. But during the reduction process one needs to rescale the spherically symmetric scalar field (see (e16)) such that the new measure in the path integral belongs to a scalar field propagating in a 2-dimensional spacetime. The field theory for the 2-dimensional field is a conformal field theory. This observations permitted us to calculate the Weyl non-invariant part of the effective action exactly. Then the problem reduces to the calculation of the $2d$-Weyl invariant part which actually is an effective action in $2d$ flat spacetime. To calculate this we developed the perturbation expansion which works well in the case of black holes and permits us to take into account the backscattering of the Hawking radiation by the gravitational field of the black hole. As an application we derived the explicit form of that part of the stress-energy tensor which leads to the Hawking radiation.

However, the range of applicability of our main results is not at all restricted to the black hole physics. They hold for arbitrary spherically symmetric backgrounds and consequently can be applied to study collapse problems, e.g. the particle production by time-dependent spherical gravitational fields.

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A Regularization of the effective action and infrared problem

The effective action Γ expressed in terms of the heat kernel is divergent. Since we are only interested in the finite part of Γ rather than the divergent one we derive here the expression for the ζ-function regularized Γ in terms of some function $W(x;\tau) \equiv W(x,x;\tau)$ which is defined via the heat kernel $K(x,y,\tau)$ in d-dimensional space as

$$K(x,y,\tau) = \frac{\mu^d}{(4\pi\tau)^{\frac{d}{2}}} \exp\Big[-\frac{\mu^2(x-y)^2}{4\tau} - \frac{W(x,y;\tau)\,\tau}{\mu^2}\Big]. \tag{53}$$

Here μ is an arbitrary mass-parameter introduced for dimensional reasons. Since this parameter can be easily restored in the final results we will set it to one to simplify the formulae. We will see how one solves the infrared problem which is usually met when one uses the Seeley-deWitt expansion for the heat kernel in the massless case. Keeping in mind the other possible applications besides the black hole physics we consider the general d-dimensional case. The formula (64) below which we need for our purposes is then gotten as particular example. Finally we show how to construct the asymptotic series for the finite part of Γ in terms of the Seeley-deWitt coefficients.

An efficient methods (which respects the diffeomorphism invariance) to calculate the effective action is the ζ-function regularization [12] in terms of which

$$\Gamma = -\frac{1}{2}\frac{d\zeta(s)}{ds}|_{s=0}. \tag{54}$$

Here $\zeta(s)$ is the meromorphic function which for $s > d/2$ has the integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \mathrm{tr}\, K(\tau) \tag{55}$$

in terms of the heat kernel (53). First, let us introduce instead of τ the new variable

$$\eta = W(x;\tau)\tau \tag{56}$$

assuming that $W(x;\tau)\tau$ increases monotonically from 0 to ∞ when τ runs through the same interval. Then the expression (55) takes the form

$$\zeta(s)=\frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(s)}\mathrm{tr}_{\,x}\int_0^\infty d\eta\, e^{-\eta}\eta^{s-1-\frac{d}{2}}W^{\frac{d}{2}-s}\Big(1-\frac{\eta W'}{W}\Big), \tag{57}$$

where now prime means differentiation with respect to η. The integral in (57) is convergent for $s>d/2$. Integrating sufficiently often by parts one gets the following expression for the analytic continuation of $\zeta(s)$ to $s\to 0$:

$$\zeta(s)=(\frac{-1}{4\pi})^{\frac{d}{2}}\frac{\Gamma(s-\frac{d}{2})}{\Gamma(s)\Gamma(s+1)}\int_0^\infty \eta^s\Big(W^{\frac{d}{2}-s}\,e^{-\eta}\Big)^{(\frac{d}{2})}d\eta \tag{58}$$ in

$d=2,4,\ldots$ dimensions and

$$\zeta(s)=\frac{1}{2\sqrt{\pi}}(\frac{-1}{4\pi})^{\frac{d}{2}}\frac{\Gamma(s-\frac{d}{2})}{\Gamma(s)\Gamma(s+\frac{1}{2})}\int_0^\infty \eta^{s-\frac{1}{2}}(W^{\frac{d}{2}-s}\,e^{-\eta})^{(\frac{d}{2}-\frac{1}{2})}d\eta \tag{59}$$ in

$d=1,3,\ldots$ dimensions. Here $(\ldots)^{(n)}$ denotes the n't derivative with respect to η. Calculating the s-derivative at $s=0$ we arrive at the following formulae for the finite parts of the effective actions

$$\begin{aligned}\Gamma \;=\;& \frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Gamma(\frac{d}{2}+1)}\sum_{k=0}^{\frac{d}{2}-1}(-1)^{\frac{d}{2}-1-k}\binom{\frac{d}{2}-1}{k}\\ &\cdot\mathrm{tr}_{\,x}\Big\{-(W^{\frac{d}{2}}\log W)_0^{(k)}+(\sum_{n=1}^{d/2}\frac{1}{n})(W^{\frac{d}{2}})_0^{(k)}\\ &+\int_0^\infty d\eta\,\frac{e^{-\eta}}{\eta}\Big[(W^{\frac{d}{2}})^{(k)}-(W^{\frac{d}{2}})_0^{(k)}\Big]\Big\}\end{aligned} \tag{60}$$ in

even dimensions and

$$\begin{aligned}\Gamma \;=\;& \frac{1}{2}\frac{\sqrt{\pi}}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Gamma(\frac{d}{2}+1)}\sum_{k=0}^{\frac{d-1}{2}}(-1)^{\frac{d}{2}-\frac{1}{2}-k}\binom{\frac{d}{2}-\frac{1}{2}}{k}\\ &\cdot\mathrm{tr}_{\,x}\int_0^\infty d\eta\frac{e^{-\eta}}{\sqrt{\eta}}(W^{\frac{d}{2}})^{(k)}\end{aligned} \tag{61}$$

in odd dimensions, where the subscript 0 means that the derivative should be taken at $\eta = 0$. Taking into account (56) we can see that the integrals in (60,61) are convergent both in the ultraviolet ($\eta \to 0$) and infrared ($\eta \to \infty$) regions. Note that even for massless fields no infrared divergences appear. Of course we assumed that the map $[0, \infty) \ni \tau \to \eta$ is bijective which in particular implies that $W(x, \tau)$ does not decay faster than $1/\tau$ for large τ. Now we shall show how to relate the finite part of the effective action to the Seeley-deWitt coefficients.

In 2-dimensions the formulae simplify considerably and the effective action (60) reads

$$\Gamma = \frac{1}{8\pi} \mathrm{tr}_x \Big[W_0 - W_0 \log W_0 + \int_0^\infty d\eta \frac{e^{-\eta}}{\eta} (W - W_0) \Big]. \tag{62}$$

For an operator

$$\hat{O} = -\triangle + V(x) \tag{63}$$

in flat space we have $W_0 = V$. The first two terms in (62) correspond then to the $2d$-Coleman-Weinberg potential and the integral gives the correction which vanishes for constant V. Expanding $W(x;\eta)$ in a Taylor series and integrating over η we find the following asymptotic series for Γ:

$$\Gamma = \frac{1}{8\pi} \mathrm{tr}_x \Big(W_0 - W_0 \log W_0 + \sum_{n=1}^{\infty} \frac{1}{n} W_0^{(n)} \Big). \tag{64}$$

Note that the derivative with respect to η is related to the τ-derivative via

$$\frac{\partial}{\partial \eta} = \frac{1}{W + \frac{\partial W}{\partial \tau} \tau} \frac{\partial}{\partial \tau}. \tag{65}$$

Thus, the series (64) can be viewed as an expansion of the effective action in terms of the τ-derivatives of $W(x;\tau)$. In particular, the first few terms in (64) can be explicitly written as

$$\begin{aligned} \Gamma \;=\; & \frac{1}{8\pi} \mathrm{tr}_x \Bigg(W_0 - W_0 \log W_0 + \frac{1}{W_0} (\frac{\partial W}{\partial \tau})_0 + \frac{1}{2} \frac{1}{W_0^2} (\frac{\partial^2 W}{\partial \tau^2})_0 \\ & - \frac{2}{W_0^3} (\frac{\partial W}{\partial \tau})_0^2 \Bigg) + O\Big(\frac{1}{W_0^3} (\frac{\partial^3 W}{\partial \tau^3})_0, \frac{1}{W_0^4} (\frac{\partial W}{\partial \tau})_0^3, \dots \Big) \end{aligned} \tag{66}$$

Expanding (53) in powers of τ and comparing it with the Seeley-deWitt expansion for the heat kernel on the diagonal

$$K(x,x;\tau) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \sum a_n \tau^n \tag{67}$$

we can express the derivatives of W at $\tau = 0$ in terms of the Seeley-deWitt coefficients as

$$\begin{aligned} W_0 &= -a_1 \\ (\frac{\partial W}{\partial \tau})_0 &= \frac{1}{2}a_1^2 - a_2 \\ (\frac{\partial^2 W}{\partial \tau^2})_0 &= 2a_1a_2 - 2a_3 - \frac{2}{3}a_1^3, \ldots \end{aligned} \tag{68}$$

Then the effective action (66) can be rewritten in terms of the a_n as

$$\begin{aligned} \Gamma &= \frac{1}{8\pi}\mathrm{tr}\,_x\Big\{a_1 - a_1 \log a_1 + (\frac{a_2}{a_1} - \frac{1}{2}a_1) \\ &\qquad (-\frac{a_3}{a_1^2} + \frac{a_2^2}{a_1^3} - \frac{1}{12}a_1) + \ldots\Big\}. \end{aligned} \tag{69}$$

In particular, for operators of the form (64) in a $2d$ flat spacetime one gets

$$\begin{aligned} \Gamma &= \frac{1}{8\pi}\mathrm{tr}\,_x\Bigg\{V - V\log V + \frac{1}{6}\frac{\triangle V}{V} \\ &\qquad -\frac{1}{12}\frac{(\nabla V)^2}{V^2} + \frac{1}{60}\frac{\triangle^2 V}{V^2} - \frac{1}{36}\frac{(\triangle V)^2}{V^3} + \ldots\Bigg\}. \end{aligned} \tag{70}$$

Note that the asymptotic expansion (64) (and correspondingly (70)) is good only if the potential V is big compared with its derivatives. In this case the formal expansion parameter is $\triangle V/V^2 \ll 1$. When this condition is not met, as for instance for the black hole metric, then we must work directly with (62).

B Perturbation theory for W

To calculate the effective action for an operator (63) in flat spacetime we need to develop some perturbation expansion for the heat kernel $\hat{K}(\tau)$ which satisfies

$$\frac{\partial \hat{K}}{\partial \tau} = -\hat{O}\hat{K} \quad \text{and} \quad \hat{K}(\tau = 0) = \hat{1}. \tag{71}$$

In the coordinate representation we write the heat kernel in the form (53) (again we set $\mu = 1$) and derive the perturbation series for W in powers of the potential V. Keeping in mind other possible applications of the perturbation expansion (e.g. in statistical mechanics [15]) we consider an arbitrary number of dimensions d. Substituting (53) into (71) we obtain the following equation for W:

$$\tau \frac{\partial W}{\partial \tau} = \triangle W \tau - (x-y)^i \nabla_i W - (\nabla W)^2 \tau^2 + V - W. \tag{72}$$

Making the 'ansatz'

$$W(x, y; \tau) = \sum b_n(x, y) \tau^n \tag{73}$$

we immediately arrive at the recurrence relations for the b_n:

$$\begin{aligned} b_0 + (x-y)^i \nabla_i b_o &= V, \\ 2b_1 + (x-y)^i \nabla_i b_1 &= \triangle b_0 \ , \end{aligned} \tag{74}$$

and for $n > 2$

$$(n+1) b_n + (x-y)^i \nabla_i b_n = \triangle b_{n-1} - \sum_{p=0}^{n-2} \nabla_i b_p \nabla^i b_{n-p-2}. \tag{75}$$

For computing the effective action or partition function it suffices to know K and correspondingly the b_n on the diagonal $x = y$. Taking this coincidence limit in (74,75) (of course, after the derivatives have been taken) we arrive at

$$\lim_{x\to y} b_n = \frac{n!}{(2n+1)!}\triangle^{(n)}V - \lim_{x\to y}\sum_{k=0}^{n-2}\frac{n!}{(2n-1-k)!}\triangle_x^{(n-2-k)}\sum_{p=0}^{n-2}\nabla_i b_p \nabla^i b_{k-p}. \tag{76}$$

The terms proportional to $\nabla^i b \nabla_i b$ are at least quadratic in the potential. Let us note that the terms which are nonlinear in V always contain products of gradients (as $\nabla_i V \nabla^i V$, $\nabla_i \triangle V \nabla^i V$ etc.). The terms linear in V in the expansion (73,76) correspond to the sum of all terms of the forms $V, V^2, \ldots, V\triangle V, \triangle^2 V$ etc. in the Seeley-deWitt expansion. Thus in the linear approximation one finds

$$W(x;\tau) \equiv W(x,x;\tau) = \sum_{n=0}^{\infty}\frac{n!}{(2n+1)!}\tau^n \triangle^{(n)}V + O(\nabla V \cdot \nabla V). \tag{77}$$

In one dimension or in the case when the potential V depends just on one variable, the series (77) can be converted into the following integral

$$W(x;\tau) = \sqrt{\frac{\pi}{4\tau}}\int_{-\infty}^{\infty} V(y)\Big[1 - \Phi(\frac{|x-y|}{\sqrt{\tau}})\Big]dy + O(\nabla V \cdot \nabla V), \tag{78}$$

where Φ is the error function. The nonlocal result (78) for W accounts for all terms which are linear in V. It is relevant for improving the Coleman-Weinberg effective potential as well as the partition function in statistical physics. It is related but not identical to a similar expression obtained by Feynman by variational method [15].

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