DIAS-STP-95-19
FSUJ TPI 4/95
July 1995

Generalized Thirring Models

I. Sachs † and A. Wipf ‡

† Dublin Institute for Advanced Studies
10 Burlington Road, D 4, Ireland
‡ Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität,
Max-Wien-Platz 1, 07743 Jena, Germany

Abstract

The Thirring model and various generalizations of it are analyzed in detail. The four-Fermi interaction modifies the equation of state. Chemical potentials and twisted boundary conditions both result in complex fermionic determinants which are analyzed. The non-minimal coupling to gravity does deform the conformal algebra which in particular contains the minimal models. We compute the central charges, conformal weights and finite size effects.
For the gauged model we derive the partition functions and the explicit expression for the chiral condensate at finite temperature and curvature. The Bosonization in compact curved space-times is also investigated.

1 Introduction

The response of physical systems to a change of external conditions is of eminent importance in physics. In particular the dependence of expectation values on temperature, the particle density, the space region, the imposed boundary conditions or external fields has been widely studied [1]. Nevertheless, many properties of such systems are poorly understood. The massless Thirring model [2], which is among the simplest interacting field theories, has already led to considerable confusion about its thermodynamic properties in the literature [3, 4, 5]. The reason is two-fold: Firstly, the computation of the fermionic determinant in the presence of a chemical potential and/or non-trivial boundary conditions is delicate, because the eigenvalues of the Dirac operator are generically complex. In section 3.1 we propose a regularization scheme via analytic continuation. We argue that the so-obtained determinant, which differs from previous results [4], leads to the correct equation of state.
The second complication originates in the infrared-sector. An elegant infrared regularization, which is particularly well suited for the study of thermodynamic properties, is to quantize the model on a torus. Harmonic contributions to the current arise then naturally and taking them into account turns out to be crucial for a correct quantization. In particular the so-obtained results differ from those gotten earlier [3] using bosonization. This is explained in section 3.2.

On another front there has been much effort to quantize self-interacting field theories in a background gravitational field [6]. For example, one is interested whether a black hole still emits thermal radiation when self-interaction is included. Due to general arguments by Gibbons and Perry [7] this question is intimately connected with the universality of the second law of thermodynamics. The Thirring model (including the gauged version of it) is still solvable in curved space-time and we can study its properties in a background gravitational field. This provides us in particular with an elegant approach to the study of its conformal structure: Correlation functions with current- and stress-tensor insertions, which are gotten by functional differentiation with respect to the gauge- and gravitational fields, contain the necessary information to characterize the underlying symmetry algebras. To familiarize the reader with our approach we first rederive the conformal structure of the original Thirring model in section 3.3. We then show how a non-minimal coupling to gravity leads in a natural way to a modification of the conformal structure. In particular, very much as for a free scalar field the central charge in the fermionic formulation of the Thirring model is not unique. Furthermore, we find that the equivalence between finite size scaling and central charge of the Virasoro algebra holds only for a particular treatment of the zero-mode sector in which a charge at infinity is generated automatically. This charge combines in a non-trivial way with the Weyl-anomaly of the determinant of the fluctuation operators to reconcile the equivalence of the finite size scaling and the central charge. For certain values of the non-minimal coupling we obtain minimal models from interacting fermions. This is the subject of sections 3.4 and 3.5.

The gauged Thirring model, which contains the Schwinger model (QED_2) as a particular limit, is no longer conformally invariant but has a mass gap: The 'photon' acquires a mass $m_\gamma^2 = e^2/(\pi + \frac{1}{2}g_2^2)$ via the Schwinger mechanism. It possesses a non-trivial vacuum structure which promotes it to an attractive toy model to mimic the complex vacuum structure in 4-d gauge theories. From our experience with the Schwinger model [8], which is supposed to share certain aspects with one-flavour QCD [9], we expect that gauge fields with winding numbers are responsible for the non-vanishing chiral condensate and in particular its temperature dependence. Configurations with windings, so called instantons, exist only for finite volumes and minimize the Euclidean action. They lead to chirality violating vacuum expectation values. For example, a non-zero chiral condensate develops which only for high temperature and large curvature vanishes exponentially.

Since for particular choices of the coupling constants the model reduces to well known and well studied exactly soluble models there are many earlier works which are related to ours. Some of them concentrated more on the gauge sector and investigated the renormalization of the electric charge by the four-Fermi interaction [10] or the non-trivial vacuum structure in the Schwinger model [8, 11]. Others concentrated on the un-gauged conformal sector. Freedman and Pilch calculated the partition function of the un-gauged Thirring model on arbitrary Riemann surfaces [4]. We do not agree with their result and in particular show that there is no holomorphic factorization for general fermionic boundary conditions. Also we deviate from Destri and deVega [5] which investigated the un-gauged model on the cylinder with twisted boundary conditions. We comment on these discrepancies in section 3.2.
Section 2 contains introductory material and in particular the classical structure of the model.

Other papers are dealing with different aspects of certain limiting cases of the model considered here. In particular in [3], the thermodynamics of the Thirring model has been studied and the Hawking radiation has been derived in [12]. The equivalence of the massive Thirring model and the Sine-Gordon model in curved space has been shown in [13]. Partition functions for scalar fields with twisted boundary conditions have been computed in [14] and more recently in [15].

2 Classical Theory

The gauged Thirring model in curved space-time has the Lagrangian

$$\mathcal{L}_{\rm Thir}[A_\mu,\bar\psi,\psi] = \bar\psi i\gamma^\mu\nabla_\mu\psi - \frac{g^2}{4}j^\mu j_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad j^\mu = \bar\psi\gamma^\mu\psi \; , \tag{1}$$

where the gamma-matrices in curved space-time are related to the ones in Minkowski space-time as $\gamma^\mu = e^\mu_a\hat\gamma^a$, $\nabla_\mu = \partial_\mu + i\omega_\mu - ieA_\mu$ is the coordinate- and gauge covariant derivative and $F_{\mu\nu}$ is the electromagnetic field strength. The gravitational field $g_{\mu\nu}$ (or rather the 2-bein e^a_μ, since the theory contains fermions) is treated as classical background field, whereas the 'photons' A_μ and fermions ψ will be quantized.
The classical theory is invariant under $U(1)$ gauge- and axial transformations and correspondingly possesses conserved vector and axial-vector currents

$$j^\mu \quad \text{and} \quad j^{5\mu} = \bar\psi\gamma^\mu\gamma_5\psi = \eta^\mu{}_\nu j^\nu . \tag{2}$$

Here $\eta_{\mu\nu} = \sqrt{-g}\,\epsilon_{\mu\nu}$ denotes the totally antisymmetric tensor. In fact, the conservation laws together with the relation (2) between the vector- and axial currents imply that the currents are free fields

$$\nabla^2 j^\mu = \nabla^2 j^{5\mu} = 0 \; , \tag{3}$$

which is the reason that accounts for the solubility of the model [16], even in the presence of gauge- and gravitational fields. Of course, for any gauge invariant regularization the axial current possesses an anomalous divergence in the quantized model. Thus the normal $U_A(1)$ Ward identities in the un-gauged Thirring model [10] become anomalous when the fermions couple to a gauge field.

The solution of the equations of motion is most easily presented by introducing auxiliary scalar- and pseudo-scalar fields, in terms of which the action takes the form

$$\begin{aligned} \mathcal{S} \;=\; \int \sqrt{-g}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar\psi\gamma^\mu(\nabla_\mu - ig_1\partial_\mu\lambda + ig_2\eta_\mu{}^\nu\partial_\nu\phi)\psi\Big] \\ &+ g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi + \partial_\mu\lambda\partial_\nu\lambda)\Big]. \end{aligned} \tag{4}$$

Note that for later use we have allowed for different couplings of the fermionic currents to the scalar- and pseudo-scalar auxiliary fields λ and ϕ, respectively. The original Thirring model is recovered for $g_1 = g_2 = g$, since then

$$\mathcal{L} = -\frac{1}{4}F^2 + i\bar{\psi}\gamma^\mu\nabla_\mu\psi + gj^\mu B_\mu + g^{\mu\nu}B_\mu B_\nu, \quad B_\mu = \partial_\mu\lambda - \eta_\mu{}^\nu\partial_\nu\phi \tag{5}$$

is classically and quantum-mechanically equivalent to (1), after elimination of the multiplier field B_μ.

By decomposing the gauge field similarly as the B_μ-field as

$$A_\mu = \partial_\mu\alpha - \eta_{\mu\nu}\partial^\nu\varphi, \qquad \text{so that} \quad F_{01} = \sqrt{-g}\nabla^2\varphi, \tag{6}$$

and choosing isothermal coordinates for which $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, the generalized Dirac operator reads

$$\begin{aligned} D\!\!\!\!/ \; &= \; e^{iF - i\gamma_5 G - \frac{3}{2}\sigma}\, \partial\!\!\!/\, e^{-iF - i\gamma_5 G + \frac{1}{2}\sigma}, \quad \text{where} \\ F \; &= \; g_1\lambda + e\,\alpha \quad \text{and} \quad G = g_2\phi + e\,\varphi\,. \end{aligned} \tag{7}$$

Hence, if $\psi_0(x)$ solves the free Dirac equation in flat Minkowski space time, then

$$\psi(x) \equiv e^{iF + i\gamma_5 G - \frac{1}{2}\sigma}\psi_0 \tag{8}$$

solves the Dirac equation of the interacting theory on curved space-time. The vector currents are related as

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \bar{\psi}_0\hat{\gamma}^\mu\psi_0 e^{-2\sigma} \equiv \frac{1}{\sqrt{-g}}j_0^\mu.$$

The same relation holds for the axial vector current.

Diffeomorphism invariance then leads covariantly conserved energy-moment um tensor

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta\mathcal{S}}{\delta g_{\mu\nu}}. \tag{9}$$

Applying the variational identities in Appendix A one obtains after a lengthy but straightforward computation

$$\begin{aligned} T^{\mu\nu} \; = \; & \frac{1}{4}g^{\mu\nu}F^{\sigma\rho}F_{\sigma\rho} - F^{\sigma\nu}F_\sigma{}^\mu + \frac{i}{2}[\bar{\psi}\gamma^{(\mu}D^{\nu)}\psi - (D^{(\mu}\bar{\psi})\gamma^{\nu)}\psi] \\ & +2\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi \quad + \quad (\phi \leftrightarrow \lambda) \\ & +\frac{1}{2}j^\mu\,(g_1\nabla^\nu\lambda - g_2\eta^{\nu\alpha}\nabla_\alpha\phi) \quad + \quad (\mu \leftrightarrow \nu) \end{aligned} \tag{10}$$

$$+g_2 g^{\mu\nu} j^\alpha \eta_{\alpha\beta} \nabla^\beta \phi - 2g_2 j^\alpha \eta_\alpha{}^{(\mu} \nabla^{\nu)} \phi \ ,$$

where we have introduced the symmetrization $A^{(\mu}B^{\nu)} = \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu)$. The first two lines are just the energy momentum of the electromagnetic field, charged fermions and free neutral (pseudo-) scalars. The remaining terms reflect the interaction between the fermionic and auxiliary fields. On shell $T^{\mu\nu}$ is conserved as required by general covariance. Using the field equations for ψ and λ its trace reads

$$T^\mu_\mu = -\frac{1}{2} F^{\sigma\rho} F_{\sigma\rho} \ . \tag{11}$$

In particular for $A_\mu = 0$ it vanishes, and the theory becomes Weyl-invariant.

Symplectic structure: In the presence of both fermions and bosons it is convenient to exploit the graded Poisson structure [17]

$$\{A(x), B(y)\} \equiv \sum_O \int dz^1 \Big(\frac{A(x)\overleftarrow{\delta}}{\delta O(z)} \frac{\overrightarrow{\delta} B(y)}{\delta \pi_O(z)} \mp \frac{A(x)\overleftarrow{\delta}}{\delta \pi_O(z)} \frac{\overrightarrow{\delta} B(y)}{\delta O(z)} \Big)\Big|_{x^0=y^0} \ .$$

The sum is over all fundamental fields $O(x)$ in the theory . The sign is minus if one or both of the fields A and B are bosonic (even) and it is plus if both are fermionic (odd) fields. The momentum densities $\pi_O(x)$ conjugate to the O-fields are given by functional left-derivatives. A simple calculation yields the following momenta

$$\pi_\psi = -i\psi^\dagger, \quad \pi_\phi = g_2 j^5_0 + 2\partial_0 \phi \quad \text{and} \quad \pi_\lambda = g_1 j_0 + 2\partial_0 \lambda.$$

In the following sections we are lead to consider the *Euclidean version* of the model. Then one must replace the Lorentzian $\gamma^\mu, g_{\mu\nu}$ and ω_μ by their Euclidean counterparts. For example, with our conventions the relation (2) becomes

$$j^{5\mu} = -i\eta^\mu{}_\nu j^\nu$$

and the generalized Dirac operator in Euclidean space-time becomes

$$D\!\!\!\!/ = e^{-2\sigma} e^{f^\dagger} \partial\!\!\!/ \ e^f, \quad \text{where} \quad f = -iF + \gamma_5 G + \frac{1}{2}\sigma \tag{12}$$

(see (7) for the definition of F and G), instead of (7). Also, to recover the Euclidean Thirring model as particular limit of (4) we must set $g_2^2 = g_1^2 = g^2$.

3 Thermodynamic- and conformal properties

In this section we analyze the quantum theory corresponding to the classical action (4) without gauge fields, in flat and curved space-time. The gauged model is then considered in the next section. Here we calculate the *partition function, ground state energy, equation of state* and determine the *conformal structure* of the un-gauged model.

To allow for a non-vanishing U(1)-charge we couple this conserved charge to a chemical potential μ. For the finite temperature model the imaginary time must vary from zero to the inverse temperature β and the bosonic and fermionic fields must obey periodic and anti-periodic boundary conditions, respectively. We enclose the system in a spatial box with length L to avoid infrared divergences.

We shall determine the dependence of the partition function and correlators on the metric. This provides us with an alternative approach to the conformal structure and its relation to finite size-effects. Also, it enables us to study the effect of non-minimal coupling to gravity in section 3.3. Hence we allow for an arbitrary metric or 2-bein $e_{\mu a}$ with Euclidean signature. We can choose (quasi) isothermal coordinates and a Lorentz frame such that

$$\begin{aligned} e_{\mu a} &= e^{\sigma}\,\hat{e}_{\mu a} \equiv e^{\sigma}\begin{pmatrix} \tau_0 & \tau_1 \\ 0 & 1 \end{pmatrix} \\ g_{\mu\nu} &= e^{2\sigma}\,\hat{g}_{\mu\nu} \equiv e^{2\sigma}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \qquad \sqrt{g} = e^{2\sigma}\tau_0, \end{aligned} \tag{13}$$

where $\tau = \tau_1 + i\tau_0$ is the Teichmueller parameter and σ the gravitational Liouville field. Space-time is then a square of length L and has volume $V = \int_0^L d^2x\sqrt{g}$. We allow for the general twisted boundary conditions for the fermions

$$\begin{aligned} \psi(x^0 + L, x^1) &= -e^{2\pi i(\alpha_0+\beta_0\gamma_5)}\psi(x^0, x^1) \\ \psi(x^0, x^1 + L) &= -e^{2\pi i(\alpha_1+\beta_1\gamma_5)}\psi(x^0, x^1). \end{aligned} \tag{14}$$

The parameters α_i and β_i represent *vectorial and chiral twists*, respectively. We could allow for twisted boundary conditions for the (pseudo) scalars as well [14, 15], e.g. $\phi(x^0+nL, x^1+mL) = \phi(x^1, x^0) + 2\pi(m+n)$. However, to recover the Thirring model for equal couplings we must assume that these fields are periodic. For $\sigma=0$, $\tau=i\beta/L$ in which case $V = \beta L$, and for $\alpha_0 = \beta_0 = 0$ the partition function has the usual thermodynamical interpretation. Its logarithm is proportional to the free energy at temperature $T=1/\beta$.

3.1 Fermionic Generating Functional

Twisted boundary conditions as in (14) require some care in the fermionic path integral. The subtleties are not related to the unavoidable ultra-violet divergences but to the transition from Minkowski- to Euclidean space-time. To see that more clearly let $\mathcal{S}^{\pm}$ denote the space of fermionic fields in *Minkowski space-time* with chirality ± 1. Since both the commutation relations and the action do not connect $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ we can consistently impose different boundary conditions on $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$. On the other hand, in the *Euclidean path-integral* for the generating functional

$$Z_F[\eta,\bar\eta]=\int\mathcal{D}\psi^\dagger\mathcal{D}\psi\; e^{\int\sqrt{g}\,\psi^\dagger i\not{D}\psi+\int\sqrt{g}\left(\bar\eta\psi+\psi^\dagger\eta\right)}, \tag{15}$$

the Dirac operator

$$\not{D}=\begin{pmatrix}0 & D_-\\ D_+ & 0\end{pmatrix}$$

exchanges the two chiral components of ψ, i.e. $\not{D}:\mathcal{S}^{\pm}\to\mathcal{S}^{\mp}$. Thus, in contrast to the situation in Minkowski space the two chiral sectors are related in the action. Of course, the eigenvalue problem for $i\not{D}$ is then not well defined. This is the origin of the ambiguity in the definition of the determinant. It is related to the ambiguities one encounters when one quantizes chiral fermions [18]. Here we reformulate this problem in such a way that the determinant with chiral twists ($\beta\neq 0$) can be obtained by analytic continuation. The resulting determinants do not factorize into (anti-) holomorphic pieces. In appendix B we give further arguments in favour of our result by calculating the determinants in a different way.

Let us now study the generating functional for fermions in an external gravitational and auxiliary field. For that we observe that on the torus we must add a harmonic piece to the auxiliary fields to which the fermionic current couples in (4). More precisely, in the Hodge-decomposition of B_μ in (5) contains a harmonic piece,

$$B_\mu=\partial_\mu\lambda-\eta_\mu{}^\nu\partial_\nu\phi+\frac{2\pi}{L}h_\mu\quad\text{with}\quad\nabla^\mu h_\mu=h_{[\mu;\nu]}=0. \tag{16}$$

More generally, allowing for arbitrary couplings of the various terms in (16) to the fermionic current, we are led to add a term

$$\frac{2\pi}{L}g_0\int\sqrt{g}\,h_\mu j^\mu+(\frac{2\pi}{L})^2\int\sqrt{g}\,h_\mu h^\mu$$

to the action (4). Note, that in isothermal coordinates, for which the metric has the form (13), the harmonics h_μ are constant. The constant h_μ couple to the harmonic part of the current and are needed to recover the Thirring model in the limit $g_0^2=g_1^2=g_2^2$. Also, we shall see that the harmonic degrees of freedom are essential to obtain the correct thermodynamic potential.

Finally we introduce a *chemical potential* for the conserved $U(1)$ charge. In the Euclidean functional approach this is equivalent to coupling the fermions to a constant imaginary gauge potential A_0 [19].

As a consequence of the above observations the scaling formula (12) (recall, that $F=g_1\lambda$ and $G=g_2\phi$ when the electromagnetic interaction is switched off) is modified to

$$\begin{aligned} \not{D} &= e^{-2\sigma}e^{f^\dagger}\,\hat{\not{D}}\,e^f, \qquad \text{where} \qquad f=-ig_1\lambda+\gamma_5 g_2\phi+\frac{1}{2}\sigma \\ \hat{\not{D}} &= \gamma^\mu\Big(\partial_\mu+i\hat{\omega}_\mu-\frac{2\pi i}{L}[g_0h_\mu+\mu_\mu]\Big) \quad \text{and} \quad \mu_\mu=-i\frac{\tau_0 L}{2\pi}\mu\,\delta_{\mu 0}. \end{aligned} \tag{17}$$

This scaling property will enable us to relate the fermionic determinants and Green's functions of $\not{D}$ and $\hat{\not{D}}$. The spin connection $\hat{\omega}$ in (17) vanishes for our choice of the reference zweibein. The dependence of $\hat{\not{D}}$ on the chemical potential μ and the constant harmonic field h_μ cannot be gotten by the anomaly equation [20]. It must be computed by direct methods. For this we expand the fermionic field in a orthonormal basis of the Hilbert space

$$\begin{aligned} \psi(x) &= \sum_n a_n\psi_{n+}(x)+\sum_n b_n\psi_{n-}(x) \\ \psi^\dagger(x) &= \sum_n \bar{a}_n\chi^\dagger_{n+}(x)+\sum_n \bar{b}_n\chi^\dagger_{n-}(x), \end{aligned} \tag{18}$$

where $a_n, b_n, \bar{a}_n, \bar{b}_n$ are independent Grassmann variables. A basis is given by

$$\psi_{n\pm}(x)=\frac{1}{\sqrt{V}}\,e^{i(p_n^\pm,x)}\,e_\pm, \quad \text{where} \quad (p_n^\pm)_i=\frac{2\pi}{L}(\frac{1}{2}+\alpha_i\pm\beta_i+n_i), \tag{19}$$

and the $e_\pm$ are the eigenvectors of γ_5. Recall that α_i and β_i represent the vectorial- and chiral twists (14) respectively. The ψ_{n+} and ψ_{n-} must obey the $\mathcal{S}^+$ and $\mathcal{S}^-$ boundary conditions, respectively. These boundary conditions fix the admissible momenta $p_n^\pm$ in (19). Since the Dirac operator maps $\mathcal{S}^\pm$ into $\mathcal{S}^\mp$ the $\chi_{n\pm}$ must then obey the same boundary conditions as the $\psi_{n\mp}$. Thus $\chi_{n\pm}(x)$ is obtained from $\psi_{n\pm}(x)$ by exchanging p_n^+ and p_n^-. It follows then that

$$i\hat{\not{D}}\,\psi_{n\pm} = \lambda_n^\pm \chi_{n\mp} \tag{20}$$

with

$$\begin{aligned} \lambda_n^+ &= \frac{2\pi}{\tau_0 L}\Big[\bar{\tau}(\frac{1}{2}+a_1+\beta_1+n_1)-(\frac{1}{2}+a_0+\beta_0+n_0)\Big] \\ \lambda_n^- &= \frac{2\pi}{\tau_0 L}\Big[\tau(\frac{1}{2}+a_1-\beta_1+n_1)-(\frac{1}{2}+a_0-\beta_0+n_0)\Big]. \end{aligned} \tag{21}$$

Here we have introduced $a_\mu \equiv \alpha_\mu - g_0 h_\mu - \mu_\mu$. To continue we recast the infinite product for the determinant in the form

$$\prod_n^\infty \lambda_n^+\lambda_n^- = \prod_{\vec{n}\in Z^2}\Big(\frac{2\pi}{L}\Big)^2 \hat{g}^{\mu\nu}(\frac{1}{2}+c_\mu+n_\mu)(\frac{1}{2}+c_\nu+n_\nu), \tag{22}$$

where $\hat{g}^{\mu\nu}$ is the inverse of the reference metric (13) and

$$c_\mu = a_\mu + i\hat{\eta}_\mu{}^\nu\beta_\nu, \quad \text{with} \quad (\hat{\eta}_\mu{}^\nu) = -\frac{1}{\tau_0}\begin{pmatrix}\tau_1 & -|\tau|^2 \\ 1 & -\tau_1\end{pmatrix}. \tag{23}$$

The logarithm of the product (22) can in turn be written as the derivative at zero argument of a generalized zeta function. Indeed one easily verifies that for

$$\zeta(s) \equiv \sum_n (\lambda_n^+\lambda_n^-)^{-s} \tag{24}$$

we have (formally)

$$\det(i\hat{\not{D}}) \equiv (\prod_n \lambda_n^+\lambda_n^-)_{reg} = \exp[-\zeta'(s)]|_{s=0}. \tag{25}$$

However $\zeta(s)$ is divergent for $s \leq 1$. These divergences can be regularized as follows: We compute $\zeta(s)$ for $s > 1$ and subsequently define its value for $s < 1$ by analytic

continuation.
Assume for the moment that c_μ is real or equivalently that there are no chiral twists β_μ and chemical potential μ. Then $\zeta(s)$ has a well defined analytic continuation to $s<1$ via a Poisson resummation [21]. Indeed, writing $\zeta(s)$ as a Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} {\sum_n}' \int dt\, t^{s-1} e^{-t\lambda_n^+ \lambda_n^-}, \tag{26}$$

the generalized Poisson resummation formula

$$\sum_Z \exp[-\pi h^{\mu\nu}(n_\mu - a_\mu)(n_\nu - a_\nu)] = \sqrt{h} \sum_Z \exp[-\pi h_{\mu\nu} n^\mu n^\nu - 2\pi i n^\mu a_\mu], \tag{27}$$

applied to the integrand in (26) yields after integration over t

$$\zeta(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \pi^{2s-1} \sqrt{g} {\sum_Z}' (g_{\mu\nu} n^\mu n^\nu)^{\frac{s-1}{2}} \exp[-2\pi i n^\mu (c_\mu + \frac{1}{2})]. \tag{28}$$

The zero mode with $n_\mu = 0$ is eliminated because for $s > 1$ it does not contribute. After this analytic continuation $\zeta(s)$ and $\zeta'(s)$ are now regular at $s=0$. More precisely $\zeta(0) = 0$ and

$$\begin{aligned} \zeta'(0) &= \pi^{-1} \sqrt{g} {\sum_Z}' (g_{\mu\nu} n^\mu n^\nu)^{-\frac{1}{2}} \exp[-2\pi i n^\mu c_\mu] \\ &= -\log\Big[\frac{1}{|\eta(\tau)|^2} \Theta\Big[\begin{matrix} -c_1 \\ c_0 \end{matrix}\Big](0,\tau)\ \bar{\Theta}\Big[\begin{matrix} -c_1 \\ c_0 \end{matrix}\Big](0,\tau)\Big]. \end{aligned} \tag{29}$$

Here we made use of $\det[c(i\hat{\not{D}})^2] = \det[(i\hat{\not{D}})^2]$, which follows from $\zeta(0)=0$.
For complex c_μ the Poisson resummation is not applicable and $\zeta'(0)$ cannot be calculated by direct means. To circumvent these difficulties we note that the infinite sum (24) defining the ζ-function for $s>1$ is a mereomorphic function in c. Thus we may first continue to $s<1$ for real c_μ and then continue the result to complex values. Using the transformation properties of theta functions the resulting determinant can be written as

$$\det(i\hat{\not{D}}) = e^{2\pi(\sqrt{\hat{g}}\hat{g}^{\mu\nu}\beta_\mu\beta_\nu - 2i\beta_1 a_0)} \frac{1}{|\eta(\tau)|^2} \Theta\Big[\begin{matrix} -a_1+\beta_1 \\ a_0 - \beta_0 \end{matrix}\Big](0,\tau) \bar{\Theta}\Big[\begin{matrix} -\bar{a}_1 - \beta_1 \\ \bar{a}_0 + \beta_0 \end{matrix}\Big](0,\tau). \tag{30}$$

This is the main result of this section.

It can be shown that this determinant is *gauge invariant*, i.e. invariant under $\alpha_\mu \to \alpha_\mu + 1$, but not invariant under chiral transformations, $\beta_\mu \to \beta_\mu + 1$, as expected. Furthermore, it transforms covariantly under modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. In other words, $\det i\hat{\not{D}}$ is invariant under modular transformations if at the same time the boundary conditions are transformed accordingly. The exponential prefactor is needed for modular covariance and is not present in the literature [4]. It correlates the two chiral sectors and will have important consequences. In Appendix B we confirm (30) with operator methods.

The last step in the calculation of the fermionic generating functional is the inclusion of the local contributions to the auxiliary- and metric field, i.e. the dependence of the determinant on λ, ϕ and σ. For this we introduce the one-parameter family of Dirac operators

$$\not{D}_\tau = \frac{\hat{g}^{1/2}}{g_\tau^{1/2}} e^{\tau f^\dagger} \hat{\not{D}} e^{\tau f}. \tag{31}$$

We take the τ-dependence of the metric as $g_\tau = e^{2\tau\sigma}\hat{g}$. With f as defined in (17), this family interpolates between $\hat{\not{D}}$ and $\not{D}$. The determinant of the full Dirac operator is then obtained by integrating the corresponding anomaly equation [22]:

$$\det i\not{D} = \det(i\hat{\not{D}}) \exp\Big(\frac{S_L}{24\pi} + \frac{g_2^2}{2\pi}\int\sqrt{\hat{g}}\phi\hat{\triangle}\phi\Big), \tag{32}$$

where

$$S_L = \int\sqrt{\hat{g}}[\hat{\mathcal{R}}\sigma - \sigma\hat{\triangle}\sigma] \tag{33}$$

is the *Liouville action.* In deriving this result we assumed that $\int\sqrt{g}\lambda = 0$. This constraint on the zero-mode of λ (and similarly of ϕ) will be discussed below. Actually, for our reference metric the Ricci scalar $\hat{\mathcal{R}}$ vanishes and the Liouville action simplifies to $-\int\sqrt{\hat{g}}\sigma\hat{\triangle}\sigma$. However, the above formulae hold for arbitrary reference metrics and arbitrary Riemannian surfaces. Furthermore, as expected for a gauge-invariant regularization, the function λ and thus the longitudinal part of B_μ does not appear in the determinant.

To complete the calculation of the generating functional we need to know the fermionic Green-functions S. Using the scaling property of the Dirac operator, eq.

(31), it is easy to see that in an arbitrary background field S is related to $\hat{S}$ by

$$S(x,y) = e^{-f(x)}\,\hat{S}(x,y)\,e^{-f^\dagger(y)}.$$

Together with the relation (32) and the explicit form (29,30) for $\det i\hat{\not{D}}$ this yields the fermionic generating functional

$$\begin{aligned} Z_F[\eta,\bar\eta] = & \frac{1}{|\eta(\tau)|^2}\Theta\Big[{-c_1 \atop c_0}\Big](0,\tau)\;\bar\Theta\Big[{-\bar c_1 \atop \bar c_0}\Big](0,\tau) \\ & e^{-\int \bar\eta(x)S(x,y)\eta(y)}\cdot\exp\Big(\frac{1}{24\pi}S_L + \frac{g_2^2}{2\pi}\int\sqrt{g}\phi\triangle\phi]\Big). \end{aligned} \tag{34}$$

By using the scaling properties of the Ricci-scalar and Laplacian (see appendix A) the exponent can be written in a manifest diffeomorphism-invariant way as

$$-\frac{1}{96\pi}\int\sqrt{g}\,\mathcal{R}\frac{1}{\triangle}\mathcal{R} + \frac{g_2^2}{2\pi}\int\sqrt{g}\phi\triangle\phi.$$

Here we used that on the torus $\mathcal{R}$ integrates to zero. On the sphere or higher genus surfaces the last formula is modified.
The Integration over the auxiliary fields then leads to the full generating functional of the Thirring model. It contains all information about the thermodynamic- and conformal properties. This is the subject of the next two sections.

3.2 Thermodynamics of the Thirring Model

In this chapter we derive the grand canonical potential, equation of state and ground state energy for the Thirring model. For this we need to compute the partition function

$$Z = \int d^2h\mathcal{D}\phi\mathcal{D}\lambda\; Z_F[\eta{=}\bar\eta{=}0]\; e^{-S_B}, \tag{35}$$

where Z_F is the fermionic generating functional (34) and S_B the bosonic action

$$S_B = (2\pi)^2\sqrt{\hat g}\hat g^{\mu\nu}h_\mu h_\nu - \int\sqrt{g}\Big(\lambda\triangle\lambda + \phi\triangle\phi\Big). \tag{36}$$

As it stands the partition function is still ill-defined unless we constrain the zero-modes artificially introduced in the Hodge decomposition of B_μ in (16). The choice of the constraints is restricted by the symmetries of the system. In particular translation invariance (or rotation invariance on the sphere) and covariance under modular transformations of the torus are symmetries which me may want to preserve by the zero-mode constraint. The constraint measure

$$\int dh_0 dh_1 \mathcal{D}\phi \mathcal{D}\lambda \delta(\bar{\phi})\delta(\bar{\lambda}) \cdots \equiv \int dh_0 dh_1 \mathcal{D}_\delta \phi \mathcal{D}_\delta \lambda \cdots, \qquad \bar{\phi} \equiv \frac{1}{V} \int \sqrt{g}\phi \tag{37}$$

(and similarly for $\bar{\lambda}$) satisfies these requirements (The normalization by the volume in the definition of $\bar{\phi}$ is needed such that the constraints and hence the partition function are both dimensionless). For example, one finds the dimensionless partition function

$$\mathcal{N}_0 \equiv \int \mathcal{D}\phi \; \delta(\bar{\phi}) \, e^{(\phi, \triangle \phi)} = \frac{\sqrt{V}}{\det'^{\frac{1}{2}}(-\triangle)} \tag{38}$$

for free bosons, where the prime indicates the omission of the zero-eigenvalue.

Integration over the harmonics: There is no restriction on the harmonic parts of the auxiliary fields and the Gaussian integral yields

$$\int\limits_{-\infty}^{\infty} d^2h \Theta\Big[{-c_1 \atop c_0}\Big] \bar{\Theta}\Big[{-\bar{c}_1 \atop \bar{c}_0}\Big] \exp[-(2\pi)^2 \sqrt{\hat{g}} \hat{g}^{\mu\nu} h_\mu h_\nu] = \frac{\Theta\big[{u \atop w}\big](\Lambda)}{4\pi\sqrt{1 + g_0^2/2\pi}}, \tag{39}$$

where

$$\Theta\Big[{u \atop w}\Big](\Lambda) = \sum_{n \in Z^2} e^{i\pi(n+u)\Lambda(n+u) + 2\pi i(n+u)w}$$

is the theta function with characteristics

$$u = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} (\alpha_1 + i\eta_1{}^\nu \beta_\nu) \quad \text{and} \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\alpha_0 + i\eta_0{}^\nu \beta_\nu - \mu_0) \tag{40}$$

and covariance

$$\Lambda = \begin{pmatrix} \tau & 0 \\ 0 & -\bar{\tau} \end{pmatrix} + i \frac{\pi g_0^2 \tau_0}{2\pi + g_0^2} \begin{pmatrix} g_0^2 & -4\pi - g_0^2 \\ -4\pi - g_0^2 & g_0^2 \end{pmatrix}. \tag{41}$$

Integration over λ and ϕ: The integral over λ, subject to the δ-constraint in (37), merely contributes one inverse square-root of the primed determinant of $-2\triangle$ to the partition function and so does the integration over ϕ. In fact, to obtain the partition function of the Thirring model we divide Z by the corresponding partition functions $\mathcal{N}_0$ of the free bosons, eq. (38). Using (39) and (34) we obtain

$$\frac{Z}{\mathcal{N}_0} = \frac{1}{|\eta(\tau)|^2}\sqrt{\frac{2\pi + g_2^2}{2\pi + g_0^2}}\,\Theta\Big[{u \atop w}\Big](\Lambda)\, e^{(1/24\pi + g_3^2)S_L}, \tag{42}$$

where we have also used the scaling formula for the primed determinant of $\triangle$ [20, 23]

$$\log\frac{\det'(-a\triangle)}{\det'(-\triangle)} = \log a\cdot\zeta(0) = \log a\cdot[\frac{1}{4\pi}\int a_1 - p], \tag{43}$$

with p being the number of zero modes of the operator. On the torus $\int a_1 = 0$ and we find

$$\det'(-a\triangle) = \frac{1}{a}\det'(-\triangle),$$

which produces the extra factor $\sqrt{2\pi + g_2}$. In the Thirring model limit $g_2 = g_0$ and the square-root in (42) disappears.

Zero-temperature limit: To investigate the thermodynamics of the model we assume space-time to be flat and that $\tau = i\beta/L$. Then

$$\Omega = -\frac{1}{\beta}\log\frac{Z}{\mathcal{N}_0}$$

is the grand canonical potential. First we analyze the low temperature limit of Ω. For $\mu = 0$ this yields the ground state energy. We observe that for $\tau = i\beta/L$ the covariance matrix Λ in (41) simplifies to

$$i\pi\Lambda = -\frac{\pi\beta}{L}\Big[\mathrm{Id} + \frac{g_0^2}{4\pi}\frac{1}{2\pi + g_0^2}\begin{pmatrix} g_0^2 & -4\pi - g_0^2 \\ -4\pi - g_0^2 & g_0^2 \end{pmatrix}\Big] \tag{44}$$

and has eigenvalues

$$\lambda_1 = -\frac{\pi\beta}{L}\frac{2\pi+g_0^2}{2\pi} \quad \text{and} \quad \lambda_2 = -\frac{\pi\beta}{L}\frac{2\pi}{2\pi+g_0^2} \tag{45}$$

with corresponding eigenvectors

$$v_1 = (-1,1) \quad \text{and} \quad v_2 = (1,1). \tag{46}$$

Also, the $\hat{\eta}$ tensor (see 23) and μ_0 (see 17) in (40) simplify to

$$\eta_\mu{}^\nu = \begin{pmatrix} 0 & \beta/L \\ -L/\beta & 0 \end{pmatrix} \quad \text{and} \quad \mu_0 = -i\frac{\beta}{2\pi}\mu.$$

For $\beta \to \infty$ the saddle point approximation to the Gaussian sum (39) defining the theta-function becomes exact and therefore using that

$$\log|\eta(\tau)|^2 \longrightarrow -\frac{\pi\beta}{6L} \qquad \text{for} \quad \beta \to \infty$$

we find

$$\begin{aligned}\Omega(\beta\to\infty) = &-\frac{\pi}{6L} - \frac{4\pi}{2\pi+g_0^2}\frac{\pi}{L}(\beta_1+\frac{\mu L}{2\pi})^2 \\ &+\frac{\pi}{2L}\min_{n\in Z^2}\Big[\frac{2\pi+g_0^2}{2\pi}\{n_2-n_1-\frac{4\pi}{2\pi+g_0^2}(\beta_1+\frac{\mu L}{2\pi})\}^2 \\ &+\frac{2\pi}{2\pi+g_0^2}\{n_1+n_2-2\alpha_1\}^2\Big]\end{aligned} \tag{47}$$

for the zero-temperature grand potential of the un-gauged model. Here the chemical potential and chiral twist enter only through the combination $\beta_1+\mu L/2\pi$. Let us now discuss the potential in the various *limiting cases.*

i) No chiral twist, $\beta_1=0$, and vanishing chemical potential: Then $\Omega(\beta\to\infty)$ coincides with the *ground state energy.* The minimum in (47) is attained for $n_1=n_2=[\frac{1}{2}+\alpha_1]$ and we find

$$E_0(L,\alpha_1,\beta_1=0) = -\frac{\pi}{6L}+\frac{2\pi}{L}\frac{2\pi}{2\pi+g_0^2}(\alpha_1-[\frac{1}{2}+\alpha_1])^2. \tag{48}$$

Only for anti-periodic boundary conditions, that is for $\alpha_1=0$, does this *Casimir energy* coincide with the corresponding result for free fermions. For $g_0^2 \geq 4\pi$ the Casimir force is always attractive whereas for $g_0^2 < 4\pi$ it can be attractive or repulsive, depending on the value of α_1. The result (48) is in agreement with the literature [5]. For example, it coincides with De Vega's and Destri's result if we make the identification $\omega_{DD} = 2\pi\alpha_1$ and $1/\beta_{DD} = 1 + g_0^2/2\pi$ in formula (42) of that paper.

ii) Small twists and chemical potential: For small β_1 and μ the minimum is assumed for $n_i = 0$ and the potential simplifies to

$$\Omega(\beta \to \infty) = -\frac{\pi}{6L} + \frac{2\pi}{L}\frac{2\pi}{2\pi + g_0^2}\alpha_1^2 \tag{49}$$

and does not depend on the chemical potential. For vanishing g_0 the minimum of (47) is attained for

$$n_1 = [\frac{1}{2} + \alpha_1 - \beta_1 - \frac{\mu L}{2\pi}] \quad \text{and} \quad n_2 = [\frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi}],$$

where $[x]$ denotes the biggest integer which is smaller or equal to x. This then leads to the following zero temperature potential

$$\begin{aligned} \Omega = & - \frac{\pi}{6L} - \frac{2\pi}{L}(\beta_1 + \frac{\mu L}{2\pi})^2 \\ & + \frac{\pi}{L}\Big\{\alpha_1 - \beta_1 - \frac{\mu L}{2\pi} - [\frac{1}{2} + \alpha_1 - \beta_1 - \frac{\mu L}{2\pi}]\Big\}^2 \\ & + \frac{\pi}{L}\Big\{\alpha_1 + \beta_1 + \frac{\mu L}{2\pi} - [\frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi}]\Big\}^2. \end{aligned} \tag{50}$$

For $\mu = \beta_1 = 0$ this reduces to the Casimir energy for free fermions with left-right symmetric twists and agrees with the results in [24].
Note, however, that for $\beta_1 \neq 0$ we disagree with [5]. The difference is due to the second term on the right in (47). Let us give two arguments in favour of our result: The discrepancy arises from the prefactor appearing in the fermionic determinant (30). As discussed earlier this prefactor implies the breakdown of holomorphic factorization, a property which has been presupposed in [5]. One can show that our results can be reproduced by starting with massive fermions and taking the limit $m \to 0$ (see appendix B).
The second argument goes as follows: Suppose that $\beta_1 = \alpha_1 = 0$. Then (50) simplifies to

$$\Omega(\beta\to\infty)=-\frac{\pi}{6L}-\frac{2\pi}{L}\Big(\frac{\mu L}{2\pi}\Big)^2+\frac{2\pi}{L}\Big(\frac{\mu L}{2\pi}-[\frac{1}{2}+\frac{\mu L}{2\pi}]\Big)^2. \tag{51}$$

For massless fermions the Fermi energy is just μ and at $T=0$ all electron states with energies less then μ and all positron states with energies less then $-\mu$ are filled. The other states are empty. Since $d\Omega/d\mu$ is the expectation value of the electric charge in the presence of μ we conclude that it must jump if μ crosses an eigenvalue of the first quantized Dirac Hamiltonian h. For vanishing twists the eigenvalues of h are just $E_n=(n-\frac{1}{2})\pi/L$. From (51) one sees by inspection that the electric charge

$$\langle Q\rangle=\frac{d\Omega}{d\mu}=2[\frac{1}{2}+\frac{\mu L}{2\pi}]=2n \qquad \text{for} \quad E_n\leq\mu<E_{n+1}$$

indeed jumps at these values of μ. Further observe, that in the thermodynamic limit $L\to\infty$ the density

$$\frac{\Omega}{L}\to-\frac{2\pi}{2\pi+g_0^2}\frac{\mu^2}{2\pi},$$

reduces for $g_0=0$ to the standard result for free electrons.

Equation of state: We wish to derive the equation of state for finite T in the infinite volume limit $L\to\infty$. This may be achieved by interchanging the roles played by L and β. More precisely, using that

$$\Theta\Big[{u \atop w}\Big](\Lambda)=\sqrt{\det(i\Lambda^{-1})}\;e^{2\pi i w\cdot u}\;\Theta\Big[{-w \atop u}\Big](i\Lambda^{-1})$$

we find in analogy with the low temperature limit, that for $L\to\infty$ the pressure is given by

$$\begin{aligned}\beta p &= \lim_{L\to\infty}\frac{1}{L}\log\frac{Z}{\mathcal{N}_0}=\frac{\pi}{6\beta}+\frac{2\pi}{\beta}\frac{2\pi+g_0^2}{2\pi}\beta_0^2\\ &- \frac{\pi}{2\beta}\min_{n\in Z^2}\Big[\frac{2\pi+g_0^2}{2\pi}\{n_1+n_2+2\beta_0\}^2\\ &\qquad +\frac{2\pi}{2\pi+g_0^2}\{n_2-n_1+2\alpha_0+2i\frac{\beta\mu}{2\pi}\}^2\Big].\end{aligned}$$

Here the minimum of the real part has to be taken. Again the minimization arises from the saddle point approximation to the theta function which becomes exact when $L \to \infty$. For small twists the minimum is assumed for $n_i = 0$ and then

$$\beta p = \frac{\pi}{6\beta} - \frac{2\pi}{\beta} \frac{2\pi}{2\pi + g_0^2} (\alpha_0 + i\frac{\beta\mu}{2\pi})^2$$

becomes independent on the chiral twist β_0. As we have interchanged the roles of the temporal and spatial twists this is consistent with the earlier result that for small twists Ω is independent of β_1. In particular, for $\alpha_0 = 0$, we find the following equation of state

$$p(\beta, \mu, \alpha_0 = 0) = \frac{\pi}{6\beta^2} + \frac{\mu^2}{2\pi} \frac{2\pi}{2\pi + g_0^2}. \tag{52}$$

This result is consistent with the renormalization of the electric charge which is conjugate to the chemical potential. It shows that the thermodynamics of the Thirring model is not just that of free fermions as has been claimed in [3]. Indeed, the zero point pressure is multiplied by a factor $2\pi/(2\pi + g_0^2)$. This modification arises from the coupling of the current to the harmonic fields. It is missed if only the local part of the auxiliary field is considered, which is the case if one quantizes the model in Minkowski space and then replaces the k_0-integral in the Green functions by the Matsubara sum. This remark should also be taken seriously in four dimensions! Furthermore, we see that the 'pressure' p is real only for $\alpha_0 = 0$, which is consistent with the finite temperature boundary conditions[1].

3.3 Conformal structure

In the first part of this section we derive the Kac-Moody and Virasoro algebras of the model (4) without gauge-interaction and prepare the ground for an extension, containing in particular the minimal models, in the second part.
Recall (11) that for $A_\mu = 0$ the theory reduces to a conformal field theory on flat Minkowski space-time. To continue it is convenient to introduce adapted light cone coordinates $x^\pm = x^0 \pm x^1$ and the chiral components of the Dirac spinor $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi$. Then after substituting the classical equations of motion

[1] This can also be observed in the Hamiltonian formalism [25].

$$T_{--} = -\frac{1}{2}(\pi_{\psi_+}\partial_-\psi_+ - \partial_-\pi_{\psi_+}\psi_+) + 2(\partial_-\lambda)^2 + 2(\partial_-\phi)^2 + i\partial_-(g_1\lambda + g_2\phi)\pi_{\psi_+}\psi_+ \quad (53)$$

depends only on x^- and is therefore the chiral Noether current. Evaluating the Poisson bracket of the symmetry generator $T_f = \int dx^- f(x^-) T_{--}$ with the different fields yields the classical structure

$$\begin{array}{ccc} \delta_f\phi = f\partial_-\phi & ; & \delta_f\lambda = f\partial_-\lambda \\ \delta_f\psi_+ = f\partial_-\psi_+ + \frac{1}{2}\psi_+\partial_- f & ; & \delta_f\psi_+^\dagger = (\delta_f\psi_+)^\dagger \\ \delta_f j_- = f\partial_- j_- + j_-\partial_- f & ; & \delta_f T_{--} = f\partial_- T_{--} + 2T_{--}\partial_- f. \end{array} \quad (54)$$

Short Distance Expansions: Let us now determine the quantum corrections to these classical results. These are computed within the Euclidean functional approach from the short-distance expansions of the relevant $n-$ point functions. We need not postulate Kac-Moody and Virasoro algebras in advance as has been done in [10, 26]. These structures are derive here. When comparing the classical with the quantum results one should keep in mind that the roles of $\psi_0^\dagger$ and $\psi_1^\dagger$ are interchanged when one switches from Minkowski to Euclidean space-time. In coordinates adapted to the holomorphic structure of the torus

$$x = i\bar{\tau}x^0 + ix^1, \quad \text{so that} \quad \partial_x = \frac{1}{2\tau_0}(\partial_{x^0} - \tau\partial_{x^1}),$$

the Dirac operator and the corresponding Greens function take the form

$$i\partial\!\!\!/ = 2i\begin{pmatrix} 0 & \partial_x \\ \partial_{\bar{x}} & 0 \end{pmatrix} \quad \text{and} \quad S(x^\alpha, y^\beta) = \frac{1}{2\pi i}\begin{pmatrix} 0 & 1/\xi \\ 1/\bar{\xi} & 0 \end{pmatrix} + O(1),$$

where $\xi = x - y$, and the chiral components of the energy momentum tensor and current are given by

$$T_{xx} = \frac{\tau_0}{2i}(\tau T^{00} + T^{01}) = \frac{\tau_0}{2i}\frac{d\hat{g}_{\mu\nu}}{d\bar{\tau}}T^{\mu\nu} \quad \text{and} \quad j_x = \frac{1}{2i}(\tau j^0 - j^1).$$

From the conformal Ward identities

$$\sum_{i=1}^{n} \langle O(x_1)\cdots \delta O(x_i)\cdots O(x_n)\rangle = \frac{1}{i}\oint dz \langle O(x_1)\cdots O(x_n) T_{zz}\rangle \tag{55}$$

we obtain the central charges and conformal weights directly from the correlation functions. However, because on the flat torus the expectation value of T_{xx} is constant, we need to compute at least the 3-point function to read off the conformal weights. As in the classical theory (see (9)) the symmetric energy momentum tensor measures the change of the effective action $\Gamma = \log Z$ under arbitrary variations of the metric. On the torus there are two independent contributions. One being due to variations of the modular parameter τ and its conjugate $\bar{\tau}$ which depend implicitly on the metric. The other is due to the variations of terms which depend explicitly on the metric. Since the chiral component T_{xx} is gotten by contracting $T^{\mu\nu}$ with $d\hat{g}_{\mu\nu}/d\bar{\tau}$ it follows that

$$\langle T_{xx}\rangle = \frac{i\tau_0}{\sqrt{g(x^\alpha)}}\Big(\frac{1}{L^2}\frac{\partial}{\partial\bar{\tau}} + \frac{d\hat{g}_{\mu\nu}}{d\bar{\tau}}\frac{\delta}{\delta g_{\mu\nu}(x^\alpha)}\Big)\,\Gamma[g,\tau,\bar{\tau}] \equiv \delta_x\Gamma[g,\tau,\bar{\tau}].$$

When doing metric variations it is always understood that we take the flat space-time limit afterwards. The $\bar{\tau}$ variation is constant and may be discarded in the short distance expansion. Thus to analyze the algebraic structure we can work on any Riemann surface. This is not true for the finite size effects, which are global properties. This aspect will be analyzed in section 3.4.
For example, taking three metric variations of the curvature dependent part of $\log Z$ with Z from (42) we find the following short distance expansions for the three point correlation function

$$\langle T_{uu}\, T_{vv}\, T_{zz}\rangle \sim\ -\frac{1}{(2\pi)^3}\frac{1}{(u-v)^2(u-z)^2(v-z)^2}.$$

Substituting this result into the Ward identity (55) we obtain the *central charge* and the conformal weight of the energy momentum tensor

$$c = 1 \quad \text{and} \quad h_{T_{xx}} =\ 2\ . \tag{56}$$

Note that the the central charge as well as the conformal weight are independent of the couplings g_1 and g_2.
The conformal weights of the fundamental fields are obtained by computing the

fermionic two point function with stress tensor insertion

$$\langle\psi_0(x)\ \psi_1^\dagger(y)\ T_{zz}\rangle = \frac{1}{Z}\delta_z\Big(Z\langle\psi_0(x)\ \psi_1^\dagger(y)\rangle\Big).$$

Since $Z \sim \exp[F(\mathcal{R}^2)]$, its metric variation vanishes after the flat space-time limit has been taken. The variation of S_{ij} can be found in appendix A. This yields

$$\begin{aligned} h_{\psi_0} &= h_{\psi_1^\dagger} = \frac{1}{2} + \frac{1}{16\pi}g_1^2 - \frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi + g_2^2} \\ \bar{h}_{\psi_0} &= \bar{h}_{\psi_1^\dagger} = \frac{1}{16\pi}g_1^2 - \frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi + g_2^2}\ . \end{aligned} \tag{57}$$

Thus we have reproduced the classical results supplemented by additional g_1 and g_2 dependent quantum corrections. In the Thirring model limit $g_2 = g_1 = g$, these terms add up to give the known anomalous dimension appearing in the Thirring model [26]. Furthermore, from (57) we may derive a condition on the couplings g_1, g_2 if we insist on unitarity, i.e. on $h \geq 0$. We find

$$g_1^2 \geq \frac{2\pi g_2^2}{2\pi + g_2^2}. \tag{58}$$

In particular for $g_1 \geq \sqrt{2\pi}$ the conformal weights are positive for any real g_2.
Next we determine the *Kac-Moody algebra* of the $U(1)$ currents. To derive the correlation functions with current insertions we couple the fermions to an external vector field, that is consider the 'gauged' model without Maxwell term. For example,

$$< j^\mu(x^\alpha)\ j^\nu(y^\beta) > = \frac{1}{e^2\sqrt{g(x^\alpha)g(y^\beta)}}\ \frac{\delta^2\Gamma[g, A]}{\delta A_\nu(x^\alpha)\delta A_\mu(y^\beta)}|_{A=0}.$$

The effective action with external vector field is then obtained by shifting the auxiliary fields in (17) as

$$g_2\phi \to g_2\phi + e\varphi \quad , \quad g_1\lambda \to g_1\lambda + e\alpha, \tag{59}$$

where $A_\mu = \eta_\mu{}^\nu\partial_\nu\varphi + \partial_\mu\alpha$ and we have neglected the harmonic contribution to the external vector field, because it does not contribute to the short distance expansion. The resulting effective action does not depend on α due to gauge-invariance. To relate the variation w.r.t. A_μ to that w.r.t. φ we use

$$\partial_\mu \phi = \eta_\mu{}^\nu A^T_\nu, \quad \text{where} \quad A^T_\mu = A_\mu - \nabla_\mu \frac{1}{\triangle} \nabla^\nu A_\nu$$

is the transverse part of A_μ. We obtain the following short distance expansion

$$\langle j_x \, j_y \rangle \; \sim -\frac{1}{2\pi} \frac{1}{2\pi + g_2^2} \frac{1}{(x-y)^2} \; .$$

We read off the value k of the *central extension* in the $U(1)$-Kac-Moody algebra

$$k = \frac{2\pi}{2\pi + g_2^2}. \tag{60}$$

The precise g_2-dependence of k (which can of course be rescaled to unity by an appropriate redefinition of the current) is related to a finite renormalization of the electric charge in the gauged Thirring-model which we will discuss in section 4.
Finally, from

$$\langle j_x \, j_y \, T_{zz} \rangle \; \sim -\frac{1}{4\pi^2} \frac{1}{2\pi + g_2^2} \frac{1}{(x-z)^2 (y-z)^2}$$

we obtain $h_j = 1$.

To see how the left and right Kac Moody currents act on the fermionic fields we notice that after the integration over the auxiliary fields the A-dependence of the fermionic Green function factorizes as

$$\langle \psi_0(x) \psi_1^\dagger(y) \rangle_A = e^{\frac{1}{2} m_\gamma \int \varphi \triangle \varphi} \cdot e^{-eg(x)} \; \langle \psi_0(x) \psi_1^\dagger(y) \rangle_{A=0} \; e^{-eg^\dagger(y)},$$

where $g(x) = -i\alpha(x) + \gamma_5 \beta \varphi(x)$, $\beta = 2\pi/(2\pi + g_2^2)$ and m_γ is the induced 'photon'-mass (see(86)). Variation w.r.t. the $A-$ field yields, after some algebraic manipulations, the $U(1)$ charges

$$q_{\psi_0} = \frac{1}{2}(1 + \frac{2\pi}{2\pi + g_2^2}) \quad \text{and} \quad \bar{q}_{\psi_0} = \frac{1}{2}(1 - \frac{2\pi}{2\pi + g_2^2}). \tag{61}$$

We have used the convention where the electric charge $q + \bar{q}$ is unity. In the Thirring model limit we can compare (61) with the results obtained in [26]. For that we need to rescale the currents such that the central extension (60) of the Kac-Moody algebra becomes unity $j_z \to \sqrt{1 + g_2^2/2\pi} \; j_z$. It is then easy to see that we agree with Furlan et al. [26] if we make the identification $\bar{g}_{Fu} = g_2^2/4\pi \sqrt{1 + g_2^2/2\pi}$.

Non-Minimal Coupling: In section 3.1 we have analyzed the fermionic determinant in the presence of twisted boundary conditions. One may ask what happens if we introduce a local twist instead, that is

$$\psi(x) \to \psi(x) \quad ; \quad \psi(x)^\dagger \to \psi(x)^\dagger e^{\alpha\lambda(x)}, \tag{62}$$

which should be interpreted as a modification of the charge neutrality condition. The computation of the fermionic determinant in the presence of such twists is similar to that for a Weyl rescaling of the background metric (31-32). Integrating the corresponding anomaly equation we find

$$\log \frac{\det(i\not{D}_\alpha)}{\det(i\not{D}_0)} \propto \alpha \int \mathcal{R}\lambda + O((\alpha\lambda)^2). \tag{63}$$

We will come back to the relation between the above determinant and charges at infinity at the end of this section. For the moment we use the analogy merely as a motivation to study the extension of the Thirring model obtained by coupling the λ-field non-minimally to the background geometry. That is we consider the model (4) again without gauge-interaction but with an extra coupling

$$g_3 \int \mathcal{R}\lambda.$$

Then T_{--} in (53) is modified,

$$T_{--} \longrightarrow \bar{T}_{--} = T_{--} + 3g_3\partial_-^2\lambda.$$

The corresponding modification of the classical conformal transformations (54) generated by the modified generator $\bar{T}_f = \int dx^- f(x^-)\bar{T}_{--}$ are

$$\begin{aligned} \bar{\delta}_f\phi &= \delta_f\phi \quad , \quad \bar{\delta}_f\lambda = \delta_f\lambda - \frac{g_3}{2}\partial_- f \\ \bar{\delta}_f\psi_+ &= \delta_f\psi_+ - \frac{i}{2}g_1g_3\psi_+\partial_- f \quad , \quad \bar{\delta}_f\psi_+^\dagger = \delta_f\psi_+^\dagger + \frac{i}{2}g_1g_3\psi_+^\dagger\partial_- f. \end{aligned} \tag{64}$$

Whereas ϕ and ψ_+ remain primary fields, λ does not. This is in fact needed for consistency. Indeed, since ψ is not a scalar under conformal transformations generated by $\bar{T}_f$, the term $\sim \int \psi^\dagger \not{D}\psi$ in the action is only conformally invariant if λ transforms inhomogeneously like a spin connection.

It may be surprising that the new symmetry transformations depend on the coupling constant g_3 which is not present in the flat space time Lagrangian. However,

the same happens for example in 4 dimensions, if one couples a scalar field conformally, that is non-minimally, to gravity. Although the Lagrangian for the minimally and conformally coupled particles are the same on Minkowski space-time, their energy momentum tensors are not. The same happens for the conformally invariant non abelian Toda theories which admit several energy momentum tensors and hence several conformal structures [27].
The current still transforms as a primary with weight 1, but the energy momentum tensor acquires a classical central charge

$$\bar{\delta}_f \bar{T}_{--} = f\partial_- \bar{T}_{--} + 2\bar{T}_{--}\partial_- f - g_3^2 \partial_-^3 f. \tag{65}$$

The corresponding commutators in the quantized theory with non-minimal coupling to gravity are calculated as explained for the minimally-coupled model. One finds that the quantum corrections to (64) are identical to those of the minimally coupled model and thus are $g_3 \neq 0$-independent.
To summarize, we have obtained the following Virasoro $\times$ Kac-Moody structure:
Central charge:

$$c = 1 + 24 g_3^2 \pi \quad \text{and} \quad h_{T_{xx}} = 2 \tag{66}$$

Kac-Moody level and charges:

$$k = \frac{2\pi}{2\pi + g_2^2} \qquad ; \qquad h_j = 1$$
$$q_{\psi_0} = \frac{1}{2}\left(1 + \frac{2\pi}{2\pi + g_2^2}\right) \qquad ; \qquad \bar{q}_{\psi_0} = \frac{1}{2}\left(1 - \frac{2\pi}{2\pi + g_2^2}\right)$$

Conformal weights:

$$\begin{aligned} h_{\psi_0} &= \frac{1}{2} + \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{i g_1 g_3}{2} = (h_{\psi_1^\dagger})^\dagger \\ \bar{h}_{\psi_0} &= \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{i g_1 g_3}{2} = (\bar{h}_{\psi_1^\dagger})^\dagger. \end{aligned} \tag{67}$$

Here some comment about unitarity is in order. It can be shown that with respect to the standard scalar product [28] reflection-positivity holds for any real g_3 [29]. However with respect to this inner product the Virasoro generators are not selfadjoint. Choosing an alternative scalar product [14] for which they are selfadjoint, positivity does not hold in general for $g_3 \neq 0$. We give a more detailed discussion about unitary subspaces in section 3.5.

3.4 Finite size effects

When quantizing a conformal field theory on a space-time with finite volume one introduces a length scale. The presence of this length scale in turn breaks the conformal invariance and gives rise to finite size effects. It has been conjectured [30] that the finite size effects on a Riemann surface are proportional to the central charge. For example, when one stretches space time, $x^\alpha \to a x^\alpha$, then the change of the effective action is proportional to c:

$$\Gamma_{ax} - \Gamma_x = -\frac{c}{6} \log a \cdot \chi, \tag{68}$$

where χ is the Euler number of the Euclidean space time. In [31] this conjecture has been proven for a wide class of conformal field theories on spaces with boundaries. The only important assumption has been that the regularization respects general covariance. In this subsection we shall see that the equivalence does hold only for a particular zero-mode treatment, which differs from (37).

The only global conformal transformations on the torus are translations which do not give rise to finite size effects. Also, the Euler number vanishes and according to (68) the finite size effects are insensitive to the value of c. For that reason we quantize the un-gauged model (4) on the sphere where the global conformal group is the Moebius group.

An effective method to compute finite size effects has been developed in [31]. It is based on the following observation: Any conformal transformation $z \to w(z)$ is a composition of a diffeomorphism (defined by the same w) and a compensating Weyl transformation $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$ with

$$e^{2\sigma} = \frac{dw(z)}{dz} \frac{d\bar{w}(\bar{z})}{d\bar{z}}, \qquad z = x^0 + i x^1.$$

Therefore, choosing a diffeomorphism invariant regularization one has

$$0 = \delta\Gamma_{Diff} = \delta\Gamma_{Conf} - \delta\Gamma_{Weyl}.$$

The change of the effective action under Weyl rescaling is

$$\delta\Gamma_{Weyl} = -\log \frac{\int \mathcal{D}(\lambda\phi) \det(i\not{D}_g) \exp(-S_B[g])}{\int \mathcal{D}(\lambda\phi) \det(i\not{D}_{\hat{g}}) \exp(-S_B[\hat{g}])},$$

where S_B is the bosonic action (36). Since on the sphere there are no harmonic vector fields the term $\sim h^2$ in S_B is not present. Imposing the conditions (37) we obtain

$$\delta\Gamma_{Weyl} = \log\frac{\hat{V}}{V} - \frac{S_L}{24\pi} + \frac{g_3^2}{4}\int \mathcal{R}\frac{1}{\triangle}(\mathcal{R} - \frac{8\pi}{V}) + \log\frac{\det'\triangle}{\det'\hat{\triangle}}. \tag{69}$$

To evaluate (69) one introduces the 1-parametric family of Laplacians

$$\triangle_\tau = e^{-2\tau\sigma}\hat{\triangle}$$

interpolating between $\hat{\triangle}$ and $\triangle$. Integrating the corresponding anomaly equation [20] we end up with

$$\delta\Gamma = \frac{g_3^2}{4}\int \sqrt{g}\mathcal{R}\frac{1}{\triangle}\Big(\mathcal{R} - \frac{8\pi}{V}\Big) - \frac{3}{24\pi}\int \sqrt{\hat{g}}\hat{\mathcal{R}}\sigma + \frac{3}{24\pi}\int \sqrt{\hat{g}}\sigma\hat{\triangle}\sigma. \tag{70}$$

Consider now a dilatation $w(z) = az$. Then, the conformal angle is constant, $\sigma = \log a$, and $(\mathcal{R} - 8\pi/V) = 0$. Then the first term in (70) vanishes and the finite size effect does not depend on g_3^2. It is given by

$$\delta\Gamma = -\frac{3}{24\pi}\log a\int \sqrt{\hat{g}}\hat{\mathcal{R}} = -\log a$$

and does not agree with (68) since c in (66) depends on g_3. On other Riemannian surfaces one would find the same result. Note that the finite size scaling comes from the middle term $\sim \log a \int \sqrt{\hat{g}}\hat{R}$ in (70) which is topological in nature, while the short-distance behaviour of the energy-momentum correlators is controlled by the remaining two terms in (70) which are insensitive to the topology. In that sense finite size scaling and the central charge are complementary. There is a way to match the two results by adding the term

$$-\frac{g_3^2}{4}\int \sqrt{g}\mathcal{R}\triangle\mathcal{R}$$

to the effective action. With this new effective action the short distance expansion of the energy-momentum correlators does not depend on g_3 any more and the corresponding central charge equals that obtained from the finite size scaling. However such a term would correspond to a non-local counter term to be added to the regularized action.

3.5 Charge neutrality and unitary subspaces

In this subsection we show how the equivalence between the central charge and finite size scaling can be restored, provided the partition function is replaced by an average over un-normalized expectation values of charges at infinity. In fact it turns out that the $g_3 \int \mathcal{R}\lambda$-term, ie. the non-minimal coupling to gravity, itself can be given the interpretation of a charge at infinity if the zero-mode constraints (37) is replaced by a non-translation invariant sum over charges at infinity.

The hint comes from inspecting the fermionic weights (67), which shows that $\psi(x)$ and $\psi_{g_3}(x) \equiv e^{-8\pi g_3 \lambda(x)}\psi(x)$ have the same conformal weights. We can therefore consistently put a charge at infinity with a corresponding modification of the charge neutrality condition. The non-vanishing two-point function is now $\langle \psi_{g_3}(x)^\dagger \psi(x)\rangle$. It's coincidence limit j_{g_3} is again a primary field with conformal weight $h_j = 1$.
On the other hand, including a charge at infinity into the definition of the partition function we have

$$\begin{aligned} Z_{g_3} &= \frac{1}{N_0}\int \mathcal{D}_\delta\phi\mathcal{D}_\delta\lambda\ Z_F[\eta = \bar\eta = 0]\ e^{-S_B}\ \ : e^{8\pi g_3 \lambda(\xi_0)} : \\ &= Z_0\ \exp[16\pi^2 g_3^2 G_0(\xi_0,\xi_0)] \qquad (\text{recall that} \quad \mathcal{D}_\delta\phi = \delta(\bar\phi)\mathcal{D}\phi). \end{aligned} \tag{71}$$

To continue we need to determine the coincidence limit of the scalar Greens function $G_0(x, y)$, i.e. to regularize the composite operator $\exp(\alpha\lambda)$ appearing in (71). The normal ordering prescription

$$: e^{\alpha\lambda(x)} := \frac{e^{\alpha\lambda(x)}}{\langle e^{\alpha\lambda(x)}\rangle}. \tag{72}$$

works well on the whole plane [32, 33]. On curved space we must be more careful when renormalizing this operator. The required wave function renormalization is not unique but it is very much restricted by the following requirements: First we take as reference system (the denominator in (72)) one with a minimal number of dynamical degrees of freedom since we do not want to loose information by our regularization. Second, the renormalized operator should have a well-defined infinite volume limit. Finally, the regularization should respect general covariance. These requirements then force us to take as reference system the infinite plane with metric $g_{\mu\nu}$. The flat metric $\delta_{\mu\nu}$ is not permitted since it leads to a ill-defined expression for $\langle\exp(\alpha\lambda)\rangle$. With this choice the normal ordering in (72) is equivalent to replacing the massless Green function in (71) by

$$G_0^{reg}(x,y) := G_0(x,y) + \frac{1}{4\pi}\log[\mu^2 s^2(x,y)]. \tag{73}$$

Here $s(x,y)$ denotes the geodesic distance between x and y. The occurrence of the arbitrary mass scale μ comes from the ambiguities in the required ultra-violet regularization. On the 2-sphere with constant Ricci scalar $\mathcal{R}$ we have

$$G_0^{reg}(x,x) = -\frac{1}{4\pi}[\log[\frac{\mathcal{R}}{8\mu^2}] + 1].$$

The expectation value $\langle : e^{8\pi g_3 \lambda(\xi_0)} : \rangle$ then transforms under a constant rescaling $z \to az$ as

$$\langle : e^{8\pi g_3 \lambda(\xi_0)} : \rangle \to \langle : e^{8\pi g_3 \lambda(\xi_0)} : \rangle \exp[8\pi g_3^2 \log(a)], \tag{74}$$

and therefore gives an extra contribution

$$\delta\Gamma_{g_3} = -\frac{24\pi g_3^2}{6}\log(a)\chi,$$

to the finite size scaling of the effective action. Adding this contribution to (70) above we see that this is precisely the piece needed to restore equivalence with the central charge for any real or imaginary g_3.

More generally we can define the functional integral as an average over all possible charges at infinity: assume g_3 imaginary. The (un-normalized) expectation values are then given by

$$\Big\langle \prod_{i=1}^n O_{\alpha_i}(x_i) \Big\rangle \equiv \frac{1}{Z}\int \mathcal{D}_\delta\phi \mathcal{D}\lambda \Big[\frac{1}{\sqrt{2\pi}}\int d\,k\ : e^{ik\lambda(\xi_0)} : \Big] \prod_{i=1}^n O_{\alpha_i}(x_i)\ e^{-S_B}. \tag{75}$$

Here α_i denotes the $U(1)$-charge of the operator O_i. In particular the partition function on S^2 is

$$Z = \frac{1}{N_0}\int \mathcal{D}_\delta\phi d\lambda_0 \mathcal{D}\lambda' \Big[\frac{1}{\sqrt{2\pi}}\int d\,k\ : e^{ik\lambda(\xi_0)} : \Big]\ : e^{-8\pi g_3 \lambda(\xi_0)} : \ \ e^{-S_B[\lambda']},$$

where λ_0 is the zero mode and λ' the excited modes of $\lambda(x)$. The middle term in the above integrand is the zero-mode part of S_B. The zero-mode integration yields a delta function $\delta(k + i8\pi g_3)$ and thus the $g_3 \int \mathcal{R}\lambda$-term itself acquires the interpretation of a charge at infinity, due to the presence of the zero mode. The 'extra' charges $e^{ik\lambda(\xi_0)}$

assure the charge neutrality of the partition function. For the general n-point function (75) the λ_0- integration yields

$$\delta(k + 8\pi i g_3 + \sum_{i=1}^{n} \alpha_i),$$

where the sum of the $U(1)$-charges of the operators in (75) enters. In particular, for neutral states, for which $(\sum \alpha_i + i8\pi g_3 = 0)$, k must be zero and no extra charge at infinity is present.
Finally, using

$$\frac{1}{\sqrt{2\pi}} \int d\, k\, e^{ik\lambda(\xi_0)} = \delta(\lambda(\xi_0)),$$

the averaging over all possible charges can also be written as

$$\mathcal{D}\lambda\delta(\lambda(\xi_0)). \tag{76}$$

It is easy to verify that if the action has translation invariance in the target space, then the constraints (76) and (37) are equivalent and the correlation functions do not depend on the chosen base-point ξ_0. However, in the present case (76) clearly breaks translation invariance (or rotation invariance on S^2) and the zero-modes constraints are inequivalent. Although we have assumed an imaginary g_3, our results apply for any g_3. For particular values we recover the (unitary) minimal models, provided screening charges [34] are included for the n-point function with $n > 2$. In particular for $g_3 = 1/\sqrt{48\pi}$ and $g_1 = g_2 = 0$ we obtain the Ising model with $h_\psi = h_{\psi^\dagger} = \frac{1}{2}$.

4 Gauged Thirring-like Models

In this section we extend the model by gauging the global $U(1)$-symmetry. Contrary to what one might think, many aspects of the gauged model are actually simpler as compared to the ungauged model. In particular the thermodynamical properties are independent of external conditions like chemical potentials and twisted boundary conditions. The reason is that the model is closely related to the Schwinger model, for which the spectrum consist solely of a neutral, massive particle. On the other hand, the gauge interaction complicates the analysis, because the $U(1)$- bundle over the torus allows for gauge field configurations with winding number, so called instantons. These, in turn, imply fermionic zero-modes which trigger a chiral symmetry breaking and therefore a non-vanishing condensate. This is the subject of the second part of this section. In the first part we discuss the partition function to which only topologically trivial configurations contribute.

To see how the fermionic generating functional (34) is modified, we decompose a general gauge potential on a torus as

$$A_\mu = A^I_\mu + \frac{2\pi}{L} t_\mu + \partial_\mu \alpha - \eta_{\mu\nu}\partial^\nu \varphi, \tag{77}$$

where the last 3 terms correspond (as for the auxiliary field B_μ) to the Hodge decomposition of the single valued part of A in a given topological sector, that is the harmonic-, exact- and co-exact pieces. The role of the toron fields t_μ has recently been emphasized within the canonical approach [35]. In the Hamiltonian formulation they are quantum mechanical degrees of freedom which are needed for an understanding of the infrared sector in gauge theories. Also, in [36] it has been argued that the Z_N-phases of hot pure Yang-Mills theories [37] should correspond to the same physical state if the toron fields are taken into account. The first term in (77) is an *instanton potential* which gives rise to a non-vanishing quantized flux. As noted above configurations with non-vanishing flux do not contribute to the partition function due to the associated fermionic zero modes. We can therefore assume $A^I_\mu = 0$ for the moment. The fermionic generating functional is obtained from (30) by simply shifting

$$g_0 h_\mu \to e t_\mu + g_0 h_\mu = H_\mu \quad , \quad g_1\lambda \to e\alpha + g_1\lambda = F \quad \text{and} \quad g_2\phi \to g_2\phi + e\varphi = G,$$

which leads to

$$\begin{aligned} Z_F[\eta,\bar\eta] = {}& e^{2\pi(\sqrt{\hat g}\hat g^{\mu\nu}\beta_\mu\beta_\nu - 2i\beta_1 a_0)} \frac{1}{|\eta(\tau)|^2} \Theta\Big[\begin{matrix} -a_1+\beta_1 \\ a_0-\beta_0 \end{matrix}\Big](0,\tau)\bar\Theta\Big[\begin{matrix} -\bar a_1-\beta_1 \\ \bar a_0+\beta_0 \end{matrix}\Big](0,\tau) \\ & e^{-\int \bar\eta(x)S(x,y)\eta(y)} \cdot \exp\Big(\frac{1}{24\pi}S_L + \frac{1}{2\pi}\int \sqrt{g} G\triangle G]\Big), \end{aligned} \tag{78}$$

with $a_\mu = \alpha_\mu - H_\mu - \mu_\mu$.

To compute the *partition function* we must switch off the sources η and $\bar\eta$ in (78) so that

$$Z_0 = J\int d^2t d^2h \mathcal{D}\varphi\mathcal{D}\phi\mathcal{D}\lambda \; Z_F[0,0]\; e^{-S_B}, \tag{79}$$

where now

$$
\begin{aligned}
S_B &= (2\pi)^2\sqrt{\hat{g}}\hat{g}^{\mu\nu}h_\mu h_\nu \\
&+ \int\sqrt{g}\Big(\frac{1}{2}\varphi\triangle^2\varphi-\lambda\triangle\lambda-\phi\triangle\phi-g_3R\lambda\Big).
\end{aligned}
\tag{80}
$$

Note that we have kept the non-minimal coupling of the λ-field to gravity as in section 3.3. Since S_B and the fermionic determinants are both gauge invariant and thus independent of the pure gauge mode α in (77), it is natural to change variables from A_μ to (φ,α,t_μ). This transformation is one to one, provided

$$
\int\sqrt{g}\varphi=\int\sqrt{g}\alpha=0 \quad \text{and} \quad et_\mu\in[0,1]. \tag{81}
$$

In contrast to the auxiliary harmonic fields h_μ in (16), the toron fields et_μ and $et_\mu+n_\mu$ with integer n_μ are to be identified, due to gauge invariance [8]. The measures are related as

$$
\mathcal{D}A_\mu=J\sum_k\, dt_0dt_1\mathcal{D}\varphi\mathcal{D}\alpha, \quad \text{where} \quad J=(2\pi)^2\det{}'(-\triangle). \tag{82}
$$

In expectation values of gauge invariant and thus α-independent operators the α-integration cancels against the normalization. This simply expresses the fact that in QED the ghosts decouple in the Lorentz gauge.
As we shall see shortly it is advantageous to integrate first over the toron fields. By using the series representation of the theta functions one computes

$$
\int_0^1 d^2(et)\Theta\Big[{-c_1 \atop c_0}\Big](0,\tau)\;\bar{\Theta}\Big[{-c_1 \atop c_0}\Big](0,\tau)=\frac{1}{\sqrt{2\tau_0}}. \tag{83}
$$

Since the result appears always together with the η-function factor in (34) it is convenient to introduce

$$
\kappa:=\frac{1}{\sqrt{2\tau_0}}\frac{1}{|\eta(\tau)|^2}
$$

in the following expressions. The result (83) does not depend on the h-field and hence the h-integration in (79) becomes Gaussian and yields a factor $1/4\pi$ so that

$$Z_0 = \pi\kappa\,\mathrm{det}'(-\triangle)\; e^{S_L/24\pi} \int \mathcal{D}_\delta(\varphi\phi\lambda)\; e^{\frac{1}{2\pi}\int\sqrt{g}G\triangle G - S_B[h=0]}, \tag{84}$$

where we inserted the explicit expression (82) for the Jacobian. Now we see why we did well integrating over the toron fields first. It has washed out the dependence on the boundary conditions and chemical potential in (83).
The integral over λ, subject to the condition in (37), decouples completely apart from the non-minimal coupling to gravity which modifies the Liouville factor and yields one inverse square-root of the determinant of $-2\triangle$ in (84). Thus

$$\begin{aligned} Z_0 &= \kappa\pi\sqrt{2V\mathrm{det}'(-\triangle)}\; e^{(g_3^2+1/24\pi)S_L} \\ &\quad \cdot \int \mathcal{D}_\delta(\varphi\phi)\; e^{\frac{1}{2\pi}\int\sqrt{g}G\triangle G - S_B[h=\lambda=0]}, \end{aligned} \tag{85}$$

where we have used (43). The *ϕ-integration* in contrast, leads to a finite renormalization of the dynamically generated 'photon' mass

$$\begin{aligned} Z_0 &= \frac{2\sqrt{\pi}\kappa eV}{m_\gamma}\; e^{(g_3^2+1/24\pi)S_L} \int \mathcal{D}\varphi e^{-\frac{1}{2}\int\sqrt{g}\varphi\left(\triangle^2 - m_\gamma^2\triangle\right)\varphi}, \\ \text{where}\quad m_\gamma^2 &= \frac{e^2}{\pi}\frac{2\pi}{2\pi+g_2^2} \end{aligned} \tag{86}$$

plays the same role as the η'-mass in QCD [41]. The determinant obtained from the $\varphi-$ integration factorizes as

$$\mathrm{det}'(\triangle^2 - m_\gamma^2\triangle) = \mathrm{det}'(-\triangle)\cdot\mathrm{det}'(-\triangle + m_\gamma^2).$$

This factorization property is not obvious since all determinants must be regulated. But it holds for commuting operators and in the zeta-function scheme. Then the partition function simplifies to

$$Z_0 = \frac{2\sqrt{\pi}\kappa eV}{m_\gamma}\Big(\mathrm{det}'(-\triangle)\mathrm{det}'(-\triangle + m_\gamma^2)\Big)^{-\frac{1}{2}}\;\exp\Big((g_3^2 + \frac{1}{24\pi})S_L\Big).$$

We can go further by using the scaling formula for the determinant of $\triangle$ [20] and the known result for the determinant of $\hat{\triangle}$ [21] which together yield

$$\mathrm{det}'^{\frac{1}{2}}(-\triangle) = \tau_0 L|\eta(\tau)|^2\sqrt{\frac{V}{\hat{V}}}\exp\Big(-\frac{1}{24\pi}S_L\Big). \tag{87}$$

Thus we obtain the following partition function for the general model (4) on curved spaces:

$$Z_0=\sqrt{2\pi V}\;\frac{e}{m_\gamma}\frac{1}{\tau_0|\eta(\tau)|^4}\;\frac{1}{\det'^{\frac{1}{2}}(-\triangle+m_\gamma^2)}\;\exp\Big((\frac{1}{12\pi}+g_3^2)\,S_L\Big). \tag{88}$$

Again we have factored out the partition function $\mathcal{N}_0$ for free auxiliary fields. The formula (88) shows explicitly that in the topologically trivial sector the theory is equivalent to a theory of free massless and massive bosons with mass m_γ, even in curved space-time [13].
The appearance of m_γ in (86) should be interpreted as *renormalization of the electric charge* induced by the interaction of the auxiliary fields with the fermions. After summing over all fermion-loops this leads to an effective coupling between the photons and the ϕ-field and in turn to a modified effective mass for the photons in (86). In the limit $g_2\to 0$ this mass tends to the well-known Schwinger model result, $m_\gamma\to e/\sqrt{\pi}$ [38].

We have already mentioned that the chemical potential coupled to the electric charge has completely disappeared from the partition function. This does not come as a surprise since the only particle in the gauged Thirring model is a neutral boson. This has no charge which may couple to the chemical potential. Also, if the partition function depended on μ then the expectation value of the charge would not vanish, in contradiction to the integrated Gauss law. The result (88) provides therefore another test for our result (30) for the fermionic determinants with chemical potential.
The final result is also independent of the chiral and non-chiral twists. The normal twists have been wiped out by the toron integration. In fact the chiral twists are equivalent to a chemical potential and therefore the above remarks concerning the chemical potential apply here as well. Did we assume holomorphic factorization for the fermionic determinant [5] then the partition function would depend on the chiral twists.
We conclude this subsection by giving the explicit formula for the partition function on the flat torus. The zeta-function regularized massive determinant is expressed by

$$\det'(-\hat{\triangle}+m_\gamma^2)^{\frac{1}{2}}=\frac{1}{m_\gamma}e^{-\frac{1}{2}\zeta'(0)},$$

where

$$\zeta'(0) = \sum_{n\neq 0} \frac{1}{\pi L} \frac{\hat{V} m_\gamma}{\sqrt{(n,n)}} K_1(m_\gamma L\sqrt{(n,n)}) - \frac{\hat{V} m_\gamma^2}{4\pi}, \tag{89}$$

and $(n,n) = \hat{g}_{ij} n^i n^j$ is the inner product taken with the reference metric, and the sum is over all $(n^i) \in Z_2$ with the origin excluded. For $g_{\mu\nu} = \delta_{\mu\nu}$, in which case the partition function has the usual thermodynamical interpretation, the result reduces to one derived previously by Ambjorn and Wolfram [39]. In addition, if L approaches infinity we recover a result in [19]. The free energy for $\tau_1 = 0$ and on flat space simplifies then to

$$F = -\frac{1}{\beta} \log Z = \frac{1}{2\beta}\, \zeta'(0).$$

with $\zeta'(0)$ from (89) and the particular choice for the parameters.

4.1 Bosonization of the gauged Thirring model

We pointed out in section 2 that for $g_1 = g_2 = g$ the classical theory (4) reduces to the gauged Thirring model. The same is true for the quantized theory on the torus if in addition we set $g_0 = g$. More precisely, the Hubbard-Stratonovich transform [40] of the Thirring model is just the derivative coupling model (4) with identical couplings. In the process of showing that we shall arrive at the Bosonization formulae for the gauged Thirring model on the curved torus. We shall see that only the non-harmonic part of the fermion current can naively be bosonized and that for this part the rules of the un-gauged model on flat space time [32] need just be covariantized.

For that we calculate the partition function (79) in a different order. First we integrate over the auxiliary fields. To understand the role of λ and ϕ we introduce sources for them. Thus we study the generating functional for the correlators of the auxiliary fields

$$Z[\xi,\zeta] = \int \mathcal{D}(\lambda\phi h\psi A_\mu) e^{-S+\int\sqrt{g}[\xi\lambda+\zeta\phi]}.$$

Here

$$S = -i\int \sqrt{g}\psi^\dagger \not{D}\psi + S_B[g_3 = 0]$$

is the action of the full theory. $\not{D}$ is the Dirac operator in (17) with all couplings set equal and S_B the bosonic action (80). Since λ and ϕ integrate to zero (see (37)) we may assume the same property to hold for the sources. The integration over the auxiliary fields is Gaussian and yields

$$Z = \mathcal{N}_0 \int \mathcal{D}(\psi A_\mu)\, e^{-S_T}\, \exp \int \sqrt{g} \Big[\frac{-1}{4} (\xi \frac{1}{\triangle} \xi + \zeta \frac{1}{\triangle} \zeta) + \frac{g}{2} (\xi \frac{1}{\triangle} j^\mu_{;\mu} + \zeta \frac{1}{\triangle} j^\mu_{5;\mu}) \Big], \quad (90)$$

where

$$S_T = -\frac{1}{4} \int \sqrt{g} \Big(F_{\mu\nu} F^{\mu\nu} - i \psi^\dagger \not{D} \psi - \frac{g^2}{4} j^\mu j_\mu \Big) \quad (91)$$

is the action of the *gauged Thirring model* on curved space-time and

$$\mathcal{N}_0 = \frac{V}{2\pi \det'(-\triangle)} \quad (92)$$

comes from the integration over the auxiliary fields.
Let us first consider the partition function, that is set the sources to zero. Comparing (90) with (86) and using (87) we easily find

$$\int \mathcal{D}(\psi t) e^{-S_T} = \sqrt{\frac{1}{2} + \frac{g^2}{4\pi}}\; e^{-\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu}} \int \mathcal{D}\gamma\, \delta(\bar{\gamma})\, e^{-S_\gamma}, \quad (93)$$

where $\bar{\gamma}$ is the mean field (see (37)) and we used (82) and (43). The action for the neutral scalar field γ is found to be

$$S_\gamma = \frac{1}{2} \int \sqrt{g} \partial_\mu \gamma \partial^\mu \gamma - \frac{ie}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \int \sqrt{g} \gamma \triangle \varphi.$$

Since (93) holds for any φ (and thus for the non-harmonic part of any A_μ, because of gauge-invariance) we read off the following *bosonization rules*:

$$\begin{aligned} j'^\mu &\longrightarrow \frac{i}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \eta^{\mu\nu} \partial_\nu \gamma \\ j'^\mu_5 &\longrightarrow -\frac{i}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \partial^\mu \gamma, \end{aligned} \quad (94)$$

where prime denotes the non-harmonic part of the currents. Thus, only the non-harmonic parts of the currents can be bosonized in terms of a single valued scalar field. To bosonize their harmonic parts one would have to allow for a scalar field γ with winding numbers. On the infinite plane the harmonic part is not present and we may leave out the primes in (94). If we further assume space time to be flat we recover the well-known bosonization rules in [32]. What we have shown then, is that for the gauged model on curved space time the bosonization rules are just the flat ones properly covariantized and with the omission of the zero-modes.

Since (93) holds for any gauge field the current correlators in the Thirring model are correctly reproduced by the bosonization rules (94). To see that more clearly we calculate the two-point functions of the auxiliary fields in the Thirring model (90-92). For that we differentiate (90) (φ is treated as external field) with respect to the sources and find

$$\begin{aligned}\langle\lambda(x)\lambda(y)\rangle &= \frac{1}{2}G_0(x,y)+\frac{g^2}{4}\int\langle G_0(x,u)j^\mu_{;\mu}(u)G_0(y,v)j^\nu_{;\nu}(v)\rangle_T \\ \langle\phi(x)\phi(y)\rangle &= \frac{1}{2}G_0(x,y)+\frac{g^2}{4}\int\langle G_0(x,u)j^\mu_{5;\mu}(u)G_0(y,v)j^\nu_{5;\nu}(v)\rangle_T,\end{aligned} \tag{95}$$

where G_0 is the free massless Green-function in curved space-time and the integrations are over the variables u and v with the invariant measure on the curved torus. Here $\langle\ldots\rangle_T$ are vacuum expectation values in the Thirring model (91). Alternatively we can calculate these expectation values from (84) and (85), where the fermionic integration has been performed and find

$$\begin{aligned}\langle\lambda(x)\lambda(y)\rangle &= \frac{1}{2}G_0(x,y) \\ \langle\phi(x)\phi(y)\rangle &= \frac{\pi m_\gamma^2}{2e^2}\,G_0(x,y)+\frac{m_\gamma^2}{2}\Big(1-\frac{\pi m_\gamma^2}{e^2}\Big)\varphi(x)\varphi(y).\end{aligned}$$

Comparing this with the result (95) we see at once that

$$\begin{aligned}\int\langle G_0(x,u)j^\mu_{;\mu}(u)G_0(y,v)j^\nu_{;\nu}(v)\rangle_T &= 0 \\ \int\langle G_0(x,u)j^\mu_{5;\mu}(u)G_0(y,v)j^\nu_{5;\nu}(v)\rangle_T &= \frac{m_\gamma^2}{e^2}\Big(m_\gamma^2\varphi(x)\varphi(y)-G_0(x,y)\Big).\end{aligned} \tag{96}$$

These correlators express the gauge invariance and the axial anomaly $\langle j^\mu_{5;\mu}\rangle=-m_\gamma\triangle\varphi$ in the gauged Thirring model. They can be correctly reproduced with the bosonization rules (94). They are not reproduced with the ones derived for the un-gauged model [32].

4.2 Chiral condensate

The chiral condensate is an order parameter for the chiral symmetry breaking. However, on the torus its expectation value, whose temperature- and curvature dependence we will here compute would vanish if topologically non-trivial gauge field configurations were absent. There is a useful classification of the gauge configurations corresponding to the number of fermionic zero modes they give rise to. If we let $k = n_+ - n_-$, where $n_\pm$ counts the number of zero-modes with positive/negative chirality, then we have

$$k = \frac{1}{2\pi}\int d^2x\,\gamma_5 a_1(\not{D}^2, x) = \frac{1}{4\pi}\int \sqrt{g}d^2x\,\eta^{\mu\nu}F_{\mu\nu} \equiv \frac{1}{2\pi}\Phi, \tag{97}$$

which establishes a relation between the number of fermionic zero modes (or, more precisely the number of zero modes with positive chirality minus the number of those with negative chirality) and the first Chern character of the bundle. Also from (97) one immediately concludes that the flux must be quantized in integer multiples of 2π. This is really a consequence of the single valuedness of the fermionic wave function (cocycle condition).

Recalling the decomposition (77) of the gauge field we now concentrate on the first term A^I_μ which is the *instanton potential* giving rise to a non-vanishing quantized flux Φ. Since 2-dimensional gauge theories are not scale or Weyl invariant, as 4-dimensional ones are, the instantons on a conformally flat space-time are not identical to the flat ones [42, 43]. As representative in the k-instanton sector we choose the, up to gauge transformations, *unique absolute minimum* of the Maxwell action in a given topological sector. It has field strength $e\,E^I = \sqrt{g}\,\Phi/V$. The corresponding potential can be chosen as

$$eA^I_\mu = e\hat{A}^I_\mu - \Phi\,\eta_\mu{}^\nu\partial_\nu\chi, \quad \text{where} \quad e\hat{A}^I = -\frac{\sqrt{\hat{g}}}{\hat{V}}\Phi(x^1, 0) \tag{98}$$

is the instanton potential on the flat torus with the same flux but field strength $\sqrt{\hat{g}}\,\Phi/\hat{V}$. The function χ is then determined (up to a constant) by

$$\sqrt{g}\frac{\Phi}{V} - \sqrt{\hat{g}}\frac{\Phi}{\hat{V}} = \sqrt{g}\triangle\chi. \tag{99}$$

The solution of this equation is given by

$$\chi(x) = -\frac{1}{\hat{V}}(\frac{1}{\triangle}e^{-2\sigma})(x) = \frac{1}{\hat{V}}\int d^2y\sqrt{g(y)}G_0(x,y)\,e^{-2\sigma(y)}, \tag{100}$$

where

$$G_0(x,y) = \langle x|\frac{1}{-\triangle}|y\rangle = \sum_{\lambda_n>0}\frac{\phi_n(x)\phi^\dagger(y)}{\lambda_n} \tag{101}$$

is the Green-function for $-\triangle$. In deriving (100) we have used that $\frac{1}{\triangle}(\Phi/V)=0$ which follows from the spectral resolution (101) for the Green function in which the constant zero mode $\phi_0=1/\sqrt{V}$ of $\triangle$ is missing.

Our choice for the instanton potential (98) corresponds to a particular trivializations of the $U(1)$-bundle over the torus [8]. In other words, the gauge potentials and fermion fields at (x^0,x^1) and (x^0,x^1+L) are necessarily related by a *nontrivial gauge transformation* with winding numbers

$$\begin{aligned} &A_\mu(x^0,x^1+L) - A_\mu(x^0,x^1) = \partial_\mu\alpha(x) \\ &\psi(x^0,x^1+L) = -e^{ie\alpha(x)}\,e^{2\pi i(\alpha_1+\beta_1\gamma_5)}\,\psi(x^0,x^1). \end{aligned} \tag{102}$$

For the choice (98) we find

$$e\alpha(x) = -\frac{\Phi}{L}\,x^0.$$

Note that A is still periodic in x^0 with period L and ψ still obeys the first boundary condition in (14). To calculate the fermionic *zero modes* we use the square of the Dirac operator

$$\not{D}^2 = \begin{pmatrix} D_-D_+ & 0 \\ 0 & D_+D_- \end{pmatrix} = \frac{1}{\sqrt{g}}D_\mu\sqrt{g}g^{\mu\nu}D_\nu - \frac{1}{4}\mathcal{R} + \frac{e}{2}\eta^{\mu\nu}F_{\mu\nu}\gamma_5 \tag{103}$$

In a pure instanton and harmonic background ($\varphi=\alpha=0$) on the flat torus (103) simplifies to

$$-\hat{\not{D}}^2 = -\hat{g}^{\mu\nu}\hat{D}_\mu\hat{D}_\nu - \frac{\Phi}{\hat{V}}\gamma_5. \tag{104}$$

In other words, $\hat{\not{D}}^2$ is the same in the left- and right-handed sectors, up to the constant

$2\Phi/\hat{V}$. Furthermore this operator commutes with the time translations which leads to the following ansatz for the zero-modes

$$\tilde{\chi}_p = e^{2\pi i c_p x^0/L}\, e^{2\pi i H_1 x^1/L}\, \xi_p(x^1)\, e_+, \quad c_p = \frac{1}{2} + p,$$

where we have assumed $k > 0$. The choice of c_p is dictated by the time-like boundary conditions in (14). Inserting this ansatz into the zero mode equation $\hat{\not{D}}^2 \tilde{\chi}_p = 0$ yields

$$\begin{aligned}(|\tau|^2 \frac{d^2}{dy^2} - \frac{\Phi^2}{L^4} y^2 - 2i\tau_1 \frac{\Phi}{L^2} y \frac{d}{dy} - i\tau \frac{\Phi}{L^2})\xi_p = 0,\\ \text{where} \quad y = x^1 + \frac{L}{k}(c_p - H_0).\end{aligned}$$

This is just the differential equation for the ground state of a generalized harmonic oscillator to which it reduces for $\tau = i\tau_0$. The solution is given by

$$\xi_p = \exp\Big[-\frac{\Phi}{2i\bar{\tau}L^2}\{x^1 + \frac{L}{k}(c_p - H_0)\}^2\Big].$$

These functions do not obey the boundary condition (102), but the correct eigenmodes can be constructed as superpositions of them. For that we observe that

$$\tilde{\chi}_p(x^0, x^1 + L) = e^{-i\Phi x^0/\beta}\, e^{2i\pi H_1}\, \tilde{\chi}_{p+k}(x^0, x^1)$$

so that the sums

$$\hat{\psi}_0^p = \frac{(2k\tau_0)^{\frac{1}{4}}}{\sqrt{|\tau|\hat{V}}} e^{\frac{\pi\mu_0^2}{k\tau_0}}\, e^{2\pi i(H_0 - \alpha_0 - \frac{1}{2})\beta_1} \sum_{n\in Z} e^{-2i\pi(n+p/k)(\frac{1}{2} - H_1)} \tilde{\chi}_{p+nk}\, e_+, \tag{105}$$

where $p = 1, \ldots, k$, obey the boundary conditions and thus are the k required zero-modes. Indeed, since $(i\hat{\not{D}})^2$ in non-negative there are no zero modes with negative chirality because of (104). With (97) we conclude then that there are exactly k zero modes with positive chirality. Modes with different p in (105) are orthogonal to each other and the overall factor normalized them to one, so that the system (105) forms an orthonormal basis of the zero-mode subspace. For $k < 0$ the zero-modes are the same if one replaces e_+ by e_-.

To compute the fermionic determinant in a given topological sector we again introduce the one-parameter family of Dirac operators

$$\not{D}_\tau = \frac{\hat{g}^{1/2}}{g_\tau^{1/2}} e^{\tau f^\dagger}\, \hat{\not{D}}\, e^{\tau f}, \qquad \hat{\not{D}} = \hat{\gamma}^\mu \Big(\partial_\mu + i\hat{\omega}_\mu - ie\hat{A}^I_\mu - \frac{2\pi i}{L}[H_\mu + \mu_\mu]\Big), \tag{106}$$

which interpolates between $\hat{\not{D}}$ and $\not{D}$, similarly as in (31). But now

$$f = -iF + \gamma_5(G + \Phi\chi) + \frac{1}{2}\sigma,$$

with F and G from (7), contains an instanton contribution. Also note that $\hat{\not{D}}$ contains the instanton part $\hat{A}^I_\mu$. To compute it's determinant we observe that the simple form (104) of $-\hat{\not{D}}^2$ allows one to reconstruct its spectrum completely [20, 8]:

$$\hat{\lambda}^2_n = \begin{cases} 0 & \text{degeneracy} = k \\ 2n\Phi/\hat{V} & \text{degeneracy} = 2k. \end{cases}$$

The corresponding determinant is [20, 8]

$$\det{}'(i\hat{\not{D}}) = \Big(\frac{\pi\hat{V}}{\Phi}\Big)^{\Phi/4\pi}. \tag{107}$$

To relate the determinants of $\hat{\not{D}}$ to that of $\not{D}$ we again integrate the anomaly equation, which now reads

$$\frac{d\log\det{}'\not{D}_\tau}{d\tau} = \int d^2x\sqrt{g_\tau}\Big(f(x)+f^\dagger(x) - \frac{d\log g_\tau}{2d\tau}\Big)\{\frac{a_1(x,\not{D}^2_\tau)}{4\pi} - P_0(x,\not{D}^2_\tau)\}, \tag{108}$$

where, due to the fermionic zero-modes, the projector onto the zero-mode-subspace,

$$P_0(x,\not{D}^2_\tau) = \sum_{pr} \psi^{(\tau)}_{p0}(x)\mathcal{N}^{-1}_{pr}(\tau)\,(\chi^{(\tau)}_r)^\dagger(x) \quad , \quad \mathcal{N}_{pr}(\tau) = (\chi^{(\tau)}_{p0}, \psi^{(\tau)}_{r0}) \tag{109}$$

appeared. For the deformed operator $\not{D}^2_\tau$ the first Seeley-deWitt coefficient is

$$a^\tau_1 = -\frac{1}{12}R^\tau + \gamma_5\tau\triangle^\tau G + \frac{1}{\sqrt{g^\tau}}\Big[(1-\tau)\sqrt{\hat{g}}\,\frac{\Phi}{\hat{V}} + \tau\sqrt{g}\,\frac{\Phi}{V}\Big]\gamma_5. \tag{110}$$

Integrating w.r.t. τ [20] one ends up with the following formula for the determinant in arbitrary background gravitational and gauge fields:

$$\det{}' i\not{D} = \det \frac{\mathcal{N}_\psi}{\hat{\mathcal{N}}_\psi} \det{}'(i\hat{\not{D}}) \exp\Big(\frac{S_L}{24\pi} + \frac{1}{2\pi}\int \sqrt{\hat{g}} G \hat{\triangle} G\Big) \\ \cdot \exp\Big(\frac{2k}{V}\int \sqrt{g} G + \frac{\Phi^2}{2\pi \hat{V}}\int \sqrt{\hat{g}}\chi\Big). \tag{111}$$

In deriving this result we used that $\int \sqrt{g}\chi = 0$.

Now we are ready to compute the chiral condensate $\langle \psi^\dagger P_+ \psi\rangle$. Observing that the fermionic Green's function anti-commutes with γ_5 one sees at once that only configuration supporting one fermionic zero-mode with positive chirality contribute to the chiral condensate

$$\langle \psi^\dagger P_+ \psi\rangle = -\frac{J}{Z_0}\frac{\delta^2}{\delta\eta_+(x)\delta\bar{\eta}_+(x)}\int \mathcal{D}(\ldots) Z_F[0,0]\, e^{-S_B},$$

where $\eta_+ = P_+\eta$. Earlier we have seen that these are the gauge fields with flux $\Phi = 2\pi$ or instanton number $k=1$. Thus the condensate becomes

$$\langle \psi^\dagger P_+ \psi\rangle = -\frac{J}{Z_0}\sqrt{\frac{\hat{V}}{2}}\int \mathcal{D}(.)\, \psi_0^\dagger(x)\psi_0(x) \exp(.)\, e^{-S_B[k=1]}, \tag{112}$$

where $\exp(\ldots)$ stands for the exponentials in (111). First we integrate over the toron field t. The t-dependence enters only through the zero mode and more specifically $\hat{\psi}_0$ in (105) with $p=1$. Using the series representation for the theta functions one finds

$$\int d^2t\, \hat{\psi}_0^\dagger(x)\hat{\psi}_0(x) = \frac{1}{\hat{V}}. \tag{113}$$

Note that the result does not depend on the chemical potential similarly as in our calculation of the partition function. To continue we observe that the term $\int \sqrt{g} G$ in $\exp(\ldots)$ vanishes because of our conditions (81) and (37) on the fields φ and ϕ. Furthermore $S_B[k=1] = S_B[k=0] + \frac{2\pi^2}{e^2 V}$. The remaining functional integrals are performed similarly as those leading to the partition function and we end up with the following formula for the condensate

$$\langle \psi^\dagger P_+ \psi\rangle = \sqrt{\frac{\tau_0}{\hat{V}}}\,|\eta(\tau)|^2 e^{-2\pi^2/e^2 V + 2\pi/\hat{V}\int \sqrt{\hat{g}}\chi}\Big\langle e^{-2(g\phi + e\varphi)(x) - \sigma(x)}\Big\rangle_{\phi\varphi}. \tag{114}$$

The expectation value is evaluated with

$$S_{eff} = \int \sqrt{g}\Big[\frac{1}{2}\varphi(\triangle^2 - \frac{e^2}{\pi}\triangle)\varphi - \frac{e^2}{\pi m_\gamma^2}\phi\triangle\phi - \frac{eg_2}{\pi}\phi\triangle\varphi\Big].$$

A formal calculation of the resulting Gaussian integrals yields

$$\begin{aligned} \langle\psi^\dagger P_+\psi\rangle &= \sqrt{\frac{\tau_0}{\hat V}}|\eta(\tau)|^2 e^{-2\pi^2/e^2 V + 2\pi/\hat V \int \sqrt{\hat g}\chi}\, e^{-\sigma(x) - 2\Phi\chi(x)} \\ &\cdot \quad \exp\big[\frac{2\pi^2 m_\gamma^4}{e^2}\,K(x,x)\big]\exp\big[\frac{2\pi g_2^2}{2\pi + g_2^2}\,G_0(x,x)\big], \end{aligned} \tag{115}$$

where

$$K(x,y) = \langle x|\frac{1}{\triangle^2 - m_\gamma^2\triangle}|y\rangle = \frac{1}{m_\gamma^2}(G_0(x,y) - G_{m_\gamma}(x,y)) \tag{116}$$

and G_m, G_0 are the massive and massless Green-functions. Here we encounter ultraviolet divergences since $G_0(x,y)$ is logarithmically divergent when x tends to y. To extract a finite answer we need to renormalize the operator $\exp(\alpha\phi)$ as explained in section 3.5. This wave function renormalization is equivalent to the renormalization of the fermion field in the Thirring model and thus is very much expected [32, 33]. The flat Green's function on the torus

$$\hat G_0(x,y) = -\frac{1}{4\pi}|\frac{1}{\eta(\tau)}\Theta\Big[\begin{matrix}\frac{1}{2}+\frac{\xi^0}{L}\\ \frac{1}{2}+\frac{\xi^1}{L}\end{matrix}\Big](0,\tau)|^2 = -\frac{1}{4\pi}|\frac{1}{\eta(\tau)}e^{i\pi\tau(\xi^0/L)^2}\Theta_1(\frac{\tau\xi^0+\xi^1}{L},\tau)|^2,$$

where $\xi = x - y$, possesses the logarithmic short distance singularity

$$\hat G_0(x,y) = -\frac{1}{4\pi}\log\frac{\hat g_{\mu\nu}\xi^\mu\xi^\nu}{\hat V} - \frac{1}{4\pi}\log(4\pi^2\tau_0|\eta(\tau)|^4) + O(\xi). \tag{117}$$

Furthermore

$$G_0(x,y) \sim \hat G_0(x,y) + 2\chi(x) - \frac{1}{\hat V}\int\sqrt{\hat g}\chi + O(\xi).$$

With the prescription explained in section 3.5 we find that on the flat torus $\hat G_0^{reg}$ has now the finite coincidence limit

$$\hat{G}_0^{reg}(x,x) = -\frac{1}{4\pi}\log\Big(\frac{4\pi^2\tau_0|\eta(\tau)|^4}{\mu^2\hat{V}}\Big). \tag{118}$$

To determine the chiral condensate we also need to determine $K(x,y)$ on the diagonal. In a first step we shall obtain it for the flat torus. Its σ-dependence is then determined in a second step. For $\sigma=0$ and $\tau=i\tau_0$ the Green function $\hat{K}$ has been computed in [8]. The generalization to arbitrary τ is found to be

$$\begin{aligned} m_\gamma^2\hat{K}(x,x) &= -\frac{1}{2m_\gamma L\tau_0}\coth\big(\frac{\pi\tau_0 a}{|\tau|^2}\big) + \frac{1}{m_\gamma^2\hat{V}} \\ &+ \frac{1}{2\pi}\Big(-\log|\eta(\frac{-1}{\tau})|^2 + F(L,\tau) - H(L,\tau)\Big), \end{aligned} \tag{119}$$

where we introduced the dimensionless constant $a = Lm_\gamma|\tau|/2\pi$ and the functions

$$\begin{aligned} F(L,\tau) &= \sum_{n>0}\Big[\frac{1}{n} - \frac{1}{\sqrt{n^2+a^2}}\Big] \\ H(L,\tau) &= \sum_{n>0}\frac{1}{\sqrt{n^2+a^2}}\Big[\frac{1}{e^{-2\pi i z_+(n)}-1} + \frac{1}{e^{2\pi i z_-(n)}-1}\Big]. \end{aligned} \tag{120}$$

We used the abbreviations

$$z_\pm = \frac{1}{|\tau|^2}(n\tau_1 \pm i\tau_0\sqrt{n^2+a^2}). \tag{121}$$

Substituting (119) and (118) into (115) with $\sigma=0$ we obtain the following *exact formula for the chiral condensate* on the torus with flat metric $\hat{g}_{\mu\nu}$:

$$\begin{aligned} \langle\psi^\dagger P_+\psi\rangle_{\hat{g}} &= \frac{1}{L|\tau|}\Big(\frac{m_\gamma L|\tau|}{2\pi}\Big)^{\frac{g_2^2}{2\pi+g_2^2}}\exp\Big(\frac{\pi^2 m_\gamma}{e^2 L\tau_0}\coth\frac{Lm_\gamma\tau_0}{2|\tau|}\Big) \\ &\cdot\ \exp\Big[\frac{\pi m_\gamma^2}{e^2}\Big(F(L,\tau)-H(L,\tau)\Big)\Big], \end{aligned} \tag{122}$$

where we used that on the flat torus $\chi=0$ and $V=\hat{V}$. Furthermore, we identified μ with the natural mass scale m_γ of the theory.

To extract the finite temperature behaviour of the chiral condensate we take $\tau = i\beta/L$ where $\beta=1/T$ is the inverse temperature. In the thermodynamic limit $L\to\infty$.

Then $\coth(\ldots)\to 1,\ \ H\to 0$ and the expression for the chiral condensate simplifies to

$$\langle\psi^\dagger P_+\psi\rangle_\beta = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}}\exp\Big[-\frac{\pi^2 m_\gamma}{e^2}T+\frac{2\pi}{2\pi+g_2^2}F\Big]. \tag{123}$$

Using

$$F(\beta)\to\gamma+\log\frac{a}{2}+\frac{1}{2a}\qquad\text{for}\quad a\to\infty,$$

where $\gamma=0.57721\ldots$ is the Euler number, we obtain the *zero temperature limit*

$$\langle\psi^\dagger P_+\psi\rangle = -\frac{m_\gamma}{4\pi}\,2^{g_2^2/(2\pi+g_2^2)}\,\exp\Big(\frac{2\pi}{2\pi+g_2^2}\gamma\Big)\qquad\text{for}\quad T\to 0. \tag{124}$$

For temperatures large compared to the induced photon mass F vanishes. Thus we obtain the *high temperature behaviour*

$$\langle\psi^\dagger P_+\psi\rangle_T = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}}\exp\Big(-\frac{\pi^2 m_\gamma}{e^2}T\Big)\quad\text{for}\quad T\to\infty \tag{125}$$

It is instructive to discuss the various limiting cases. For all $g_i=0$, i.e. the Schwinger model limit, the exact result (123) simplifies to

$$\langle\psi^\dagger P_+\psi\rangle_T = -T\,e^{-\frac{\pi}{m_\gamma}T+F(\beta)}\longrightarrow\begin{cases}-\frac{m_\gamma}{4\pi}e^\gamma & T\to 0\\ -T\,e^{-\pi T/m_\gamma} & T\to\infty,\end{cases} \tag{126}$$

where now $m_\gamma^2=e^2/\pi$ is the induced photon mass in the Schwinger model. This formula for the temperature dependence of the chiral condensate in QED_2 agrees with the earlier results in [8].

Next we wish to investigate how the self interaction of the fermions affect the breaking. For large coupling g_2 and fixed temperature the exponent in (123) vanishes so that

$$\langle\psi^\dagger P_+\psi\rangle_T\sim\frac{1}{\sqrt{2\pi+g_2^2}}\qquad\text{for}\quad T\quad\text{fixed},\quad g_2\to\infty.$$

Hence, for very large current-current coupling the chiral condensate vanishes. Or in other words, the electromagnetic interaction which is responsible for the chiral condensate, is shielded by the pseudo scalar-fermion interaction.

For intermediate temperature and coupling g_2 we must retreat to numerical evaluations of the sums defining the chiral condensate in (123). The numerical results are depicted in Fig. 1

How does the gravitational field affect the chiral condensate? To answer this question we need to know the massive Green's function, entering in (116), for non-trivial gravitational fields (for simplicity we assume $T=0$). Let us first consider a space with constant negative curvature. Then G_{m_γ} has been computed explicitly in [44]. Here we only need its short distance expansion, given by

$$G_{m_\gamma}(x,y) = -\frac{1}{4\pi}\{2\gamma + \log\big(\frac{-s^2\mathcal{R}}{8}\big) + \psi(\frac{1}{2}+\alpha) + \psi(\frac{1}{2}-\alpha) + O(s^2)\}, \tag{127}$$

where $\alpha^2 = \frac{1}{4} + \frac{2m_\gamma^2}{\mathcal{R}}$ and $\psi(z)$ is the Digamma function. Substituting (127) into (116) we end up with the exact formula for the *chiral condensate for constant curvature*

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[\frac{\pi}{2e^2}m_\gamma^2\{\log\big(\frac{-\mathcal{R}}{2m_\gamma^2}\big) + \psi(\frac{1}{2}+\alpha) + \psi(\frac{1}{2}-\alpha)\}\Big]. \tag{128}$$

The asymptotic expansions for *large-and small curvature* are easily worked out inserting the corresponding expansions for the Digamma function [45]. We find

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[\frac{\pi}{12e^2}\mathcal{R}\Big] \quad \text{for} \quad \frac{|\mathcal{R}|}{e^2} \ll 1 \tag{129}$$

and

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\big(\frac{\mathcal{R}}{2m_\gamma^2}\big)^{\frac{\pi}{2\pi+g_2^2}}\exp\Big[\frac{\pi}{4e^2}\mathcal{R} - \frac{\pi m_\gamma^2}{4e^2}\gamma\Big] \quad \text{for} \quad \frac{|\mathcal{R}|}{e^2} \gg 1. \tag{130}$$

Hence the chiral condensate decays exponentially for large curvature analogous to the high temperature behaviour. However, the pseudo-scalars do not suppress the effect of the curvature in contrast to (125). Comparing the exponentials in (130) to (125) we may define the curvature induced effective temperature as

$$T_{eff} = \frac{-\mathcal{R}}{4\pi m_\gamma}. \tag{131}$$

In passing we note that if we compare the prefactors, rather than the exponentials, we would write

$$T_{eff} = \frac{(-\mathcal{R})^{\frac{1}{2}}}{4\pi\sqrt{2}}. \tag{132}$$

The latter identification actually coincides (up to factor of 2) with the Hawking temperature of free scalars in de Sitter space [6]. The correct identification involves the (dynamical) mass of the gauge field and is therefore not universal. From this observation we learn that the temperature associated with curvature depends on the matter content. Note finally that the non-minimal coupling (g_3) has no effect on the chiral condensate. In Fig. 2 we have plotted the chiral condensate for arbitrary constant values of the curvature.

For gravitational backgrounds with non-constant curvature we have to refer to perturbative methods for the calculation of the massive Green's function. Again we only need the short distance expansion of G_{m_γ}. For geodesic distances s small compared to m_γ^{-1} the massive Green's function may be approximated by the Seeley-DeWitt expansion [46]

$$G_m(x,y) \sim \frac{1}{4i}\sum_{j=0}^{\infty} a_j(x,y)(-\frac{\partial}{\partial m^2})^j\, H_0^{(2)}(ms), \tag{133}$$

where $H_0^{(2)}$ is the Hankel function of the second kind and order zero. In particular

$$H_0^{(2)}(z) \to \frac{2}{i\pi}[\log\frac{z}{2}+\gamma] \quad \text{for} \quad z\to 0.$$

Inserting (133) into (116) we end up with the following expansion for the *chiral condensate in an arbitrary background*

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[-\frac{\pi}{2}(\frac{m_\gamma}{e})^2\sum_{1}^{\infty} a_j(x)\frac{(j-1)!}{m^{2j}}\Big], \tag{134}$$

where we have used that $a_0(x)=1$. The first order contribution involves $a_1(x)=-\frac{1}{6}\mathcal{R}$ and reproduces the asymptotic behaviour (129). Higher order contributions must be taken into account to uncover the effect of variable curvature. For this one has to substitute is the corresponding Seeley DeWitt coefficients a_j into (134). These have been computed up to $j=5$ [47].

5 Conclusions

In this paper we have elaborated on various features of the Thirring model as well as some of its extensions. In particular we found the dependence of the partition function on the chemical potential and the non-trivial boundary conditions for the fermions

on the torus. For that a careful analysis of fermionic determinants has been crucial. We have found that the familiar chiral anomaly of the UV-regularized two point function is also seen in the IR-sector as a breakdown of holomorphic factorization. This fact, which has not been properly taken into account previously, together with the presence of harmonic contributions to the current, leads to a modification of the equation of state due to the current-current interaction. We believe that our results could also be obtained in the bosonized theory, provided the usual bosonization rules are modified to include scalar fields with winding numbers, i.e. scalar fields with values in a compactified target space.
Furthermore, we have deformed the conformal structure by allowing for different couplings in the transversal- and the longitudinal parts of the current-current interaction. This does not change the Virasoro- and Kac-Moody algebra, but modifies the conformal weights of the primaries and in particular of the fermionic fields. Not all values of the coupling constants belong to physical theories, since positivity of the scalar product imposes certain restrictions on them. Our approach allows also for a non-minimal coupling of the longitudinal sector to gravity. While such a coupling may seem to be ad-hoc we gave some arguments that it might arise naturally when quantizing fermions in presence of a a background charge. We find that the central charge of the Virasoro algebra is sensitive to the non-minimal coupling. In particular $c < 1$ occurs for certain values of the coupling constant. However, we have not been able to derive constraints on this extra coupling without referring to the result by Friedan, Qiu and Shenker. We believe that an independent derivation of their result within a fermionic model would be most welcome. We have also established that the central charge controls the finite size effects only for a particular treatment of the zero-modes of the auxiliary fields which is equivalent to an average over charges at infinity.
Finally we have considered the *gauged Thirring model* in curved space-time. We find that the partition function is independent of vectorial as well as chiral twists and the chemical potential. This result, which technically is due to the harmonic contributions to the gauge-fields, is in fact expected as a consequence of Gauss's law. Furthermore, using the (probably not so obvious) factorization property of the zeta-function regularized determinants of commuting operators we find that the partition function can be expressed completely in terms of a single massive scalar field. The gauged Thirring model shows a chiral symmetry breaking which originates in the existence of fermionic zero-modes and thus in configurations with winding number (instantons). We have obtained explicit expressions for these instantons as well as the expectation value of the chiral condensate as a function of temperature and curvature. The con-

densate is exponentially suppressed for high temperatures and/or big curvature which is interpreted as an almost restoration of the chiral symmetry under these extreme conditions. Although temperature and curvature have qualitatively the same effect they cannot be identified. In particular the identification with the Hawking temperature for free scalar fields in de Sitter space does not hold in the present situation. It follows from general arguments that the chiral symmetry can not be restored for any finite temperature or curvature so an exponential suppression is most we can expect. In fact, it has been argued earlier, that the axial $U(1)$-symmetry in 4 dimensional QCD also shows an almost restoration as a function of the temperature [49]. Our results on the curvature dependence could motivate a corresponding investigation in QCD. Finally we note that the chiral condensate is linearly suppressed for large current-current couplings.

Acknowledgments: This work has been partially supported by the Swiss National Science Foundation and the ETH-Zürich, where part of the research leading to the present results has been done. We wish to thank Arne Dettki for his collaboration at the beginning of this work and J. Fröhlich, K. Gawedzky, D. O'Connor and C. Nash for helpful discussions.

A Conventions and Variational Formulae

Our conventions for the metric and curvature agree with those of Birrell and Davies [6]. We use the chiral representation $\hat{\gamma}^0_M = \sigma_1, \hat{\gamma}^1_M = i\sigma_2$ for flat space with Lorentzian signature and $\hat{\gamma}^0_E = \sigma_1, \hat{\gamma}^1_E = -\sigma_2$ in Euclidean space. Furthermore $\hat{\gamma}_5 = \gamma_5 = \sigma_3$.
In what follows we derive some variational formulae used in the text. Here D_μ denotes the space-time and Lorentz covariant derivative.
Using the definition of the Christoffel symbols it is straightforward to show that

$$\begin{aligned} \delta g_{\mu\nu} = \delta e_\mu{}^a e_{\nu a} + e_\mu{}^a \delta e_{\nu a} \quad &; \quad \delta\sqrt{g} = \frac{1}{2}\sqrt{g} g^{\mu\nu}\delta g_{\mu\nu} \\ \delta\gamma^\mu = -\gamma^\nu e^\mu{}_a \delta e_\nu{}^a \quad &; \quad \delta\eta_\mu{}^\nu = \frac{1}{2}(\eta^{\alpha\nu}\delta g_{\mu\alpha} - \eta_\mu{}^\sigma g^{\nu\rho}\delta g_{\sigma\rho}) \\ \delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}&(D_\nu \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}). \end{aligned} \tag{135}$$

For some formulae related to the variation of the tetrad let us refer to [48]

$$\delta e^{\mu}{}_{a} = \frac{1}{2} e_{\nu a} \delta g^{\mu\nu} - t_{a}{}^{b} e^{\mu}{}_{b} \quad ; \quad \delta e_{\mu}{}^{a} = \frac{1}{2} e^{\nu a} \delta g_{\mu\nu} - t^{a}{}_{b} e_{\mu}{}^{b},$$
$$\text{where} \quad t^{a}{}_{b} = \frac{1}{2}(e^{\nu a} \delta e_{\nu b} - e^{\nu}{}_{b} \delta e_{\nu}{}^{a}). \tag{136}$$

In addition we have

$$\delta\omega_{\mu ab} = D_{\mu} t_{ab} - \alpha_{\mu ab} \quad ; \alpha_{\mu ab} = \frac{1}{2} e^{\alpha}{}_{a} e^{\beta}{}_{b} (D_{\alpha} \delta g_{\beta\mu} - D_{\beta} \delta g_{\alpha\mu}). \tag{137}$$

When performing the variation of curvature dependent expressions we have used the identities

$$g^{\mu\nu} \delta \mathcal{R}_{\mu\nu} = \omega^{\alpha}_{;\alpha}\,, \quad \text{where} \quad \omega^{\alpha} = g^{\mu\nu} \delta \Gamma^{\alpha}_{\mu\nu} - g^{\alpha\nu} \delta \Gamma^{\mu}_{\mu\nu}$$
$$\text{and} \quad \int \sqrt{g}\, \omega^{\alpha} A_{\alpha} = \int \sqrt{g} \{ g^{\alpha\beta} \nabla_{\mu} A^{\mu} - \nabla^{\alpha} A^{\beta} \} \delta g_{\alpha\beta}\ . \tag{138}$$

Depending on the topology of space-time, the reference curvature $\hat{\mathcal{R}}$ may be different from zero. In this case it is not possible to express the conformal angle σ in terms of the curvature scalar. Nevertheless, to perform variations of σ-dependent expressions, the identity

$$\delta(\sqrt{g}\mathcal{R}) = -2\delta(\sqrt{g}\triangle\sigma) \tag{139}$$

proves to be useful.
Taking the variations of the equations

$$\sqrt{g}\Box G(x,y) = -\delta(x-y) \quad \text{and} \quad \sqrt{g}\, i \not{D} S(x,y) = \delta(x-y) \tag{140}$$

for the scalar and fermionic Greens functions we may derive (up to contact terms) the following variational formulae

$$\delta G = \int \ (- \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} + g^{\alpha\mu} g^{\beta\nu}) \partial_{\alpha} G(x,u)\, \partial_{\beta} G(u,y) \sqrt{g} \delta g_{\mu\nu}$$
$$\delta S = \frac{i}{4} \int \ \Big(2S(x,u) \gamma^{\mu} D^{\nu} S(u,y) - D_{\alpha} [S(x,u) \gamma^{\delta} \eta_{\delta}{}^{\mu} \eta^{\nu\alpha} S(u,y)] \Big) \sqrt{g} \delta g_{\mu\nu}.$$

Here all arguments and derivatives which are not made explicit in the integral refer to the coordinate u over which is integrated. Finally, we need the following formula for the variation of the inverse Laplacian

$$\delta\left(\frac{1}{\triangle}f\right)=\frac{1}{\triangle}\left(\delta f-\delta(\triangle)\frac{1}{\triangle}f\right)-\frac{1}{2V}\int\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}\frac{1}{\triangle}f, \tag{141}$$

where V is the volume of space-time and f an arbitrary function. To prove this identity we note that for $f\in(\mathrm{Kern}\triangle)^{\perp}$ we have

$$\triangle\frac{1}{\triangle}f=f.$$

Varying this equation yields

$$\triangle(\delta\frac{1}{\triangle}f)=\delta f-(\delta\triangle)\frac{1}{\triangle}f$$

which may be inverted to give

$$\delta\left(\frac{1}{\triangle}f\right)=\frac{1}{\triangle}\left(\delta f-\delta(\triangle)\frac{1}{\triangle}f\right)+\frac{1}{V}\int\sqrt{g}\delta\left(\frac{1}{\triangle}f\right)\ . \tag{142}$$

Varying the identity

$$\frac{1}{V}\int\sqrt{g}\frac{1}{\triangle}f=\ 0$$

allows to replace the last term of (142) to obtain the required result (141).

B Canonical Approach to the Partition Function

In this appendix we compute the partition function for massive Dirac fermions in the canonical formalism. In the limit $m\to 0$ we confirm the result (30) for the fermionic determinant with chemical potential in chapter 3. For massive fermions one cannot consistently impose chirally twisted boundary conditions. However, from the explicit eigenvalues (21) one sees at once that the chiral twist β_1 and the chemical potential are equivalent. One can easily verify that this equivalence holds also for massless fermions in the canonical approach and that $\beta_1\sim\mu L/2\pi$. Let us therefore compute the partition function

$$Z(\beta)=Tr[e^{-\beta:(H-\mu Q):}] \tag{143}$$

for massive Dirac fermions with chemical potential μ on a cylinder with (non chiral) twisted boundary conditions

$$\psi(x+L,t) = -e^{-2i\pi\alpha_1}\psi(x,t). \tag{144}$$

For massive particles it is more convenient to use the Dirac representation

$$\gamma^0 = \sigma_3 \quad \gamma^1 = -i\sigma_2, \quad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1. \tag{145}$$

The Dirac field is expanded in terms of the eigenmodes of the first quantized Hamiltonian

$$h = \begin{pmatrix} m & i\partial_x \\ i\partial_x & -m \end{pmatrix} \tag{146}$$

as

$$\Psi(x,t) = \sum_n \psi_{n,+} b_n + \sum_n \psi_{n,-} d_n^\dagger, \tag{147}$$

where the $\psi_{n,+}$ and $\psi_{n,-}$ are the positive and negative energy modes,

$$\begin{aligned} &\psi_{n,+} = e^{-i\omega_n t - i\lambda_n x} c_n, \quad \psi_{n,-} = e^{i\omega_n t - i\lambda_n x}\gamma_1 c_n, \\ &c_n = (2\omega_n(\omega_n+m)L)^{-\frac{1}{2}} \begin{pmatrix} \omega_n + m \\ \lambda_n \end{pmatrix}. \end{aligned} \tag{148}$$

The momenta λ_n and frequencies ω_n are determined by the boundary condition (144) to be

$$\lambda_n = \frac{2\pi}{L}(n - \frac{1}{2} - \alpha_1) \quad \text{and} \quad \omega_n = \sqrt{m^2 + \lambda_n^2}. \tag{149}$$

After normal ordering the 'positron' operators with respect to the Fock vacuum defined by H we find

$$(H - \mu Q) = \sum_n (\omega_n - \mu) b_n^\dagger b_n + \sum_n (\omega_n + \mu) d_n^\dagger d_n - \sum_n (\omega_n + \mu), \tag{150}$$

where the last c-number term represents the infinite vacuum contribution which must be regularized. To do that we employ the zeta function regularization. That is we define the zeta-function for $s > 1$ by the sum

$$\zeta(s) = \sum_n (\omega_n + \mu)^{-s},$$

which in turn defines an analytic function on the whole complex s-plane up to a simple pole at $s=1$. The analytic continuation is constructed by a Poisson resummation

$$\sum_n (\omega_n+\mu)^{-s} = \frac{L^s}{2\pi}\sum_n F(n), \tag{151}$$

where

$$F(\xi) = e^{2\pi i\xi(\frac{1}{2}-\alpha_1)}\int dy\; e^{i\xi y}[\sqrt{\tilde{m}^2+y^2}\;+\tilde{\mu}]^{-s} \tag{152}$$

and $\tilde{m}=Lm$, $\tilde{\mu}=L\mu$. Taking the Mellin transform of (152) we find

$$\begin{aligned} F(\xi) &= e^{2\pi i\xi(\frac{1}{2}-\alpha_1)}\frac{1}{\Gamma(s)}\int dy\; e^{i\xi y}\int dt\; t^{s-1}e^{-t\sqrt{\tilde{m}^2+y^2}-t\tilde{\mu}} \\ &= -\frac{2}{\Gamma(s)}e^{2\pi i\xi(\frac{1}{2}-\alpha_1)}\int dt\; t^{s-1}e^{-t\tilde{\mu}}\;\frac{d}{dt}K_0(\tilde{\mu}\sqrt{\xi^2+t^2}) \\ &= \frac{2\tilde{m}}{\Gamma(s)}e^{2\pi i\xi(\frac{1}{2}-\alpha_1)}\int dt\; t^{s}e^{-t\tilde{\mu}}\frac{K_1(\tilde{\mu}\sqrt{\xi^2+t^2})}{\sqrt{\xi^2+t^2}}. \end{aligned} \tag{153}$$

F diverges at $\xi=0$ since the Kelvin function $K_1(z)\sim 1/z$ for small z. It follows that the $n=0$ term in (151) diverges. This divergence is regularized by subtracting the ground state energy of the infinite volume system. Indeed, because of the exponential decay of the Bessel function for large arguments, only the $n=0$ term contributes for infinite volume. So we find for the regularized sum

$$\sum_n (\omega_n+\mu)^{-s} = \frac{\tilde{m}L^s}{\Gamma(s)\pi}\sum_{n\neq 0}\int dt\; e^{2\pi in(\frac{1}{2}-\alpha_1)}t^s e^{-t\tilde{\mu}}\frac{K_1(\tilde{m}\sqrt{n^2+t^2})}{\sqrt{n^2+t^2}}. \tag{154}$$

Now we perform the limit $m\to 0$. Only the most singular term in the expansion of the Bessel function contributes, hence

$$\begin{aligned} \sum_n (\omega_n+\mu)^{-s} &= \frac{L^s}{\Gamma(s)\pi}\sum_{n\neq 0}\int dt\; e^{2\pi in(\frac{1}{2}-\alpha_1)}t^s e^{-t\tilde{\mu}}\frac{1}{(n^2+t^2)} \\ &= \frac{sL^s}{\pi}\sum_{n\neq 0}e^{2\pi in(\frac{1}{2}-\alpha_1)}\sqrt{\tilde{\mu}}n^{s-\frac{1}{2}}S_{-s-\frac{1}{2};\frac{1}{2}}(\tilde{\mu}n), \end{aligned} \tag{155}$$

where $S_{a;b}(z)$ is the Lommel function [50]. In particular for $s=-1$ this function is $S=1/z$ so that finally

$$\sum_n (\omega_n + \mu)^{reg} = -\frac{1}{\pi L} \sum_{n\neq 0} \frac{(-)^n}{n^2} e^{-2\pi i n\alpha_1} = \frac{\pi}{6L} - \frac{2\pi}{L}(\alpha_1 - [\alpha_1 + \frac{1}{2}])^2. \tag{156}$$

Inserting this into (150) then yields the regularized expression

$$: H - \mu Q := \sum_n (\omega_n - \mu) b_n^\dagger b_n + \sum_n (\omega_n + \mu) b_n^\dagger d_n - \frac{\pi}{6L} + \frac{2\pi}{L}\Big(\alpha_1 - [\alpha_1 + \frac{1}{2}]\Big)^2. \tag{157}$$

For small μ the normal ordering is μ-independent so that

$$\langle 0| : H - \mu Q : |0\rangle = -\frac{\pi}{6L} + \frac{2\pi}{L}\Big(\alpha_1 - [\alpha_1 + \frac{1}{2}]\Big)^2 = \langle 0| : H : |0\rangle \tag{158}$$

is independent of μ and coincides with the Casimir energy [24].

Let us now compute the partition function. Using (158) we easily find

$$\begin{aligned} Z(\beta) &= \text{tr}\,[e^{-\beta :(H-\mu Q):}] = q^{[\alpha_1^2 - \frac{1}{12}]} \\ &= \prod_{n>[\frac{1}{2}+\alpha_1]}^{\infty} (1 + q^{(n-\frac{1}{2}-\alpha_1)} e^{\beta\mu}) \prod_{n>-[\frac{1}{2}+\alpha_1]}^{\infty} (1 + q^{(n-\frac{1}{2}+\alpha_1)} e^{\beta\mu}) \cdot \\ &\quad \prod_{n>[\frac{1}{2}-\alpha_1]}^{\infty} (1 + q^{(n-\frac{1}{2}+\alpha_1)} e^{-\beta\mu}) \prod_{n>-[\frac{1}{2}-\alpha_1]}^{\infty} (1 + q^{(n-\frac{1}{2}-\alpha_1)} e^{-\beta\mu}) \\ &= \frac{1}{|\eta(\tau)|^2} \Theta\Big[{-\alpha_1 \atop i\mu\frac{\beta}{2\pi}}\Big](0,\tau)\ \bar{\Theta}\Big[{-\alpha_1 \atop -i\mu\frac{\beta}{2\pi}}\Big](0,\tau), \end{aligned} \tag{159}$$

where we have used the product representation of the theta functions in the last identity and that $q = e^{2\pi i\tau} = e^{-2\pi\beta/L}$. A non-vanishing chiral twist β_1 can now be included by shifting the chemical potential. Thus we have confirmed the formula (30) in the text.

Note that for $\mu \neq 0$ the zero-temperature limit of the grand potential is not equal to the vacuum expectation value of $: H - \mu Q :$. For $\mu \neq 0$ all states up to the μ-dependent Fermi energy are filled. For example, for $\omega_1 < \mu < \omega_2$ in the limit $\beta \to \infty$, Ω reduces to the expectation value of $: H - \mu Q :$ in the one-electron state.

References

[1] see e.g. W. Dittrich and M. Reuter, "Effective Lagrangians in QED", Lecture notes in Physics, Springer, Heidelberg, 1984 and references therein.

[2] W.E. Thirring, *Ann. Phys. (NY)* **3** (1958), 91; V. Glaser, Nuovo Cim. **9**, (1958), 1007.

[3] H. Yokota, Prog. Theor. Phys. **77** (1987), 1450.

[4] D.Z. Freedman and K. Pilch, Phys. Lett. **213B** (1988), 331; *Ann. Phys. (NY)* **192** (1989), 331; S. Wu, Comm. Math. Phys. **124** (1989), 133.

[5] C. Destri and J.J. deVega, Phys. Lett. **223B** (1989), 365.

[6] N.D. Birrell and P.C.W. Davies, "Quantum fields in curved space", Cambridge Univ. Press, 1982.

[7] G.W. Gibbons and M.J. Perry, Proc. R. Soc. London **A 358** (1978), 467.

[8] I. Sachs and A. Wipf, Helv. Phys. Acta **65** (1992), 653.

[9] H. Leutwyler and V. Smilga, Phys. Rev. **D46** (1992), 5607.

[10] K. Johnson, Nuovo Cim. **20** (1964), 773.

[11] C. Jayewardena, Helv. Phys. Acta **61** (1988), 636; H. Joos, Nucl. Phys. **B17** (Proc. Suppl.) (1990), 704; Helv. Phys. Acta **63** (1990), 670; K.D. Rothe and J.A. Swieca, *Ann. Phys (N.Y.)* **117** (1979), 382; J. Verbaarschot et.al, Nucl. Phys. **425** (1994), 553.

[12] N.D. Birrell and P.C.W. Davies, Phys. Rev. **D18** (1978), 4408.

[13] O.J.P. Eboli, Phys. Rev. **D36** (1987), 2408.

[14] K. Gawedzky, "Conformal field theory", to appear in Birkhäuser, Basel.

[15] C. Nash and D. O'Connor, *Preprint* DIAS-STP-95-24.

[16] B. Klaiber, "Lecture notes in Phys. XA", Gordon and Breach, New York, 1968.

[17] R. Casalbuoni, Nuovo Cim. **A33** (1976), 115

[18] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234** (1983), 269; H. Leutwyler, Phys. Lett. **153B** (1985), 65; L. Alvarez-Gaume et. al, Comm. Math. Phys. **112** (1987), 503; E. D'Hoker and D.H. Phong, Nucl. Phys. **B278** (1 986), 226; K. Harada and I. Tsutsui, Phys. Lett. **B183** (1987), 311.

[19] A. Actor, Fortschritte der Phys. **35** (1987), 793; K. Tsokos, Phys. Lett. bf 157B (1985), 413.

[20] S. Blau, M. Visser and A. Wipf, Int. J. Mod. Phys. **A4** (1989), 1467.

[21] S. Blau, M. Visser and A. Wipf, Int. J. Mod. Phys. **A6** (1991), 5409.

[22] see e.g. A.M. Polyakov, "Gauge fields and Strings", Harwood Academic Publishers, 1987.

[23] C. Wiesendanger and A. Wipf, *Ann. of Phys. (NY)* **233** (1994), **233**, 125.

[24] C. Kiefer and A. Wipf, *Ann. of Phys. (NY)* **236** (1994), 241; S. Iso and H. Murayama, Progr. Theor. Phys. **84** (1990), 142.

[25] D. Lüst and S. Theisen, "Lectures on string theory", Lecture notes in physics, Springer, 1989.

[26] P. Furlan, G.M. Sotkov and I.T. Todorov, Riv. Nuovo Cim. **12** (1989), 1.

[27] L. O'Raifeartaigh and A. Wipf, Phys. Lett. **251B** (1990), 361.

[28] see eg. J. Glimm and A. Jaffe, "Quantum Physics", second edition, Springer, New York, 1987.

[29] I. Sachs, PhD. Thesis, ETH-No. 10728 (1994), *unpublished.*

[30] J.L. Cardy, *in* "Fields, Strings and Statistical Mechanics", Les Houches, 1988.

[31] A. Dettki and A. Wipf, Nucl. Phys. **B377** (1992), 252.

[32] S. Coleman, Phys. Rev. **D11** (1975), 2088.

[33] A.J. da Silva, M. Gomes and R. Köberle, Phys. Rev. **D34** (1986), 504; M. Gomes and A.J, da Silva, Phys. Rev. **D34** (1986), 3916.

[34] see eg. T. Banks, *in* Proceedings of the Santa Fe TASI-87, Vol. II, 1987.

[35] F. Lenz, H.W.L. Naus and M. Thies, *Ann. of Phys. (NY)* **233** (1994), 17.

[36] A.V. Smilga, *Ann. of Phys. (NY)* **234** (1994), 1.

[37] L. Susskind, Phys. Rev. **D20** (1979), 2610; N. Weiss, Phys. Rev. **D24** (1981), 475.

[38] J. Schwinger, Phys. Rev. **128** (1962), 2425.

[39] J.Ambjorn and S. Wolfram, *Ann. of Phys. (NY)* **147** (1983), 1.

[40] J. Hubbard, Phys. Rev. Lett. **3** (1958), 77; R.L. Stratonovich, Sov. Phys. Dokl. **2** (1958), 416.

[41] C. Gattringer and E. Seiler, *Ann. of Phys. (NY)* **233** (1994), 97.

[42] I. Sachs and A. Wipf, Phys. Lett. **B326** (1994), 105.

[43] S.V. Ketov and O. Lechtenfeld, Phys. Lett. **B353** (1995), 463.

[44] T.S. Bunch and P.C.W Davies, Proc. R. Soc. Lond. **A360** (1978), 117.

[45] M. Abramowitz and I.S. Stegun, "Handbook of Mathematical Functions", Dover Publications Inc. NY, 1972.

[46] S.M. Christensen, Phys. Rev. **D14** (1976), 2490.

[47] I.G. Avramidi, Phys. Lett. **B238** (1990), 92.

[48] H. Leutwyler and S. Mallik, Z. Phys. **C33** (1986), 205.

[49] E.V. Shuryak, Phys. Lett. **79B** (1978), 135; R.D. Pisarski and L.G. Yaffe, Phys. Lett., **97B** (1980), 110.

[50] I.S. Gradshteyn and I.M. Ryzhik, "Table of Integrals, Series and Products", Academic, London, 1980.