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Supersymmetry and the Dirac Equation

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We discuss in detail two supersymmetries of the 4-dimensional Dirac operator $\not{D}^2$ where $\not{D} = \not{\partial} - ie\not{A}$, namely the usual chiral supersymmetry and a separate complex supersymmetry. Using SUSY methods developed to categorize solvable potentials in 1-dimensional quantum mechanics we systematically study the cases where the spectrum, eigenfunctions, and S-matrix of $\not{D}^2$ can be obtained analytically. We relate these solutions to the solutions of the ordinary massive Dirac equation in external fields. We show that whenever a Schrödinger equation for a potential $V(x)$ is exactly solvable, then there always exists a corresponding static scalar field $\varphi(x)$ for which the Jackiw–Rebbi type $(1+1)$-dimensional Dirac equation is exactly solvable with $V(x)$ and $\varphi(x)$ being related by $V(x) = \varphi^2(x) + \varphi'(x)$. We also discuss and exploit the supersymmetry of the path integral representation for the fermion propagator in an external field.

I. INTRODUCTION

Recently there has been a renewed interest in understanding the zero modes and complete spectrum of the square of the Dirac operator for a Euclidean fermionic theory interacting with background gauge fields. This operator can be considered

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[1] the Hamiltonian of a supersymmetric quantum mechanical system, with a superalgebra identical to that of Witten's original $N=1$ supersymmetric quantum mechanical model [2]. This supersymmetry was first successfully exploited in the context of the study of chiral anomalies. The reason for this is that the partition function for the supersymmetric quantum mechanics associated with the squared Dirac operator

$$Z_5(\beta)=\int dx\, \langle x|\, \mathrm{Tr}(\gamma_5 e^{-\beta H})\, |x\rangle, \qquad H=-(\not{D})^2 \tag{1.1}$$

is related in the limit $\beta=0$ to the chiral $U(1)$ anomaly [3].

Another reason for studying the spectrum of the Dirac operator is the recent experiments which explore the behavior of QED in the presence of strong external fields [4]. In these experiments conducted at GSI some curious results have been found concerning narrow peaks in the energy of electron–positron pairs coming from heavy-ion scattering. One interpretation of the results of this experiment is that in strong external fields, QED undergoes a phase transition [5].

The starting point for studying QED in strong external fields is the determination of the fermion propagator in an external field. Using the Fock–Nambu–Schwinger proper time formalism [6] one obtains for the fermion propagator in Euclidean space

$$S(x, x', A)=(i\gamma D+m)(-i)\int_0^\infty ds \exp(-m^2 s)\langle x|\, \exp(-Hs)\, |x'\rangle. \tag{1.2}$$

The determination of this quantity clearly requires the wavefunctions as well as energy eigenvalues of H. As we will show both Z_5 and $S(x, x', A)$ have path integral formulations which are explicitly supersymmetric. We will also show that by introducing fermionic degrees of freedom, one can greatly simplify the determination of the path integral. This path integral representation for S has been studied in a different context by Rajeev who was interested in reformulating quantum electrodynamics as a supersymmetric theory of loops [7].

Using the chiral decomposition of the Dirac operator

$$\not{D}=\tfrac{1}{2}(1+\gamma_5)\not{D}+\tfrac{1}{2}(1-\gamma_5)\not{D}\equiv Q_+ + Q_- \tag{1.3}$$

to obtain a supersymmetric representation of H, Alvarez Gaumè [8] constructed a simple proof of the Atiyah–Singer index theorem on compact spaces. This chiral supersymmetry was also exploited by Forgacs *et al.* to evaluate the partition function and thus the $U(1)$ anomaly for the Dirac operator on both compact and non-compact 2-dimensional manifolds [9]. The calculation of the anomaly proceeds in a fashion identical to that of the determination of the Witten index in SUSY quantum mechanics [10]. The index is determined by the difference in the density of states of the two partner Hamiltonians. This in turn is related to the derivative of the difference in phase shifts. However, the phase shifts of the partner

Hamiltonians are related by SUSY, which allows a simple calculation of the anomaly.

In the past people have studied the Dirac equation in particular configurations without any strategy for finding exact solutions. In a paper written in 1967, Stanciu [11] points out that until then, only six configurations of the external field were known to be solvable. He then discusses some new solutions where the problem reduces to a known 1-dimensional Schrödinger equation. In his case only one component of the gauge field A was non-zero, and it depended on only one cartesian coordinate.

Given the renewed interest in the external field problem we have undertaken a systematic study of two different supersymmetric structures of the Dirac equation—the chiral supersymmetry discussed above and another, complex supersymmetry possessed by H [12]. In SUSY quantum mechanics it is now understood that all exactly solvable potentials can be obtained by exploiting the supersymmetry, factorization of the Hamiltonian [13], and a discrete reparametrization invariance—shape invariance [14].

The purpose of this paper is to discuss the supersymmetric structure of H and to see how the methods used to obtain analytic solutions for the Schrödinger equation can be extended to finding the eigenfunctions and eigenvalues of H (and therefore also of $\not{D}$). For the case of the Dirac equation in a 3D Coulomb field Sukumar [15] showed how to exploit the supersymmetry along with factorization and "shape invariance" to obtain the complete energy spectrum and eigenfunctions of the Dirac equation. Here we are more interested in the Euclidean Dirac operator.

We will show that all previously found solvable external electromagnetic field configurations can be found by our methods. We also find some configurations which were not known before. We find that in four dimensions the complex supersymmetry leads to a formulation of the solvability problem that is more useful than the chiral supersymmetry.

The rest of the paper is divided as follows: in Section II we discuss the supersymmetric structure of the Dirac equation in 4 dimensions with respect to both the chiral and the complex supersymmetries. We also show how to trivially obtain zero modes using the complex supersymmetry. In Section III we discuss the Dirac equation in two Euclidean space-time dimensions as a preliminary exercise. Because $O(4) = SU(2) \otimes SU(2)$ the 2D solutions are a subset of the solutions to the Euclidean 4-dimensional Dirac equation. We then find all shape invariant potentials in 2 dimensions for which the Dirac equation is exactly solvable. In Section IV we present our strategy for finding solutions for the 4-dimensional Euclidean Dirac operator and find analytic solutions when the superpotential is a function of one variable or the sum of two functions of single variables. In Section V we discuss the path integral formulation of both the partition function and the fermion propagator in an external field and show how to obtain trivially previously known results for the constant external field case. We review in Appendix I how one uses shape invariance and factorization to obtain the exact wavefunctions for the Schrödinger equation. The subset of these solutions relevant to the Dirac problem are given. In

Appendix II we show that whenever the 1-dimensional Schrödinger equation with potential $V(x)$ is exactly solvable then there always exists a corresponding static scalar field $\varphi(x)$ for which the Jackiw–Rebbi type Dirac problem in $1+1$ dimensions can be exactly solved. It turns out that $\varphi(x)$ is the superpotential corresponding to the Schrödinger potential $V(x)$.

II. Supersymmetric Structures of the Dirac Operator in 4 Dimensions

In 4 dimensions one can show that the square of the Dirac operator possesses two different supersymmetric decompositions. The first of these is the well known chiral supersymmetry.

Defining

$$Q_{\pm} = \tfrac{1}{2}(1 \pm \gamma_5)\,\gamma_\mu D_\mu, \tag{2.1}$$

where

$$D_\mu = \partial_\mu - iA_\mu,$$

one finds that the Dirac operator can be written as

$$\not{D} = Q_+ + Q_-. \tag{2.2}$$

Because Q_+ and Q_- are nilpotent, $Q_+^2 = Q_-^2 = 0$, the square of the Dirac operator, together with Q_+ and Q_- yields the usual $N=1$ supersymmetry algebra

$$H = -(\gamma_\mu D_\mu)^2 = \{Q_+, Q_-\}, \qquad [Q_+, H] = [Q_-, H] = 0 \tag{2.3}$$

first discussed in the context of SUSY quantum mechanics.

If we take the following representation of the Euclidean γ matrices,

$$\gamma_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \tag{2.4}$$

where the σ are the usual Pauli matrices, then

$$\gamma_\mu D_\mu = \begin{bmatrix} 0 & iD_0 + \sigma \cdot D \\ -iD_0 + \sigma \cdot D & 0 \end{bmatrix} = \begin{bmatrix} 0 & L \\ L^\dagger & 0 \end{bmatrix} \tag{2.5}$$

and

$$-H = (\gamma_\mu D_\mu)^2 = D_\mu D_\mu + \begin{bmatrix} \sigma \cdot (B+E) & 0 \\ 0 & \sigma \cdot (B-E) \end{bmatrix} = \begin{bmatrix} LL^\dagger & 0 \\ 0 & L^\dagger L \end{bmatrix}. \tag{2.6}$$

Here we have used the convention

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \qquad B_i=\tfrac{1}{2}\varepsilon_{ijk}F_{jk}, \qquad E_i=F_{0i}. \tag{2.7}$$

We see that if we have

$$-L^\dagger L\Psi_1=E_1\Psi_1, \tag{2.8}$$

then

$$-LL^\dagger(L\Psi_1)=E_1(L\Psi_1). \tag{2.9}$$

This shows that eigenfunctions of H with energy E_1 are given by

$$\Psi=\begin{bmatrix}0\\ \Psi_1\end{bmatrix} \quad \text{and} \quad \Psi=\begin{bmatrix}L\Psi_1\\ 0\end{bmatrix}. \tag{2.10}$$

Once we have an eigenfunction of H with a non-zero eigenvalue E_1 we can easily obtain the eigenfunctions $\Phi_\pm$ of $\not{D}$ with eigenvalues $\pm(E_1)^{1/2}$ by the construction

$$\Phi_\pm=[\not{D}\pm(E_1)^{1/2}]\Psi. \tag{2.11}$$

In the appendix we also show how to obtain the solutions of the usual massive Dirac equation from Ψ.

The chiral supersymmetry still requires that we solve the two component equation [(2.8) or (2.9)] to obtain solutions.

When A_μ has the special form

$$A_\mu=f_{\mu\nu}\partial_\nu\chi, \tag{2.12}$$

where the constant 4 by 4 matrix $f_{\mu\nu}$ has the properties

$$f_{\mu\nu}=-f_{\nu\mu}, \tag{2.13}$$

$$f_{\mu\nu}f_{\nu\alpha}=-\delta_{\mu\alpha}, \tag{2.14}$$

then there exists another breakup of the Dirac operator $\not{D}$ which leads to the so-called complex supersymmetry. In particular, for all self-dual or anti-self-dual fields the potential can be chosen to have this form. However, (2.12) allows solutions which are not restricted to being dual or anti-self-dual.

The complex supersymmetry relies on the fact that the Dirac matrices can be interpreted as fermionic creation and annihilation operators. This will generalize the form of Q_- and Q_+ which was found for the SUSY quantum mechanics of the Schrödinger equation. There the matrices $\sigma_\pm$ found in the definition of Q_- and Q_+ were interpreted as the fermion creation and annihilation operators, $\psi^\dagger$ and ψ.

In SUSY quantum mechanics one has

$$H=p^2+W^2+\sigma_3W'=\{Q_+,Q_-\}, \tag{2.15}$$

where

$$\begin{aligned} &W=\chi', \qquad \psi^\dagger=\sigma_-, \qquad \psi=\sigma_+ \\ &Q_+=\psi^\dagger(\partial_x-W)=\sigma_- e^{\chi}(\partial_x)e^{-\chi} \\ &Q_-=\psi(\partial_x+W)=\sigma_+ e^{-\chi}(\partial_x)e^{\chi}. \end{aligned} \tag{2.16}$$

The zero energy ground state wavefunctions could therefore be automatically found for any χ since for the ground state

$$Q_-\Psi=0 \Rightarrow \Psi=e^{-\chi}. \tag{2.17}$$

The introduction of the fermionic degrees of freedom allows a Lagrangian formulation of SUSY quantum mechanics and of the Hamiltonian $(\not{D})^2$ in which the SUSY transformations can be seen to be bose-fermi transformations. The Lagrangian formulation is exploited in Section V.

To see the complex supersymmetry of the Dirac operator it is simplest to choose a particular basis for the fermion creation and annihilation operators. The general structure of the complex supersymmetry of the Dirac operator in 2D dimensions is discussed in Ref. (12). Let us introduce the two complex variables

$$u=x_0+ix_3 \qquad \text{and} \qquad v=x_1+ix_2, \tag{2.18}$$

and the fermionic operators

$$b_u^\pm=\tfrac{1}{2}(\gamma_0 \pm i\gamma_3), \qquad b_v^\pm=\tfrac{1}{2}(\gamma_1 \pm i\gamma_2). \tag{2.19}$$

We need the complex derivatives $(\partial_1=\partial/\partial_{x_1})$,

$$\begin{aligned} \partial_u&=\tfrac{1}{2}(\partial_0-i\partial_3), \qquad \partial_v=\tfrac{1}{2}(\partial_1-i\partial_2), \\ \bar{\partial}_u&=\tfrac{1}{2}(\partial_0+i\partial_3), \qquad \bar{\partial}_v=\tfrac{1}{2}(\partial_1+i\partial_2), \end{aligned} \tag{2.20}$$

and define (the identical equation holds for v)

$$D_u=\partial_u-iA_u, \qquad \bar{D}_u=\bar{\partial}_u-i\bar{A}_u. \tag{2.21}$$

One can then show that we can write as in (2.2)

$$\not{D}=2(\hat{Q}_+ + \hat{Q}_-), \tag{2.22}$$

where now, however, $\hat{Q}_+$ and $\hat{Q}_-$ are defined by

$$\hat{Q}_+=b_u^+D_u+b_v^+D_v, \qquad \hat{Q}_-=b_u^-\bar{D}_u+b_v^-\bar{D}_v. \tag{2.23}$$

The condition that $H=\{\hat{Q}_+,\hat{Q}_-\}$ is that

$$\hat{Q}_+^2=\hat{Q}_-^2=0. \tag{2.24}$$

This imposes the integrability condition on the gauge field A_μ, the solution of which is Eq. (2.12). For our particular decomposition of $\not{D}$ we have that $f_{\mu\nu}$ is given by

$$\begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}. \tag{2.25}$$

The Hilbert space breaks up into three subspaces labelled by the fermion number, N, where

$$N = b_u^+ b_u^- + b_v^+ b_v^- = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.26}$$

The chiral supersymmetry operators Q_+ and Q_- defined in (2.1) connect the $N=2$ plus the $N=0$ sector to the $N=1$ sector. On the other hand, the complex supersymmetry operator $\hat{Q}_+$ defined by (2.23) takes the zero particle sector into a particular linear combination of the one particle states which is orthogonal to the linear combination obtained by applying $\hat{Q}_-$ to a state in the $N=2$ sector. These subspaces depend on the superpotential χ.

In terms of the superpotential χ we can write $\hat{Q}_+$ and $\hat{Q}_-$ in a manner analogous to Eq. (2.16). That is we have

$$\hat{Q}_+ = e^{-\chi}(b_i^+ \partial_i) e^{\chi}, \qquad \hat{Q}_- = e^{\chi}(b_i^- \bar{\partial}_i) e^{-\chi}. \tag{2.27}$$

Here i is summed over the complex coordinates u and v. We will use this form later to find the zero modes of the Dirac operator, just as we found the zero eigenstates of the Schrödinger equation, in (2.17).

If we analyze the restriction [(2.12) and (2.25)] of having the electromagnetic field A_μ derived from a superpotential we find that for our decomposition

$$E_1 = -B_1, \qquad E_2 = -B_2, \tag{2.28}$$

with E_3 and B_3 unrestricted. Another decomposition would have given the result

$$E_1 = B_1, \qquad E_2 = B_2 \tag{2.29}$$

and E_3 and B_3 unrestricted. All other decompositions are just 4-dimensional rotations of these two possibilities. We define the states of the Hamiltonian by defining the vacuum to be

$$b_u^- b_v^- \,|0_v 0_u\rangle = 0. \tag{2.30}$$

The Hilbert space of solutions decomposes according to the Fermion number operator

$$\mathfrak{H} = \mathfrak{H}_0 + \mathfrak{H}_1 + \mathfrak{H}_2, \tag{2.31}$$

and the one particle states also have a natural decomposition for a given superpotential χ,

$$\mathfrak{H}_1 = \mathfrak{H}_1^{(0)} + \mathfrak{H}_1^{(2)}. \tag{2.32}$$

The states in $\mathfrak{H}_1^{(0)}$ are obtained by applying $\hat{Q}_+$ to a zero particle state

$$\Psi_1^{(0)} = \hat{Q}_+(\Psi_0 \, |0\,0\rangle). \tag{2.33}$$

Since $\hat{Q}_+$ is a fermionic operator, we have

$$\hat{Q}_+ \Psi_1^{(0)} = 0. \tag{2.34}$$

Similarly the states in $\mathfrak{H}_1^{(2)}$ are obtained from

$$\Psi_1^{(2)} = \hat{Q}_-(\Psi_2 \, |1\,1\rangle) \qquad \text{or} \qquad \hat{Q}_- \Psi_1^{(2)} = 0. \tag{2.35}$$

In the zero and two particle sectors the Dirac equation reads

$$-\hat{Q}_- \hat{Q}_+ \Psi_0 = E\Psi_0, \qquad -\hat{Q}_+ \hat{Q}_- \Psi_2 = E\Psi_2. \tag{2.36}$$

If we solve these two equations, then we can reconstruct the whole solution either from the chiral supersymmetry or by using the fact that $\Psi_1^{(0)}$ and $\Psi_1^{(2)}$ as defined by (2.33) and (2.35) are then solutions in $\mathfrak{H}_1^{(0)}$ and $\mathfrak{H}_1^{(2)}$ with the same energy. We have then a twofold degeneracy in the space of solutions as seen from either the chiral or the complex supersymmetry.

The explicit projections of the Dirac operator squared in the zero and two particle sectors are given by

$$\begin{aligned} \hat{Q}_- \hat{Q}_+ |_{\mathfrak{H}_0} &= \tfrac{1}{4}\{\nabla^2 + \nabla^2\chi - (\nabla\chi)^2 + 2i\,\nabla\chi \circ \mathbf{f} \circ \nabla\} \\ \hat{Q}_+ \hat{Q}_- |_{\mathfrak{H}_2} &= \tfrac{1}{4}\{\nabla^2 - \nabla^2\chi - (\nabla\chi)^2 + 2i\,\nabla\chi \circ \mathbf{f} \circ \nabla\}, \end{aligned} \tag{2.37}$$

where ∇ is the gradient operator with respect to the ordinary cartesian variables and $\mathbf{f}$ is the 4×4 matrix defined by (2.13) and given explicitly for our choice of complex coordinates in (2.25).

If the condition $\nabla^2\chi = 0$ is satisfied, then the differential equation is identical in the two sectors and we end up with a fourfold degeneracy of all energy eigenvalues. If we take $\mathbf{f}$ as given by (2.25) this condition implies that the electromagnetic field is anti-self-dual. In the other inequivalent decomposition of the Dirac operator we would instead be in the self-dual sector when $\nabla^2\chi = 0$.

One of our strategies to find analytic solutions to the 4-dimensional Dirac equation is to see if there is an *additional* supersymmetry to relate the solution in the zero particle sector to the solution in the two particle sector.

In the chiral symmetry language, this is the same as the problem of finding cases where the *2 by 2 matrix* Hamiltonian $H_1 = LL^\dagger$ defined in Eq. (2.6) itself contains *another* SUSY so that it can be written as

$$H_1 = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix}. \tag{2.38}$$

If we can do that then we can find the solvable external field configurations by knowing solvable quantum mechanical potentials. For example, we can use the technique of factorization and shape invariance to obtain analytic solutions as described in the appendix.

Let us first show how one finds the zero energy solutions. If we assume that these solutions are in the $|0\,0\rangle$ sector, then we have the equation

$$\hat{Q}_+(\Psi_0\,|0\,0\rangle) = 0 \tag{2.39}$$

or

$$\partial_u(e^\chi \Psi_0)\,|0\,1\rangle + \partial_v(e^\chi \Psi_0)\,|1\,0\rangle = 0.$$

This can be satisfied only if

$$e^\chi \Psi_0 = f(\bar{u}, \bar{v}) \qquad \text{or} \qquad \Psi_0 = e^{-\chi} f(\bar{u}, \bar{v}), \tag{2.40}$$

where f is an antianalytic function. Thus candidate solutions are of the form

$$(x_0 - ix_3)^m\,(x_1 - ix_2)^n\,e^{-\chi}. \tag{2.41}$$

Only those m, n for which the wavefunction is normalizable are bona fide zero mode solutions. If instead the zero modes are in the two particle sector, we obtain, by a similar argument, that the zero modes are of the form

$$(x_0 + ix_3)^m\,(x_1 + ix_2)^n\,e^{\chi}. \tag{2.42}$$

If $\chi = \chi(\rho_1, \rho_2)$, where $\rho_1^2 = x_0^2 + x_3^2$, $\rho_2^2 = x_1^2 + x_2^2$, then m and n are the conserved angular momenta in the two 2-dimensional planes spanned by (x_0, x_3) and (x_1, x_2). This corresponds to the $SU(2) \otimes SU(2)$ decomposition of $O(4)$.

Thus we see that when the vector potential A_μ is derivable from the superpotential χ, then the method for obtaining zero modes is quite analogous to what happens in the Schrödinger case.

III. The Spectrum of the 2D Dirac Operator

Although our primary interest is in the 4-dimensional Dirac equation, we first study the 2-dimensional Dirac equation because of its simpler structure and because one set of solutions to the 4-dimensional equation is just the solution of the 2D equation and another is just a product of two 2-dimensional solutions.

In two dimensions, the chiral and complex supersymmetries are equivalent since we can always by a choice of gauge write

$$A_\mu = \varepsilon_{\mu\nu} \partial_\nu \chi. \tag{3.1}$$

We use the representation

$$\gamma_0 = \sigma_1, \qquad \gamma_1 = \sigma_2. \tag{3.2}$$

In that representation we have that

$$\gamma \cdot D = \begin{bmatrix} 0 & L \\ L^\dagger & 0 \end{bmatrix}, \qquad (\gamma \cdot D)^2 = \begin{bmatrix} LL^\dagger & 0 \\ 0 & L^\dagger L \end{bmatrix}, \tag{3.3}$$

where

$$L = D_0 - iD_1, \qquad L^\dagger = D_0 + iD_1. \tag{3.4}$$

We find that $(B = F_{01})$

$$(\gamma \cdot D)^2 = -(p_\mu - A_\mu)^2 + \sigma_3 B \tag{3.5}$$

from which one immediately reads off $LL^\dagger$ and $L^\dagger L$ from (3.3).

The complex supersymmetry leads to the same result. For the complex supersymmetry we introduce

$$\begin{aligned} b_u^\pm &= \sigma^\pm, & \partial_u &= \tfrac{1}{2}(\partial_0 - i\partial_1) \\ \hat{Q}_+ &= b_u^+ D_u, & \hat{Q}_- &= b_u^- \bar{D}_u, \end{aligned} \tag{3.6}$$

which yield $\bar{D}_u = L$, $D_u = L^\dagger$ as given above.

The complex supersymmetry for $(\gamma \cdot D)^2$ is always ensured in 2 dimensions since we can always choose the gauge:

$$A_\mu = \varepsilon_{\mu\nu} \partial_\nu \chi. \tag{3.7}$$

In analogy with finding exact solutions to SUSY quantum mechanics we look for potentials that lead to shape invariance. Using (3.7) we find

$$LL^\dagger = \{\nabla^2 - \nabla^2\chi - (\nabla\chi)^2 + 2i\, \nabla\chi \circ \boldsymbol{\varepsilon} \circ \nabla\} \tag{3.8}$$

$$L^\dagger L = \{\nabla^2 + \nabla^2\chi - (\nabla\chi)^2 + 2i\, \nabla\chi \circ \boldsymbol{\varepsilon} \circ \nabla\}. \tag{3.9}$$

These equations are similar in structure to those of SUSY quantum mechanics except that we now have a partial differential equation instead of an ordinary differential equation. If we can find a system of orthogonal coordinates and the superpotential χ depends only on one of the two variables, then we can reduce this problem to a 1-dimensional quantum mechanical one. For example, in cartesian coordinates, if we let $\chi = \chi(x_0)$, then the linear momentum p_1 is conserved. Letting

$$\Psi(x_0, x_1) = e^{ip_1 x_1} \varphi(x_0), \tag{3.10}$$

we find

$$\begin{aligned} -LL^\dagger &= -\partial_0^2 + [p_1 + \chi']^2 + \chi'' \\ -L^\dagger L &= -\partial_0^2 + [p_1 + \chi']^2 - \chi''. \end{aligned} \tag{3.11}$$

This is precisely the structure of SUSY quantum mechanics with the SUSY superpotential

$$W = p_1 + \chi'. \tag{3.12}$$

For a physical electromagnetic field, χ' cannot depend on p_1. Thus only those solvable quantum mechanical potentials of the form $W = p_1 + f(x_0)$, where $f(x_0)$ is independent of p_1, lead to solvable Dirac equations with

$$A_1(x_0) = -f(x_0). \tag{3.13}$$

As shown in the appendix, when we solve the shape invariance equation with the above form there are only *three* solutions: the Morse potential with $W(x) = C_1 - C_2 e^{-\alpha x}$, the Rosen–Morse potential with $W(x) = C_1/C_2 + C_2 \tanh(\alpha x)$, and the oscillator potential $W(x) = \omega x + C_1$.

Let us next introduce polar coordinates via $x_0 = \rho \cos\theta$, $x_1 = \rho \sin\theta$. If $\chi = \chi(\rho)$ then we obtain using (3.1)

$$A_\theta = -\rho\chi', \qquad A_\rho = 0, \tag{3.14}$$

where the prime denotes differentiation with respect to ρ.

The electromagnetic field F_{01} we denote as before as B,

$$B(\rho) = \partial_0 A_1 - \partial_1 A_0 = -\chi'' - \chi'/\rho = \rho^{-1}\, \partial A_\theta/\partial\rho. \tag{3.15}$$

The operators L and $L^\dagger$ which factorize the Hamiltonian take the forms

$$\begin{aligned} L &= e^{-i\theta}[\partial_\rho - i\rho^{-1}(\partial_\theta - iA_\theta)], \\ L^\dagger &= e^{i\theta}[\partial_\rho + i\rho^{-1}(\partial_\theta - iA_\theta)]. \end{aligned} \tag{3.16}$$

Following Akhoury and Comtet [10], we take for the wavefunction

$$\Psi = \begin{bmatrix} f_n \\ g_n \end{bmatrix}, \quad f_n = \rho^{-1/2} \Phi_-^{m,n}(\rho) \exp[i(m - \tfrac{1}{2})\theta], \quad g_n = \rho^{-1/2} \Phi_+^{m,n}(\rho) \exp[i(m + \tfrac{1}{2})],$$

where m takes on half integer values.

Then we have that the pairing equations

$$L^\dagger f_n = \sqrt{E_n}\, g_n, \qquad L g_n = \sqrt{E_n}\, f_n \tag{3.18}$$

become

$$\begin{aligned} -(\partial_\rho - W)\, \Phi_+^{m,n} &= \sqrt{E_n}\, \Phi_-^{m,n} \\ (\partial_\rho + W)\, \Phi_-^{m,n} &= \sqrt{E_n}\, \Phi_+^{m,n}, \end{aligned} \tag{3.19}$$

where $W = (m - A_\theta)/\rho$.

Since we want A_θ to be independent of m, we see from the appendix that there are only two shape invariant Schrödinger potentials that are relevant—namely the free problem in 2-dimensional radial coordinates (with shifted angular momentum) $W = (m - \Phi)/\rho$ and the 2-dimensional harmonic oscillator $W = -\frac{1}{2}B\rho + m/\rho$.

The first case is connected with the vortex. Asymptotically the vortex field goes like

$$A = \Phi\, \nabla\theta \tag{3.20}$$

so that $A_\theta = \Phi$, $l = m - \Phi$, leading to a shifted angular momentum free problem and corresponding to $\chi = -\Phi \ln(\rho)$, $W = (m - \Phi)/\rho$. A non-singular vortex solution can be obtained [16] by assuming

$$\begin{aligned} \chi &= -\tfrac{1}{4}B\rho^2 && \text{for} \quad \rho < R \\ \chi &= -\tfrac{1}{4}BR^2(1 + \ln(\rho/R)) && \text{for} \quad \rho \geqslant R. \end{aligned} \tag{3.21}$$

The second case corresponds to the constant field problem so that

$$A_\theta = \tfrac{1}{2}B\rho^2, \qquad \chi = -\tfrac{1}{4}B\rho^2. \tag{3.22}$$

Thus we see that for the 2-dimensional Dirac equation we find only five analytically solvable potentials—three for the case of a conserved linear momentum and two for the case of conserved angular momentum.

IV. The Spectrum of the 4-Dimensional Dirac Equation

The relevant equations that we need to solve in 4 Euclidean dimensions are Eqs. (2.36), where in the $N = 0$ and $N = 2$ fermion sectors, $\hat{Q}_-\hat{Q}_+$ and $\hat{Q}_+\hat{Q}_-$ are given by (2.37).

The simplest way of solving the pair of equations (2.36) is to have χ a function of a single variable. The solutions of Stanciu [11] arise when we choose that variable to be one of the cartesian coordinates, say x_1. One then chooses for Ψ_0 and Ψ_2 the form

$$\Psi = e^{i(p_0 x_0 + p_2 x_2 + p_3 x_3)}\, \Phi(x_1). \tag{4.1}$$

One has that $A_2 = -\chi'(x_1)$ is the only non-zero component of the field and we have

$$4H_0 = p_0^2 + p_3^2 - \partial_1^2 + (p_2 + \chi')^2 - \chi'' \tag{4.2}$$

$$4H_2 = p_0^2 + p_3^2 - \partial_1^2 + (p_2 + \chi')^2 + \chi''. \tag{4.3}$$

We see that H_0 and H_2 are SUSY partners with

$$W = p_2 - A_2(x_1). \tag{4.4}$$

Equations (4.2)–(4.3) are essentially the same as (3.11); thus again because we need A_2 independent of p_2 there are only three solvable potentials—the Morse potential with $W(x) = C_1 - C_2 e^{-\alpha x}$, the Rosen–Morse potential with $W(x) = C_1/C_2 + C_2 \tanh(\alpha x)$, and the oscillator potential $W(x) = \omega x + C_1$. The eigenvalues and eigenfunctions of these potentials are given in the appendix. These potentials were the ones considered by Stanciu [11] in solving the massive Dirac equation. Because we are in Euclidean space, these solutions to the squared Dirac operator in Euclidean space are magnetic field solutions.

The next simplest choice for χ is that $\chi = \chi(\rho)$, where $\rho^2 = x_1^2 + x_2^2$ for example. Then the magnetic field is just a function of ρ and we obtain the same result in the $(1, 2)$ plane as in the discussion of the Euclidean 2D Dirac equation. We obtain wavefunctions which are those functions of (ρ, m) found there times plane waves in the 0 and 3 directions. Thus we have in that case that the only solutions are a constant magnetic field F_{12} or a vortex field F_{12}. The solutions are similar to those of the 2D equation with appropriate changes of coordinates.

For $\chi = \chi(\rho_3)$, where $\rho_3^2 = x_1^2 + x_2^2 + x_3^2$ we get the possibility of Coulomb solutions. In a beautiful paper, Sukumar [15] has analyzed in detail how supersymmetry and shape invariance allow one to solve this problem.

The most interesting 1-dimensional χ is $\chi = \chi(r)$, where $r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. In that case

$$2i\,\nabla\chi \cdot f \cdot \nabla = -2r^{-1}\chi'(r)(M_{03} + M_{12}), \tag{4.5}$$

where M_{ij} are the $O(4)$ angular momentum generators. Since $O(4) = SU(2) \otimes SU(2)$ we can simultaneously diagonalize M_{03}, M_{12}, and M^2. Letting the eigenvalues of $M_{03} = m$ and $M_{12} = n$, $M^2 = j(j+2)$, we find that, after scaling out a factor of $r^{-3/2}$, the radial equations for H_0 and H_2 become

$$\begin{aligned} 4H_0 &= -\partial_r^2 + r^{-2}[j(j+2) + \tfrac{3}{4}] + (\chi')^2 - \chi'' + r^{-1}\chi'(2m + 2n - 3) \\ 4H_2 &= -\partial_r^2 + r^{-2}[j(j+2) + \tfrac{3}{4}] + (\chi')^2 + \chi'' + r^{-1}\chi'(2m + 2n + 3). \end{aligned} \tag{4.6}$$

There are only two solutions where χ is independent of m, n, j such that H_0 and H_2 are SUSY partners and the potential is shape invariant; namely

$$\chi'(r) = \tfrac{1}{2}\omega r \text{ (harmonic oscillator)}, \qquad \chi = \omega r^2/4 \tag{4.7}$$

which is the constant magnetic field case with

$$F_{\mu\nu} = \omega f_{\mu\nu} \tag{4.8}$$

and f given by (2.25), and a vortex like potential

$$\chi'(r) = \alpha/r, \qquad \chi = \alpha \ln r + C \tag{4.9}$$

which leads to the field

$$A_\mu = \alpha f_{\mu\nu} x_\nu / r \tag{4.10}$$

which as in 2 dimensions corresponds to a shifted 4D angular momentum. As in 2 dimensions one can also solve exactly the case where we combine these two solutions at $r = R$. So that

$$\begin{aligned} \chi &= \omega r^2 && \text{for} \quad r < R \\ \chi &= \omega R^2(1 + \ln(r/R)) && \text{for} \quad r \geqslant R. \end{aligned} \tag{4.11}$$

The other shape invariant solutions which are possible in the 4D Dirac equation are those for which χ is the sum of two functions, each of which is a function of one variable only.

For example, for the particular choice (2.25) for $f_{\mu\nu}$ the two 2-dimensional planes (x_0, x_3) and (x_1, x_2) are distinguished. If we choose

$$\chi = \chi_0(x_0) + \chi_1(x_1), \tag{4.12}$$

then p_3 and p_2 are conserved momenta. Using a product wavefunction of the form

$$\Psi = e^{i(p_2 x_2 + p_3 x_3)} \varphi_0(x_0)\, \varphi_1(x_1) \tag{4.13}$$

the strutures of H_0 and H_2 become

$$\begin{aligned} 4H_0 &= -\partial_0^2 + (p_3 + \partial_0 \chi_0)^2 - \partial_0^2 \chi_0 - \partial_1^2 + (p_2 + \partial_1 \chi_1)^2 - \partial_1^2 \chi_1 \\ 4H_2 &= -\partial_0^2 + (p_3 + \partial_0 \chi_0)^2 + \partial_0^2 \chi_0 - \partial_1^2 + (p_2 + \partial_1 \chi_1)^2 + \partial_1^2 \chi_1. \end{aligned} \tag{4.14}$$

After separating variables, we get two 1-dimensional SUSY Schrödinger equations in the x_0 and x_1 coordinate systems. Thus one can have different shape invariant potentials in say the x_0 and x_1 directions and still solve analytically the Dirac equation. Each of these equations is exactly the equation we had for the 2-dimensional Dirac equation. Thus we have the restriction again that for χ to be independent of the conserved p_i only those χ corresponding to the 1-dimensional Morse, oscillator, and Rosen–Morse potentials are exactly solvable.

Similarly because $O(4) = SU(2) \otimes SU(2)$ we can simultaneously diagonalize say

M_{03}, M_{12}, and M^2. Thus we will get a similar structure in the two planes ρ_1, φ_1 and ρ_2, φ_2 if we choose

$$\chi = \chi_1(\rho_1) + \chi_2(\rho_2), \tag{4.15}$$

where $\rho_1^2 = x_0^2 + x_3^2$, $\rho_2^2 = x_1^2 + x_2^2$. Taking product solutions we can again separate the equation into two copies of the problem that we had for the 2D Dirac equation. So now we can have following the arguments from Eqs. (3.14)–(3.21) four possibilities with χ_1 and χ_2 being of the form ρ^2 or $\ln \rho$.

This exhausts the solutions we have found for the Dirac equation which relied on our knowing exact solutions to the Schrödinger equation.

V. Path Integral Formulation of the Fermion Propagator and the Witten Index

The solvability of the Dirac operator is of course related to the ability to integrate the path integral. In this section we show that by introducing fermionic degrees of freedom, the path integral for the Dirac operator, which is initially a path ordered integral, is reduced to an ordinary path integral. By integrating over the fermionic degrees of freedom, we obtain a purely bosonic path integral. For the case of a constant external field strength $F_{\mu\nu}$, the bosonic path integral is a Gaussian, which allows one to obtain trivially the effective potential for QED as well as the fermion propagator $S(x, x, A)$ and the associated index I.

As stated in the Introduction, the quantities we calculate are the Green's functions,

$$\begin{aligned} G(x, x, \tau) &= \langle x|\, e^{-H\tau}\, |x'\rangle|_{x=x'} \\ G_5(x, x, \tau) &= \langle x|\, \mathrm{Tr}(\gamma_5 e^{-H\tau})\, |x'\rangle|_{x=x'}. \end{aligned} \tag{5.1}$$

By using Schwinger's proper time formalism we can determine $S((x\,|\,A)$ from G. The index Z_5 is just the spatial integral over G_5. Once we introduce fermionic variables then G is determined by choosing antiperiodic boundary conditions for the fermions. If instead we use periodic boundary conditions on the fermions we obtain instead G_5. The index of the Dirac operator which is related to the axial anomaly as shown in the papers of Friedan and Windey [1] and Freedman and Cooper [10] is obtained by integrating G_5 over dx as in (1.1).

The square of the Dirac operator is given by

$$H = (p - eA)^2 + \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}. \tag{5.2}$$

If we do not introduce auxiliary fermions, then the related matrix valued Lagrangian would be

$$\begin{aligned} L(x, \dot{x}) &= \tfrac{1}{4}\dot{x}_\mu \dot{x}_\mu + ieA_\mu \dot{x}_\mu - \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}, \\ \sigma^{\mu\nu} &= (2i)^{-1}\,[\gamma^\mu, \gamma^\nu] \end{aligned} \tag{5.3}$$

and we would obtain for Feynman's path integral representation

$$\langle x|\, e^{-H\tau}\, |x'\rangle = \int \mathfrak{D}x_\mu(t)\, P \exp\left[-\int_0^\tau d\tau'\, L(x, \dot{x})\right], \tag{5.4}$$

where P denotes path ordering. In analogy with what one does in SUSY quantum mechanics [2], several authors [1, 7] have notied that one can introduce the Grassman variables ψ_μ via

$$\psi_\mu = 2^{-1/2}\gamma_\mu, \qquad \{\psi_\mu, \psi_\nu\} = \delta_{\mu\nu}. \tag{5.5}$$

Then one can write H as

$$H = (p - eA)^2 - \tfrac{1}{2} i\psi_\mu F_{\mu\nu}\psi_\nu. \tag{5.6}$$

The Lagrangian now becomes

$$L_{ss} = \tfrac{1}{4}\dot{x}_\mu \dot{x}_\mu + ieA_\mu \dot{x}_\mu + \tfrac{1}{2}\psi_\mu(\partial_t \delta_{\mu\nu} + F_{\mu\nu})\psi_\nu \tag{5.7}$$

which is invariant under the SUSY transformations

$$\delta x_\mu = -i\varepsilon\psi_\mu; \qquad \delta\psi_\mu = \varepsilon\dot{x}_\mu. \tag{5.8}$$

We now obtain for the path integral

$$G(x, x; \tau) = \langle x|\, e^{-H\tau}\, |x\rangle = \int \mathfrak{D}x_\mu(t)\, \mathfrak{D}\psi_\mu \exp\left[-\int_0^\tau d\tau'\, L_{ss}(x, \dot{x})\right], \tag{5.9}$$

where we impose antiperiodic boundary conditions on the fermions at 0 and τ. To determine G_5 we have exactly the same path integral except one imposes periodic boundary conditions on ψ at $t = 0$ and τ. Since the fermion path integral is quadratic, we can perform the functional integral over the fermionic degrees of freedom exactly for arbitrary $F_{\mu\nu}$.

Consider first the quantum mechanics case where F is replaced by an ordinary function W. After performing the path integral, we must evaluate

$$\det[\partial_t - W] = \prod_m \lambda_m, \tag{5.10}$$

where the λ_m satisfy

$$(\partial_t - W(x))\Psi_m = \lambda_m \Psi_m, \tag{5.11}$$

so that

$$\Psi_m = C_m \exp \int_{\tau_0}^{\tau} d\tau'\, [\lambda_m + W(x(\tau'))]. \tag{5.12}$$

Imposing antiperiodic boundary conditions $\Psi_m(\tau) = -\Psi_m(0)$ yields

$$\begin{aligned}\lambda_m &= \tau^{-1}[i(2m+1)\pi - \varepsilon(\tau)]\\ \varepsilon(\tau) &= \int_0^\tau d\tau'\, W(x(\tau'))\end{aligned} \tag{5.13}$$

and the fermion determinant is

$$\prod_m \frac{\lambda_m(g)}{\lambda_m(0)} = \cosh \int_0^\tau d\tau' \frac{1}{2} W(x(\tau')), \tag{5.14}$$

whereas imposing periodic boundary conditions $\Psi_m(\tau) = \Psi_m(0)$ yields

$$\lambda_m = \tau^{-1}[i(2m)\pi - \varepsilon(\tau)] \tag{5.15}$$

so that for G_5 we have that the determinant is given by

$$\frac{\prod_{-\infty}^{\infty} \lambda_m(g)}{\prod_{m\neq 0} \lambda_m(g=0)} = -\frac{\varepsilon}{\tau}\prod_1^\infty \left(1 + \left[\frac{\varepsilon}{2m\pi}\right]^2\right) = -\frac{2}{\tau}\sinh\left(\frac{1}{2}\int_0^\tau d\tau'\, W(x(\tau'))\right). \tag{5.16}$$

To generalize to the case of an antisymmetric matrix $F_{\mu\nu}$, we can follow the work of Alvarez Gaumè in his Bonn Lectures [16]. $F_{\mu\nu}$ is antisymmetric, so it can be skew-diagonalized:

$$F_{\mu\nu} \sim \begin{bmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & -x_2 & 0 \end{bmatrix}. \tag{5.17}$$

In terms of the "eigenvalues" of F, x_n we obtain for the fermion path integral

$$\int \mathfrak{D}\psi_\mu \exp\left[\int_0^\tau \frac{i}{2}\psi_\mu(\partial_t\delta_{\mu\nu} + F_{\mu\nu})\psi_\nu\, d\tau'\right] = \frac{\det^{1/2}(i\partial_\tau\delta_{\mu\nu} + F_{\mu\nu})}{\det'^{1/2}(i\partial_\tau\delta_{\mu\nu})}. \tag{5.18}$$

The prime in det′ denotes that we have omitted the zero mode contribution.

For antiperiodic boundary conditions the ratio of determinants yields

$$\prod_i \left[\cosh\frac{1}{2}\int_0^\tau x_i\, d\tau\right], \tag{5.19}$$

whereas for periodic boundary conditions relevant to the Witten index we obtain instead

$$\left(\frac{2}{\tau}\right)^n \prod_i \left[-\sinh\int_0^\tau d\tau \frac{x_i}{2}\right]. \tag{5.20}$$

Thus we can always perform the integration over fermions to get a sum of bosonic path integrals for G or G_5. Whenever the path integrals can be transformed into Gaussian form, then we can explicitly evaluate the fermion propagator, and the index.

Let us now calculate G and G_5 in the constant field case for which

$$A_\mu = -\tfrac{1}{2}F_{\mu\nu}x_\nu \tag{5.21}$$

with $F_{\mu\nu}$ a constant matrix. The Fermion determinant is independent of x and can now be factored out of the path integral. The remaining path integral is Gaussian and can be performed trivially,

$$\begin{aligned} \tilde{Z} &= \int \mathfrak{D}x_\mu(t) \exp\left[- \int_0^\tau d\tau' \frac{1}{2}\dot{x}_\mu \dot{x}_\mu - ie\frac{1}{2}\dot{x}_\mu F_{\mu\nu} x_\nu \right] \\ &= \frac{\det'^{1/2}(d^2\delta_{\mu\nu}/dt^2)}{\det'^{1/2}(-(d^2/dt^2)\delta_{\mu\nu} + iF_{\mu\nu}(d/dt))} \\ &= \prod_i \frac{x_i/2}{\sinh(x_i\tau/2)}. \end{aligned} \tag{5.22}$$

In 2 dimensions only $F_{01} = B$ exists and we can write H in diagonal form,

$$H = (p - eA)^2 + \sigma_3 B, \tag{5.23}$$

so that it resembles SUSY quantum mechanics in that

$$\begin{aligned} Z &= \operatorname{Tr} e^{-H_+\tau} + \operatorname{Tr} e^{-H_-\tau} \\ Z_5 &= \operatorname{Tr} e^{-H_+\tau} - \operatorname{Tr} e^{-H_-\tau}, \end{aligned} \tag{5.24}$$

where

$$H_\pm = (p - eA)^2 \pm B.$$

Using the above equations we obtain, since $x_1 = B$,

$$Z = B\tanh(B\tau); \qquad Z_5 = B = \tfrac{1}{2}\varepsilon_{\mu\nu}F_{\mu\nu}, \tag{5.25}$$

a result obtained with more difficulty by Akhoury and Comtet [10] using heat kernel methods and the exact eigenvalues and eigenfunctions.

In 4 dimensions one can use the trick of Schwinger [6] to find the eigenvalues of F. Using the fact that we can write

$$F_{\mu\nu}F^*_{\nu\lambda} = \mathfrak{G}\delta_{\mu\lambda}, \qquad \mathfrak{G} = \tfrac{1}{4}(FF^*) \tag{5.26}$$

$$F^*_{\mu\nu}F^*_{\nu\lambda} + F_{\mu\nu}F_{\nu\lambda} = 2\mathfrak{F}\delta_{\mu\lambda}, \qquad \mathfrak{F} = \tfrac{1}{4}(FF). \tag{5.27}$$

we find

$$x_1 = [\mathfrak{F} + (\mathfrak{F}^2 - \mathfrak{G}^2)^{1/2}]^{1/2}, \qquad x_2 = [\mathfrak{F} - (\mathfrak{F}^2 - \mathfrak{G}^2)^{1/2}]^{1/2}. \tag{5.28}$$

Thus

$$\prod_i x_i = \mathfrak{G} = \tfrac{1}{4}(FF^*). \tag{5.29}$$

For the path integral we therefore obtain

$$G(x, x; \tau) = \langle x|\, e^{-H\tau}\, |x\rangle = F_{\mu\nu} F^*_{\mu\nu} \prod_{i=1}^{2} \frac{\cosh(x_i \tau/2)}{\sinh(x_i \tau/2)} \tag{5.30}$$

which is just the result of Schwinger rewritten in Euclidean space.

For G_5 we obtain

$$G_5(x, x; \tau) = \langle x|\, \mathrm{Tr}(\gamma_5 e^{-H\tau})\, |x\rangle = \mathfrak{G} = \tfrac{1}{4}(FF^*). \tag{5.31}$$

Once we have obtained G, the effective action is obtained by integration,

$$S_{\text{eff}} = \int dx\, L(x), \qquad L(x) = \int_{\varepsilon}^{\infty} ds\, s^{-1} e^{-m^2 s} G(s). \tag{5.32}$$

This gives an example of the use of the auxiliary fermion variables.

APPENDIX I: SUSY, Shape Invariance, and Solvable Potentials

In this appendix we briefly sketch the procedure for obtaining exact energy eigenvalues, eigenfuctions, and S-matrix elements for a class of Dirac Hamiltonians. There are two separate problems that we are interested in. First, in order to determine field theory quantities such as $S(x, y{:}A)$ or the index for particular external fields, it is sufficient to know the eigenvalues and eigenfunctions (continuous and discrete) of $\not{D}^2$. Then one can directly evaluate Z_5 and S in Eqs. (1.1) and (1.2) by brute force as was done in two dimensions by Akhoury and Comtet [10].

Another problem of interest is to find solutions to the massive Dirac equation. Since we are in Euclidean space, we can only discuss magnetic field solutions. As shown by Feynman and Gell-Mann [18] and Brown [19] the solution of the four component Dirac equation in the presence of an external electromagnetic field can be generated from the solution of a two component relativistically invariant equation. The connection is if Ψ obeys the two component equation,

$$[(\mathbf{P} - e\mathbf{A})^2 + m^2 + e\boldsymbol{\sigma} \circ (\mathbf{B} - \mathbf{E})]\boldsymbol{\Psi} = (\bar{E} - eA_0)^2 \boldsymbol{\Psi}, \tag{A1}$$

then the four component spinors that are solutions of the massive Dirac equation are generated from the two component Ψ via

$$\Psi_D = \begin{bmatrix} (\sigma \circ (p - eA) + (\bar{E} - A_0) + m)\Psi \\ (\sigma \circ (p - eA) + (\bar{E} - A_0) - m)\Psi \end{bmatrix}. \tag{A2}$$

For the case that $A_0 = A_1 = A_3 = 0$ and $A_2 = A_2(x_1)$ we find that the two component $\boldsymbol{\Psi}$ satisfies the equation

$$[-\partial_1^2 + p_3^2 + (p_2 - eA_2(x_1))^2 + m^2 + e\sigma_3 B_3]\boldsymbol{\Psi} = \bar{E}^2 \boldsymbol{\Psi}. \tag{A3}$$

A comparison with Eqs. (4.3) and (4.4) shows that one can determine $\boldsymbol{\Psi}$ from our solutions to (2.36), (4.3), and (4.4) with the identification

$$\bar{E}_n + m = 2E_n + p_0 \qquad \text{and} \qquad \bar{E}_n - m = 2E_n - p_0, \tag{A4}$$

where E_n is an eigenvalue of $H = -\not{D}^2$, in Eq. (2.36).

Once we have reduced the problem to a 1-dimensional Schrödinger equation such as (A3) then one can determine the wavefunctions, eigenvalues, and S-matrix of such an equation using supersymmetry, factorization, and shape invariance.

It is well known that all 1-dimensional Schrödinger Hamiltonians possessing a ground state wavefunction Ψ_0 with energy E_0 can be factorized [13, 14]:

$$H_1 \Psi_n^{(1)} = [-d^2/dx^2 + V_1(x, a_1)]\, \Psi_n^{(1)} = [A^\dagger A + E_0^{(1)}]\, \Psi_n^{(1)} = E_n^{(1)} \Psi_n^{(1)}, \tag{A5}$$

where

$$\begin{gathered} A^\dagger = -d/dx + W(x, a_1), \qquad A = d/dx + W(x, a_1) \\ V_1 = W^2(x, a_1) - dW(x, a_1)/dx + E_0^{(1)}. \end{gathered} \tag{A6}$$

The a_1 are the parameters describing the potential, n labels the states, $n = 0, 1, \ldots,$ and $n = 0$ denotes the ground state. The superpotential $W(x, a_1)$ is given in terms of the ground state wavefunction $\Psi_0^{(1)}(x, a_1)$ as

$$W(x, a_1) = -d(\ln \Psi_0^{(1)}(x, a_1))/dx. \tag{A7}$$

For simplicity let us set $E_0^{(1)} = 0$. Then the partner Hamiltonian

$$\begin{aligned} H_2 &= AA^\dagger = -d^2/dx^2 + V_2, \\ V_2 &= W^2(x, a_1) + dW(x, a_1)/dx \end{aligned} \tag{A8}$$

gives the same spectrum as $H_1 = A^\dagger A$ but with the ground state missing; that is,

$$E_0^{(1)} = 0, \qquad E_n^{(2)} = E_{n+1}^{(1)}. \tag{A9}$$

In addition the eigenfunctions of H_1 and H_2 with the same energy are related,

$$\begin{aligned}\Psi^{(1)}_{n+1}(x) &= [E^{(1)}_{n+1}]^{-1/2} A^{\dagger}\Psi^{(2)}_{n}(x)\\ \Psi^{(2)}_{n}(x) &= [E^{(2)}_{n}]^{-1/2} A\Psi^{(1)}_{n+1}(x).\end{aligned} \tag{A10}$$

It is well understood by now that the degeneracy in the two spectra is due to a supersymmetry [2]. We define the super-Hamiltonian H and the supercharges Q and $Q^{\dagger}$ as

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \qquad Q^{\dagger} = \begin{bmatrix} 0 & A^{\dagger} \\ 0 & 0 \end{bmatrix}. \tag{A11}$$

These operators are the 2-dimensional representation of the sl(1/1) superalgebra:

$$[Q, H] = [Q^{\dagger}, H] = 0, \qquad \{Q^{\dagger}, Q\} = H, \qquad \{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0. \tag{A12}$$

The fact that the supercharge commutes with H gives the energy degeneracy.

Clearly this process can be continued. Using factorization, V_2 can also be written as $A_2^{\dagger}A_2 + E_0^{(2)}$, and the partner Hamiltonian of this can be constructed. This produces a ladder of potentials, V_n. If the partner potentials are "shape invariant" in that V_2 has the same functional form as V_1 but different parameters describing the strength and shape of the potential (except for an additive constant), then as noted by Gendenshtein and Infeld and Hull [14] the full ladder of potentials will be shape invariant,

$$V_{n+1}(x, a_n) = V_n(x, a_{n+1}) + C(a_n), \tag{A13}$$

and the energy spectrum of the original potential can be determined algebraically:

$$E^{(1)}_n = \sum_{k=2}^{n+1} C(a_k), \qquad E^{(1)}_0 = 0. \tag{A14}$$

Making use of (A10) Dutt *et al.* [14] then showed that the energy eigenfunctions could also be obtained algebraically:

$$\Psi^{(1)}_n(x, a_1) = \prod_{k=1}^{n} A^{\dagger}(x, a_k)\, \Psi^{(1)}_0(x, a_{n+1}). \tag{A15}$$

In fact by using relations (A10) and interpreting them as recursion relations for special functions, explicit wavefunctions have been obtained for all known shape invariant potentials [20]. Furthermore, on analytic continuation of these wavefunctions to imaginary values of k, the scattering matrix has also been obtained for these potentials [22].

Let us now come to the question of categorizing the shape invariant potentials, i.e., those which satisfy the condition (A13). This is a highly non-trivial problem as (A13) is a non-linear Riccati type of equation. However, a small number of shape invariant potentials have been found and categorized by assuming that the superpotential $W(x, a)$ factorizes [14, 21]:

$$W(w, a) = \sum f_i(a)\, g_i(x). \tag{A16}$$

In this paper we have seen that there are basically two different types of relevant 1-dimensional problems depending on whether one separates variables in cartesian coordinates or in polar coordinates. From (2.37) and (3.8) we find that the equation for the square of the Dirac operator is most simple when written in terms of the potential $\chi(x_i)$ which determines the electromagnetic field via (2.12). If χ depends on one cartesian coordinate [$\chi = \chi(x_1)$, say] then we obtain a 1-dimensional quantum mechanical problem where the superpotential has the form [see (3.4), (4.3), and (4.4)]

$$W(x_1) = p_2 - A_2(x_1). \tag{A17}$$

If χ is the sum of two functions of one variable of the form [see (4.14)]

$$\chi = \chi(x_1) + \chi(x_3) \tag{A18}$$

then one can separate variables and one has a 1-dimensional problem in both the 1 and 3 directions with W having the form of Eq. (A17) (with different variables x_i, etc.).

For 1-dimensional problems having the form

$$W(x) = p_i - A_i(x) \tag{A19}$$

one needs that the electromagnetic field $A_i(x)$ is independent of the conserved linear momentum p_i. This puts severe constraints on the acceptable solutions of the shape invariance condition:

$$W^2(x, a_1) + W'(x, a_1) = W^2(x, a_0) - W'(x, a_0) + C(a_0). \tag{A20}$$

If we let $W(x, p) = p + f(x)$ we obtain the equation for $f(x)$ ($a_0 = p$, $a_1 = p - \alpha$),

$$2f'(x) - 2\alpha f = p^2 - (p-\alpha)^2 - C(p). \tag{A21}$$

In order for $f(x)$ to be independent of p, the r.h.s. of (A21) must be independent of p. Thus we choose

$$C(p_2) = p^2 - (p-\alpha)^2 - 2k \tag{A22}$$

and obtain

$$f(x) = k/\alpha + be^{-\alpha x} \tag{A23}$$

with k, α, and b arbitrary parameters. This gives the Morse potential:

$$W(x) = p_1 + C_1 + C_2\alpha^{-1}(1 - e^{-\alpha x}). \tag{A24}$$

For the special case where $\alpha = 0$ we obtain instead the oscillator potential

$$W(x) = p_1 + \omega x + C. \tag{A25}$$

The other way of obtaining a solution independent of p is to assume that

$$W(x) = B/A + Af(x), \tag{A26}$$

where $B/A = p$, and we want A and $f(x)$ independent of p. Now we choose $a_0 = A$, $a_1 = A - \alpha$ under our shape invariance condition, and we find that the solution is

$$f(x) = \tanh(\alpha x), \tag{A27}$$

giving rise to the Rosen–Morse potential

$$W(x) = p + A\tanh(\alpha x) + C. \tag{A28}$$

The next type of problem was that where the potential was of the form $\chi = \chi(\rho_1)$ or $\chi = \chi(\rho_1) + \chi(\rho_3)$, where $\rho_1^2 = x_1^2 + x_2^2$, $\rho_3^2 = x_0^2 + x_2^2$. In that case we found that the problem reduced to knowing the solution to the 2-dimensional Euclidean Dirac problem (see (3.19)) with $W(\rho)$ having the form

$$W(m, \rho) = m/\rho - A_\vartheta(\rho)/\rho = m/\rho - f(\rho). \tag{A29}$$

From the shape invariance equation with $a_0 = m$, $a_1 = m + 1$, we obtain the differential equation

$$\frac{df}{d\rho} + \frac{f}{\rho} = \frac{C}{2}. \tag{A30}$$

In order for A_ϑ to be independent of m we need C to be independent of m. The general solution is

$$f = \frac{C\rho}{4} - \frac{\alpha}{\rho}, \tag{A31}$$

where α is an integration constant. For $\alpha = 0$ we obtain the 3-dimensional oscillator potential with $C = 2\omega$,

$$W(\rho) = m/\rho + \tfrac{1}{2}\omega\rho, \tag{A32}$$

which corresponds to a constant magnetic field. For $C=0$ we obtain instead a shifted angular momentum free theory

$$W(\rho)=(m+\alpha)/\rho, \tag{A33}$$

which corresponds to a vortex solution.

In 4-dimensional problems with spherical symmetry we found (see (4.6)) that W was of the form

$$W(r)=\alpha/r+\chi'. \tag{A34}$$

This is exactly the same form as that in 2 dimensions (except ρ is replaced by r) so that the same analysis applies.

We now tabulate the eigenfunctions and the energy eigenvalues associated with these particular shape invariant potentials using Eqs. (A10), (A14), (A15). The method of obtaining these results is discussed in detail in Refs. [14, 20, 21]:

i. *Morse Potential.*

$$\begin{gathered} W(x)=A-Be^{-\alpha x}; \qquad V_{-}(x)=A^2+B^2e^{-2\alpha x}-2B(A+\alpha)e^{-\alpha x}; \\ E_n=A^2-(A-n\alpha)^2; \\ \Psi_n(y)=y^{s-n}e^{-y/2}L_n^{2s-2n}(y); \qquad y=\frac{2B}{\alpha}e^{-\alpha x}, \quad s=\frac{A}{\alpha}. \end{gathered} \tag{A35}$$

ii. *Rosen–Morse Potential.*

$$\begin{gathered} W(x)=A\tanh\alpha x+B/A; \quad V_{-}(x)=A^2+B^2/A^2+2B\tanh\alpha x-A(A+\alpha)\operatorname{sech}^2(\alpha x); \\ E_n=A^2-(A-n\alpha)^2+B^2/A^2-B^2/(A-n\alpha)^2; \\ \Psi_n(y)=(1-y)^{(s-n+a)/2}\,(1+y)^{(s-n-a)/2}\,P_n^{(s-n+a,s-n-a)}(y); \\ y=\tanh\alpha x, \quad s=A/\alpha, \quad \lambda=B/\alpha^2, \quad a=\lambda/(s-n). \end{gathered} \tag{A36}$$

iii. *Shifted* 1D *Oscillator.*

$$\begin{gathered} W(x)=\tfrac{1}{2}\omega x-b; \qquad V_{-}(x)=\tfrac{1}{4}\omega^2(x-2b/\omega)^2-\tfrac{1}{2}\omega; \\ E_n=n\omega; \\ \Psi_n(y)=e^{-y^2/2}H_n(y); \qquad y=\sqrt{\frac{\omega}{2}}\left(x-\frac{2b}{\omega}\right). \end{gathered} \tag{A37}$$

iv. 3-*Dimensional Oscillator.*

$$\begin{gathered} W(r)=\tfrac{1}{2}\omega r-(l+1)/r; \qquad V_{-}(r)=\tfrac{1}{4}\omega^2r^2+l(l+1)/r^2-(l+\tfrac{3}{2})\omega; \\ E_n=2n\omega; \\ \Psi_n(y)=e^{-y/2}y^{(l+1)/2}L_n^{l+1/2}(y); \qquad y=\tfrac{1}{2}\omega r^2. \end{gathered} \tag{A38}$$

v. *Shifted Angular Momentum (Bessel Functions).*

In this case there are no bound states and $C(m)$ is zero. The equation

$$Z'' - (m^2 - \tfrac{1}{4})Z/r^2 + \lambda Z = 0 \tag{A39}$$

is factorized with

$$W(r, m) = (m - \tfrac{1}{2})/r. \tag{A40}$$

The operators A and $A^\dagger$, as noted by Infeld and Hull [14], give the two recurrence formulas for the Bessel functions,

$$\begin{aligned} Z_{m+1} &= \lambda^{-1/2}\{(m+\tfrac{1}{2})/r - d/dr\}\, Z_m \\ Z_m &= \lambda^{-1/2}\{(m+\tfrac{1}{2})/r + d/dr\}\, Z_{m+1}, \end{aligned} \tag{A41}$$

from which we realize

$$Z_m = r^{1/2} J_m(\lambda^{1/2} r). \tag{A42}$$

Given the above wavefunctions we are now in a position to obtain the eigenfunctions and eigenvalues of $H = -\not{D}^2$ as well as the solutions of the massive Dirac equation.

For example, to solve Eqs. (2.36) with W given by (4.5) and a Rosen–Morse potential,

$$A_2 = -eH_0\alpha^{-1}\tanh(\alpha x_1), \tag{A43}$$

one has that the wavefunction corresponding to the eigenfunction with eigenvalue $\lambda_n = 4E_n - p_0^2 - p_3^2$ in the $\mathfrak{H}_0$ sector is

$$\Psi_0^{(n)} = \begin{bmatrix} 0 \\ e^{i(p_0x_0 + p_2x_2 + p_3x_3)}\Phi_n(y(x_2)) \\ 0 \\ 0 \end{bmatrix}, \tag{A44}$$

where Φ_n and y are given in (A36) with the identification that

$$A = eH_0/\alpha, \qquad B = p_2 eH_0/\alpha, \qquad \lambda_n = A^2 - (A - n\alpha)^2. \tag{A45}$$

In the $\mathfrak{H}_2$ sector a similar wavefunction having the same eigenvalue λ_n is obtained, being non-zero only in the first column of the four component spinor, which because it is the SUSY partner of the previous wavefunction has A replaced by $A - \alpha = eH_0/\alpha - \alpha$. To obtain the other two solutions with the same λ_n, one can use either the complex SUSY operators $\hat{Q}_\pm$ and Eqs. (2.23) and (2.25) or the chiral SUSY operator Q_+ (see (2.1) and (2.5)),

$$Q_+ = \begin{bmatrix} 0 & 0 \\ -iD_0 + \sigma \cdot D & 0 \end{bmatrix} \tag{A46}$$

to reconstruct the other two solutions.

If instead we want solutions to the massive Dirac equation we consider instead the two component Ψ defined by

$$\Psi_0^{(n)} = \begin{bmatrix} 0 \\ e^{i(p_0 x_0 + p_2 x_2 + p_3 x_3)} \Phi_n(y(x_1)) \end{bmatrix} \tag{A47}$$

as input in Eq. (A2) and uses the identification of (A4).

APPENDIX II: Exact Solutions of the (1 + 1)-Dimensional Dirac Equation with a Scalar Field

In this appendix we sketch the procedure for obtaining exact energy eigenvalues, eigenfunctions, and hence the S-matrix for the Dirac equation involving a scalar field in $1+1$ dimensions. These models are useful in the context of the phenomenon of fermion number fractionalization which has ben seen in certain polymers like polyacetylene.

For these systems the Dirac Lagrangian is given by

$$L = i\bar{\Psi}(x)\, \gamma^\mu \partial_\mu \Psi(x) - \bar{\Psi}(x)\, \varphi(x)\, \Psi(x), \tag{B1}$$

where $\varphi(x)$ is the *static*, finite energy solution corresponding to the scalar field Lagrangian

$$L_\varphi = \tfrac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - V(\varphi). \tag{B2}$$

Note that in (B1) and (B2) we have absorbed the coupling constant in φ and $V(\varphi)$, respectively. We would now like to address the question What are the various forms of the scalar field $\varphi(x)$ for which the Dirac equation corresponding to (B1) can be exactly solved? We will show now that whenever the 1-dimensional Schrödinger equation is exactly solvable for a given potential $V(x)$ then there always exists a corresponding scalar field $\varphi(x)$ for which the Dirac equation is also exactly solvable.

The Dirac equation following from (B1) is

$$i\gamma^\mu \partial_\mu \Psi(x, t) - \varphi(x)\, \Psi(x, t) = 0. \tag{B3}$$

Let

$$\Psi(x, t) = e^{-i\omega t}\Psi(x) \tag{B4}$$

so that (B3) reduces to

$$\gamma^0 \omega \Psi(x) + i\gamma^1 \, d\Psi(x)/dx - \varphi(x)\, \Psi(x) = 0. \tag{B5}$$

Following Jackiw and Rebbi [23] we choose

$$\gamma^0 = \sigma^1, \qquad \gamma^1 = i\sigma^3, \qquad \Psi(x) = \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix} \tag{B6}$$

so that we have the coupled equations

$$\Psi_1'(x) + \varphi(x)\, \Psi_1(x) = \omega \Psi_2(x) \tag{B7a}$$

$$\Psi_2'(x) - \varphi(x)\, \Psi_2(x) = -\omega \Psi_1(x). \tag{B7b}$$

These equations are easily decoupled yielding

$$-\Psi_1''(x) + [\varphi^2(x) - \varphi'(x)]\, \Psi_1(x) = \omega^2 \Psi_1(x) \tag{B8a}$$

$$-\Psi_2''(x) + [\varphi^2(x) + \varphi'(x)]\, \Psi_2(x) = \omega^2 \Psi_2(x). \tag{B8b}$$

These are precisely the Schrödinger like equations corresponding to the supersymmetric partner potentials

$$V_{\pm}(x) \equiv \varphi^2(x) \pm \varphi'(x). \tag{B9}$$

As shown in Ref. (21), not only the shape invariant potentials but also the entire class of solvable Natanzon potentials (these are potentials whose wavefunctions are hypergeometric and confluent hypergeometric functions) can be cast into the SUSY form (B9), with $\varphi(x)$ being related to the ground state wavefunction of the related quantum mechanics problem via (A7) [here W is replaced by φ]. These potentials have wavefunctions and spectra that can be determined algebraically by exploiting supersymmetry, shape invariance, and hidden shape invariance [14, 21]. Using the construction given above one can then immediately find the solution of the corresponding Dirac Lagrangian (B1).

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References

1. L. Alvarez Gaumè, *J. Phys. A* **16** (1983), 4177; A. Niemi, IAS report (1984), unpublished; D. Friedan and P. Windey, *Physica D* **15** (1985), 71; R. Musto, L. O'Raifeartaigh, and A. Wipf, *Phys. Lett. B* **175** (1986), 433; R. J. Hughes, V. A. Kostelecky, M. M. Nieto, *Phys. Rev. D* **34** (1986), 1100.

2. E. Witten, *Nucl. Phys. B* **185** (1981), 513; P. Salomonson and J. W. van Holten, *Nucl. Phys. B* **196** (1982), 509; F. Cooper and B. Freedman, *Ann. Phys. (N.Y.)* **146** (1983), 262.
3. A. K. Fujikawa, *Phys. Rev. D* **21** (1980), 2848.
4. T. Cowan *et al., Phys. Rev. Lett.* **56**, (1986), 444.
5. D. G. Caldi and A. Chodos, *Phys. Rev. D* **36** (1987), 2876.
6. J. Schwinger, *Phys. Rev.* **82** (1951), 664.
7. S. G. Rajeev, *Ann. Phys. (N.Y.)* **173** (1987), 249.
8. L. Alvarez Gaumè, *Comm. Math. Phys.* **90** (1983), 161.
9. P. Forgacs, L. O'Raifeartaigh, and A. Wipf, *Nucl. Phys. B* **293** (1987), 559.
10. D. Boyanovsky and R. Blankenbecler, *Phys. Rev. D* **31** (1985), 3234; R. Akhoury and A. Comtet, *Nucl. Phys. B* **246** (1984), 253; B. Freedman and F. Cooper, *Physica D* **15** (1985), 138.
11. G. Stanciu, *J. Math. Phys.* **8** (1967), 2043.
12. A. Wipf, in preparation; see also R. Jackiw, *Phys. Rev. D* **29** (1984), 2375; A. A. Andrianov, N. V. Borisov, M. I. Eides, and M. V. Ioffe, *Phys. Lett. A* **109** (1985), 143.
13. C. V. Sukumar, *J. Phys. A* **18** (1985), L57; A. A. Andrianov, N. V. Borisov, and M. V. Ioffe, *Phys. Lett. A* **105** (1984), 19.
14. L. Infeld and T. D. Hull, *Rev. Mod. Phys.* **23** (1951), 21; L. Gendenshtein, *JETP Lett.* **33** (1983), 356; R. Dutt, A. Khare and U. P. Sukhatme, *Amer. J. Phys.*, in press; F. Cooper, G. Ginocchio, and A. Wipf, Los Almos Preprint LAUR-87-3498 (1987).
15. C. V. Sukumar, *J. Phys. A* **18** (1985), 697.
16. J. Kiskis, *Phys. Rev. D* **15** (1977), 2329; D. Bolle, P. Dupont, and D. Roekarks, University of Leuven Preprint KUL TF-87/1.
17. L. Alvarez Gaumè, "Supersymmetry and Index Theory," Bonn Summer School Lectures, 1984, unpublished.
18. R. P. Feynman and M. Gell-Mann, *Phys. Rev.* **109** (1958), 193.
19. L. M. Brown, *Phys. Rev.* **111** (1958), 957.
20. J. W. Dabrowska, A. Khare, and U. P. Sukhatme, Univ. of Illinois at Chicago, Preprint UIC-87-31 (to be published in *J. Phys. A*).
21. F. Cooper, J. N. Ginocchio, and A. Khare, *Phys. Rev. D* **36** (1987), 2458.
22. A. Khare and U. P. Sukhatme, Univ. of Illinois at Chicago, Preprint UIC-88-33.
23. R. Jackiw and C. Rebbi, *Phys. Rev. D* **13** (1976), 3358; R. Jackiw and J. R. Schrieffer, *Nucl. Phys. B* **190**[F53] (1981), 253; D. K. Campbell and A. R. Bishop, *Nucl. Phys. B* **200**[F59] (1982), 293, and references therein.