Abelian Projection on the Torus for general Gauge Groups[1]

C. Ford, T. Tok and A. Wipf[2]
Theor.–Phys. Institut, Universität Jena
Fröbelstieg 1, D–07743 Jena, Germany

Abstract

We consider Yang-Mills theories with general gauge groups G and twists on the four torus. We find consistent boundary conditions for gauge fields in all instanton sectors. An extended Abelian projection with respect to the Polyakov loop operator is presented, where A_0 is independent of time and in the Cartan subalgebra. Fundamental domains for the gauge fixed A_0 are constructed for arbitrary gauge groups. In the sectors with non-vanishing instanton number such gauge fixings are necessarily singular. The singularities can be restricted to Dirac strings joining magnetically charged defects. The magnetic charges of these monopoles take their values in the co-root lattice of the gauge group. We relate the magnetic charges of the defects and the windings of suitable Higgs fields about these defects to the instanton number.

PACS numbers: 11.10Wx, 11.15Tk, 11.15Kc, 12.38Aw

Keywords: Gauge field theory at finite temperature, gauge fixing, abelian projection, magnetic monopoles, instanton number

1 Introduction

Confinement and chiral symmetry breaking are supposed to follow from the dynamics of Yang-Mills fields. These phenomena are highly non-perturbative and still have not been derived from first principles. In this paper we will follow the strategy put forward by 't Hooft [1] who considered Yang-Mills theories on a Euclidean space-time torus $\mathbb{T}^4$. The torus provides a gauge invariant infrared cut-off. Its non-trivial topology gives rise to a non-trivial structure in the space of Yang-Mills fields which yields additional information on the possible phases of Yang-Mills theories. Compared to other Riemannian 4-dimensional compact manifolds the torus has many advantages (besides being the 'space-time' used in lattice simulations):

- one can use a flat metric in which case curvature effect do not mix with finite size effects,

[1]Supported by the Deutsche Forschungsgemeinschaft, DFG-Wi 777/3-2
[2]e–mails: Ford, Tok and Wipf@tpi.uni-jena.de

- the circumference L_0 in the temporal direction can be identified with the inverse temperature β [2, 3],
- gauge invariant periodic fields on $\mathbb{R}^4$ can be viewed as fields on $\mathbb{T}^4$,
- one may calculate non-perturbative quantities from finite size effects [4]; the string constant is directly related to the energy of a string winding around the torus [1],
- one keeps the relevant part of the supersymmetry in SUSY-YM theories.

Even the less ambitious goal to demonstrate confinement of static quarks without reliance on numerical simulations has not been achieved yet. Without dynamical fermions the relevant observables are products of Wilson-loops [5]. At finite temperature $T = 1/\beta$ the gauge fields in the functional integrals are periodic in Euclidean time i.e.

$$A_\mu(x^0 + \beta, \vec{x}) = A_\mu(x^0, \vec{x}).$$

and one may use Polyakov loops [6]

$$P(\vec{x}) = \text{Tr } R(\mathcal{P}(\beta, \vec{x})), \quad \text{where} \quad \mathcal{P}(x^0, \vec{x}) = \mathcal{P} \exp\left[i \int_0^{x^0} d\tau A_0(\tau, \vec{x})\right] \tag{1.1}$$

as order parameters for confinement. Here R is the representation of the gauge group which acts on the matter fields. We shall assume that the *gauge group G is simply connected*, e.g. $G = SU(2)$ rather than $SO(3) = \text{Ad(SU(2))}$. But since we allow for arbitrary representations R of G our results apply to general gauge groups $R(G)$, for example to $SO(3)$.

The Polyakov loop $P(\vec{x})$ is invariant under gauge transformations which are periodic in time. Since it is a functional of A_0 only, one is motivated to seek a gauge fixing where A_0 is as simple as possible. Note that the Weyl gauge, $A_0 = 0$, is not compatible with time-periodicity. In a previous paper [7] we discussed an extended Abelian projection for $SU(2)$ gauge theories on the four torus in which A_0 is time independent and in the Cartan subalgebra. The gauge fixing procedure hinges on the diagonalization of the path ordered exponential, $\mathcal{P}(\beta, \vec{x})$, whose trace is the Polyakov loop. In contrast to the two dimensional case investigated in [8] the diagonalization procedure has unavoidable singularities [9, 10]. The singularities can be interpreted as Dirac strings [11] joining magnetically charged 'defects'. Here we understand defects as points, loops (not to be confused with the Dirac strings!), sheets and lumps where $\mathcal{P}(\beta, \vec{x})$ has degenerate eigenvalues. For the gauge group $SU(2)$, the eigenvalues of $\mathcal{P}(\beta, \vec{x})$ are degenerate when $\mathcal{P}(\beta, \vec{x}) = \pm \mathbb{1}$. Thus there are two types of defect according to whether $\mathcal{P}(\beta, \vec{x})$ is plus or minus the identity. Associated with the gauge fixing procedure one can define an Abelian magnetic potential A_{mag} on $\mathbb{T}^3$ [9]. In [7] we showed that the total magnetic charge of $\mathcal{P} = \mathbb{1}$ defects is equal to the instanton number q. Moreover, the total magnetic charge of all defects is zero, i.e. the total magnetic charge of $\mathcal{P} = -\mathbb{1}$ defects is minus that of the $\mathcal{P} = \mathbb{1}$ defects. The relationship between magnetic charges and the instanton number was considered earlier by Christ and

Jackiw [12], Gross et.al. [2] and Reinhardt [13] who worked on $S^1 \times \mathbb{R}^3$ or $\mathbb{R}^4$. Though here one requires 'charges at infinity' to have overall magnetic charge neutrality. For an explicit discussion of the singularities emerging in the gauge fixing procedure at point like monopoles see the recent paper by Jahn and Lenz [14].

In this paper we extend the defect analysis to gauge theories on $\mathbb{T}^4$ with arbitrary gauge groups G of rank r. We also consider arbitrary twists [1], which allows us to treat matter transforming according to any representation of the gauge group. One has $r+1$ types of *basic defects* associated with the $r+1$ faces constituting the boundary of a 'fundamental domain' (these are essentially compactified Weyl chambers) in the root space. Since the magnetic potential lies in the Cartan subalgebra $\mathcal{H}$ we now have a matrix $Q_M \in \mathcal{H}$ of magnetic charges. The possible magnetic charges are quantized and are in one to one correspondence with the points of the integral co-root lattice. For a basic defect, Q_M is an integer multiple of a fixed matrix. Much as in the $SU(2)$ analysis there is a simple linear relation between the total magnetic charge of a given type of defect and the instanton number q. We have overall charge neutrality on $\mathbb{T}^3$ unless there are non-orthogonal magnetic and electric twists.

The paper is organized as follows. In the remainder of this section we recall some basic facts concerning gauge fields on $\mathbb{T}^4$. Next we present a set of transition functions (i.e. boundary conditions for the gauge fields) where the instanton number is equal to the winding number of the mapping $\mathcal{P}(\beta,\vec{x}) : \mathbb{T}^3 \to G$. These transition functions serve as the starting point for our gauge fixing. In section three we construct 'fundamental domains' for all gauge groups. Our Lie algebra conventions are stated here. Then we explain precisely what we mean by 'defects'. In the next section we define the magnetic charge of the defects. Our key result is given in section six. Here we obtain the relationship between the magnetic charges and the instanton number. Next we rewrite $\mathcal{P}(\beta,\vec{x})$ in terms of 'Higgs fields'. This enables us to tie up a loose end from section six, and also allows us to interpret the magnetic charges as Higgs winding numbers. In section eight we show how the ideas apply to $SU(3)$ and give our conclusions in section nine. Technicalities regarding our transition functions (including a construction of magnetic twist eaters for all gauge groups) can be found in Appendix A. Finally, an identity quoted in section six is derived in Appendix B.

We view the four torus as $\mathbb{R}^4$ modulo the lattice generated by four orthogonal vectors $b_\mu,\ \mu = 0,1,2,3$, for a recent review see [15]. The Euclidean lengths of the b_μ are denoted by L_μ (we may identify L_0 with the inverse temperature β). Local gauge invariants such as $\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}$ are periodic with respect to a shift by an arbitrary lattice vector. However, the gauge fields have to be periodic only up to gauge transformations. In order to specify boundary conditions for gauge potentials A_α on the torus one requires a set of *group valued transition functions* $U_\mu(x)$, which are defined on the whole of $\mathbb{R}^4$. The periodicity properties of A_α are as follows

$$A_\alpha(x+b_\mu) = U_\mu^{-1}(x)A_\alpha(x)U_\mu(x) + iU_\mu^{-1}(x)\partial_\alpha U_\mu(x), \quad \alpha,\mu = 0,1,2,3,$$

where the summation convention is *not* applied. It follows at once, that the path ordered exponential $\mathcal{P}(x^0,\vec{x})$ in (1.1) has the following periodicity properties

$$\mathcal{P}(x^0+L_0,\vec{x}) = \mathcal{P}(x^0,\vec{x})\mathcal{P}(L_0,\vec{x}), \qquad \mathcal{P}(x^0,\vec{x}+b_i) = U_i^{-1}(x^0,\vec{x})\mathcal{P}(x^0,\vec{x})U_i(0,\vec{x}). \quad (1.2)$$

The transition functions $U_\mu(x)$ satisfy the cocycle conditions [1]

$$U_\mu(x)U_\nu(x+b_\mu) = z_{\mu\nu}U_\nu(x)U_\mu(x+b_\nu), \qquad z_{\mu\nu} = z_{\nu\mu}^{-1}, \tag{1.3}$$

where the *twists* $z_{\mu\nu}$ lie in the center $\mathcal{Z}$ of the group. From now on *we assume that the transition functions belong to the universal covering group.* In general, our matter fields will not transform according to the covering group. However, a matter field in some representation is equivalent to matter transforming according to the covering group *provided we place suitable restrictions on the twists.* More precisely, consider a matter field which transforms under some representation $R(G)$ of the gauge group. A center element $z \in \mathcal{Z}$ is an allowed twist if $R(z) = \mathbb{1}$. For example if we have matter fields in the defining representation of $SU(3)$ all the twists must be the identity, since the other two center elements are faithfully represented. By contrast, if the matter fields are in the adjoint representation of any group then there is no restriction on the twists.

Under a gauge transformation, $V(x)$, the pair (A, U) is mapped to

$$A_\alpha^V(x) = V^{-1}(x)A_\alpha(x)V(x) + iV^{-1}(x)\partial_\alpha V(x), \quad U_\mu^V(x) = V^{-1}(x)U_\mu(x)V(x+b_\mu). \tag{1.4}$$

The twists, $z_{\mu\nu}$, are gauge invariant. We define the topological charge or instanton number as follows

$$q = \frac{1}{32\pi^2}\int\limits_{\mathbb{T}^4} \epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\,F_{\mu\nu}F_{\alpha\beta}, \tag{1.5}$$

where the trace corresponds to the canonically normalized scalar product in the Lie algebra[3]. Note that q is fully determined by the transition functions [17]. In particular, if we take all the transition functions to be the identity (i.e. we assume the gauge fields are periodic in all directions) then the instanton number is zero. Accordingly, if we are to describe the non-perturbative sectors, one must consider non-trivial transition functions. For a given q and set of twists, $z_{\mu\nu}$, we only require *one* set of transition functions. If we have two sets of transition functions with the same instanton number and twists then they are gauge equivalent [17].

2 Transition functions, the Polyakov loop operator and gauge fixing

First we construct a convenient set of transition function such that the instanton number is equal to the winding number of the map $\mathcal{P}(\beta,\vec{x}) : \mathbb{T}^3 \to G$. Then we find the (in general

[3]It is equal to half the trace in the adjoint representation divided by the dual Coxeter number.

singular) gauge transformation which transforms A_0 into a time-independent field in the Cartan subalgebra.
In the untwisted case, $z_{\mu\nu} = 1\!\!1$, we may assume that the transition functions have the following properties

$$U_0 = 1\!\!1, \quad U_i(x^0\!=\!0, \vec{x}) = 1\!\!1, \qquad i = 1,2,3, \quad \text{so that} \quad U_i(x + b_0) = U_i(x). \tag{2.1}$$

In [7] it was shown by explicit construction that there exist untwisted (i.e. $z_{\mu\nu} = 1\!\!1$) transition functions satisfying (2.1) in all instanton sectors. The condition that $U_0 = 1\!\!1$ is simply the statement that our gauge fields are periodic in time. Since the transition functions are trivial on the time slice $x^0 = 0$, and hence with (2.1) also on the time slice $x^0 = \beta$, the path ordered exponential $\mathcal{P}(\beta, \vec{x})$ is periodic in the three spatial directions (see (1.2)).

In the presence of magnetic twists (i.e. at least one of the $z_{ij} \neq 1\!\!1$) it is no longer possible to attain (2.1). However, one can still arrange for the transition functions to be independent of $\vec{x}$ on the time slice $x^0 = 0$. In appendix A we prove that there exist transition functions with the following properties

$$U_0 = 1\!\!1, \quad U_i(x^0 = 0, \vec{x}) = \omega_i, \qquad \text{so that} \quad U_i(x^0 = \beta, \vec{x}) = \omega_i z_{0i}, \tag{2.2}$$

where the ω_i are independent of $\vec{x}$ and satisfy the 'twist eating' conditions

$$\omega_i \omega_j = z_{ij} \omega_j \omega_i, \quad i,j = 1,2,3, \tag{2.3}$$

which follow from the cocycle conditions for the U_i at time $x^0 = 0$. For example, consider $SU(2)$ gauge theory with the following magnetic twists $z_{12} = -1\!\!1$, $z_{23} = z_{31} = 1\!\!1$. Then a possible choice of ω_i's is $\omega_1 = i\sigma_1$, $\omega_2 = i\sigma_2$, $\omega_3 = 1\!\!1$, where the σ_i are the Pauli matrices. Twist eaters satisfying (2.3) are known to exist for arbitrary twists in $SU(N)$ gauge theories [16]. Twist eaters for the other simple Lie groups are constructed in appendix A.

Now we use the properties of the transition functions to obtain a relation for the instanton number in terms of the Polyakov loop. Consider the following gauge transformation

$$V(x^0, \vec{x}) = \mathcal{P}(x^0, \vec{x}),$$

where $\mathcal{P}(x^0, \vec{x})$ is the path ordered exponential in (1.1) which in general is non-periodic in time. For brevity we use the notation

$$\mathcal{P}(\vec{x}) := \mathcal{P}(\beta, \vec{x}). \tag{2.4}$$

Using (1.2,1.4,2.1), the gauge transformed transition functions are

$$U_0^V = \mathcal{P}(\vec{x}), \qquad U_i^V = \omega_i.$$

The new U_0 is simply the path ordered exponential $\mathcal{P}(\vec{x})$, while the transformed spatial transition functions are constant matrices. Applying the well know formula for the instanton number in terms of the transition functions [17] yields

$$q = \frac{1}{24\pi^2} \int\limits_{\mathbb{T}^3} \epsilon_{0ijk} \mathrm{Tr}\left[(\mathcal{P}^{-1}\partial_i \mathcal{P})(\mathcal{P}^{-1}\partial_j \mathcal{P})(\mathcal{P}^{-1}\partial_k \mathcal{P})\right], \tag{2.5}$$

where $\mathcal{P} = \mathcal{P}(\vec{x})$, and $\mathbb{T}^3 = \{x \in \mathbb{T}^4 | x^0 = 0\}$. We emphasize that (2.5) is only valid when the (original) transition function satisfy (2.2). Another useful consequence of (2.2) is that $\mathcal{P}(\vec{x})$ has very simple periodicity properties

$$\mathcal{P}(\vec{x} + b_i) = z_{0i}\, \omega_i^{-1} \mathcal{P}(\vec{x})\, \omega_i, \quad i = 1, 2, 3. \tag{2.6}$$

In particular, $\mathcal{P}(\vec{x})$ is completely periodic in the absence of twists.

Now we follow [18, 19, 20, 7, 8] and seek a (time-periodic) gauge transformation, $V(x)$, for which the gauge transformed A_0 is independent of time and in the Cartan subalgebra. Consider the time-periodic gauge transformation

$$V(x^0, \vec{x}) = \mathcal{P}(x^0, \vec{x})\, \mathcal{P}^{-x^0/\beta}(\vec{x})\, W(\vec{x}), \tag{2.7}$$

where $\mathcal{P}(x^0, \vec{x})$ is the path ordered exponential (1.1), and $W(\vec{x})$ diagonalizes $\mathcal{P}(\vec{x})$, i.e.

$$\mathcal{P}(\vec{x}) = W(\vec{x}) D(\vec{x}) W^{-1}(\vec{x}), \qquad D(\vec{x}) = \exp[2\pi i\, h(\vec{x})], \tag{2.8}$$

with $h(\vec{x})$ in the Cartan subalgebra $\mathcal{H}$. The fractional power of $\mathcal{P}$ is defined via the diagonalization of $\mathcal{P}$. It follows at once that the gauge transformed A_0 reads

$$A_0^V = \frac{2\pi}{\beta} h(\vec{x}), \tag{2.9}$$

which is indeed independent of time and in the Cartan subalgebra. Whereas $\mathcal{P}(\vec{x})$ is smooth the factors $W(\vec{x})$ and $D(\vec{x})$ in the decomposition (2.8) are in general not. The classification and implications of these singularities are investigated in sections 4-7.

3 Fundamental domains

The mapping $h(\vec{x}) \to D(\vec{x})$ in (2.8) from the Cartan subalgebra to the toroidal (Cartan) subgroup is not one to one. In this section we shall find domains $\mathcal{M}$ in the Cartan subalgebra such that this mapping becomes bijective. We shall choose domains which are left invariant under the action of the Weyl group $\mathcal{W}$. If w is a Weyl reflection, then Ww diagonalizes $\mathcal{P}$ in (2.8) if W does. We shall fix this residual gauge freedom, under which $D \to wDw^{-1}$, by restricting h to one Weyl chamber. The intersection of a Weyl chamber with the 'Weyl invariant' domain $\mathcal{M}$ defines our fundamental domain $\mathcal{F}$. $\mathcal{F}$ is in one to one correspondence with the toroidal subgroup modulo Weyl transformations or equivalently with the conjugacy classes of G. The main result of this section is that $\mathcal{F}$ is the simplicial box with the extremal points (3.7).

Our Lie algebra conventions are as follows: Let $H_k,\ k=1,\ldots,r$ be an orthogonal basis of the Cartan subalgebra $\mathcal{H}$,

$$\mathrm{Tr}\, H_k H_l = \frac{|\alpha_L|^2}{2}\delta_{kl},$$

which are diagonal in a given representation[4],

$$H_k|\mu\rangle = \mu_k|\mu\rangle \quad \text{and} \quad [H_k, E_\alpha] = \alpha_k E_\alpha.$$

We normalize the roots such that the long roots have length $\sqrt{2}$, i.e. $(\alpha_L, \alpha_L) = 2$, and the H_k become orthonormal. *Throughout this paper we identify* $\sum \rho^k H_k = \rho \cdot H \in \mathcal{H}$ *with* $\rho \in \mathbb{R}^r$. Let

$$\alpha_{(i)}\ ,\quad \mu_{(i)}\ ,\quad \alpha^\vee_{(i)} = \frac{2\alpha_{(i)}}{(\alpha_{(i)},\alpha_{(i)})} \quad \text{and} \quad \mu^\vee_{(i)} = \frac{2\mu_{(i)}}{(\alpha_{(i)},\alpha_{(i)})}\ , \qquad i=1,\ldots,r \tag{3.1}$$

be the simple roots, fundamental weights, co-roots and co-weights, respectively:

$$(\alpha_{(i)},\alpha^\vee_{(j)}) = K_{ij}, \quad (\alpha^\vee_{(i)},\mu_{(j)}) = (\alpha_{(i)},\mu^\vee_{(j)}) = \delta_{ij}, \quad (\mu_{(i)},\mu^\vee_{(j)}) = (K^{-1})_{ij}. \tag{3.2}$$

We used that the simple roots and fundamental weights are related by the Cartan matrix,

$$\alpha_{(i)} = \sum_{j=1}^r K_{ij}\ \mu_{(j)}.$$

The fundamental weight-states (which are the highest weight states of the r fundamental representations) and states in the adjoint representation obey

$$\alpha^\vee_{(i)}\cdot H\ |\mu_{(j)}\rangle = \delta_{ij}|\mu_{(j)}\rangle \quad \text{and} \quad \mu^\vee_{(i)}\cdot H\ |\alpha_{(j)}\rangle = \delta_{ij}|\alpha_{(j)}\rangle. \tag{3.3}$$

The most negative root $\alpha_{(0)}$ and its co-root $\alpha^\vee_{(0)}$ define the integral *Coxeter numbers* n_i and *dual Coxeter numbers* $n_i^\vee$:

$$0 = \alpha_{(0)} + \sum_1^r n_i\alpha_{(i)} \equiv \sum_{\sigma=0}^r n_\sigma\alpha_{(\sigma)} \quad \text{and} \quad 0 = \alpha^\vee_{(0)} + \sum_1^r n_i^\vee\alpha^\vee_{(i)} \equiv \sum_{\sigma=0}^r n_\sigma^\vee\alpha^\vee_{(\sigma)},$$

where we have defined $n_0 = n_0^\vee = 1$. The (dual) Coxeter numbers are listed in appendix A. For later convenience we assign to $\alpha_{(0)}$ the co-weight $\mu^\vee_{(0)} = 0$.

The fundamental domains we seek are intimately related to the center elements of the group. Thus it is useful to find conditions on $\rho\cdot H \in \mathcal{H}$ such that $\exp(2\pi i\rho\cdot H)$ is in the center $\mathcal{Z}$. Center elements are the identity in the adjoint representation. Because of the second set of equations in (3.3) they must be powers of

$$z_i = \exp\left(2\pi i\mu^\vee_{(i)}\cdot H\right).$$

[4]We use the same symbol H_k for H_k in any representation.

In an irreducible representation a center element acts the same way on all states. Hence, a necessary and sufficient condition for $z_i \neq 1\!\!1$ is that

$$z_i\,|\mu_{(j)}\rangle = \exp\Big(2\pi i K^{-1}_{ji}\Big)|\mu_{(j)}\rangle \neq |\mu_{(j)}\rangle, \quad \text{or that} \quad K^{-1}_{ji} \notin \mathbb{Z}$$

for at least one fundamental weight $\mu_{(j)}$. Here we have used that the inner products of the weights with the co-weights yield the inverse Cartan matrix, see (3.2). The order of the center group is just $\det(K)$. The centers and their generators are listed in appendix A.
Let us now find a suitable domain in the Cartan subalgebra which is mapped bijectively into the toroidal subgroup. The elements

$$\exp\Big(2\pi i \rho\cdot H\Big)$$

in the toroidal subgroup are the identity if ρ is in the integral co-root lattice, i.e. the lattice spanned by the simple co-roots $\alpha^\vee_{(i)}$ (see (3.2)). Thus, the convex region $\mathcal{M}$ defined by the intersecting half-spaces $(\rho,\alpha)\leq 1$, where α is an arbitrary root, is in one to one[5] correspondence with the toroidal subgroup of the gauge group[6]. This set is invariant under the action of the Weyl group $\mathcal{W}$ and is given by

$$\mathcal{M} = \{\rho|\ \ (\rho,\alpha)\leq 1 \quad \text{for all roots} \quad \alpha\}. \tag{3.4}$$

Now we may fix the residual Weyl reflections by further assuming that $\rho\sim\rho\cdot H$ is in the Weyl chamber defined by

$$\{\rho|\ \ (\rho,\alpha_{(i)})\geq 0 \qquad \text{for all simple roots} \quad \alpha_{(i)}\}. \tag{3.5}$$

The inner product of a vector ρ in this Weyl chamber with the highest root $-\alpha_{(0)}$ is always greater or equal to the inner product with any other root. It follows that the conditions (3.4,3.5), which define the *fundamental domain* $\mathcal{F}$, simplify to

$$\mathcal{F} = \Big\{\rho|\ \ (\rho,\alpha_{(i)})\geq 0, \quad -(\rho,\alpha_{(0)})\leq 1\Big\}. \tag{3.6}$$

$\mathcal{F}$ is a simplex bounded by $r+1$ hyperplanes orthogonal to the roots $\{\alpha_{(\sigma)}\} = \{\alpha_{(0)},\alpha_{(i)}\}$. In what follows we call the plane orthogonal to $\alpha_{(\sigma)}$ the σ-plane, $\sigma\in\{0,i\}$. The i-planes all meet at the origin. Since $\alpha_{(0)}$ is a long root the last condition in (3.6) means that the 0-plane orthogonal to $\alpha_{(0)}$ goes through $-\alpha^\vee_{(0)}/2$. The roots $\alpha_{(\sigma)}$ point inside the box.

An equivalent definition of $\mathcal{F}$ is that $\mathcal{F}$ is the convex set with extremal points

$$\{0, \frac{1}{n_1}\mu^\vee_{(1)}, \frac{1}{n_2}\mu^\vee_{(2)},\dots,\frac{1}{n_r}\mu^\vee_{(r)}\}. \tag{3.7}$$

This can be seen by expanding ρ in terms of the co-weights

[5] On the boundary of the so defined set we have to identify points differing by a vector $\alpha^\vee$, i.e. we have to remove half of the boundary to get a one to one correspondence.

[6] The hyperplane $(\rho,\alpha)=1$ is orthogonal to $\alpha^\vee$ and goes through $\alpha^\vee/2$.

$$\mathcal{F} = \Big\{\rho = \sum_i \xi_i\, \mu^\vee_{(i)} |\;\; \xi_i \geq 0, \quad (n,\xi) \leq 1\Big\}, \tag{3.8}$$

where $n = (n_1, \dots, n_r)$ being the r-vector formed from the Coxeter labels. For example, the fundamental domains $\mathcal{F}$ for the A_r and C_r groups are the simplicial boxes with extremal points $\{0, \mu_{(i)},\ i = 1, \dots, r\}$ (recall, that we have chosen $|\alpha_L|^2 = 2$). Also, if α_1 and α_r are the long and short roots at the endpoints of the B_r-Dynkin-diagram, the fundamental domain for B_r is the convex set with extremal points

$$\{0, \mu_{(1)},\ \frac{1}{2}\mu_{(2)},\ \frac{1}{2}\mu_{(3)},\ \dots \frac{1}{2}\mu_{(r-1)},\ \mu_{(r)}\}.$$

The fundamental domains $\mathcal{F}$ and the center elements for the gauge groups of rank 2 are depicted in fig.1. The fundamental domain of A_2 is an equilateral triangle, that of B_2 half a square, that of G_2 half of an equilateral triangle and that of $A_1 \times A_1$ is a square. The reflections on the r walls of $\mathcal{F}$ through 0 generate the Weyl group $\mathcal{W}$ of G and give rise to $\mathcal{M}$.

Since $(\alpha_{(0)}, \alpha_{(i)}) \leq 0$, the highest root $-\alpha_{(0)}$ is always inside the Weyl chamber (3.6) or on its boundary. Indeed, for all groups with the exception of A_2 $\;-\alpha_{(0)}$ lies on the boundary of $\mathcal{F}$. From the *extended Dynkin diagram*[7](see fig.2) one reads off that for all but the A_r algebras the highest root is orthogonal to $r-1$ simple roots. Hence it must be proportional to the weight $\mu_{(i)}$ corresponding to the simple root $\alpha_{(i)}$ with $(\alpha_{(i)}, \alpha_{(0)}) \neq 0$.

Although our strategy is to work in the covering group with suitably restricted twists rather than directly dealing with arbitrary representations, we could in principle do without twists if we used transition functions and fundamental domains $\mathcal{F}_R$ appropriate to the representation R. Actually it is quite straightforward to construct domains $\mathcal{F}_R$ for any representation. The volume of such domains is always less than or equal to that of $\mathcal{F}$; more precisely

$$\mathrm{Vol}(\mathcal{F}_R) = \frac{\mathrm{Vol}(\mathcal{F})}{|\mathcal{C}_R|},$$

where $\mathcal{C}_R$ is the subgroup of the center $\mathcal{C}$ which is mapped to the identity by going from the covering group to the representation R and $|\mathcal{C}_R|$ is its order. For a given group, the domain with the smallest volume is that for the adjoint representation since the center is trivial in this case. The fundamental domains for the adjoint representation for the rank two groups are shown in figure 1.

4 Defects

Although the Polyakov loop operator itself is smooth for smooth gauge potentials the factors $W(\vec{x})$ and $D(\vec{x})$ in the decomposition (2.8) are in general not. In this section we

[7]One adds the most negative root $\alpha_{(0)}$ to the system of simple roots $\alpha_{(i)}$ and uses the well-known rules to draw the Dynkin diagram of this extended system of roots.

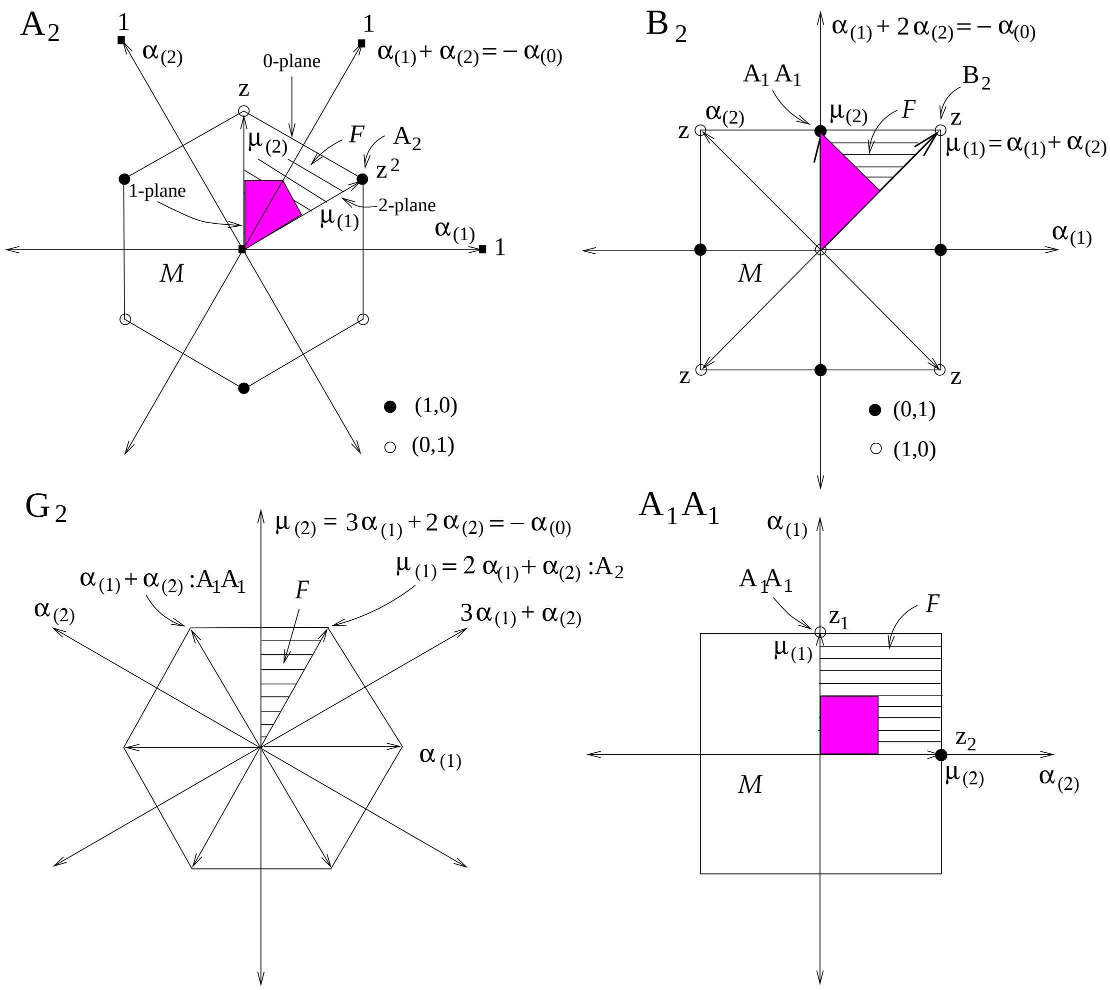

Figure 1: *Roots, fundamental weights, center elements, centralizer subgroups and fundamental domains $\mathcal{F}$ for the rank 2 case shown. The shaded regions inside $\mathcal{F}$ are the fundamental domains for the adjoint representations.*

shall see that singularities (so called defects) occur at points $\vec{x}$ at which $h(\vec{x})$ is on the boundary of the fundamental domain $\mathcal{F}$. At such defects the residual gauge freedom is enlarged. We shall explicitly determine the residual gauge groups at the various defects.

From now on we shall assume that $h(\vec{x})$ is in the fundamental domain $\mathcal{F}$. Then (2.8) assigns a unique $D(\vec{x})$ (and thus a unique $h(\vec{x}) \in \mathcal{F}$) to each Polyakov loop operator since we have fixed the Weyl reflections. However, the diagonalizing matrix $W(\vec{x})$ in (2.8) is determined only up to right-multiplication with an arbitrary matrix commuting with $D(\vec{x})$

$$W(\vec{x}) \longrightarrow W(\vec{x})V(\vec{x}), \quad V(\vec{x})D(\vec{x})V^{-1}(\vec{x}) = D(\vec{x}), \qquad D(\vec{x}) = e^{2\pi i h(\vec{x})}. \tag{4.1}$$

At each point the residual gauge transformations $V(\vec{x})$ form a subgroup of G, the centralizer of $D(\vec{x})$ in G, denoted by $\mathcal{C}_{D(\vec{x})}(G)$. The centralizer contains the toroidal subgroup of G.

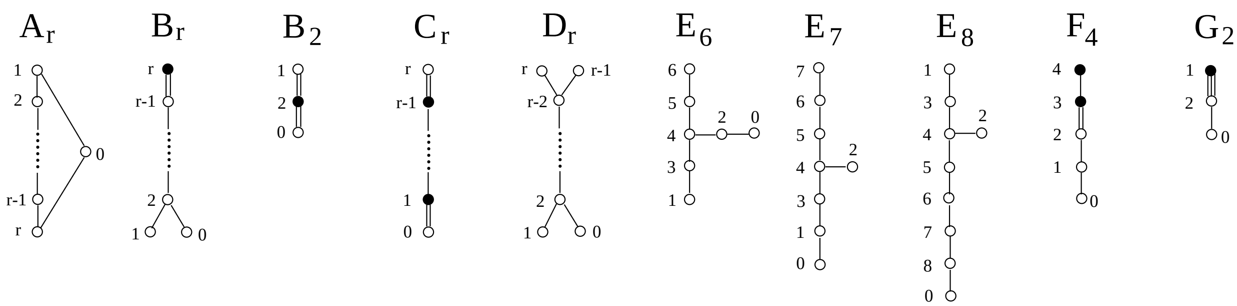

Figure 2: *The extended Dynkin diagrams, ○: long roots, •: short roots,* 0*: most negative roots (vertices are labelled as in [22]).*

At points where the centralizer is just the toroidal subgroup we can smoothly diagonalize the Polyakov loop operator.
However, at points where the centralizer is non-Abelian $\mathcal{P}(\vec{x})$ has degenerate eigenvalues and there are obstructions to diagonalizing $\mathcal{P}(\vec{x})$ smoothly [7, 9, 10]. For what follows it is useful to define the *defect manifold*

$$\mathcal{D} = \{\vec{x} \in \mathbb{T}^3 | \mathcal{C}_{D(\vec{x})}(G) \neq U^r(1)\} \tag{4.2}$$

on which the centralizer is non-Abelian. In the special case $G = SU(2)$ the defect manifold is $\mathcal{D} = \{\vec{x} \in \mathbb{T}^3 | \mathcal{P}(\vec{x}) = \pm \mathbb{1}\}$. A *defect* $\mathcal{D}_i$ is understood to be a connected subset of $\mathcal{D}$. In the neighborhood of a defect the diagonalization is in general not smoothly possible and the gauge fixing will be singular. Note that $\mathcal{D}$ is invariant under time-periodic gauge transformations so that the positions of the defects are gauge invariant.
Now we are going to classify the various defects which arise in our gauge fixing. To do that we expand $h(\vec{x})$ in (4.1) into a basis of the Lie algebra as $h(\vec{x}) = \rho(\vec{x}) \cdot H$ so that

$$D(\vec{x}) E_\alpha D(\vec{x})^{-1} = e^{2\pi i \left(\rho(\vec{x}), \alpha\right)} E_\alpha.$$

We see that $D(\vec{x})$ commutes with the subgroup $SU(2)$ corresponding to α if and only if $(\rho(\vec{x}), \alpha)$ is integer-valued. For $\rho \in \mathcal{F}$ in (3.6) this can only happen if ρ lies on the boundary of the fundamental domain. We parametrize $\rho(\vec{x})$ as in (3.8) so that

$$D(\vec{x}) E_{\alpha_{(i)}} D(\vec{x})^{-1} = e^{2\pi i \xi_i(\vec{x})} E_{\alpha_{(i)}} \quad \text{and} \quad D(\vec{x}) E_{\alpha_{(0)}} D(\vec{x})^{-1} = e^{-2\pi i (\xi(\vec{x}), n)} E_{\alpha_{(0)}}.$$

Therefore D commutes with the $SU(2)$-subgroup corresponding to the simple roots $\alpha_{(i)}$ if and only if $\xi_i = 0$ and it commutes with the $SU(2)$-subgroup corresponding to $\alpha_{(0)}$ if and only if $(\xi, n) = 1$. In other words, the centralizer contains the $SU(2)$ corresponding to $\alpha_{(\sigma)}$ if the defect is on the σ-plane, i.e. the plane perpendicular to $\alpha_{(\sigma)}$.
The centralizer of $D(\vec{x})$ generated by these $SU(2)$ subgroups can be read off from the extended Dynkin diagram (see fig.2) as follows: keep the vertex σ assigned to the root $\alpha_{(\sigma)} \in \{\alpha_{(0)}, \alpha_{(i)}\}$ in the extended Dynkin diagram if and only the defect lies on the σ-plane. Remove the other vertices and bonds attached to them. The remaining diagram

is then just the Dynkin diagram belonging to the semisimple factor of the centralizer. To obtain the complete centralizer group we must multiply with as many $U(1)$-factors as are needed to get a group of rank r.
Let us illustrate how this works for the simply laced groups $G = A_r$ for which the fundamental domains $\mathcal{F}$ can be parametrized as

$$\rho = \sum_1^r \xi_i \, \mu^\vee_{(i)}, \qquad \xi_i \geq 0, \quad \xi_0 \equiv 1 - \sum_1^r \xi_i \geq 0.$$

The extremal points of the fundamental domain are $\{\mu^\vee_{(\sigma)}\}$ and they correspond to the r+1 center elements of A_r. If just one ξ_σ vanishes then ρ lies inside the $(r-1)$-dimensional σ-plane. and we must keep the vertex σ in the extended Dynkin of A_r, i.e. the leftmost diagram in fig.2. The corresponding centralizer is $A_1 \times U^{r-1}(1)$. We call such defects with minimal non-Abelian centralizers *basic defects*. If ξ_σ and $\xi_{\sigma'}$ vanish in which case the defect lies both on the σ- and σ'-plane, then we must keep the two vertices σ and σ' in the extended Dynkin diagram. If they are neighbors in figure 2, then the centralizer is $A_2 \times U^{r-2}(1)$, otherwise it is $A_1 \times A_1 \times U^{r-2}(1)$. In the extreme case where just one ξ_σ does not vanish (then ρ is one of the extremal points of $\mathcal{F}$) we must retain all vertices with the exception of the vertex σ. We get the Dynkin diagram of A_r and the centralizer is the whole gauge group. By scanning the whole boundary of $\mathcal{F}$ comprising of $r-1$-dimensional, $r-2$-dimensional,...,1-dimensional simplices and the extremal points we obtain all stabilizer subgroups of G.

5 Quantization of the magnetic charges

In this section we define the Abelian magnetic potential $A_{mag}(\vec{x})$ associated with the partial gauge fixing and show that the magnetic charge of any defect is quantized. Away from the defects the centralizer of $D(\vec{x})$ is $U^r(1)$ and $W(\vec{x})$ in (2.8) is unique up to a residual Abelian gauge transformation (4.1):

$$W(\vec{x}) \longrightarrow W(\vec{x})V(\vec{x}) \quad \text{with} \quad V(\vec{x}) = e^{-i\lambda(\vec{x})} \in U^r(1) \quad \text{on} \quad \mathcal{D}^c. \tag{5.1}$$

If we append to each point in $\mathcal{D}^c$ the set of all diagonalizing matrices $W(\vec{x})$ we obtain a $U^r(1)$ principal bundle over $\mathcal{D}^c$. If we can find a smooth global section in this bundle then the diagonalization is smoothly possible outside of the defects, see also [23]. To investigate the structure of the bundle we employ the Abelian $U^r(1)$ gauge potential, $A_{mag}(\vec{x})$, obtained by projecting the pure gauge $A(\vec{x}) = iW^{-1}(\vec{x})dW(\vec{x})$ onto the Cartan subalgebra, i.e.

$$A_{mag}(\vec{x}) := A_c(\vec{x}),$$

where the subscript c denotes projection onto the Cartan subalgebra of G. This potential is singular at the defects and on Dirac strings joining the defects. Under a residual gauge transformation (5.1) the gauge potentials transform as

$$A_{mag} \longrightarrow A_{mag} + i(V^{-1}dV)_c = A_{mag} + d\lambda \quad \text{on} \quad \mathcal{D}^c.$$

Since A is pure gauge the corresponding field strength is given by

$$F_{mag} = \mathrm{d}A_{mag} = i(A \wedge A)_c, \tag{5.2}$$

and it is invariant under residual $U^r(1)$-gauge transformations.
Next we will show that a defect may carry r quantized magnetic charges [24]. For each defect these charges form a matrix Q_M in the Cartan subalgebra $\mathcal{H}$,

$$Q_M = \frac{1}{2\pi}\int_{\mathcal{S}} F_{mag}. \tag{5.3}$$

Here $\mathcal{S}$ is a surface surrounding the defect $\mathcal{D}_i$. Excluding walls extending over the whole

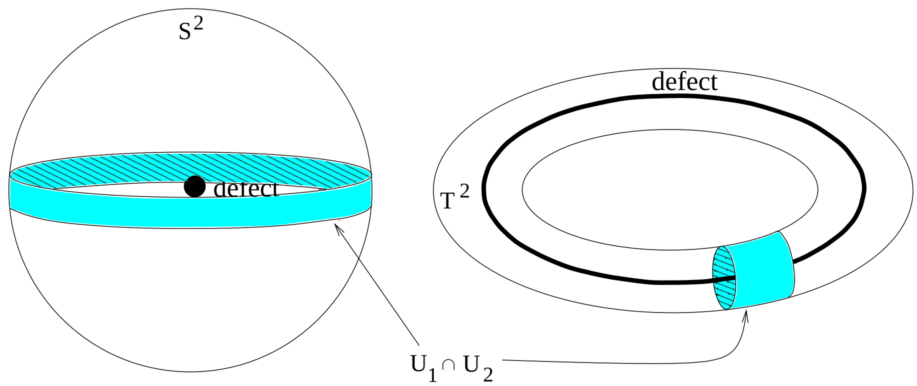

Figure 3: *Two typical defects: a monopole and a magnetic loop with surrounding surfaces and overlap regions.*

3-torus this surface is either a 2-sphere or a 2-torus (see fig.3). For each $U(1)$ the magnetic charge is just the instanton number of an Abelian gauge model on S^2 or $\mathbb{T}^2$ [25, 26] and hence is quantized. More explicitly, the magnetic charges are the winding numbers of the map $\exp(i\lambda) : S^1 \longrightarrow U^r(1)$,

$$Q_M = \frac{1}{2\pi}\oint_{S^1} \mathrm{d}\lambda,$$

where S^1 is in the overlap of the two patches U_i one needs to cover S^2 or $\mathbb{T}^2$. Since the gauge transformation $\exp(-i\lambda)$ is single valued on the overlap, $Q_M \in \mathcal{H}$ must satisfy

$$e^{2\pi i Q_M} = 1\!\!1 \quad \text{for each defect.} \tag{5.4}$$

For simply connected G this equality must hold on all states $|\mu\rangle$ and we find

$$Q_M = \alpha^\vee \cdot H, \qquad \text{where} \quad \alpha^\vee \in \text{co-root lattice.} \tag{5.5}$$

Thus we obtain the same magnetic charge quantization as uncovered by Goddard, Nuyts and Olive [27] in their pioneering work on electric-magnetic duality in Yang-Mills-Higgs theories.

6 Instantons and magnetic monopoles

In this section we work with the simply connected covering group and exclude twists[8]. Depending on the residual gauge symmetry in the defects we get different types of magnetic monopoles. There are $r+1$ kinds of basic monopoles with minimal non-Abelian centralizer $SU(2)\times U^r(1)$, corresponding to the $r+1$ hyperplanes which make up the boundary of the fundamental domain. We will show that a basic defect on the σ-plane has magnetic charge

$$Q_M = n\,\alpha^\vee_{(\sigma)}\cdot H, \qquad \sigma\in\{0,1,\ldots,r\} \tag{6.1}$$

with integer n. If we have a defect which is on two or more of the hyperplanes (which means that the Polyakov loop has more than two degenerate eigenvalues) then the magnetic charge of this defect is an integer combination of the co-roots perpendicular to these hyperplanes. Below we argue that in general the total magnetic charge of the defects on a given face gives the instanton number. For example, the magnetic charge of a defect on the 0-plane is $Q_M = (n\alpha^\vee_{(0)} + \beta^\vee)\cdot H$, $n\in\mathbb{Z}$, where $\beta^\vee$ is in the co-root lattice. This decomposition of the magnetic charge is unique, see below. Now the instanton number is simply

$$q = -\sum_{\text{defects on 0-plane}} n \tag{6.2}$$

This is our main result. Some illustrative examples of the use of this formula are given in section 8.

To derive the results (6.1,6.2) we assume that:

- There are no wall defects[9]
- Inside a defect the centralizer $\mathcal{C}_{D(\vec{x})}$ is uniform.

The first assumption is a reflection of the fact that one cannot surround a wall defect with a closed surface and so it is not obvious how to define the magnetic charge of such a defect. The second assumption is made to avoid the complication of 'defects within defects'. It may be possible to drop this requirement.

Our arguments are based on the observation that

$$l\int_{\mathbb{T}^3}\mathrm{Tr}(\mathcal{P}^{-1}\mathrm{d}\mathcal{P})^3 = \int_{\mathbb{T}^3}\mathrm{Tr}(P^{-l}dP^l)^3 \tag{6.3}$$

and furthermore

[8]See section 8 where we included twists for the relevant example $G=SU(3)$.

[9]We can formally define the absence of walls as follows. Consider the extension of the defect manifold to $\mathbb{R}^3$, i.e. $\tilde{\mathcal{D}} = \{x\in\mathbb{R}^3|\mathcal{C}_{D(\vec{x})}\neq U^r(1)\}$ There are no walls if $\tilde{\mathcal{D}}^c = \mathbb{R}^3\setminus\tilde{D}$ is connected.

$$\mathrm{Tr}(P^{-l}dP^{l})^3 = \mathrm{d}\mathcal{A}^{(\sigma)}, \qquad \sigma \in \{0, i\} \tag{6.4}$$

where the 2-forms are

$$\mathcal{A}^{(\sigma)} = -12l\,\pi i\,\mathrm{Tr}\left[A \wedge A\left(h - \frac{1}{n_\sigma}\mu^\vee_{(\sigma)} \cdot H\right)\right] + 3\,\mathrm{Tr}\left[AD^{-l} \wedge AD^{l}\right]. \tag{6.5}$$

Here l is the least common multiple of the Coxeter labels n_i and as before $\mu^\vee_{(0)} \equiv 0$ and $n_0 \equiv 1$. We prove this crucial identity in appendix B. These 2-forms are well defined outside the defects, because they are invariant under the residual Abelian gauge transformations (5.1). Both terms in (6.5) may be singular at defects. However, in the following section we will show that $\mathcal{A}^{(\sigma)}$ can be singular only at defects on the σ-plane or equivalently at defects whose centralizers have $\alpha_{(\sigma)}$ as root,

$$\mathcal{A}^{(\sigma)} \text{ singular } \iff \text{ defect is on } \sigma \text{ plane} \iff \alpha_{(\sigma)} \text{ is a root of defect centralizer.} \tag{6.6}$$

Actually, in (6.5) we could have subtracted an arbitrary constant Lie algebra element from $h(\vec{x})$ and (6.3) would still hold true. But the smoothness conditions (6.6) only hold if this constant element is an extremal point of the fundamental domain and if

$$\exp\left(2\pi i\frac{l}{n_\sigma}\mu^\vee_{(\sigma)} \cdot H\right)$$

is a center element. Thus we take for l in (6.3) the least common multiple of the Coxeter labels n_i. For example $l = 1$ for the A_r series and $l = 2$ for the other classical groups.

Now we make use of (6.3) to relate the magnetic charges of the defects on the 0-plane to the instanton number. Away from defects on the 0-plane $\mathcal{A}^{(0)}$ is regular. Now we surround each defect D on the 0-plane with a closed surface $\mathcal{S}$ and pick a two form $\mathcal{A}^{(i)}$ which is smooth inside $\mathcal{S}$, see fig.4. Since a defect can lie on at most r of the $r+1$ faces constituting the boundary of $\mathcal{F}$ there is always at least one such regular two form. With (1.5,6.3) the

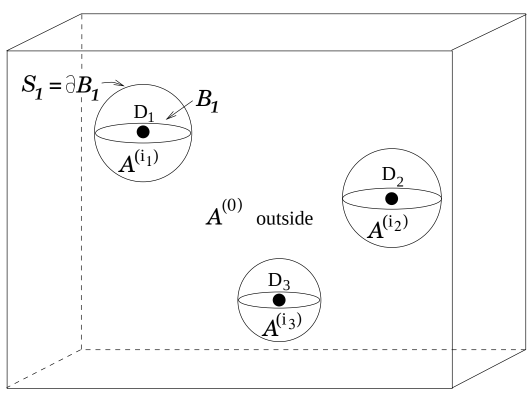

Figure 4: *We must choose two forms $\mathcal{A}^{(i_p)}$ which are regular inside spheres S_p containing a defect on the inhomogeneous 0-face.*

instanton number reads

$$q = \frac{1}{24\pi^2 l} \int\limits_{\text{outside}} d\mathcal{A}^{(0)} + \frac{1}{24\pi^2 l} \sum_p \int\limits_{\mathcal{B}_p} d\mathcal{A}^{(i_p)} = \frac{1}{24\pi^2 l} \sum_p \int\limits_{\mathcal{S}_p} (\mathcal{A}^{(i_p)} - \mathcal{A}^{(0)}), \tag{6.7}$$

where, since $\mathcal{A}^{(0)}$ is periodic on $\mathbb{T}^3$, we get no contributions from the 'boundary of the torus'[10]. Using (6.5) we obtain

$$\mathcal{A}^{(i)} - \mathcal{A}^{(0)} = \frac{12\pi i\, l}{n_i} \mathrm{Tr}\left(A \wedge A\, \mu^\vee_{(i)} \cdot H\right).$$

Since the magnetic field F_{mag} is the projection to the Cartan of $iA \wedge A$ we find

$$\mathcal{A}^{(i)} - \mathcal{A}^{(0)} = \frac{12\pi\, l}{n_i} \mathrm{Tr}(F_{mag}\, \mu^\vee_{(i)} \cdot H) \tag{6.8}$$

and end up with

$$q = \sum_{D_p} \frac{1}{n_{i_p}} \mathrm{Tr}\left(Q_M\, \mu^\vee_{i_p} \cdot H\right), \tag{6.9}$$

where we used (5.3). The sum extends over defects on the inhomogeneous 0-plane. Let us have a closer look at the contribution

$$\frac{1}{n_i} \mathrm{Tr}\left(Q_M\, \mu^\vee_{(i)} \cdot H\right) \tag{6.10}$$

of a given defect on the 0-plane. Consider first a *basic defect* with minimal non-Abelian centralizer. Then all two forms $\mathcal{A}^{(i)}$, $i \in \{1, \ldots, r\}$ are regular and must lead to the same contribution (6.10). We see at once that the magnetic charge must be proportional to $\alpha^\vee_{(0)}$,

$$Q_M = n\alpha^\vee_{(0)} \cdot H, \quad n \in \mathbb{Z}$$

and it contributes n to the instanton number.

A non-basic defect on the inhomogeneous face must also lie on at least one of the homogeneous faces, say the i-plane. For such a defect we must not take the corresponding singular $\mathcal{A}^{(i)}$ in (6.7) or $\mu^\vee_{(i)}$ in (6.10). We see that Q_M may be an integer linear combination of $\alpha^\vee_{(0)}$ and $\alpha^\vee_{(i)}$. More generally, if the defect lies on the 0-plane and several homogeneous planes, then

$$Q_M = \left(n\alpha^\vee_{(0)} + \sum m_i \alpha^\vee_{(i)}\right) \cdot H, \qquad m_i \neq 0 \quad \text{if defect is not on plane} \quad i. \tag{6.11}$$

Since a defect on the 0-plane can at most sit on $r-1$ of the r homogeneous planes, the representation (6.11) for the magnetic charge is unique.

[10]For twisted gauge fields there are surface contributions, see section 8.

Outside of the defects we could have taken any $\mathcal{A}^{(\sigma)}$ instead of $\mathcal{A}^{(0)}$. Then only defects on the σ-plane would contribute to the instanton number and we would find

$$q = \sum_{\text{defects on } \sigma\text{-plane}} \mathrm{Tr}\left(Q_M(\mu^\vee_{(\rho)} - \mu^\vee_{(\sigma)}) \cdot H\right).$$

Again the contribution of a given defect must not depend on ρ if the corresponding two form $\mathcal{A}^{(\rho)}$ is regular on the defect. As above we conclude that the magnetic charge of a defect is in the co-root lattice of the defect centralizer,

$$Q_M = \left(n\alpha^\vee_{(\sigma)} + \sum m_\rho \alpha^\vee_{(\rho)}\right) \cdot H, \qquad m_\rho \neq 0 \quad \text{if defect is not on plane} \quad \rho, \tag{6.12}$$

and that the instanton number is

$$q = -\sum_{\text{defects on } \sigma\text{-plane}} n.$$

7 Higgs fields

In this section we consider a parametrization of $\mathcal{P}(\vec{x})$ in terms of static 'Higgs' fields. This may seem to be a backward step since we are encoding a smooth group-valued object, $\mathcal{P}(\vec{x})$, in terms of $r+1$, in general singular, Lie algebra-valued fields. However the Higgs fields facilitate a very direct proof that the $\mathcal{A}^{(\sigma)}$ 2-forms introduced in the previous section have the stated smoothness properties. Moreover, we shall see that the magnetic charges of the defects can be related to Higgs winding numbers around the defects.

One can define a 'basic' Higgs field, $\phi^{(0)}$, as follows

$$\mathcal{P}(\vec{x}) = \exp\left[2\pi i \phi^{(0)}(\vec{x})\right] \quad \text{with} \quad \phi^{(0)}(\vec{x}) = W(\vec{x}) h(\vec{x}) W^{-1}(\vec{x}). \tag{7.1}$$

Now, $\phi^{(0)}(\vec{x})$, is smooth everywhere except for the inhomogeneous 0-plane. This follows because the centralizer of $D(\vec{x})$ commutes with $h(\vec{x})$ unless $(\rho, \alpha_{(0)}) = -1$. It is possible to define 'alternative' Higgs fields which are smooth on the 0-plane, but singular on one of the homogeneous i-planes, i.e. consider

$$\phi^{(i)} = W(\vec{x}) \left(h(\vec{x}) - \frac{1}{n_i}\mu^\vee_{(i)} \cdot H\right) W^{-1}(\vec{x}), \quad i = 1, 2, ..., r. \tag{7.2}$$

n_i being the i'th Coxeter label. The field $\phi^{(i)}$ is smooth everywhere except points on the i-plane. The relation between the Polyakov loop and the alternative Higgs fields is as follows

$$[\mathcal{P}(\vec{x})]^{n_i} z_i = \exp\left[2\pi i n_i \phi^{(i)}(\vec{x})\right],$$

where z_i is the center element $\exp[2\pi i \mu^\vee_{(i)} \cdot H]$. The $r+1$ Higgs fields $\phi^{(\sigma)}$, $\sigma \in \{0, i\}$ 'cover' the group in the sense that it is possible to partition $\mathbb{T}^3$ into patches, so that in each patch at least one of the Higgs fields is smooth.

In the previous section we wrote $\mathrm{Tr}(\mathcal{P}^{-l}d\mathcal{P}^{l})^3$ as the derivative of two forms $\mathcal{A}^{(\sigma)}$. We claimed that $\mathcal{A}^{(\sigma)}$ is only singular on the σ-plane. In other words, wherever $\phi^{(\sigma)}$ is smooth $\mathcal{A}^{(\sigma)}$ is smooth. This is obvious in the light of the following identity

$$\mathcal{A}^{(\sigma)} = 12\pi^2 l^2 \int_0^1 ds(s-1)\mathrm{Tr}\left[\exp(2\pi isl\phi^{(\sigma)})d\phi^{(\sigma)} \wedge \exp(-2\pi isl\phi^{(\sigma)})d\phi^{(\sigma)}\right], \qquad (7.3)$$

where as before l is the least common multiple of the Coxeter labels[11].

We now show that the magnetic field, F_{mag} can be written in terms of the Higgs fields. Using the fields $\phi^{(i)}$ one can construct normalized Higgs fields $\hat{\varphi}^{(i)}$ as follows

$$\hat{\varphi}^{(i)}(\vec{x}) = \phi^{(0)}(\vec{x}) - \phi^{(i)}(\vec{x}) = W(\vec{x})\,\frac{\mu^\vee_{(i)}}{n_i}\cdot H\,W(\vec{x})^{-1}.$$

In terms of the normalized Higgs fields, the magnetic fields are

$$-\frac{l}{n_i}\mathrm{Tr}(F_{mag}\mu^\vee_{(i)}\cdot H) = \pi l^2 \int_0^1 ds(s-1)\mathrm{Tr}\left[\exp\left(2\pi isl\hat{\varphi}^{(i)}\right)d\hat{\varphi}^{(i)} \wedge \exp\left(-2\pi isl\hat{\varphi}^{(i)}\right)d\hat{\varphi}^{(i)}\right]$$

If the Coxeter label n_i is unity, the integral reduces to

$$\mathrm{Tr}(F_{mag}\,\mu^\vee_{(i)}\cdot H) = i\mathrm{Tr}\left(\hat{\varphi}^{(i)}d\hat{\varphi}^{(i)} \wedge d\hat{\varphi}^{(i)}\right). \qquad (7.4)$$

Let $\mathcal{S}$ be a closed surface surrounding a defect. Since the centralizer of $\mu^\vee_{(i)}\cdot H$ in G is $K\times U(1)$, where K is semi-simple, the normalized Higgs field $\hat{\varphi}^{(i)}$ induces a map from $\mathcal{S}$ into a coset space $\mathcal{C}_i = G/(K\times U(1))$ with $\pi_2(\mathcal{C}_i) = \mathbb{Z}$. That is to each normalized Higgs field $\hat{\varphi}^{(i)}$ there is *one* associated winding number which can be identified with $\mathrm{Tr}\left(Q_M(\mathcal{S})\mu^\vee_{(i)}\cdot H\right)$.

For $SU(N)$ all the Coxeter labels are unity, and so

$$F_{mag} = i\sum_{i=1}^{N-1}\alpha_{(i)}\cdot H\,\mathrm{Tr}\left(\hat{\varphi}^{(i)}d\hat{\varphi}^{(i)} \wedge d\hat{\varphi}^{(i)}\right).$$

For the groups B_r, C_r, D_r, E_6 and E_7 it seems that the magnetic field cannot be written trilinearly in normalised fields since (7.4) only applies if the relevant Coxeter label is one. For example the gauge group E_7 has only one unit Coxeter label, n_7. However, the Weyl orbit of $\mu^\vee_{(7)}$ contains a linearly independent basis of the root space. To make this more concrete, consider the field

$$\hat{\varphi}_X = W(\vec{x})\,X\cdot H\,W^{-1}(\vec{x}).$$

A simple calculation shows that

$$\mathrm{Tr}(F_{mag}X\cdot H) = i\mathrm{Tr}\left(\hat{\varphi}_X\,d\hat{\varphi}_X \wedge d\hat{\varphi}_X\right),$$

[11] One can prove this identity by inserting $\phi = WDW^{-1}$ into the integral and comparing with equation (6.5). Alternatively, one can get it from the identity $\mathrm{Tr}(e^{-\psi}de^{\psi})^3 = 3d\left[\int_0^1 ds\,(s-1)\mathrm{Tr}\left(e^{-s\psi}d\psi \wedge e^{s\psi}d\psi\right)\right]$.

if and only if

$$(X,\alpha)^3 = (X,\alpha) \qquad \text{for all roots } \alpha. \tag{7.5}$$

Clearly, $X = \mu^\vee_{(i)}$ is a solution of (7.5) if and only if $n_i = 1$. But there are other solutions of (7.5) apart from the co-weights with unit Coxeter; these correspond to Weyl reflections of the co-weights. In fact for B_r, C_r, D_r, E_6 and E_7 one can always find r linearly independent solutions of (7.5) which we denote by $X_i,\ i = 1, 2, ..., r$. Thus we have

$$F_{mag} = i \sum_{i=1}^{r} Y^i \cdot H \ \mathrm{Tr}\left(\hat\varphi^{(i)}\, d\hat\varphi^{(i)} \wedge d\hat\varphi^{(i)}\right),$$

where now $\hat\varphi^{(i)} = \varphi_{X_i}$, and the Y^i are dual to the X_i in the sense that $(Y^i, X_j) = \delta^i{}_j$ (the Y^i are roots or Weyl reflections thereof). To each normalized Higgs field $\hat\varphi^{(i)}$ there is *one* associated winding number which can be identified with $\mathrm{Tr}\,(Q_M\, X_i \cdot H)$.

For the groups E_8, F_4 and G_2 no solutions of (7.5) exist.

8 $SU(3)$

In this section we illustrate the ideas of the previous sections by considering the relevant gauge group $SU(3)$. In the instanton number calculation of chapter 6 we assumed that our matter transformed according to the covering group. Here we will also consider the case of matter in the adjoint representation by allowing for twists.

First we consider $SU(3)$ with untwisted gauge fields, i.e. the Polyakov loop operator in the defining representation. The fundamental domain $\mathcal{F}$ has been depicted in figs.(1a,5). The magnetic charges of the three types of defects corresponding to the three edges of $\mathcal{F}$ are integer multiples of

$$\alpha^\vee_{(1)} \cdot H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha^\vee_{(2)} \cdot H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \alpha^\vee_{(0)} \cdot H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because of overall charge neutrality the magnetic charges of all defects must add up to zero,

$$\sum_{\text{all defects}} Q_M = 0.$$

Any cluster of magnetic monopoles connected by a Dirac string has vanishing magnetic charge. For example, if a monopole pair is uncharged no Dirac string, besides the one connecting the two monopoles, is needed. Since defects on the 0-plane for which $Q_M = n\,\alpha^\vee_{(0)} \cdot H$ (ignoring 'higher defects') contribute to the instanton number as

$$q = \sum_{\text{defects on 0-plane}} \mathrm{Tr}\Big(Q_M\, \mu^\vee_{(1)} \cdot H\Big)$$

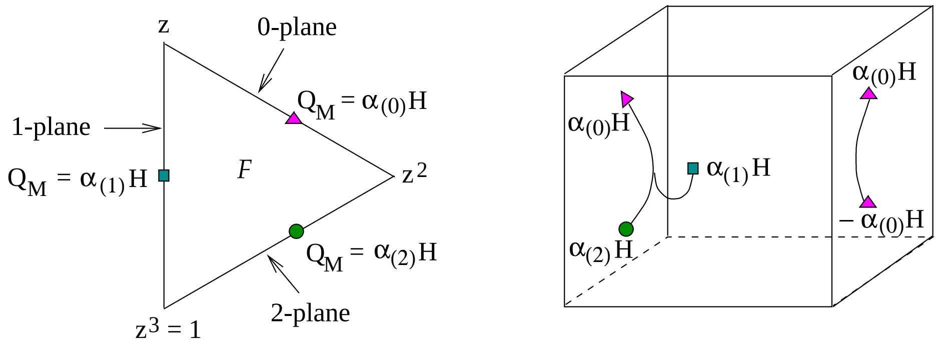

Figure 5: *The fundamental domain $\mathcal{F}$ for $G = A_2$ with elementary magnetic charges corresponding to the defects and a string network connecting different basic monopoles. Shown is a network with instanton number -1*

the monopole pair connected by a string in fig.5 does not contribute to the instanton number. The three monopoles connected by a Dirac string contribute -1 to the instanton number.

What about defects with larger centralizers? If $\mathcal{P}(\vec{x}) = z$, in which case $h(\vec{x})$ lies at an extremal point of $\mathcal{F}$ in fig.5, then $\mathcal{P}$ has maximal degeneracy and the centralizer is A_2. Such a defect has magnetic charge

$$Q_M = n_1\, \alpha^\vee_{(1)} \cdot H + n_2\, \alpha^\vee_{(0)} \cdot H = \Big((n_1 - n_2)\alpha^\vee_{(1)} - n_2\alpha^\vee_{(2)}\Big) \cdot H, \qquad n_i \quad \text{integers}$$

and contributes with $-n_2$ to the instanton number.

Finally, let us switch to the adjoint representation. In principle we could do this by restricting $h(\vec{x})$ to the fundamental domain for the adjoint representation, see the shaded regions in fig.1. However this would lead to walls on which $W(x)$ is not smooth. A much easier approach is to work in the covering group *but now with arbitrary twists.* In general this leads to a fractional instanton number. Such fractional instanton numbers are related to a loss of charge neutrality and nonperiodicity of $\mathcal{P}(\vec{x})$ engendered by the twists.

For example consider the following set of twists $z_{01} = \exp[4\pi i/3]\mathbb{1} = \exp[2\pi i\mu^\vee_{(1)} \cdot H], z_{23} = \exp[2\pi i/3]\mathbb{1} = \exp[2\pi i\mu^\vee_{(2)} \cdot H]$, and all other twists the identity. This is an example of non-orthogonal twists, and leads to an instanton number of the form $q = \frac{1}{3} + n$ where $n \in \mathbb{Z}$. From the periodicity properties of $\mathcal{P}(\vec{x})$

$$\mathcal{P}(\vec{x} + b_i) = z_{0i}\omega_i^{-1}\mathcal{P}(\vec{x})\omega_i\,, \quad i = 1, 2, 3$$

we obtain periodicity properties of $W(\vec{x})$, $D(\vec{x}) = \exp[2\pi i h(\vec{x})]$ and $h(\vec{x})$. In our example we get

$$h(\vec{x} + b_1) = w\left(h(\vec{x}) - \mu^\vee_{(2)} \cdot H\right) w^{-1},$$

where $\exp(-2\pi i\mu^\vee_{(2)} \cdot H) = z_{01}$ and w corresponds to an element of the Weyl group, here a rotation of $2\pi/3$. The equation can be understood as follows. By multiplying $D(\vec{x})$ with

z_{01} we shift $h(\vec{x})$ by the vector $-\mu^\vee_{(2)} \cdot H$. Then we have to Weyl reflect this shifted vector back into the fundamental domain $\mathcal{F}$ with w. In $\mathcal{F}$ itself this corresponds to a rotation with angle $2\pi/3$ around the center of the equilateral triangle $\mathcal{F}$. It follows that we get charge neutrality in the 'tripled' torus obtained by taking three adjoining tori in the x_1-direction. If we have in the first torus a defect of one type then in the adjoining torus in the x_1 direction we have a defect with the next type of charge and so on, see fig.6. In the x_2-

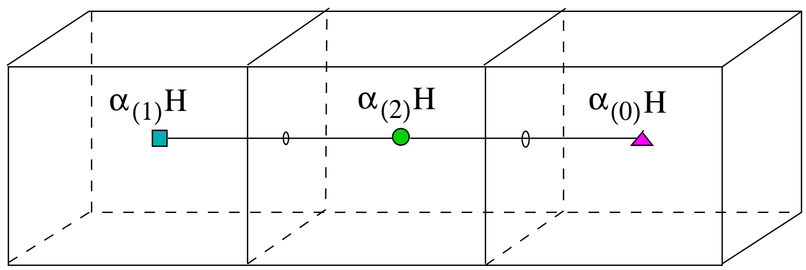

Figure 6: *In the twisted sector with $q = 1/3$ there maybe just one basic monopole in the torus. In the tripled torus we have charge neutrality.*

and x_3-directions $h(\vec{x})$ is periodic ($z_{02} = z_{03} = \mathbb{1}$). The periodicity properties of $W(\vec{x})$ are given by

$$W(\vec{x} + b_1) = W(\vec{x})w^{-1}R_1(\vec{x}) \quad \text{and} \quad W(\vec{x} + b_i) = \omega_i W(\vec{x})R_i(\vec{x}), \quad i = 2, 3,$$

where ω_1 and ω_2 are twist eaters such that $\omega_2\omega_3 = \omega_3\omega_2 z_{23}$ and R_i are functions with values in the Cartan subgroup[12]. From these conditions we obtain periodicity of the magnetic field strength $F_{mag} = iA \wedge A$ in the x_2- and x_3-directions and $F_{mag}(\vec{x}+b_1) = wF_{mag}(\vec{x})w^{-1}$. To calculate the topological index q we may use the 2-forms $\mathcal{A}^{(\sigma)}$, but now we will get contributions from the 'boundary' of the torus. This is in contrast to the non twisted case where we have had no contributions from the boundary because of the periodicity of $\mathcal{P}(\vec{x})$. We assume that there are no defects on the boundary. Then we can integrate $\mathcal{A}^{(0)}$ over the boundary. One easily checks that $\mathcal{A}^{(0)}$ is periodic in the x_2- and x_3-directions. Therefore we end up with

$$q_b = \frac{1}{24\pi^2}\int_{\partial\mathbb{T}^3} \mathcal{A}^{(0)} \quad = \quad \frac{1}{24\pi^2}\int_{x_1=0} \mathcal{A}^{(0)}(\vec{x}+b_1) - \mathcal{A}^{(0)}(\vec{x}) = \frac{1}{2\pi}\int_{x_1=0} \mathrm{Tr}(F_{mag}\mu^\vee_{(2)} \cdot H).$$

This shows the relation between the noninteger boundary contribution[13] to the instanton number and the total magnetic flux through the torus which results from the loss of charge neutrality on $\mathbb{T}^3$. In our example the element w of the Weyl group is a rotation of $2\pi/3$ in the Cartan subalgebra. Therefore $r^3 = \mathbb{1}$ which shows together with the periodicity properties of F_{mag} that in the tripled torus we have no boundary contributions to the topological index.

[12]In general the functions R_i can not be chosen smooth on the whole torus.

[13]By writing $F_{mag} = dA$ and using the cocycle condition for R_2 and R_3 one easily sees that q_b is indeed noninteger.

9 Conclusions

In this paper we have considered gauge-fixing of Yang-Mills theory on the four torus for arbitrary gauge groups, instanton sectors and twists. We have generalized our earlier results [7, 8] on the extended Abelian projection with respect to the Polyakov loop operator on the four torus. We have constructed a complete set of non-Abelian transition functions, which encode the 'boundary conditions' for the gauge potentials, for all instanton sectors and arbitrary twists. With these transition functions the path ordered exponential, $\mathcal{P}(\vec{x})$, which is central to the gauge fixing, is periodic up to multiplication by constant matrices, even though of course the gauge field itself is non-periodic. Then we found an explicit gauge transformation which transforms A_0 into the Cartan subalgebra and hence the Polyakov loop operator into the toroidal subgroup of G. The resulting gauge fixed A_0 is time independent. We have fixed the freedom in choosing the gauge transformation by restricting A_0 to a fundamental domain in the Cartan subalgebra.

In the sectors with non-vanishing instanton number the final gauge fixed potential must have singularities [9]. These singularities are due to ambiguities in the diagonalization of $\mathcal{P}(\vec{x})$ at points where the centralizer of $\mathcal{P}(\vec{x})$ is non-Abelian. There is a close analogy between these defects and magnetic charges in Yang-Mills-Higgs theories. The defects are classified according to the non-Abelian centralizer subgroups of $\mathcal{P}(\vec{x})$. A point $\vec{x}$ belongs to a defect if the gauge fixed $A_0(\vec{x})$ lies on the boundary of the fundamental domain. Here the results for $SU(2)$ may be misleading; at the defects the Polyakov loop operator need not be in the center of the gauge group as it must for $SU(2)$. For example, for $G \in \{E_8, F_4, G_2\}$ the center is trivial but there are many different types of defects corresponding to the different faces of the fundamental domain. The magnetic charges of the defects are quantized and linearly related to the points of the integral co-root lattice. For all groups with nontrivial centers we have constructed r normalized Higgs fields which wind around the magnetized defects. Finally we generalized earlier results in [12, 2, 13, 7] and related the magnetic charges of a given type of defect to the instanton number q. In particular, if $q \neq 0$ then all possible magnetic defects must appear.

One may view our gauge fixing as the 'nearest' fixing to the Weyl gauge compatible with time periodicity. Yet unlike the Weyl gauge we find monopole like singularities. This is gratifying, since in those theories where we analytically understand confinement, the latter is due to the condensation of monopoles; these examples are compact QED [29] and supersymmetric Yang-Mills theories [30]. Of course, there is a long way from the picture of condensed magnetic monopoles to real QCD.

The treatment given here has been purely classical. The next step would be to study the path integral within this gauge fixing. At this point one would need a sensible approximation [31]. The balancing of the energy and the entropy of monopoles (and/or loops) may explain the occurrence of the deconfinement transition in QCD. It would be interesting to clarify the role of the center of the gauge groups. There are gauge groups with trivial centers but many different types of monopoles and other magnetic defects.

Acknowledgements

We are grateful to Falk Bruckmann, Jan Pawlowski and Hugo Reinhardt for helpful discussions. We thank R. Jackiw for bringing [12], in which the relation between magnetic charges and the instanton number has been discovered, to our attention.

A Transition functions and twist

We prove that for arbitrary twists and instanton number there exist transition functions with the following property

$$U_0 = \mathbb{1}, \quad U_i(x^0 = 0) = \omega_i, \tag{A.1}$$

where the ω_i are twist eaters satisfying

$$\omega_i \omega_j = z_{ij} \omega_j \omega_i, \quad z_{ij} \in \mathcal{Z}. \tag{A.2}$$

We now start off with *Abelian* transition functions

$$U_\mu = \exp\left[2\pi i \sum_{\nu=0}^{3} \frac{\mathbf{n}_{\mu\nu} x^\nu}{L_\nu}\right], \tag{A.3}$$

where $\mathbf{n}_{\mu\nu}$ is a Cartan sub-algebra valued lower triangular matrix

$$\mathbf{n}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{n}^1 & 0 & 0 & 0 \\ \mathbf{n}^2 & \mathbf{m}^3 & 0 & 0 \\ \mathbf{n}^3 & -\mathbf{m}^2 & \mathbf{m}^1 & 0 \end{pmatrix}. \tag{A.4}$$

With this choice of $\mathbf{n}_{\mu\nu}$ we have $U_0 = \mathbb{1}$. The cocycle condition ensures that the $\mathbf{n}^i$ and $\mathbf{m}^i$ satisfy the constraints

$$e^{2\pi i \mathbf{n}^i} = z_{0i}, \quad e^{2\pi i \mathbf{m}^1} = z_{23} \quad \text{and cyclic permutations.} \tag{A.5}$$

The instanton number is simply

$$q = \mathrm{Tr}\,(\mathbf{n}^1\mathbf{m}^1 + \mathbf{n}^2\mathbf{m}^2 + \mathbf{n}^3\mathbf{m}^3).$$

Now we claim that there exists a time-independent gauge transformation $V(\vec{x})$ with the following properties

$$V^{-1}(\vec{x}) U_i(x^0 = 0, \vec{x}) V(\vec{x} + b_i) = \omega_i. \tag{A.6}$$

To prove this consider the following two sets of transition functions. Firstly take the Abelian transition functions (A.3) but with the $\mathbf{n}^i$ all set to zero. Secondly take the set of transition functions $U_0 = \mathbb{1}, \quad U_i = \omega_i$, where the ω_i are defined as in (A.2). Now both sets of transition functions have instanton number zero and an identical set of (magnetic) twists. Hence they must be gauge equivalent [17]. This establishes the existence of a smooth $V(\vec{x})$ satisfying (A.6). Now we perform this gauge transformation on the original Abelian transition functions (i.e. with the $\mathbf{n}^i$ not necessarily zero)

$$U_0^V = 1\!\!1, \quad U_i^V = V^{-1}(\vec{x}) \exp\left[2\pi i \sum_{\nu=0}^{3} \frac{\mathbf{n}_{i\nu} x_\nu}{L_\nu}\right] V(\vec{x}+b_i). \tag{A.7}$$

These transition functions have the stated properties.
This proof hinges on two assumptions:

- The existence of Abelian transition functions for arbitrary twists and instanton number.
- The existence of twist eaters for all possible magnetic twists z_{ij}.

It is well known that the first assumption breaks down in the odd instanton sectors of untwisted $SU(2)$ gauge theory. This special case has been addressed in ref. [7]. In [16] it was shown that the second assumption is valid for $SU(N)$. We will show the existence of magnetic twist eaters also for the other simple Lie groups. For every group (with the exception of the D_{2r}-series, which will be considered separately) the cyclic center is generated by

$$z = \exp\left(2\pi i \mu_{(z)}^{\vee} \cdot H\right).$$

In the table below we list the co-weights $\mu_{(z)}^{\vee}$ generating the centers. We now argue that magnetic twist eaters can be constructed from an Abelian element A and an element w in the Weyl group. The Abelian element A is given by

$$A = \exp\left[\frac{2\pi i}{g} \delta_w \cdot H\right], \tag{A.8}$$

where $g = 1 + \sum n_i$ is the Coxeter number (see the table below) and δ_w is the Weyl vector

$$\delta_w = \sum_i \mu_{(i)} = \frac{1}{2} \sum_{\alpha>0} \alpha\,, \qquad |\delta_w|^2 = \frac{\mathrm{dim} G}{24} g\, |\alpha_L|^2.$$

The element w is fixed by the requirement that

$$w^{-1}(\delta_w \cdot H) w = \delta_w \cdot H - g\mu_{(z)} \cdot H. \tag{A.9}$$

Such a Weyl group element w exists for *all* groups. For example for $G = SU(N)$ and $\mu_{(z)} = \mu_{(r)}$ it is

$$w = w_1 w_2 \ldots w_{N-1},$$

where w_i is the fundamental reflection on the plane orthogonal to the simple root $\alpha_{(i)}$,

$$w_i^{-1}(\mu\cdot H)w_i=\sigma_{\alpha_{(i)}}(\mu)\cdot H.$$

The Weyl word w_1w_2 first reflects on the plane orthogonal to $\alpha_{(1)}$ and then on the plane orthogonal to $\alpha_{(2)}$. A and w have the basic property

$$w^{-1}Aw=z^{-1}A\quad\text{so that}\quad w^{-p}A^qw^p=z^{-pq}A^q.$$

To prove this property we first note, that we may replace the weight $\mu_{(z)}$ in (A.9) by the corresponding co-weight, since $\alpha_{(z)}$ is always a long root. Now we conclude that

$$w^{-1}Aw=\exp\left[\frac{2\pi i}{g}w^{-1}\delta_w\cdot Hw\right]=\exp\big(-2\pi i\mu^\vee_{(z)}\cdot H\big)\,A=z^{-1}A,$$

as required.

Now we prove that for given magnetic twists $z_{ij}=z^{\epsilon_{ijk}t_k},\ t_k\in\mathbb{Z}$ we can find twist eaters ω_i satisfying equation (A.2). We make the ansatz

$$\omega_i=w^{p_i}A^{q_i}\quad\text{such that}\quad\omega_i\omega_j=z^{p_iq_j-p_jq_i}\omega_j\omega_i.$$

It follows that equation (2.3) is equivalent to

$$\vec n\equiv\vec p\wedge\vec q\mod(|\mathcal{Z}|),\tag{A.10}$$

where $|\mathcal{Z}|$ is the order of the center group. If all twists are the identity (all n_i are zero) the solution is trivial. So let us assume that at least one n_i, say n_3 is not zero. Then we choose

$$\vec p=\begin{pmatrix}0\\1\\p\end{pmatrix},\quad\vec q=\begin{pmatrix}-n_3\\0\\n_1\end{pmatrix}\quad\text{so that}\quad\vec n=\begin{pmatrix}n_1\\-pn_3\\n_3\end{pmatrix}.$$

It remains to be shown that for a given n_2 and $n_3\neq0$ we can solve

$$n_2=-pn_3\mod(|\mathcal{Z}|).\tag{A.11}$$

If the order of the center is a prime number, as it is for all but the A and D groups, then we can always find a p solving this equation. For the D_r groups with odd r the order of the center is not prime but 4. If only one n_i, say again n_3 is odd then we can again solve (A.11). In the other case all n_i must be even and (A.11) can again be solved. This proves the existence of twist eaters for all but the D_r-groups with even rank.

For the D_r-groups with even rank the center comprises of

$$\mathbb{1},\quad z_1=e^{2\pi i\mu^\vee_{(1)}\cdot H},\quad z_2=e^{2\pi i\mu^\vee_{(r)}\cdot H}\quad\text{and}\quad z_3=e^{2\pi i\mu^\vee_{(r-1)}\cdot H},$$

where $z_iz_j=\delta_{ij}\mathbb{1}+\epsilon_{ijk}z_k$. As before one can find commuting Weyl words $w_{(i)}$ such that for each center element

$$w_{(i)}^{-1} A w_{(i)} = z_i^{-1} A = z_i A \quad \text{and} \quad w_{(i)} w_{(j)} = w_{(j)} w_{(i)}. \tag{A.12}$$

For example,

$$w_{(1)} = w_1 w_2 \cdots w_{2r} w_{2r-2} w_{2r-3} \cdots w_1.$$

Now we make a case by case analysis to show the existence of twist eaters for arbitrary given twists. Using (A.12) one finds the following solution for the possible choices for z_{ij} in (A.2):

case	ω_1	ω_2	ω_3	z_{12}	z_{13}	z_{23}
one twist	A	$w_{(i)}$	$1\!\!1$	z_i	$1\!\!1$	$1\!\!1$
two twists	A	$w_{(i)}$	$w_{(j)}$	z_i	z_j	$1\!\!1$
3 different twists	$w_{(i)}A$	$w_{(j)}A$	$w_{(k)}A$	$\epsilon_{ijk} z_k$	$\epsilon_{ikj} z_j$	$\epsilon_{jki} z_i$
2 or 3 identical twists	$w_{(i)}$	$w_{(j)}A$	A	z_i	z_i	z_j

Together with the result in [16] this finally proves the existence of magnetic twist-eaters for all gauge groups.
In the main body of the paper we needed the centers, (dual) Coxeter labels and Coxeter numbers of the various gauge groups. For completeness we have listed these in the tables below.

group	A_r	B_r	C_r	$D_r,\ r$ even	$D_r,\ r$ odd
$\mathcal{Z}$	$\mathbb{Z}_{r+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$
$\mu^\vee_{(z)}$	$\mu^\vee_{(1)}$	$\mu^\vee_{(1)}$	$\mu^\vee_{(r)}$	$\mu^\vee_{(1)}, \mu^\vee_{(r)}$	$\mu^\vee_{(r)}$
n_i	$1, \ldots, 1$	$1, 2, \ldots, 2, 2$	$2, \ldots, 2, 1$	$1, 2, \ldots, 2, 1, 1$	$1, 2, \ldots, 2, 1, 1$
$n^\vee_i$		$1, 2, \ldots, 2, 1$	$1, \ldots, 1, 1$		
g	$r+1$	$2r$	$2r$	$2r-2$	$2r-2$

Table 1a: *Centers $\mathcal{Z}$, generators $\mu^\vee_{(z)}$ of the centers: $z = \exp(2\pi i \mu^\vee_{(z)})$, Coxeter labels n_i, dual Coxeter labels $n^\vee_i$ and Coxeter number g of the classical groups*

group	E_6	E_7	E_8	F_4	G_2
$\mathcal{Z}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$1\!\!1$	$1\!\!1$	$1\!\!1$
$\mu^\vee_{(z)}$	$\mu^\vee_{(1)}$	$\mu^\vee_{(7)}$			
n_i	$1, 2, 2, 3, 2, 1$	$2, 2, 3, 4, 3, 2, 1$	$2, 3, 4, 6, 5, 4, 3, 2$	$2, 3, 4, 2$	$3, 2$
$n^\vee_i$				$2, 3, 2, 1$	$1, 2$
g	12	18	30	12	6

Table 1b: *Centers $\mathcal{Z}$, generators $\mu^\vee_{(z)}$ of the centers: $z = \exp(2\pi i \mu^\vee_{(z)})$, Coxeter labels n_i, dual Coxeter labels $n^\vee_i$ and Coxeter number g of the exceptional groups*

B Proof of (6.3)

For $l = 1$ this formula is easily checked if one uses $\mathcal{P} = WDW^{-1}$ and the definitions $A = iW^{-1}dW$ and $\log D = 2\pi ih$. To prove the formula for $l > 1$ is less trivial. As a first step consider two group valued fields P_1, P_2. Then

$$\mathrm{Tr}\Big((P_1P_2)^{-1}d(P_1P_2)\Big)^3 = \sum_i \mathrm{Tr}(P_i^{-1}dP_i)^3 - 3\,d\,\mathrm{Tr}(P_1^{-1}dP_1 \wedge dP_2P_2^{-1}).$$

If the P_i are smooth and periodic then

$$\int\limits_{\mathbb{T}^3} Tr\Big((P_1P_2)^{-1}d(P_1P_2)\Big)^3 = \sum_i \int\limits_{\mathbb{T}^3} \mathrm{Tr}(P_i^{-1}dP_i)^3.$$

With our choice for the transition functions the Polyakov loop operator is indeed periodic and we conclude that

$$\int\limits_{\mathbb{T}^3} \mathrm{Tr}\Big(\mathcal{P}^{-l}d(\mathcal{P}^l)\Big)^3 = l\int\limits_{\mathbb{T}^3} \mathrm{Tr}(\mathcal{P}^{-1}d\mathcal{P})^3. \tag{B.13}$$

Now we can relate the instanton number in (1.5) to the winding of $\mathcal{P}^l$ as follows

$$q = \frac{1}{24\pi^2 l}\int\limits_{\mathbb{T}^3} \mathrm{Tr}\Big(\mathcal{P}^{-l}\,d\mathcal{P}^l\Big)^3$$

Since $\mathcal{P}^l = WD^lW^{-1}$ we can now apply formula (6.3) with D replaced by D^l. This then leads to

$$q = \sum_\sigma \frac{1}{24\pi^2 l}\int\limits_{M_\sigma} d\mathcal{A}^{(\sigma)}\,, \quad \bigcup_\sigma M_\sigma = \mathbb{T}^3\,, \quad M_\sigma \cap M_{\sigma'} = \emptyset\,, \text{ if } \sigma \neq \sigma',$$

where $\mathcal{A}^{(\sigma)}$ is smooth in M_σ and has been defined in (6.5). This proves (6.3) for $l > 1$ as required.

References

[1] G. 't Hooft, Nucl. Phys. B153 (1979) 141; Acta Phys. Austria CA Suppl. XXII (1980) 1063; Phys. Scr. 24 (1981) 841

[2] D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43

[3] J.I. Kapusta, *Finite-temperature field theory*, Cambridge University Press, 1989

[4] M. Lüscher, Phys. Lett. 118B (1982) 391

[5] K.G. Wilson, Phys. Rev. D10 (1974) 2445

[6] A.M. Polyakov, Phys. Lett. 72B (1978) 477; L. Susskind, Phys. Rev. D20 (1979) 2610

[7] C. Ford, U.G. Mitreuter, T. Tok, A. Wipf and J.M. Pawlowski, *Monopoles, Polyakov loops and gauge fixing on the torus*, preprint FSUJ-TPI-98-03, hep-th/9802191, Annals Phys., in press

[8] U. G. Mitreuter, J. M. Pawlowski and A. Wipf, Nucl. Phys. B514 (1998) 381

[9] G. 't Hooft, Nucl. Phys. B190 (1981) 455

[10] A.S. Kronfeld, G. Schierholz and U.J. Wiese, Nucl. Phys. B293 (1987) 461

[11] P.A.M. Dirac, Proc. Roy. Soc. A133 (1931) 60

[12] N. Christ and R. Jackiw, Phys. Lett. 91B (1980) 228

[13] H. Reinhardt, Nucl. Phys. B503 (1997) 505

[14] O. Jahn and F. Lenz, Phys. Rev. D58 (1998) 085006

[15] A. Gonzalez-Arroyo, *Yang-Mills Fields on the 4-dimensional torus. Part I: Classical Theory*, preprint FTUAM-97/18, hep-th/9807108

[16] J. Ambjørn and H. Flyvbjerg, Phys. Lett. 97B (1980) 241

[17] P. van Baal, Comm. Math. Phys. 85 (1982) 529

[18] N. Weiss, Phys. Rev. D24 (1981) 475

[19] E. Langmann, M. Salmhofer and A. Kovner, Mod. Phys. Lett. A9 (1994) 2913

[20] F. Lenz, H.W.L. Naus and M. Thies, Annals Phys. 233 (1994) 317

[21] C. Itzykson, Int. Journal of Mod. Phys. A1 (1986) 65

[22] N. Bourbaki, Groupes et algebres de Lie, Chap 7 and 8, Hermann, Paris, 1968; LiE sofware package, CWI, Amsterdam, 1997

[23] H. W. Grießhammer, *Magnetic Defects Signal Failure of Abelian Projection Gauges in QCD*, preprint FAU-TP3-97/6, hep-ph/9709462

[24] G. 't Hooft, Nucl. Phys. B79 (1974) 276; A.M. Polyakov, JETP lett. 20 (1974) 194

[25] C. Jayewardena, Helv. Phys. Acta 61 (1988) 636

[26] I. Sachs and A. Wipf, Helv. Phys. Acta 65 (1992) 653; Annals Phys. 249 (1996) 380

[27] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125 (1977) 1

[28] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry - Methods and Applications*, Springer Verlag New York, 1992

[29] A.M. Polyakov, Phys. Lett. 59B (1975) 82; T. Banks et.al, Nucl. Phys. B129 (1977) 493

[30] N. Seiberg and E. Witten, Nucl. Phys. B341 (1994) 484

[31] M. Engelhardt and H. Reinhardt, Phys. Lett. 430B (1998) 161; P. van Baal and J. Keller, Annals Phys. 174 (1987) 299; Di Giacomo, preprint IFUP-TH 9/98, hep-lat/9802008 and references therein