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Post-Newtonian Generation of Gravitational Waves in a Theory of Gravity with Torsion

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Abstract

We adapt the post-Newtonian gravitational-radiation methods developed within general relativity by Epstein and Wagoner to the gravitation theory with torsion, recently proposed by Hehl et al., and show that the two theories predict in this approximation the same gravitational radiation losses. Since they agree also on the first post-Newtonian level, they are at the present time–observationally–indistinguishable.

§(1): *Introduction*

The binary pulsar data obtained by Taylor and coworkers (Taylor [1]) have already ruled out many alternative gravitation theories. Will [2] has adopted the post-Newtonian gravitational-radiation methods developed within general relativity by Epstein and Wagoner [3] and by Wagoner and Will [4] to alternative metric theories of gravitation and thus has shown that in most (if not all) of these dipole gravitational radiation can also exist. The dipole radiation of a two-body system, say, is generated by the varying dipole moment of the gravitational binding energy and is typically much too large for the binary pulsar PSR 1913+16. There are examples (e.g. the Rosen bimetric theory) where the dipole gravitational radiation causes the system even to gain energy at a relatively high rate and strong dissipative mechanisms would have to be invented to account for the observed decrease.

In an earlier paper [5] we have shown that the "Poincaré gauge theory of gravity" proposed by Hehl *et al.* [6] agrees with general relativity on the first post-Newtonian level. The purpose of the present work is to demonstrate that the "dipole catastrophe" mentioned above does not occur in this theory and

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that—within the approximation scheme of [3]—the gravitational radiation loss is the same as in general relativity. From the observational point of view it is, therefore, practically impossible to distinguish the theory of Hehl et al. from general relativity.

In Section 2, the theory of Hehl et al. (slightly generalized) is briefly summarized in the language of differential forms. In Section 3, we recast the field equations into a form from which one can deduce easily that the first post-Newtonian approximation of the theory agrees with that of general relativity. In Section 4, we determine the energy-momentum forms of the gravitational field, and finally we demonstrate in Section 5 that the post-Newtonian generation of gravitational radiation agrees also with general relativity. The results are summarized in Section 6.

§(2): *The Poincaré Gauge Theory of Hehl et al.*

In this section, we give a short description of the gravitation theory which was proposed by Hehl et al. [6]. In contrast to Reference 6 we use Cartan's calculus of differential forms and follow the conventions of Trautman [7]. (See also [5, 8, 9].) Let θ^α denote an orthonormal tetrad field of 1-forms, and $\omega^\alpha{}_\beta$ the connection forms of a metric connection with torsion. The exterior covariant derivatives of θ^α and $\omega^\alpha{}_\beta$ are the torsion forms Θ^α and the curvature forms $\Omega^\alpha{}_\beta (c = 1, 8\pi G = 1)$. The 1-forms $(\theta^\alpha; \omega^\alpha{}_\beta)$ can be considered as gauge potentials of the Poincaré group because they determine a connection in the Poincaré bundle (which is a subbundle of the affine bundle). In this interpretation, the 2-forms $(\Theta^\alpha; \Omega^\alpha{}_\beta)$ are the corresponding field strengths. A slight generalization of the gravitational Lagrangian which was chosen in [6] reads as follows:

$$\mathcal{L}_g = \mathcal{L}_{\text{transl}} + \mathcal{L}_{\text{rot}} \tag{2.1}$$

$$\mathcal{L}_{\text{transl}} = -\frac{1}{2l^2}\left[(\Theta^\alpha \wedge \theta^\beta) \wedge *(\Theta_\beta \wedge \theta_\alpha) - \frac{\lambda}{2}(\Theta^\alpha \wedge \theta_\alpha) \wedge *(\Theta^\beta \wedge \theta_\beta)\right] \tag{2.2}$$

$$\mathcal{L}_{\text{rot}} = -\frac{1}{2k}\Omega^{\alpha\beta} \wedge *\Omega_{\alpha\beta} \tag{2.3}$$

Here l is the Planck length and k is a dimensionless coupling constant. Hehl et al. use $\lambda = 0$ in (2.2). To the Lagrangian (2.1) we have to add a matter term, L_m. Independent variations of $L_g + L_m$ with respect to θ^α and $\omega^\alpha{}_\beta$ give the field equations. The variation of L_m with respect to $\omega^\alpha{}_\beta$ determines the spin density which is for macroscopic matter very small compared to the energy-momentum density. For this reason Hehl et al. choose for astronomical systems the "translational gauge limit" in which the curvature vanishes:

$$\Omega^\alpha{}_\beta = 0 \tag{2.4}$$

Consequently the rotational (Yang-Mills type) part (2.3) in the Lagrangian (2.1) has to be dropped. The resulting theory has an obvious geometric interpretation in a Weitzenböck space. Relative to a teleparallel tetrad the connection forms $\omega^{\alpha}{}_{\beta}$ also vanish and the exterior covariant derivative reduces the ordinary exterior derivative. The torsion Θ^{α} is then equal to $d\theta^{\alpha}$. Thus, we use the following total Lagrangian relative to an arbitrary (orthonormal) frame:

$$\mathcal{L} = -\frac{1}{2}(\Theta^{\alpha} \wedge \theta^{\beta}) \wedge *(\Theta_{\beta} \wedge \theta_{\alpha}) + \frac{\lambda}{4}(\Theta^{\alpha} \wedge \theta_{\alpha}) \wedge *(\Theta^{\beta} \wedge \theta_{\beta}) + \mathcal{L}_m \quad (2.5)$$

which reduces to

$$\mathcal{L} = -\frac{1}{2}(d\theta^{\alpha} \wedge \theta^{\beta}) \wedge *(d\theta_{\beta} \wedge \theta_{\alpha}) + \frac{\lambda}{4}(d\theta^{\alpha} \wedge \theta_{\alpha}) \wedge *(d\theta^{\beta} \wedge \theta_{\beta}) + \mathcal{L}_m \quad (2.6)$$

for a teleparallel frame.

Now we note that the Hilbert-Einstein Lagrangian can be written in the following way (up to an exact differential):

$$\mathcal{L}_E = -\tfrac{1}{2}(d\theta^{\alpha} \wedge \theta^{\beta}) \wedge *(d\theta_{\beta} \wedge \theta_{\alpha}) + \tfrac{1}{4}(d\theta^{\alpha} \wedge \theta_{\alpha}) \wedge *(d\theta^{\beta} \wedge \theta_{\beta}) \quad (2.7)$$

This can easily be shown by putting $d\theta^{\alpha} = \frac{1}{2} C^{\alpha}_{\beta\gamma}\theta^{\beta} \wedge \theta^{\gamma}$ and expressing both sides of (2.7) in terms of $C^{\alpha}{}_{\beta\gamma}$. Thus the gravitational part of (2.6) reduces to the Hilbert-Einstein Lagrangian for $\lambda = 1$. This remark is crucial for all that follows and was also used in [5]. It is not surprising that the gravitational Lagrangian is invariant under local Lorentz transformations

$$\theta^{\alpha}(x) \longrightarrow \Lambda^{\alpha}{}_{\beta}(x)\,\theta^{\beta}(x), \qquad \Lambda(x) \in \mathcal{L}^{\uparrow}{}_{+} \quad (2.8)$$

for $\lambda = 1$ only.

Variation of the tetrad fields leads to the field equations. We introduce the notations:

$$\delta\mathcal{L}_E = \delta\theta^{\alpha} \wedge \epsilon^{E}_{\alpha} \quad (2.9)$$

$$\delta\mathcal{L}_m = \delta\theta^{\alpha} \wedge t_{\alpha} \quad (2.10)$$

$$\delta(\Delta\mathcal{L}) = \delta\theta^{\alpha} \wedge \Delta\epsilon_{\alpha} \quad (2.11)$$

where $(\lambda - 1)\,\Delta\mathcal{L}$ is the difference between the gravitational part of (2.6) and (2.7). The explicit expressions of ϵ^{E}_{α} and $\Delta\epsilon_{\alpha}$ are

$$\epsilon^{\alpha}_{E} = -d[\theta^{\beta} \wedge *(d\theta_{\beta} \wedge \theta^{\alpha})] - d\theta^{\beta} \wedge *(d\theta^{\alpha} \wedge \theta_{\beta}) + \tfrac{1}{2}\, d\,\{\theta^{\alpha} \wedge *(d\theta^{\beta} \wedge \theta_{\beta})\} + \tfrac{1}{2}\, d\theta^{\alpha} \wedge *(d\theta^{\beta} \wedge \theta_{\beta}) \quad (2.12)$$

$$\Delta\epsilon^{\alpha} = d\theta^{\alpha} \wedge *(d\theta^{\beta} \wedge \theta_{\beta}) - \tfrac{1}{2}\,\theta^{\alpha} \wedge d*(d\theta^{\beta} \wedge \theta_{\beta}) \quad (2.13)$$

and the field equations read as follows:

$$\epsilon^{\alpha}_{E} + (\lambda - 1)\,\Delta\epsilon^{\alpha} = -t^{\alpha} \quad (2.14)$$

t^α are the energy-momentum 3-forms of matter. The components of the energy-momentum tensor, $T^{\alpha\beta}$, relative to θ^α are given by

$$*t^\alpha = T_\beta{}^\alpha \theta^\beta \tag{2.15}$$

The field equations (2.14) and the Einstein equations have a large family of common solutions. For any metric of the diagonal form

$$ds^2 = a_0^2\,(dx^0)^2 - a_1^2\,(dx^1)^2 - a_2^2\,(dx^2)^2 - a_3^2\,(dx^3)^2$$

with arbitrary functions a_i the forms $\Delta\epsilon^\alpha$ vanish for the orthonormal basis

$$\theta^0 = a_0\,dx^0\,, \qquad \theta^1 = a_1\,dx^1\,, \qquad \theta^2 = a_2\,dx^2\,, \qquad \theta^3 = a_3\,dx^3$$

because $d\theta^\alpha \wedge \theta_\alpha = 0$. This proves in particular that all spherically symmetric solutions of the Einstein equations are also solutions of (2.14) and thus a large body of astrophysical applications remains unchanged. The Kerr solution is, however, no vacuum solution of (2.14) for $\lambda \neq 1$. It is an open problem to generalize the stationary black hole solution to this case.

§(3): *The First Post-Newtonian Approximation*

For all the further discussions, it will turn out to be useful to split the field equations (2.14) into symmetrized and antisymmetrized parts. In contrast to ϵ_E^α and t^α the forms $\Delta\epsilon^\alpha$ are not symmetric:

$$\Delta\epsilon^\alpha \wedge \theta^\beta \neq \Delta\epsilon^\beta \wedge \theta^\alpha$$

because $\Delta\mathcal{L}$ is not locally Lorentz invariant. The symmetric and antisymmetric parts of $\Delta\epsilon^\alpha$ are

$$\Delta\epsilon_s^\alpha = -\tfrac{1}{2} * [(d\theta^\alpha \wedge \theta^\beta + d\theta^\beta \wedge \theta^\alpha) \wedge * (d\theta^\gamma \wedge \theta_\gamma)]\ \eta_\beta \tag{3.1}$$

$$\Delta\epsilon_a^\alpha = -\tfrac{1}{2} * \{d\,[\theta^\alpha \wedge \theta^\beta \wedge * (d\theta^\gamma \wedge \theta_\gamma)]\}\ \eta_\beta \tag{3.2}$$

where $\eta^\beta = * \theta^\beta$. Hence the field equations (2.14) are equivalent to the pair of equations

$$\epsilon_E^\alpha - \frac{(\lambda - 1)}{2} * [(d\theta^\alpha \wedge \theta^\beta + d\theta^\beta \wedge \theta^\alpha) \wedge * (d\theta^\gamma \wedge \theta_\gamma)]\ \eta_\beta = -t^\alpha \tag{3.3}$$

$$(\lambda - 1)\, d[\theta^\alpha \wedge \theta^\beta \wedge * (d\theta^\gamma \wedge \theta_\gamma)] = 0 \tag{3.4}$$

Now we expand the teleparallel frames θ^α in terms of a coordinate basis

$$\theta^\alpha = dx^\alpha + \Phi^\alpha{}_\beta\, dx^\beta \tag{3.5}$$

and decompose $\Phi_{\alpha\beta}$ into its symmetric and antisymmetric pieces

$$\Phi_{\alpha\beta} = \phi_{\alpha\beta} + \mathrm{a}_{\alpha\beta} \tag{3.6}$$

$$\phi_{\alpha\beta} = \Phi_{(\alpha\beta)}, \qquad a_{\alpha\beta} = \Phi_{[\alpha\beta]} \tag{3.7}$$

Let us first consider the linearized approximations of (3.3) and (3.4). The second term on the left-hand side of (3.3) contains obviously no linear terms. Furthermore, the antisymmetric field $a_{\alpha\beta}$ drops out identically in the linearized part of ϵ_E^α (as a consequence of the local Lorentz invariance of (2.7)). Thus equation (3.3) is reduced to the linearized Einstein equation for $\phi_{\alpha\beta}$. For $\lambda = 1$ (i.e., general relativity) equation (3.4) is empty; $a_{\alpha\beta}$ is just a gauge degree of freedom. For $\lambda \neq 1$ equation (3.4) leads in the linearized approximation to the decoupled source free equation

$$\Box\, a^{\alpha\beta} + a^{\lambda\alpha,\beta}{}_{\lambda} - a^{\lambda\beta,\alpha}{}_{\lambda} = 0 \tag{3.8}$$

for $a^{\alpha\beta}$ which is invariant under the gauge transformation

$$a_{\alpha\beta} \longrightarrow a_{\alpha\beta} + \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{3.9}$$

Imposing the gauge condition $a^{\alpha\beta}{}_{,\beta} = 0$, we are left with $\Box\, a^{\alpha\beta} = 0$. In the Newtonian approximation $a_{\alpha\beta}$ vanishes in this gauge and the theory gives the correct Newtonian limit. Next we show that the first post-Newtonian approximation of (3.3) and (3.4) agrees with that of general relativity. (This was shown already in [5], but the following discussion is simpler.) Since the term proportional to $(\lambda - 1)$ in (3.3) is of higher order in $\Phi_{\alpha\beta}$, it contains in the first post-Newtonian approximation at most quadratic expressions of the Newtonian approximation of $\phi_{\alpha\beta}$. But these vanish identically because $d\theta^\alpha \wedge \theta_\alpha$ is already quadratic in $\phi_{\alpha\beta}$. (The linear term vanishes identically.) Hence (3.3) reduces to the first post-Newtonian approximation of general relativity. For the discussion of equation (3.4), we note first that the Newtonian approximation, $\phi_{\alpha\beta}^{(N)}$, of $\phi_{\alpha\beta}$ is diagonal in a suitable coordinate system: $\phi_{\alpha\beta}^{(N)} = \delta_{\alpha\beta}\Phi$, Φ = Newtonian potential. From this, one concludes easily that the quadratic terms in $\phi_{\alpha\beta}^{(N)}$ in (3.4) vanish identically and thus the first post-Newtonian approximation of (3.4) reduces to equation (3.8) for the post-Newtonian approximation, $a_{\alpha\beta}^{(\mathrm{PN})}$, of $a_{\alpha\beta}$. This implies that $a_{\alpha\beta}^{(\mathrm{PN})}$ vanishes also in a suitable gauge and hence our claim is proven. Hehl and Nitsch [10] have shown that the post-post-Newtonian approximation no longer agrees with that of general relativity. The deviations are, however, unmeasurably small.

§(4): *Conservation Laws*

We start by writing the symmetric field equation (3.3) in arbitrary (not necessarily teleparallel or orthonormal) frames:

$$\epsilon_E^\alpha - \frac{(\lambda - 1)}{2} * [(\Theta^\alpha \wedge \theta^\beta + \Theta^\beta \wedge \theta^\alpha) \wedge * (\Theta^\gamma \wedge \theta_\gamma)]\, \eta_\beta = -t^\alpha \tag{4.1}$$

For ϵ_E^α we use the "Landau decomposition" derived in [11]:

$$\epsilon_E^\alpha = [2(-g)^{1/2}]^{-1} \cdot d\,[(-g)^{1/2}\, \overset{\mathrm{LC}}{\omega_\beta{}^\gamma} \wedge \eta^{\alpha\beta}{}_\gamma] + t_{\mathrm{LL}}^\alpha \tag{4.2}$$

Here $\overset{\mathrm{LC}}{\omega_{\alpha\beta}}$ are the Levi-Cività connection forms and t_{LL}^α are the Landau-Lifschitz energy-momentum forms of the metric field given explicitly by

$$t_{LL}^\alpha = -\tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}(\overset{\mathrm{LC}}{\omega_{\sigma\beta}} \wedge \overset{\mathrm{LC}}{\omega^\sigma{}_\gamma} \wedge \theta_\delta - \overset{\mathrm{LC}}{\omega_{\beta\gamma}} \wedge \overset{\mathrm{LC}}{\omega_{\sigma\delta}} \wedge \theta^\sigma) \tag{4.3}$$

Relative to a coordinate basis $\theta^\alpha = dx^\alpha$ the 3-forms t_{LL}^α are symmetric. Inserting (4.2) into (4.1) the symmetric field equation takes the form

$$-\tfrac{1}{2}\,d[(-g)^{1/2}\, \overset{\mathrm{LC}}{\omega_{\beta\gamma}} \wedge \eta^{\alpha\beta\gamma}] = (-g)^{1/2}\,\tau^\alpha \tag{4.4}$$

where

$$\tau^\alpha = t^\alpha + t_{\mathrm{LL}}^\alpha - (\lambda - 1)\,\Delta t^\alpha \tag{4.5}$$

$$\Delta t^\alpha = \tfrac{1}{2} * [(\Theta^\alpha \wedge \theta^\beta + \Theta^\beta \wedge \theta^\alpha) \wedge *(\Theta^\gamma \wedge \theta_\gamma)]\ \eta_\beta \tag{4.6}$$

It may be useful to note that the left-hand side of (4.4) relative to a coordinate basis can be expressed in terms of the Landau-Lifschitz superpotential as follows:

$$d[(-g)^{1/2}\, \overset{\mathrm{LC}}{\omega_{\beta\gamma}} \wedge \eta^{\mu\beta\gamma}] = (-g)^{1/2}\ \mathrm{H}^{\mu\alpha\nu\beta}{}_{,\alpha}\ \eta_\nu$$

where

$$\mathrm{H}^{\mu\alpha\nu\beta} = \hat{g}^{\mu\nu}\hat{g}^{\alpha\beta} - \hat{g}^{\mu\beta}\hat{g}^{\nu\alpha}$$

with

$$\hat{g}^{\mu\nu} = (-g)^{1/2} g^{\mu\nu}$$

As in general relativity, the τ^α are interpreted as the total energy-momentum forms. By construction, they are symmetric relative to a coordinate basis:

$$\tau^\alpha \wedge dx^\beta = \tau^\beta \wedge dx^\alpha \tag{4.7}$$

From (4.4) we conclude that the field equations imply the conservation laws

$$d[(-g)^{1/2}\tau^\alpha] = 0 \tag{4.8}$$

The last two equations imply

$$d[(-g)^{1/2}\ \mathrm{M}^{\alpha\beta}] = 0 \tag{4.9}$$

where

$$\mathrm{M}^{\alpha\beta} = x^\alpha \tau^\beta - x^\beta \tau^\alpha \tag{4.10}$$

is the total angular momentum density. For isolated systems the total momentum

$$\mathrm{P}^\alpha = \int_\Sigma (-g)^{1/2}\tau^\alpha \tag{4.11}$$

and the total angular momentum

$$\mathsf{J}^{\alpha\beta} = \int_{\Sigma} (-g)^{1/2} \mathsf{M}^{\alpha\beta} \tag{4.12}$$

(Σ: spacelike surface) can be expressed with the help of the field equations (4.4) in terms of flux integrals at infinity:

$$\mathsf{P}^{\alpha} = -\tfrac{1}{2} \oint (-g)^{1/2} \overset{\mathrm{LC}}{\omega_{\beta\gamma}} \wedge \eta^{\alpha\beta\gamma} \tag{4.13}$$

$$\mathsf{J}^{\alpha\beta} = \tfrac{1}{2} \oint (-g)^{1/2} [(x^{\alpha}\eta^{\beta}{}_{\sigma\gamma} - x^{\beta}\eta^{\alpha}{}_{\sigma\gamma}) \wedge \overset{\mathrm{LC}}{\omega^{\sigma\gamma}} + \eta^{\alpha\beta}] \tag{4.14}$$

As usual, the coordinates have to be asymptotically Lorentzian. P^{α} and $\mathsf{J}^{\alpha\beta}$ transform like Lorentz tensors under coordinate transformations which preserve this property. Clearly, the flux integrals (4.13) and (4.14) are the same as in general relativity. Since we have not seen equation (4.14) in the literature, we derive it in the Appendix.

§(5): *Post-Newtonian Generation of Gravitational Radiation*

In this section we adapt the method of Epstein and Wagoner [3] and show that the post-Newtonian generation of gravitational radiation is the same as in general relativity.

First we have to bring the field equations (3.3) and (3.6) into a convenient form by separating explicitly the linear terms in the fields $\phi_{\alpha\beta}$ and $a_{\alpha\beta}$.

We have already noted in Section 3 that the term proportional to $(\lambda - 1)$ in (3.3) contains no linear terms and that the field $a_{\alpha\beta}$ does not appear in the linearized part of ϵ_E^{α}. Hence we only have to split off the linear part in the Einstein form

$$\epsilon_E^{\alpha} = \epsilon_L^{\alpha} + \epsilon_Q^{\alpha} \tag{5.1}$$

The linear part, ϵ_L^{α}, is given by

$$\epsilon_L^{\alpha} = -G_L^{\alpha\beta}\eta_{\beta} \tag{5.2}$$

where $G_L^{\alpha\beta}$ is the linearized Einstein tensor

$$-G_L^{\alpha\beta} = \Box\,(\phi^{\alpha}{}_{\beta} - \delta^{\alpha}{}_{\beta}\phi^{\gamma}{}_{\gamma}) + \delta^{\alpha}{}_{\beta}\phi^{\lambda\sigma}{}_{,\lambda\sigma} + \phi^{\lambda}{}_{\lambda}{}^{,\alpha}{}_{\beta} - (\phi^{\alpha\lambda}{}_{,\beta} + \phi^{\lambda}{}_{\beta}{}^{,\alpha})_{,\lambda} \tag{5.3}$$

The quadratic and higher-order terms are easily obtained from (2.12):

$$\epsilon_Q^{\alpha} = \Delta\epsilon_S^{\alpha} + \{A^{\beta\alpha} + B^{\beta\alpha} + C^{\beta\alpha}\}\,\eta_{\beta} \tag{5.4a}$$

where

$$A^{\nu\mu} = 2 * [\theta^\nu \wedge d\theta^\alpha \wedge * (d\theta_\alpha \wedge \theta^\mu)] \tag{5.4b}$$

$$B^{\nu\mu} = \tfrac{1}{2} * [\theta^\nu \wedge d\theta^\alpha \wedge * d(\theta^\mu \wedge \theta_\alpha) + \theta^\mu \wedge d\theta^\alpha \wedge * d(\theta^\nu \wedge \theta_\alpha)] \tag{5.4c}$$

$$\begin{aligned} C^{\nu\mu} = -\tfrac{1}{2} * [&\theta^\nu \wedge \theta^\alpha \wedge d * (d\theta_\alpha \wedge \Phi^\mu{}_\lambda\, dx^\lambda) + (dx^\nu \wedge \Phi^\alpha{}_\sigma\, dx^\sigma + \Phi^\nu{}_\lambda\, dx^\lambda \wedge dx^\alpha \\ &+ \Phi^\nu{}_\lambda \Phi^\alpha{}_\sigma\, dx^\lambda \wedge dx^\sigma) \wedge d * (d\theta_\alpha \wedge dx^\mu) \\ &+ \theta^\mu \wedge \theta^\alpha \wedge d * (d\theta_\alpha \wedge \Phi^\nu{}_\lambda\, dx^\lambda) + (dx^\mu \wedge \Phi^\alpha{}_\lambda\, dx^\lambda + \Phi^\mu{}_\sigma\, dx^\sigma \wedge dx^\alpha \\ &+ \Phi^\mu{}_\sigma \Phi^\alpha{}_\lambda\, dx^\sigma \wedge dx^\lambda) \wedge d * (d\theta_\alpha \wedge dx^\nu)] \end{aligned} \tag{5.4d}$$

The decomposition (5.4) is equivalent to equation (31) of Reference 3 and is quite useful for practical (post-Newtonian) calculations.

Inserting (5.1) into (3.3) gives

$$G_L^{\alpha\beta} = T_{\mathrm{eff}}^{\alpha\beta} \tag{5.5}$$

with

$$T_{\mathrm{eff}}^{\alpha\beta}\eta_\beta = t^\alpha + \epsilon_Q^\alpha + (\lambda - 1)\, \Delta\epsilon_S^\alpha \tag{5.6}$$

where $\Delta\epsilon_S^\alpha$ is given by (3.1) (and contains no linear terms). The last term in (5.6) is absent in general relativity. We also rewrite (3.4) (for $\lambda \neq 1$) in a similar way:

$$\Box\, \mathrm{a}^{\alpha\beta} + \mathrm{a}^{\lambda\alpha,\beta}{}_\lambda - \mathrm{a}^{\lambda\beta,\alpha}{}_\lambda = A_{\mathrm{eff}}^{\alpha\beta} \tag{5.7}$$

where $A_{\mathrm{eff}}^{\alpha\beta}$ collects the quadratic and higher-order terms of

$$\tfrac{1}{2}\, d[\theta^\alpha \wedge \theta^\beta \wedge * (d\theta^\gamma \wedge \theta_\gamma)]$$

For outgoing-wave boundary conditions, we obtain from (5.5) and (5.7) the integral equations

$$\phi_{\alpha\beta}(t, \mathbf{x}) = -\frac{1}{4\pi}\int \frac{\hat{T}_{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' + \xi_{\alpha,\beta} + \xi_{\beta,\alpha} \tag{5.8}$$

$$(\hat{T}^\alpha{}_\beta = T_{\mathrm{eff}}{}^\alpha{}_\beta - \tfrac{1}{2}\, \delta^\alpha{}_\beta\, T_{\mathrm{eff}}{}^\lambda{}_\lambda)$$

$$\mathrm{a}_{\alpha\beta}(t, \mathbf{x}) = \frac{1}{4\pi}\int \frac{A_{\alpha\beta}^{\mathrm{eff}}\,(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' + \eta_{\alpha,\beta} - \eta_{\beta,\alpha} \tag{5.9}$$

The gauge terms added on the right-hand sides of (5.8) and (5.9) are, of course, not determined.

Let us now consider the fields $\phi_{\alpha\beta}$ and $\mathrm{a}_{\alpha\beta}$ far from the source ($r = |\mathbf{x}| >> R$ = "size" of source, $|\phi_{\alpha\beta}| << 1$, $|\mathrm{a}_{\alpha\beta}| << 1$) where the radiation is detected. In the approximation scheme of Reference 3, one first expands the $1/r$ terms with respect to the retardation within the source (slow motion). In the next step, one constructs a post-Newtonian expansion of the sources $T^{\alpha\beta}_{\text{eff}}$ and $A^{\alpha\beta}_{\text{eff}}$. The necessary orders in the post-Newtonian expansion parameter which are needed for the various pieces are described in [3] and will not be repeated. In this approximation, the effective sources are contained within the near zone. We may then employ (5.8) and (5.9) for field points within this region as well, again in terms of a post-Newtonian expansion. But the near-zone post-Newtonian expansion was already studied in Section 3. To the necessary orders $a_{\alpha\beta}$ vanishes and $\phi_{\alpha\beta}$ agrees with general relativity if we impose suitable gauge conditions. [$A^{\alpha\beta}_{\text{eff}}$ and the last term in (5.6) for $T^{\alpha\beta}_{\text{eff}}$ both vanish in the near-zone post-Newtonian approximation.] Hence, we conclude from (5.8) and (5.9) that far away from the source $a_{\alpha\beta}$ is just equal to a guage term and that $\phi_{\alpha\beta}$ agrees also there with general relativity. But in this weak field region, the equation for the a field decouples and has a separate gauge invariance. Thus the a field can be gauged away.

Finally we note that the modified energy-momentum $t^{\alpha}_{\text{LL}} - (\lambda - 1)\,\Delta t^{\alpha}$ [see equation (4.5)] does not change the gravitational energy loss because Δt^{α} is cubic in $\phi_{\alpha\beta}$. This follows from (4.6) and from the fact that the factor $d\theta^{\gamma} \wedge \theta_{\gamma}$ in (4.6) is already quadratic in $\phi_{\alpha\beta}$. (The linear term vanishes identically.) Taken all together, this proves our claim at the beginning of this section.

§(6): *Summary*

The gravitational theory with torsion corresponding to the one-parameter family of Lagrangians (2.6) has many exact solutions in common with general relativity. We have shown that they agree also with general relativity on the first post-Newtonian level (but not in higher orders.). The main new result of this paper is contained in Section 5, where we demonstrate that even the post-Newtonian generation of gravitational waves (developed within general relativity by Epstein and Wagoner [3]) is the same as in general relativity. In particular, the "dipole catastrophe," described in the Introduction, which occurs in many alternative metric theories of gravitation, is absent. Therefore, the results obtained in [3] and [4] also hold in the theory by Hehl et al. [6]. At the present time, this theory can thus not be distinguished observationally from general relativity.

This is, of course, only true if the approximation scheme of Reference 3 is numerically reliable. Various authors (see, e.g., [12] and references therein) have criticized the presently existing approximation methods for treating the radiation problem. We are aware of the critical questions that have been raised, but we think that the method used here is physically plausible.

Apart from aesthetic arguments, we see no way to favor the Lagrangian with $\lambda = 1$ in (2.6), i.e., general relativity.

Appendix

In this Appendix, we derive the expression (4.14) for the angular momentum. From the definition (4.10) and the field equations in the form (4.4) we conclude that

$$(-g)^{1/2}\,\mathrm{M}^{\sigma\alpha} = \tfrac{1}{2}\,(x^\sigma\, dh^\alpha - x^\alpha\, dh^\sigma)$$
$$= \tfrac{1}{2}\, d(x^\sigma h^\alpha - x^\alpha h^\sigma) - \tfrac{1}{2}\,(dx^\sigma \wedge h^\alpha - dx^\alpha \wedge h^\sigma) \tag{A.1}$$

where

$$h^\alpha = -(-g)^{1/2}\,(\omega^{\beta\gamma} \wedge \eta^\alpha{}_{\beta\gamma}) \tag{A.2}$$

In this Appendix $\omega^\alpha{}_\beta$ denote always the Levi-Cività connection forms. Now we write also the last term in (A.1) as an exact differential. We have

$$dx^\sigma \wedge h^\alpha - dx^\alpha \wedge h^\sigma = (-g)^{1/2}\,\omega^{\beta\gamma} \wedge dx^\sigma \wedge \eta^\alpha{}_{\beta\gamma} - (\alpha \longleftrightarrow \sigma)$$
$$= (-g)^{1/2}\,(\omega_\beta{}^\sigma \wedge \eta^{\alpha\beta} + \omega^\sigma{}_\beta \wedge \eta^{\beta\alpha}$$
$$- \omega_\beta{}^\alpha \wedge \eta^{\sigma\beta} - \omega^\alpha{}_\beta \wedge \eta^{\beta\sigma})$$

Here we use

$$d\eta^{\sigma\alpha} + \omega^\sigma{}_\beta \wedge \eta^{\beta\alpha} + \omega^\alpha{}_\beta \wedge \eta^{\sigma\beta} = 0$$

and obtain

$$dx^\sigma \wedge h^\alpha - dx^\alpha \wedge h^\sigma = (-g)^{1/2}\,[\omega_\beta{}^\sigma \wedge \eta^{\alpha\beta} - (\sigma \longleftrightarrow \alpha) - d\eta^{\sigma\alpha}]$$

But

$$\omega_\beta{}^\sigma \wedge \eta^{\alpha\beta} = \Gamma^{\;\sigma}_{\beta\mu}\, dx^\mu \wedge \eta^{\alpha\beta} = \Gamma_\beta{}^{\beta\sigma}\,\eta^\alpha - \Gamma_\beta{}^{\alpha\sigma}\eta^\beta$$

and hence

$$dx^\sigma \wedge h^\alpha - dx^\alpha \wedge h^\sigma = (-g)^{1/2}\,(\Gamma^{\beta\sigma}_{\beta}\eta^\alpha - \Gamma^{\beta\alpha}_{\beta}\eta^\sigma - d\eta^{\sigma\alpha})$$

If we use in this expression

$$\Gamma^{\beta\sigma}_{\beta} = (-g)^{1/2} \cdot g^{\mu\sigma}\partial_\mu(-g)^{1/2}$$

then we find easily

$$dx^\sigma \wedge h^\alpha - dx^\alpha \wedge h^\sigma = -d[(-g)^{1/2} \cdot \eta^{\sigma\alpha}] \tag{A.3}$$

With this result and (A.2) equation (A.1) becomes

$$(-g)^{1/2}\,\mathrm{M}^{\sigma\alpha} = \tfrac{1}{2}\, d\,\{(-g)^{1/2}\,[\eta^{\sigma\alpha} + (x^\sigma\eta^\alpha{}_{\beta\gamma} - x^\alpha\eta^\sigma{}_{\beta\gamma}) \wedge \omega^{\beta\gamma}]\} \tag{A.4}$$

Inserting this into (4.14) and using Stokes' theorem finally gives equation (4.14).

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References

1. Taylor, J. H., Hulse, R. A., Fowler, L. A., Gullahorn, G. E., and Rankin, J. M. (1976). *Astrophys. J.*, **206**, L53; Taylor, J. H. (1978). Talk given at the Ninth Texas Symposium on Relativistic Astrophysics (München, 1978); Taylor, J. H., Fowler, L. A., and McCullough, P. M. (1979). *Nature*, **277**, 437.
2. Will, C. M. (1977). *Astrophys. J.*, **214**, 826-839; (1978). Talk given at the Ninth Texas Symposium on Relativistic Astrophysics (München, 1978).
3. Epstein, R., and Wagoner, R. V. (1975). *Astrophys. J.*, **197**, 717.
4. Wagoner, R. V., and Will, C. M. (1976). *Astrophys. J.*, **210**, 764.
5. Schweizer, M., and Straumann, N. (1979). *Phys. Lett.*, **71A**, 493.
6. Hehl, F. W., Ne'eman, Y., Nitsch, J., and Von der Heyde, P. (1978). *Phys. Lett.*, **788**, 102; Hehl, F. W., Nitsch, J., and Von der Heyde, P. (1980). *Einstein Commemorative Volume*, ed. Held, A. Plenum, New York (to appear); Hehl, F. W. (1980). Four Lectures on Poincaré Gauge Field Theory, Proceedings of the International School of Cosmology and Gravitation, Erice, May 1979, eds. Bergmann, P. G. and de Sabbata, V. Plenum, New York (to appear).
7. Trautman, A. (1973). *Symposia Mathematica*, Vol. 12, Academic, New York, p. 139.
8. Rumpf, H. (1978). *Z. Naturforsch.*, **33a**, 1224.
9. Wipf, A. (1979). Diplomarbeit, "Analyse einer Poincaré Eichfeldtheorie der Gravitation und Emission von Gravitationsstrahlung in derselben," Institute for Theoretical Physics, Schönberggasse 9, University of Zürich, CH-8001 Zürich.
10. Hehl, F. W., and Nitsch, J. (1979). University of Cologne, preprint; Nitsch, J. (1980). Seminar Talk, Proceedings of the International School of Cosmology and Gravitation, Erice, May 1979, eds. Bergmann, P. G., and de Sabbata, V. Plenum, New York (to appear).
11. Thirring, W. (1978). *Lehrbuch der Mathematischen Physik*, Vol. 2, Springer-Verlag, Wein.
12. Ehlers, J. (1978). "Isolated Systems in General Relativity," Talk given at the Ninth Texas Symposium on Relativistic Astrophysics, (München, 1978).