DETERMINANTS OF CONFORMAL WAVE OPERATORS IN FOUR DIMENSIONS

Steven K. BLAU, Matt VISSER
Theoretical Division, T8, Mail Stop B-285, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

and

Andreas WIPF
Max-Planck-Institut für Physik und Astrophysik - Werner-Heisenberg-Institut für Physik, Postfach 401212, D-8000 Munich 40, Fed. Rep. Germany

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We consider conformally coupled wave operators in four dimensions. Such operators are associated with conformally coupled massless scalars, massless spin $\frac{1}{2}$ particles, and abelian gauge bosons. We explicitly calculate the change in the determinant of these wave operators as a function of conformal deformations of the background metric. This variation is given in terms of a geometrical object, the second Seeley–de Witt coefficient.

1. Introduction

In this paper we discuss the determinants of wave operators associated with conformally coupled particles, focusing on four dimensions. (The two-dimensional situation is extensively discussed in ref. [1].) The results obtained are applicable to photons, massless neutrinos, and massless conformally coupled scalars. We find that conformal deformations of the metric induce a change in the determinant (equivalently, the one-loop effective action) that is proportional to the second Seeley–de Witt coefficient (a_2). Furthermore, this change in the determinant my be cast into an explicit form involving the background Riemann tensor and derivatives of the conformal deformation.

An application to cosmology is considered. We extract the effect of the scale factor on the one-loop effective action, obtaining a result in terms of the Hubble and decelleration parameters. The expression for the one-loop effective action contains coefficients which depend only on the spatial three-geometry. These coefficients are easily calculated for the standard Friedmann–Robertson–Walker cosmologies.

2. Conformally coupling wave operators

Consider the following wave operators on d-dimensional spacetime

$$\Delta_0 = \nabla^2 + [(d-2)/4(d-1)]R\,, \quad \Delta_{1/2} = \not{D}^2\,, \quad \Delta_{(d-2)/2} = \delta d\,, \tag{1}$$

which act on scalars, spinors, and $\frac{1}{2}(d-2)$ forms, respectively. These operators have very simple transformation properties under a conformal deformation of the metric. If $g_{\alpha\beta}(\tau) = \exp[2\tau\sigma(x)]\, g_{\alpha\beta}(0)$ then (see e.g. Birrell and Davies [2], Blau, Visser, and Wipf [1], or Buchbinder et al. [3]):

$$\Delta_0(\tau) = \exp[-\tfrac{1}{2}(d+2)\tau\sigma]\, \Delta_0(0) \exp[\tfrac{1}{2}(d-2)\tau\sigma]\,, \tag{2}$$

$$\not{D}(\tau)=\exp[-\tfrac{1}{2}(d+1)\tau\sigma]\,\not{D}(0)\exp[\tfrac{1}{2}(d-1)\tau\sigma]\,,$$

$$\Delta_{(d-2)/2}(\tau)=\exp(-2\tau\sigma)\,\Delta_{(d-2)/2}(0)\,. \tag{2 cont'd}$$

Using these simple transformation properties, and the zeta function definition of the determinant [4], a minor generalization of the analysis in ref. [1] yields

$$\frac{\mathrm{d}[\det\Delta(\tau)]}{\mathrm{d}\tau}=-\det\Delta(\tau)\frac{2}{(4\pi)^{d/2}}\int\sqrt{g_\tau}\,\sigma(x)a_{d/2}[\Delta(\tau)]\,\mathrm{d}^dx\,, \tag{3}$$

where Δ is any one of the operators in (1), and whenever $a_{d/2}$ is matrix valued a trace is implied. This equation may be easily integrated to obtain

$$\det\Delta(\tau)=\det\Delta(0)\exp\left(-\frac{2}{(4\pi)^{d/2}}\int_0^\tau\mathrm{d}\tau'\int\sqrt{g_{\tau'}}\,\sigma(x)a_{d/2}[\Delta(\tau')]\,\mathrm{d}^dx\right). \tag{4}$$

In any odd number of dimensions $a_{d/2}=0$ and $\det\Delta$ is a conformal invariant. Eq. (4) is valid both in the presence and absence of zero modes for the operator $\Delta(\tau)$. If zero modes are present a stronger statement is possible by making the substitution $\det\Delta\mapsto\det'\Delta/\det\kappa$. κ is a matrix describing the relative normalizations of the τ-dependent zero modes, fully described in ref. [1]. For notational simplicity we suppress explicit mention of zero modes henceforth, but bear in mind that zero mode information may always be recovered by the above substitution. It should be pointed out that eq. (4) is, after several changes in notation (and modulo the discussion of zero modes), equivalent to eq. (6.128) of Birrell and Davies [2]. A major point of this note is to point out that the functional integration over conformal deformations can be performed exactly (in our notation this is just the τ integration).

In two dimensions the τ integration in (4) is trivial [1]. In any even dimension $a_{d/2}$ is a homogeneous polynomial in the Riemann tensor, its contractions, and its covariant derivatives. Specialising to four dimensions gives

$$a_2[\Delta]=A(\mathrm{Weyl})^2+B[(\mathrm{Ricci})^2-\tfrac{1}{3}R^2]+C\nabla^2R+DR^2\,. \tag{5}$$

The coefficients A through D are spin dependent and are given by (for instance) Birrell and Davies [2]. For the operators of interest to us $D=0$. For spin 0, $(A, B, C)=(-1, -1, +1)$; for spin $\tfrac{1}{2}$, $(A, B, C)=(-\tfrac{7}{4}, -\tfrac{11}{2}, +3)$; and, for spin 1, $(A, B, C)=(+13, -62, -18)$.

A standard computation gives for the τ dependence of the curvature

$$[\mathrm{Weyl}]^\alpha{}_{\beta\gamma\delta}(\tau)=[\mathrm{Weyl}]^\alpha{}_{\beta\gamma\delta}(0)\,,$$

$$[\mathrm{Ricci}]^\alpha{}_\beta(\tau)=\exp(-2\tau\sigma)\{[\mathrm{Ricci}]^\alpha{}_\beta(0)+2\tau\nabla^\alpha\nabla_\beta\sigma-2\tau^2\nabla^\alpha\sigma\nabla_\beta\sigma+[\tau\nabla^2\sigma+2\tau^2(\nabla\sigma)^2]\delta^\alpha{}_\beta\}\,,$$

$$R(\tau)=\exp(-2\tau\sigma)\{R(0)+6[\tau\nabla^2\sigma+\tau^2(\nabla\sigma)^2]\}\,. \tag{6}$$

All covariant derivatives ∇ are now with respect to the undeformed metric $g_{\alpha\beta}(0)$. We compute

$$\sqrt{g_\tau}[\mathrm{Weyl}(\tau)]^2=\sqrt{g_0}[\mathrm{Weyl}(0)]^2, \tag{7}$$

$$\sqrt{g_\tau}\{[\mathrm{Ricci}(\tau)]^2-\tfrac{1}{3}[R(\tau)]^2\}=\sqrt{g_0}\{[\mathrm{Ricci}(0)]^2-\tfrac{1}{3}[R(0)]^2+\tau[4\mathrm{Ricci}(0)\,\nabla\nabla\sigma-2R(0)\nabla^2\sigma]$$
$$+\tau^2[-4\nabla\sigma\,\mathrm{Ricci}(0)\,\nabla\sigma-4(\nabla^2\sigma)^2+4(\nabla\nabla\sigma)^2]+\tau^3[-4\nabla^2\sigma(\nabla\sigma)^2-8\nabla\sigma\nabla\nabla\sigma\nabla\sigma]\}\,, \tag{8}$$

$$\sqrt{g_\tau}[\nabla^2R](\tau)=\sqrt{g_0}[\nabla^2\{R(0)+6[\tau\nabla^2\sigma+\tau^2(\nabla\sigma)^2]\}-2\tau\nabla(\nabla\sigma\{R(0)+6[\tau\nabla^2\sigma+\tau^2(\nabla\sigma)^2]\})]\,, \tag{9}$$

$$\sqrt{g_\tau}[R(\tau)^2]=\sqrt{g_0}\{R(0)+6[\tau\nabla^2\sigma+\tau^2(\nabla\sigma)^2]\}^2\,. \tag{10}$$

This implies that the combination $\sqrt{g_\tau}a_2[\Delta(\tau)]$ is a *polynomial* in τ. Since $D=0$ this polynomial reduces to one of third order. The τ integration in (4) is easily performed using a generalization of Simpson's rule [#1]:

$$\det \Delta(\tau) = \det \Delta(0)$$

$$\times \exp\left(-\frac{1}{8\pi^2} \tau \int \mathrm{d}^4x\, \sigma(x) \{ \tfrac{1}{8}[\sqrt{g}a_2](0) + \tfrac{3}{8}[\sqrt{g}a_2](\tfrac{1}{3}\tau) + \tfrac{3}{8}[\sqrt{g}a_2](\tfrac{2}{3}\tau) + \tfrac{1}{8}[\sqrt{g}a_2](\tau) \} \right). \tag{11}$$

This formula, though compact, is still implicit. An explicit formula may be obtained by directly integrating over τ. The appearance of the final result may be considerably improved upon by use of several (spacetime) integrations by parts:

$$\det \Delta(\tau) = \det \Delta(0) \exp\left(-\frac{1}{8\pi^2} \int \mathrm{d}^4x \sqrt{g} [A\tau\sigma[\mathrm{Weyl}]^2 + B(\tau\sigma([\mathrm{Ricci}]^2 \right.$$

$$- \tfrac{1}{3}R^2) + \tfrac{1}{2}\tau^2(4\sigma\,\mathrm{Ricci}\,\nabla\nabla\sigma - 2R\sigma\nabla^2\sigma) + \tfrac{1}{3}\tau^3[-4\sigma\nabla\sigma\,\mathrm{Ricci}\,\nabla\sigma + 6(\nabla^2\sigma)(\nabla\sigma)^2] + \tfrac{1}{4}\tau^4[+4(\nabla\sigma)^4])$$

$$\left. + C(\tau\sigma\nabla^2 R + \tfrac{1}{2}\tau^2[6(\nabla^2\sigma)^2 + 2R(\nabla\sigma)^2] + \tfrac{1}{3}\tau^3[18(\nabla^2\sigma)(\nabla\sigma)^2] + \tfrac{1}{4}\tau^4\{+12[(\nabla\sigma)^4]\}))] \right). \tag{12}$$

3. Cosmological models

We would like to compare the effective action of conformal wave operators in the "cosmological bckground"

$$g_4 = (\mathrm{d}t)^2 - [a(t)]^2 g_3 = (\mathrm{d}t)^2 - \exp[2\sigma(t)]\, g_3, \quad t \in [0, T], \tag{13}$$

with that in the reference background $g_{4,0} = (\mathrm{d}t)^2 - g_3$. While we phrase the discussion in terms of cosmology, note that our results are applicable to any arbitrary three-geometry which is permitted to fluctuate with a time-dependent scale factor. In the static spacetime described by $g_{4,0}$, the effective action is proportional to the time interval and so may be written $S_{\mathrm{eff}}(g_{4,0}) = TE_{\mathrm{eff}}(g_3)$. Since only the spatial sections see the conformal factor, eq. (4) is not directly applicable. Therefore we define the conformal time by $\mathrm{d}\eta = a^{-1}\,\mathrm{d}t$ with range $\eta \in [0, \eta_0]$. Theorem (4) does apply to the family of metrics

$$\tilde{g}_{4,\tau} = \exp[2\tau\sigma(\eta)]\,[(\mathrm{d}\eta)^2 - g_3] = \exp[2\tau\sigma(\eta)]\,\{\exp[-2\sigma(\eta)\,(\mathrm{d}t)^2 - g_3\}. \tag{14}$$

In order to avoid the difficulties associated with boundaries and boundary conditions [#2], we shall think of the universe as being periodic in time with period T in the t coordinate (i.e. period η_0 in the η coordinate). This is merely a technical convenience; the physics is unaffected. Bearing this in mind, we now consider the one-loop effective action $[S_{\mathrm{eff}} = \ln \det \Delta]$ for conformally coupled particles, and use our theorem (4) to write

$$S_{\mathrm{eff}}(g_4; T) = S_{\mathrm{eff}}(\tilde{g}_{4,\tau=1}; \eta_0) = \eta_0 E_{\mathrm{eff}}(g_3) - \frac{1}{8\pi^2} \int_0^1 \mathrm{d}\tau \int_0^{\eta_0} \mathrm{d}\eta \int \mathrm{d}^3x \sqrt{\tilde{g}_{4,\tau}}\, \sigma(\eta)\, a_2[\tilde{\Delta}_{4,\tau}]. \tag{15}$$

As in section 2 the τ integration is polynomial. In the current situation there are considerable simplifications; note that $\nabla\sigma \mapsto \partial\sigma/\partial\eta \equiv \partial_\eta\sigma$, and $R_{00} = R_{0i} = 0$. We introduce additional definitions: $\Omega = \int \mathrm{d}^3x \sqrt{g_3} =$ Volume of space, $\mathscr{R} = \int \mathrm{d}^3x \sqrt{g_3} R$, and $A_2 = \int \mathrm{d}^3x \sqrt{g_3}\, a_2(g_3)$. Using eq. (12) we obtain

[#1] For any third-order polynomial: $\int_0^\tau P_3(\tau')\,\mathrm{d}\tau' = \frac{1}{8}\tau[P_3(0) + 3P_3(\frac{1}{3}\tau) + 3P_3(\frac{2}{3}\tau) + P_3(\tau)]$.

[#2] Boundaries complicate (and alter) both the statement and proof of our main theorem by introducing surface terms b_2 similar to the volume term a_2 used in (4). These boundary contributions depend on the nature of the boundary conditions imposed, see e.g. Birrell and Davies [2] pp. 223, 224.

$$S_{\mathrm{eff}}(g_4;\,T)=E_{\mathrm{eff}}(g_3)\,\eta_0-\frac{1}{8\pi^2}\int_0^{\eta_0}\mathrm{d}\eta\,(A_2\sigma(\eta)+\mathscr{R}[B+C](\partial_\eta\sigma)^2+\Omega\{[B+3C](\partial_\eta\sigma)^4+3C(\partial_\eta^2\sigma)^2\})\,. \tag{16}$$

We may return to "physical time" t by noting: $\mathrm{d}\eta=a^{-1}(t)\,\mathrm{d}t$; $\sigma=\ln a$; $\partial_\eta\sigma=a\,\partial_t\ln a=\dot a$; and $(\partial_\eta^2\sigma)=a\ddot a$. Then

$$S_{\mathrm{eff}}(g_4;\,T)=E_{\mathrm{eff}}(g_3)\int_0^T\frac{\mathrm{d}t}{a(t)}-\frac{1}{8\pi^2}\int_0^T\frac{\mathrm{d}t}{a(t)}\,(A_2\ln a(t)+\mathscr{R}[B+C](\dot a)^2+\Omega\{[B+3C](\dot a)^4+3C(a\ddot a)^2\})$$

$$\equiv\int_0^T\mathrm{d}t\,\mathscr{E}_{\mathrm{eff}}(t;g_4)\,, \tag{17}$$

which defines the effective energy $\mathscr{E}_{\mathrm{eff}}$ for non-static spacetimes. We introduce [5] the Hubble parameter $H=\dot a/a$, and the decelleration parameter $q=-a\ddot a/(\dot a)^2=-(\ddot a/a)\,H^{-2}$ to write

$$\mathscr{E}_{\mathrm{eff}}(t;g_4)=\frac{E_{\mathrm{eff}}(g_3)}{a(t)}-\frac{1}{8\pi^2}\left(A_2\frac{\ln a}{a}+\mathscr{R}[B+C]aH^2+\Omega([3C+B]a^3H^4+3Ca^3H^4q^2)\right). \tag{18}$$

This, finally, is an expression for the contribution (at one loop) to the effective energy of the universe due to conformally coupled matter such as neutrinos and photons. Even better, the final form is given in terms of *observables* such as the Hubble and decelleration parameters. The constants A_2 and $\mathscr{R}$ are calculable for any particular choice of spatial geometry, while Ω is merely a normalization constant.

If the spacetime of eq. (13) is of the Friedmann–Robertson–Walker type, then the spatial metric g_3 depends on a single scale factor Λ:

$$g_{3,\Lambda}=\frac{(\mathrm{d}r)^2}{1\pm r^2/\Lambda^2}+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) \tag{19}$$

with the + (−) sign corresponding to an open (closed) universe [#3]. The Λ dependence of $\mathscr{E}_{\mathrm{eff}}$ is in E_{eff} and the geometric quantities $\mathscr{R}$ and Ω (note that A_2 vanishes).

Since $a_2(\tilde\Delta_{4,\tau=0})$ vanishes, it is particularly easy to compute the scale dependence of $E_{\mathrm{eff}}(g_{3,\Lambda})$. The closed and open cases are slightly different because only the former corresponds to a compact spacetime. With cartesian spatial coordinates the closed Friedmann–Robertson–Walker metric is

$$\tilde g_{4,\tau=0}=(\mathrm{d}\eta)^2-\left(\mathrm{d}\boldsymbol{x}^2+\frac{(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2/\Lambda^2}{1-\boldsymbol{x}^2/\Lambda^2}\right). \tag{20}$$

Define $\boldsymbol{y}\equiv\boldsymbol{x}/\Lambda$ and $\eta'\equiv\eta/\Lambda$ so

$$\tilde g_{4,\tau=0}=\Lambda^2\left[(\mathrm{d}\eta')^2-\left(\mathrm{d}\boldsymbol{y}^2+\frac{(\boldsymbol{y}\cdot\mathrm{d}\boldsymbol{y})^2}{1-\boldsymbol{y}^2}\right)\right]. \tag{21}$$

The coordinate ranges are $\eta'\in[0,\eta_0/\Lambda]$ and $y^2\in[0,1]$. Since A_2 vanishes we may read off from (18)

$$E_{\mathrm{eff}}(\Lambda)=E_{\mathrm{eff}}(1)/\Lambda\,. \tag{22}$$

We compactify open universes by enclosing the spatial sections in a (periodic) box of volume $\Omega(\Lambda)$ which may depend upon Λ. Then, much like the closed universe case we obtain

$$E_{\mathrm{eff}}(\Lambda)=\Lambda^{-4}[\Omega(\Lambda)/\Omega(1)]\,E_{\mathrm{eff}}(1)\,. \tag{23}$$

[#3] Usually Λ is set equal to unity, but this involves a redefinition of time at each scale. We prefer not to redefine time but rather to fix its range to be $t\in[0,T]$.

Defining the effective energy density by $\rho_{\mathrm{eff}} \equiv E_{\mathrm{eff}}(\Lambda)/\Omega(\Lambda)$ gives $\rho_{\mathrm{eff}}(\Lambda)=\rho_{\mathrm{eff}}(1)\,\Lambda^{-4}$ independently of the prescription used to define $\Omega(\Lambda)$.

As a final comment, consider a flat Friedmann–Robertson–Walker universe, then $A_2 = \mathscr{R} = 0$. Defining a (spatially averaged) effective energy density by $\varrho_{\mathrm{eff}}(t) = \mathscr{E}_{\mathrm{eff}}(t)/\Omega(t) = \mathscr{E}_{\mathrm{eff}}(t)\, a^{-3}(t)/\Omega$ we find

$$\varrho_{\mathrm{eff}}(t) = \rho_{\mathrm{eff}}\, a^{-4} - [(B+3C)H^4 + 3CH^4 q^2]/8\pi^2 \,. \tag{24}$$

Observe that the term involving ρ_{eff} scales in the manner normally expected for massless particles. The term proportional to H^4 is negligibly small at the present epoch. During inflation, on the other hand, $H^2 = \Lambda_{\mathrm{eff}}/M^2_{\mathrm{Planck}}$, so $H^4 = \Lambda^2_{\mathrm{eff}}/M^4_{\mathrm{Planck}}$. This is still rather small compared to Λ_{eff}, but, depending on the details of the GUT phase transition, may be appreciable.

4. Conclusion

The results of this paper can be understood at a number of levels. The results for the determinant may be viewed as pure mathematical technology. At another level, these results give us information about one-loop quantum effects, and, as we have seen in the discussion of simple cosmological models, these effects are potentially interesting.

The analysis of this paper leaves open some important questions. Probably most significantly, one should understand the effect of particle masses on $\det(\Delta + m^2)$. Particle masses destroy the argument leading to eq. (4), and it is not yet clear to us if any suitable extension of the argument is possible. On another front, it would be very nice to be able to discuss deformations of the metric more general than conformal deformations.

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