Conformal bridge between freedom and confinement

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Abstract

We construct a non-unitary transformation that relates a given "asymptotically free" conformal quantum mechanical system H_f with its confined, harmonically trapped version H_c. In our construction, Jordan states corresponding to the zero eigenvalue of H_f, as well as its eigenstates and Gaussian packets are mapped into the eigenstates, coherent states and squeezed states of H_c, respectively. The transformation is an automorphism of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra of the nature of the fourth-order root of the identity transformation, to which a complex canonical transformation corresponds on the classical level being the fourth-order root of the spatial reflection. We investigate the one- and two-dimensional examples which reveal, in particular, a curious relation between the 2D free particle and the Landau problem.

1 Introduction

A well known deficiency of the conformal quantum mechanics is that it has no invariant ground state that is annihilated by the generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, and in this sense its conformal symmetry is spontaneously broken. In the context of the AdS/CFT correspondence, this phenomenon can be related with a peculiar nature of a usual evolution variable which is not a good global coordinate on AdS_2, whose isometry is conformal symmetry [1, 2, 3]. The problem emerges from the fact that the generators of the time translation H, the dilatation D, and the special conformal transformations K are non-compact generators of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra, and the spectrum of the Hamiltonian operator H of the system is the open interval $(0,\infty)$. The deficiency, however, can be cured from the perspective of Dirac's different forms of dynamics [4], by considering as a new Hamiltonian a linear combination of these generators to be of compact topological nature. For example, one can take the operator $H + m\omega^2 K$, which has an equidistant spectrum bounded from below, where m and ω are the mass and frequency parameters.

This "regularization" was first considered by de Alfaro, Fubini and Furlan (AFF) in their seminal work [5], where by means of a canonical transformation of the spatial and time coordinates, the conformal mechanics action describing the asymptotically free trajectories is transformed into a modified action principle with an additional confining term in the form of a harmonic trap. The AFF conformal mechanics model finds diverse interesting applications including the black hole physics [6, 7], cosmology [8, 9], and holographic QCD [10].

A similar transformation was considered earlier by Niederer [11] as the canonical transformation by which the conformal-invariant classical dynamics of a free non-relativistic particle on the whole real line can be related to the dynamics of the harmonic oscillator. A generalization of the construction allows to transform time-dependent Schrödinger equation of the free particle into that for the quantum harmonic oscillator. However, a relation between stationary states of the corresponding pairs of the "asymptotically free" and "confined" quantum systems remains to be unclear. To establish such a relation, we construct here a non-unitary similarity transformation in a form of a "conformal bridge between the freedom and confinement".

Before we pass over to the discussion of our constructions, it is instructive to recall some aspects of the Darboux transformations [12], a comparison with which will be useful in what follows.

The Darboux transformation and its generalizations allow to generate new quantum systems, sometimes called superpartners, from a given one. Any generated superpartner is completely or almost completely isospectral to the initial system, and the energy eigenstates of both systems are related to each other. The extended quantum system composed from two superpartners is described by a linear or nonlinear supersymmetry. An important class of exactly solvable quantum systems to which Darboux transformations are applied to produce new nontrivial systems includes, in particular,

(i) the free particle,

(ii) the harmonic oscillator,

(iii) conformal mechanics model without confining term (two-particle Calogero model with omitted center of mass coordinate [13]),

(iv) the AFF conformal mechanics model with a confining harmonic trap,

(v) particle in the infinite potential well.

By applying Darboux transformations to the free particle (i), reflectionless quantum systems can be obtained. The covariance of the Lax representation of integrable systems allows then to promote reflectionless potentials to soliton solutions of the Korteweg-de Vries (KdV) equation. The construction admits a further generalization to the case of finite-gap solutions to the KdV equation [14].

Proceeding from the systems (ii) and (iv), an interesting class of rational extensions of the harmonic oscillator and conformal mechanics is constructed, which have a discrete finite-gap type spectral structure described by exceptional orthogonal polynomials, and are characterized by an extended deformed conformal symmetry [15, 16, 17]. Exceptional orthogonal polynomials also are generated by applying Darboux transformations to the system (v) [18].

Darboux transformations are usually constructed by using physical or formal, non-physical eigenstates of the original system. Via the appropriate confluent limit of Darboux transformations, Jordan states of the original system enter into the construction [12, 19, 20]. Such states are used, particularly, in rational deformations of the conformal mechanics with confining trap (iv) [21], and in the construction of the extreme type wave solutions to the complexified KdV equation based on the system (i) and PT-regularized conformal mechanics (iii) [22].

Some pairs of the systems (i)—(v) can be related among themselves by using appropriate singular Darboux transformations. In this way, conformal mechanics models (iii) with certain values of the coupling constant and model (v) can be generated from the system (i), while the AFF model (iv) can be obtained from the system (ii). The systems (i) and (ii), however, are not related by Darboux transformations. The same is true for the pair of the systems (iii) and (iv). One may ask whether the quantum systems in the two indicated pairs can be related by an alternative differential transformation.

In the present paper we show how the indicated systems can be connected by employing conformal symmetry. A priori it is obvious that, as for the Darboux transformations, the sought for transformations have to be non-unitary. We shall see, however, that being similarity transformations, they can be related to the unitary Bargmann-Segal transformation in the case of the pair of systems (i) and (ii), where a non-unitary Weierstrass transformation plays an essential role. At the same time, in correspondence with the above indicated modification of conformal symmetry, the transformations effectively relate Dirac's different forms of dynamics with respect to the conformal $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$ symmetry. Also, we will observe the essential role played by the Jordan as well as coherent and squeezed states in our constructions. Our "conformal bridge transformation" can be generalized to higher-dimensions, and as an example we consider a relation of the 2D free particle system with the planar isotropic harmonic oscillator and the Landau problem.

The paper is organized as follows. In Section 2, we investigate a relation between the quantum free particle and the harmonic oscillator by exploting the structure of the inverse Weierstrass transformation and the generating function for Hermite polynomials in the light of conformal symmetry. In Section 3 we generalize the obtained results in the context of the Dirac's different forms of dynamics. In Section 4 we explore the meaning of our "conformal bridge" transformation on the classical level. In Section 5 we apply the transformation to establish a relation between the conformal mechanics model and the AFF model. This way we generate the Schrödinger odd cat states from eigenstates of the two-particle Calogero system. In Section 6 the "conformal bridge" transformations are applied to two-dimensional systems to establish a relation of certain quantum states of the 2D free particle system with eigenstates and coherent states of the planar isotropic harmonic oscillator as well as the Landau problem. Section 7 is devoted to the discussion and outlook. Two Appendices include some technical details.

2 Free particle and harmonic oscillator

In this section we investigate the transformation, based on conformal symmetry, by which the free particle and quantum harmonic oscillator systems can be related.

Consider the generating function for Hermite polynomials $H_n(x)$,

$$G(x;t) = \exp\big(2xt - t^2\big) = e^{x^2} e^{-(x-t)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)\,. \tag{2.1}$$

Using it, one can obtain a chain of various representations for $H_n(x)$:

$$\begin{gathered} H_n(x) = \tfrac{\partial^n}{\partial t^n} G(x;t)|_{t=0} = (-1)^n e^{x^2} \tfrac{d^n}{dx^n} e^{-x^2} = \\ \big(-e^{x^2} \tfrac{d}{dx} e^{-x^2}\big)^n \cdot 1 = \big(2x - \tfrac{d}{dx}\big)^n \cdot 1 = e^{\frac{1}{2}x^2} (x - \tfrac{d}{dx})^n e^{-\frac{1}{2}x^2}\,. \end{gathered} \tag{2.2}$$

Normalized eigenfunctions of the quantum harmonic oscillator of mass m and frequency ω described by the Hamiltonian operator $\hat{H}_{\rm osc} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2 q^2$ are

$$\psi_n(q) = C_n e^{-\frac{1}{2}x^2} H_n(x)\,, \qquad C_n = \frac{1}{\sqrt{\ell_0}} \frac{1}{\sqrt{\pi^{1/2} 2^n n!}}\,, \tag{2.3}$$

where $\ell_0 = \sqrt{\frac{\hbar}{m\omega}}$ and $x = q/\ell_0$ is a dimensionless variable. Take the units with $\hbar = m = \omega = 1$, and introduce the ladder operators $a^+ = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$, $a^- = (a^+)^\dagger = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$, $[a^-, a^+] = 1$. Then the last equality in Eq. (2.2) multiplied from the left by $e^{-\frac{1}{2}x^2}$ gives, up to a normalization, the eigenfunctions (2.3), that correspond to the coordinate representation of the Fock states $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$ generated from the ground state of the quantum harmonic oscillator.

Based on the relation $\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) e^{2xt} = -t^2 e^{2xt}$, one can also represent the generating function (2.1) as follows,

$$G(x,t) = \exp\left(-\tfrac{1}{4}\tfrac{d^2}{dx^2}\right) e^{2xt} = \textstyle\sum_{n=0}^{\infty} \frac{2^n t^n}{n!} \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) x^n\,. \tag{2.4}$$

Comparison of (2.4) with (2.1) yields yet another representation of Hermite polynomials via the formal inverse of the Weierstrass transform [23, 24],

$$H_n(x) = 2^n \exp\left(-\tfrac{1}{4}\tfrac{d^2}{dx^2}\right) x^n\,. \tag{2.5}$$

Eq. (2.5) allows us to generate the eigenfunctions (2.3) of the harmonic oscillator by applying the operator

$$\hat{\mathfrak{S}}_0 = e^{-\hat{K}} e^{\frac{1}{2}\hat{H}_0} \tag{2.6}$$

to the monoms x^n, $n = 0, 1, \ldots$,

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} n!}} \hat{\mathfrak{S}}_0 \left(\sqrt{2}x\right)^n\,. \tag{2.7}$$

The operator (2.6) is constructed from the operators

$$\hat{H}_0 = -\tfrac{1}{2}\tfrac{d^2}{dx^2}\,, \qquad \hat{K} = \tfrac{1}{2}x^2\,, \tag{2.8}$$

which together with the dilatation operator

$$\hat{D} = \tfrac{1}{4}(x\hat{p} + \hat{p}x) = -\tfrac{i}{2}\left(x\tfrac{d}{dx} + \tfrac{1}{2}\right) \tag{2.9}$$

generate the dynamical conformal symmetry of the quantum free particle system,

$$[\hat{H}_0, \hat{D}] = -i\hat{H}_0\,, \qquad [\hat{H}_0, \hat{K}] = -2i\hat{D}\,, \qquad [\hat{K}, \hat{D}] = i\hat{K}\,. \tag{2.10}$$

The operators $\hat{K}$ and $\hat{D}$ are the $t = 0$ form of the explicitly time-dependent integrals of motion $\hat{\mathcal{K}} = \frac{1}{2}\hat{\mathcal{X}}^2$ and $\hat{\mathcal{D}} = \frac{1}{4}(\hat{p}\hat{\mathcal{X}} + \hat{\mathcal{X}}\hat{p})$, where $\hat{\mathcal{X}} = (x - \hat{p}t)$ is the generator of Galileo transformations of the free particle, $\frac{d}{dt}\hat{\mathcal{X}} = \frac{\partial}{\partial t}\hat{\mathcal{X}} - i[\hat{\mathcal{X}}, \hat{H}_0] = 0$. The set $\hat{H}_0$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{K}}$ generates the same conformal algebra (2.10). Its extension by the integrals $\hat{p}$, $\hat{\mathcal{X}}$ and central element 1 (in the chosen units) corresponds to the Schrödinger symmetry of the free particle system.

In (2.7), $\chi_0 = x^0 = 1$ is an eigenstate of the lowest, zero eigenvalue of the free particle Hamiltonian $\hat{H}_0$, while $\chi_1 = x$ is a non-physical, unbounded at infinity, eigenstate of $\hat{H}_0$ of the same zero eigenvalue. The states given by wave functions $\chi_n(x) = x^n$ with $n \geq 2$ are the Jordan states [22] of the free particle corresponding to the zero energy $E = 0$:

$$(\hat{H}_0)^n\chi_{2n} = (-\tfrac{1}{2})^n(2n)!\,\chi_0\,, \qquad (\hat{H}_0)^n\chi_{2n+1} = (-\tfrac{1}{2})^n(2n+1)!\,\chi_1\,, \tag{2.11}$$

$$(\hat{H}_0)^{n+1}\chi_{2n} = (\hat{H}_0)^{n+1}\chi_{2n+1} = 0\,, \qquad n = 1, \ldots\,. \tag{2.12}$$

The $\chi_n(x)$ are, at the same time, the formal eigenstates of the operator $2i\hat{D}$ with the same eigenvalues as the eigenfunctions $\psi_n(x)$ of the quantum harmonic oscillator Hamiltonian,

$$2i\hat{D}\chi_n = \left(n + \tfrac{1}{2}\right)\chi_n\,. \tag{2.13}$$

Behind the last observation lies the fact that the non-unitary operator $\hat{\mathfrak{S}}_0$ intertwines generators of the conformal $\mathfrak{sl}(2,\mathbb{R})$ symmetry of the quantum free particle system with generators of the Newton-Hooke symmetry of the quantum harmonic oscillator by changing the form of dynamics. In details, this can be traced out as follows. Using the relations $\hat{a}^-\psi_n(x) = \sqrt{n}\psi_{n-1}(x)$ and $\hat{a}^+\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x)$, we obtain from (2.7) that

$$\hat{a}^- = \hat{\mathfrak{S}}_0\left(\tfrac{1}{\sqrt{2}}\tfrac{d}{dx}\right)\hat{\mathfrak{S}}_0^{-1}\,, \qquad \hat{a}^+ = \hat{\mathfrak{S}}_0\left(\sqrt{2}x\right)\hat{\mathfrak{S}}_0^{-1}\,. \tag{2.14}$$

Therefore, the non-unitary operator $\hat{\mathfrak{S}}_0$ intertwines the free particle momentum and coordinate operators multiplied by $i/\sqrt{2}$ and $\sqrt{2}$, respectively, with the annihilation and creation operators of the quantum harmonic oscillator,

$$\hat{\mathfrak{S}}_0\left(\tfrac{1}{\sqrt{2}}\tfrac{d}{dx}\right) = \hat{a}^-\hat{\mathfrak{S}}_0\,, \qquad \hat{\mathfrak{S}}_0\left(\sqrt{2}x\right) = \hat{a}^+\hat{\mathfrak{S}}_0\,. \tag{2.15}$$

Then we find that the same operator intertwines the generators of conformal symmetry of the free particle with the generators of the Newton-Hooke symmetry [25] of the quantum harmonic oscillator,

$$\hat{\mathfrak{S}}_0\hat{H}_0 = (-2\hat{J}_-)\hat{\mathfrak{S}}_0\,, \qquad \hat{\mathfrak{S}}_0\hat{K} = \tfrac{1}{2}\hat{J}_+\hat{\mathfrak{S}}_0\,, \qquad \hat{\mathfrak{S}}_0(\hat{D}) = -\tfrac{i}{2}\hat{H}_{\rm osc}\hat{\mathfrak{S}}_0\,, \tag{2.16}$$

where

$$\hat{J}_- = \tfrac{1}{2}(\hat{a}^-)^2\,, \qquad \hat{J}_+ = \tfrac{1}{2}(\hat{a}^+)^2\,, \qquad \hat{H}_{\rm osc} = \hat{a}^+\hat{a}^- + \tfrac{1}{2} = 2\hat{J}_0\,. \tag{2.17}$$

In this picture, the zero energy eigenstates, $\chi_0 = 1$ and $\chi_1 = x$ of the free particle, together with the Jordan states $\chi_n = x^n$, $n \geq 2$, corresponding to the same zero energy (being at the same time formal eigenstates of the non-compact $\mathfrak{sl}(2,\mathbb{R})$ generator $\hat{D} = \hat{J}_2$ multiplied by $2i$) are transformed by $\hat{\mathfrak{S}}_0$ into eigenstates of the harmonic oscillator Hamiltonian. As a consequence of the intertwining relations (2.15) and equations (2.13) and (2.16), the operators $\frac{1}{\sqrt{2}}\frac{d}{dx}$ and $\sqrt{2}x$ act on the described states $\chi_n(x)$, $n = 0, 1, \ldots$, of the free particle in the same way as the ladder operators $\hat{a}^-$ and $\hat{a}^+$ act on the energy eigenstates of the harmonic oscillator.

The ladder operators connect the states from the two irreducible $\mathfrak{sl}(2,\mathbb{R})$-representations of the discrete type series D^+_α with $\alpha = 1/4$ and $\alpha = 3/4$, in which the Casimir operator (A.5) takes the same value $\hat{C} = -\alpha(\alpha-1) = 3/16$, while the compact generator $\hat{J}_0 = \frac{1}{2}\hat{H}_{\rm osc}$ has the values $j_{0,n} = \alpha + n$, $n = 0, 1, \ldots$. These two irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ are realized on subspaces spanned by even, $\{\psi_{2n}(x)\}$, and odd, $\{\psi_{2n+1}(x)\}$, $n = 0, 1, \ldots$, eigenstates of the quantum harmonic oscillator. Together they constitute an irreducible representation of the $\mathfrak{osp}(1|2)$ superconformal algebra with $\hat{a}^-$ and $\hat{a}^+$ as odd generators [26]. The indicated separation of eigenstates of the harmonic oscillator corresponds to a separation of the set of Jordan states in (2.11) with odd and even values of the index.

The generating function (2.1) is related with a unitary map from the Hilbert space $L^2(\mathbb{R})$ to the Fock-Bargmann space [27, 28],

$$G(x, z/\sqrt{2}) = (\pi)^{1/4}\, e^{\frac{1}{2}x^2}U(x,z)\,. \tag{2.18}$$

Here

$$U(x,z) = \sum_{n=0}^{\infty}\overline{\psi_n(x)}f_n(z)\,, \tag{2.19}$$

with $\psi_n(x)$ being the energy eigenstates of the harmonic oscillator in the coordinate representation, while $f_n(z) = \frac{z^n}{\sqrt{n!}}$, $z \in \mathbb{C}$, describe the same orthonormal states in the Fock-Bargmann representation with scalar product

$$(f,g) = \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-|z|^2}\,\overline{f(z)}g(z)d^2z\,, \qquad d^2z = d(\mathrm{Re}\,z)d(\mathrm{Im}\,z)\,. \tag{2.20}$$

The explicit form of (2.19) is

$$U(x,z) = \pi^{-1/4}\exp\left(-\tfrac{1}{2}(x^2 - 2\sqrt{2}zx + z^2)\right) = \langle x|z\rangle_S\,, \tag{2.21}$$

that corresponds to the Schrödinger non-normalized coherent state in the coordinate representation [29]. Equations (2.4) and (2.18) yield then the relation

$$U(x,z) = \hat{\mathfrak{S}}_0 \phi_z(x)\,, \quad \text{where} \quad \phi_z(x) = \pi^{-1/4} e^{\sqrt{2}zx}\,. \tag{2.22}$$

The wave function $\phi_z(x)$ with $z \in \mathbb{C}$ corresponds to a formal eigenstate of the operator $\hat{H}_0$ with eigenvalue $-z^2$, which at pure imaginary values $z = ik/\sqrt{2}$ of z is a plane wave eigenstate e^{ikx} of the free particle[1]. Therefore, in addition to relation (2.7) we find that the non-unitary operator $\hat{\mathfrak{S}}_0$ maps (formal in general case of $z \in \mathbb{C}$) plane wave type eigenstates of the free particle into the Schrödinger coherent states of the quantum harmonic oscillator. The standard, or canonical Schrödinger-Klauder-Glauber coherent states of the quantum harmonic oscillator in coordinate representation [28, 30, 31, 32],

$$\langle x|z\rangle = e^{-\frac{1}{2}|z|^2} U(x,z) := \psi_z(x)\,, \tag{2.23}$$

are generated from the rescaled formal plane wave eigenstates of the free particle:

$$\psi_z(x) = \hat{\mathfrak{S}}_0 \left(e^{-\frac{1}{2}|z|^2} \phi_z(x) \right)\,. \tag{2.24}$$

Thus, the eigenstates of the quantum harmonic oscillator are obtained from the Jordan states and eigenstates of the free particle corresponding to zero energy by applying to them the non-unitary operator $\hat{\mathfrak{S}}_0$. These free particle's states are also formal eigenstates (with purely imaginary eigenvalues) of the dilatation operator $\hat{D}$. On the other hand, in correspondence with the first relation in (2.15), the plane wave eigenstates of the free particle, being eigenfunctions of the momentum operator, are mapped by $\hat{\mathfrak{S}}_0$ into coherent states of the quantum harmonic oscillator (QHO) being eigenstates of its annihilation operator. The free particle plane wave eigenstates are produced then by action of the inverse operator $\hat{\mathfrak{S}}_0^{-1} = e^{-\frac{1}{2}H_0} e^{K}$ on the coherent states of the QHO.

Another important class of the states of the quantum harmonic oscillator corresponds to the squeezed states [31, 28]. The so called single-mode squeezed states $|r\rangle = \hat{S}(r)|0\rangle$ are obtained from the ground state $|0\rangle$ by acting with the unitary operator $\hat{S}(r) = \exp\Big(r\hat{J}_- - r\hat{J}_+\Big) = \exp\Big(-2ir\hat{J}_2\Big)$ on it, see Eqs. (2.17) and (A.2). In the coordinate representation it is an infinite linear combination of eigenfunctions of the QHO,

$$\psi_r(x) = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} (-\tanh r)^n \frac{\sqrt{(2n)!}}{2^n n!} \psi_{2n}(x)\,. \tag{2.25}$$

Using relation (2.7), we find that the pre-image of (2.25) under the action of the non-unitary operator $\hat{\mathfrak{S}}_0$ is given by a Gaussian wave packet. Namely,

$$\hat{\mathfrak{S}}_0 \left(\tfrac{1}{\sqrt{\pi^{1/2}\cosh r}} \exp(-\tanh r \cdot x^2) \right) = \psi_r(x)\,. \tag{2.26}$$

Therefore, the Gaussian wave packets of the quantum free particle system correspond to the single-mode squeezed states of the QHO under the similarity transformation generated by the non-unitary operator $\hat{\mathfrak{S}}_0$.

[1]Note that linear combination of the states $\phi_z(x)$ with real z are used as seed states for Darboux transformations to generate soliton solutions to the KdV equation. These states with complex z are employed in the construction of the KdV soliton Baker-Akhiezer function [33].

3 Generalization for further applications

The similarity transformation which allowed us to establish a "bridge" between quantum free particle system and quantum harmonic oscillator is given by the non-unitary operator (2.6) constructed from generators of the conformal algebra. The generators of this algebra correspond to the dynamical symmetry $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$ of the quantum free particle system. This transformation generates a change of dynamics in the sense of Dirac [4]. The initial system is described by a Hamiltonian operator being a non-compact linear combination of the generators $\hat{J}_\mu$ of the Lorentz algebra $\mathfrak{so}(2,1)$, $\hat{H}_0 = \hat{J}_0 + \hat{J}_1$. The dynamics of the final system being the QHO is described by the compact generator $2\hat{J}_0 = \hat{H}_{\rm osc} = \hat{H}_0 + \hat{K}$ of the same algebra. The similarity transformation produced by $\hat{\mathfrak{S}}_0$ does not change the conformal algebra, but being non-unitary operator, it intertwines $\mathfrak{sl}(2,\mathbb{R})$ generators of different topological nature according to Eq. (2.16).

One can modify the similarity transformation by changing operator (2.6) to

$$\hat{\mathfrak{S}} = e^{-\hat{K}} e^{\frac{1}{2}\hat{H}_0} e^{i\ln 2\cdot\hat{D}} = e^{-\hat{K}} e^{i\ln 2\cdot\hat{D}} e^{\hat{H}_0} \,. \tag{3.1}$$

As a result of the inclusion of the additional unitary factor $e^{i\ln 2\cdot\hat{D}}$, operator (3.1) acts on the coordinate and momentum $\hat{p} = -i\frac{d}{dx}$ operators in a more symmetric way. Instead of (2.14), we have

$$\hat{\mathfrak{S}}\hat{p}\hat{\mathfrak{S}}^{-1} = -i\hat{a}^- \,, \qquad \hat{\mathfrak{S}}x\hat{\mathfrak{S}}^{-1} = \hat{a}^+ \,. \tag{3.2}$$

This is a non-unitary canonical transformation which is identified as fourth order root of the space reflection operator,

$$\hat{\mathfrak{S}} : (x,\hat{p},\hat{a}^+,\hat{a}^-) \rightarrow (\hat{a}^+,-i\hat{a}^-,-i\hat{p},x) \,, \qquad \hat{\mathfrak{S}}^2 : (x,\hat{p},\hat{a}^+,\hat{a}^-) \rightarrow (-i\hat{p},-ix,-\hat{a}^-,\hat{a}^+) \,,$$
$$\hat{\mathfrak{S}}^4 : (x,\hat{p},\hat{a}^+,\hat{a}^-) \rightarrow (-x,-\hat{p},-\hat{a}^+,-\hat{a}^-) \,. \tag{3.3}$$

Therefore, from the point of view of the quantum phase space, transformation (3.2) is the eighth order root of the identity transformation.

The operator (3.1) can also be factorized as

$$\hat{\mathfrak{S}} = \exp\Big(\hat{J}_1 - \hat{J}_0\Big) \cdot \exp\Big(i\ln 2 \cdot \hat{J}_2\Big) \cdot \exp\Big(\hat{J}_0 + \hat{J}_1\Big) \tag{3.4}$$

involving the $\mathfrak{sl}(2,\mathbb{R})$ generators $\hat{J}_\mu$. It produces the following non-unitary automorphism of $\mathfrak{sl}(2,\mathbb{R})$:

$$\hat{\mathfrak{S}}\hat{J}_0\hat{\mathfrak{S}}^{-1} = i\hat{J}_2 \,, \qquad \hat{\mathfrak{S}}\hat{J}_1\hat{\mathfrak{S}}^{-1} = -\hat{J}_1 \,, \qquad \hat{\mathfrak{S}}\hat{J}_2\hat{\mathfrak{S}}^{-1} = -i\hat{J}_0 \,. \tag{3.5}$$

The action of the squared non-unitary similarity transformation on the $\mathfrak{sl}(2,\mathbb{R})$ generators is a rotation by π about J_1, $\hat{\mathfrak{S}}^2 : (\hat{J}_0,\hat{J}_1,\hat{J}_2) \rightarrow (-\hat{J}_0,\hat{J}_1,-\hat{J}_2)$, such that the action of $\hat{\mathfrak{S}}$ on $\hat{J}_\mu$ is identified as fourth order root of the identity.

Eqs. (3.4) and (3.5) can be used to relate other pairs of the quantum systems described by different reaizations of the conformal $\mathfrak{sl}(2,\mathbb{R})$ dynamics in the sense of Dirac.

4 Classical picture

In this section, we consider the classical picture underlying the quantum transformation based on conformal symmetry and relating the quantum mechanical systems of the free particle and harmonic oscillator. This will allow us, in particular, to obtain an interesting reinterpretation of some aspect of the quantum "conformal bridge" transformation in relation with the unitary Bargmann-Segal transformation and the Neumann-Stone theorem.

The classical analogs of the free particle's generators of conformal symmetry, $H_0 = \frac{1}{2}p^2$, $D = \frac{1}{2}xp$ and $K = \frac{1}{2}x^2$, satisfy the Poisson bracket relations

$$\{D, H_0\} = H_0\,, \qquad \{D, K\} = -K\,, \qquad \{K, H_0\} = 2D\,, \tag{4.1}$$

which correspond to the quantum $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$ algebra (A.1) under the identification

$$J_0 = \frac{1}{2}(H_0 + K)\,, \qquad J_1 = \frac{1}{2}(H_0 - K)\,, \qquad J_2 = D\,. \tag{4.2}$$

The classical analog of the Casimir element (A.5) is presented here by $C = -H_0K + D^2$, and takes zero value, $C = 0$. Since $J_0 \geq 0$, the dynamics of both the free particle and the harmonic oscillator takes place on the upper cone surface in coordinates of the $\mathfrak{sl}(2,\mathbb{R})$ conformal algebra generators (J_0, J_1, J_2).

The phase space functions H_0, K and D generate the following canonical transformations of x and p, see Appendix B:

$$T_{H_0}(\alpha)(x) = x - \alpha p\,, \qquad T_{H_0}(\alpha)(p) = p\,, \tag{4.3}$$

$$T_K(\beta)(x) = x\,, \qquad T_K(\beta)(p) = p + \beta x\,, \tag{4.4}$$

$$T_D(\gamma)(x) = e^{-\frac{1}{2}\gamma}x\,, \qquad T_D(\gamma)(p) = e^{\frac{1}{2}\gamma}p\,. \tag{4.5}$$

Then the composition

$$T_{\beta\alpha\gamma} := T_K(\beta) \circ T_{H_0}(\alpha) \circ T_D(\gamma) = T_K(\beta) \circ T_D(\gamma) \circ T_{H_0}(2\alpha) \tag{4.6}$$

transforms the canonical variables as follows,

$$T_{\beta\alpha\gamma}(x) = e^{-\frac{1}{2}\gamma}\left(x(1-\alpha\beta) - \alpha p\right) := \widetilde{x}\,, \qquad T_{\beta\alpha\gamma}(p) = e^{\frac{1}{2}\gamma}\left(p + \beta x\right) := \widetilde{p}\,. \tag{4.7}$$

The choice

$$\alpha = \frac{i}{2}\,, \qquad \beta = -i\,, \qquad \gamma = -\ln 2 \tag{4.8}$$

gives

$$\widetilde{x} = a^+\,, \qquad \widetilde{p} = -ia^-\,, \qquad \widetilde{a}^+ = -ip\,, \qquad \widetilde{a}^- = x\,, \tag{4.9}$$

where $a^- = \frac{1}{\sqrt{2}}(x + ip)$ and $a^+ = \frac{1}{\sqrt{2}}(x - ip)$, $\{a^-, a^+\} = -i$, are the classical analogs of the creation and annihilation operators. This shows that the transformation $T_{\beta\alpha\gamma}$ with the parameters fixed as in (4.8) is the classical analog of the operator (3.1).

The same complex canonical transformation applied to the generators of conformal symmetry gives $\widetilde{H}_0 = -\frac{1}{2}(a^-)^2$, $\widetilde{K} = \frac{1}{2}(a^+)^2$, $\widetilde{D} = -\frac{i}{2}a^+a^-$. The $2iD$ transforms into the

Hamiltonian of the classical harmonic oscillator, and we find that it generates the complex flow in the phase space given by $x' := \{x, 2iD\} = ix$, $p' := \{p, 2iD\} = -ip$. Taking into account relations (4.9) and $2i\widetilde{D} = H_{\rm osc}$, one concludes that these relations correspond exactly to the classical equations of motion for the harmonic oscillator, $\dot{a}^{\pm} = \{a^{\pm}, H_{\rm osc}\} = \pm ia^{\pm}$.

Being a similarity transformation, (3.2) preserves the commutation relations, but operator (3.1) is not unitary with respect to the scalar product

$$(\psi_1, \psi_2) = \int_{-\infty}^{+\infty} \overline{\psi_1(x)}\psi_2(x)dx\,. \tag{4.10}$$

This is the case since the parameters α and β in the corresponding classical canonical transformation are purely imaginary, as a result of which the transformed canonical variables $\widetilde{x} = a^+$ and $\widetilde{p} = -ia^-$ are complex variables, while their quantum analogs (3.2) are not Hermitian operators with respect to the scalar product (4.10). This deficiency can be "cured" and reinterpreted in correspondence with the Neumann-Stone theorem [27]. The transformed classical coordinate takes complex values, $\tilde{x} = a^+ \in \mathbb{C}$, and its complex conjugation is

$$\overline{\tilde{x}} = a^- = i\tilde{p}\,. \tag{4.11}$$

In correspondence with these properties, at the quantum level we pass over from the coordinate representation to representation in which $\hat{\tilde{x}} = \hat{a}^+$ acts as the operator of multiplication by a complex variable z, $\hat{a}^+\psi(z) = z\psi(z)$, while the operator $i\hat{\tilde{p}} = \hat{a}^-$ acts as differential operator, $\hat{a}^-\psi(z) = \frac{d}{dz}\psi(z)$. Hence the quantum relation $[\hat{a}^-, \hat{a}^+] = 1$ is the analog of the classical relation $\{\tilde{x}, \tilde{p}\} = 1$. Replacing finally the scalar product (4.10) by the scalar product (2.20),

$$(\psi_1, \psi_2) = \frac{1}{\pi}\int_{\mathbb{R}^2} \overline{\psi_1(z)}\psi_2(z)e^{-\bar{z}z}d^2z, \qquad d^2z = d({\rm Re}\, z)d({\rm Im}\, z)\,, \tag{4.12}$$

we arrive at the Fock-Bargmann representation, in which the operators $\hat{a}^+ = z$ and $\hat{a}^- = \frac{d}{dz}$ satisfy the relation $(\hat{a}^+)^\dagger = \hat{a}^-$, that corresponds to the classical identity (4.11). In this representation

$$2i\hat{\widetilde{D}} = \hat{H}_{\rm osc} = \left(z\frac{d}{dz} + \frac{1}{2}\right), \qquad \hat{\widetilde{H}}_0 = -\frac{1}{2}\frac{d^2}{dz^2}\,, \qquad \hat{\widetilde{K}} = \frac{1}{2}z^2\,, \tag{4.13}$$

where $\hat{\widetilde{D}}$ is constructed from the phase space function $\widetilde{D}$ via an anti-simmetrization. In other words, the transformed operators in the Fock-Bargmann representation can be obtained from the corresponding initial generators of conformal symmetry of the quantum free particle by a formal change of x to z. The change of the scalar product from (4.10) to (4.12) transmutes then the non-unitary similarity transformation (3.2) into the unitary transformation from the coordinate to the holomorphic representation for the Heisenberg algebra in correspondence with the Neumann-Stone theorem, to which the kernel (2.19) corresponds to.

5 Conformal mechanics bridge

In this section, we construct in a similar way a "conformal bridge" between the quantum conformal mechanics systems (iii) and (iv). The peculiarity in comparison with the previous case of the pair of the systems (i) and (ii) is that here the operators $\hat{x}$ and $\hat{p}$ are not observables anymore. Thus we work in terms of an operator analogous to (3.4) and its associated relations (3.5).

The Hamiltonian operator of the two-body Calogero model with omitted center of mass degree of freedom is

$$\hat{H}_\nu = -\frac{1}{2}\frac{d^2}{dx^2} + \Delta_\nu \,, \qquad \Delta_\nu = \frac{\nu(\nu+1)}{2x^2}\,, \tag{5.1}$$

where $\nu \geq -1/2$, $x \in \mathbb{R}^+$, and we impose the Dirichlet boundary condition $\psi(0) = 0$ for the wave functions. The case $\nu = 0$ corresponds to the free particle on the half-line $\mathbb{R}^+$ with eigenstates $\psi_\kappa(x) = \sin \kappa x$ of energies $E = \frac{1}{2}\kappa^2 > 0$. The limit $\lim_{\kappa\to 0} \sin \kappa x/\kappa = x$ gives a non-physical, unbounded zero energy eigenstate, which simultaneously is eigenstate of the dilatation operator (2.9), $2i\hat{D}x = \frac{3}{2}x$. In the general case of $\nu \geq -1/2$, solutions of the stationary equation $\hat{H}_\nu \psi_{\kappa,\nu} = E_\nu(\kappa)\psi_{\kappa,\nu}$ with $E_\nu(\kappa) = \frac{1}{2}\kappa^2$, $\kappa > 0$, are

$$\psi_{\kappa,\nu}(x) = \sqrt{x}\mathcal{J}_{\nu+1/2}(\kappa x)\,, \tag{5.2}$$

where

$$\mathcal{J}_\alpha(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{\eta}{2}\right)^{2n+\alpha} \tag{5.3}$$

is the Bessel function of the first kind. The solutions (5.2) include $\nu = 0$ as a particular case. The Hamiltonian operator $\hat{H}_\nu$ is invariant under the change $\nu \to -(\nu+1)$, but the same transformation applied to (5.2) produces eigenfunctions of $\hat{H}_\nu$ not satisfying the boundary condition $\psi(0) = 0$ [21]. At $\nu = -1/2$, we have $\nu = -(\nu+1)$, and the eigenstates (5.2) correspond to a particular case of a family of self-adjoint extensions of $\hat{H}_{-1/2}$ [21, 34].

Consider the set of wave functions $x^{\nu+1+2n}$, $n = 0, 1, \ldots$. The function with $n = 0$ represents a formal, diverging at infinity, eigenstate of the differential operator $\hat{H}_\nu$ with $\nu \geq -1/2$ of eigenvalue $E = 0$, which follows from solutions (5.2) by applying to them the same limit as in the case of $\nu = 0$. The wave functions with $n \geq 1$ are the Jordan states of rank n corresponding to the same eigenvalue $E = 0$ of $\hat{H}_\nu$. The functions $x^{\nu+1+2n}$ are at the same time eigenstates of the operator $2i\hat{D}$ with eigenvalues $\nu + 2n + 3/2$. The Jordan states with $n \geq 1$ satisfy the relations

$$(\hat{H}_\nu)^j x^{\nu+1+2n} = \frac{(-2)^j\Gamma(n+1)}{\Gamma(n+1-j)}\frac{\Gamma(n+\nu+3/2)}{\Gamma(n+\nu+3/2-j)}x^{\nu+1+2(n-j)}\,, \quad j = 0, 1, \ldots, n\,, \tag{5.4}$$

which can be proved by induction. Eq. (5.4) extends to the case $j = n+1$ giving $(\hat{H}_\nu)^{n+1}x^{\nu+1+2n} = 0$ due to the appearing simple pole in the denominator.

The operator $\hat{H}_\nu$ together with the operators $\hat{K}$ and $\hat{D}$ defined by Eqs. (2.8) and (2.9) satisfy the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.10) as in the case of the free particle on the whole real line. Now we can define the direct analog of the operator $\hat{\mathfrak{S}}$ in the form[2]

$$\hat{S}_\nu := e^{-\hat{K}} e^{\frac{1}{2}\hat{H}_\nu} e^{i\ln 2\cdot\hat{D}} = e^{-\hat{K}} e^{i\ln 2\cdot\hat{D}} e^{\hat{H}_\nu} . \tag{5.5}$$

A similarity transformation with this non-unitary operator produces, analogously to (2.16), the relations

$$\hat{S}_\nu \hat{H}_\nu \hat{S}_\nu^{-1} = -\hat{J}_- , \qquad \hat{S}_\nu \hat{K} \hat{S}_\nu^{-1} = \hat{J}_+ , \qquad \hat{S}_\nu(\hat{D})\hat{S}_\nu^{-1} = -\tfrac{i}{2}\hat{H}_\nu^{\mathrm{AFF}} , \tag{5.6}$$

where now instead of (2.17),

$$\hat{J}_- = \tfrac{1}{2}(\hat{a}^-)^2 - \Delta_\nu , \qquad \hat{J}_+ = \tfrac{1}{2}(\hat{a}^+)^2 - \Delta_\nu , \qquad 2\hat{J}_0 = \hat{H}_\nu^{\mathrm{AFF}} = \hat{a}^+\hat{a}^- + \tfrac{1}{2} + \Delta_\nu . \tag{5.7}$$

The dilatation operator multiplied by $2i$ transforms in this case into the Hamiltonian $\hat{H}_\nu^{\mathrm{AFF}}$ of the conformal mechanics model of de Alfaro, Fubini and Furlan [5]. At the same time, we note that the operators x and $\frac{d}{dx}$ are transformed by the operator $\hat{S}_\nu$ into nonlocal operators corresponding to the square root of the operators $\hat{J}_+$ and $\hat{J}_-$ in (5.7), whose action violates the boundary condition $\psi(0) = 0$.

Application of $\hat{S}_\nu$ to the states $x^{\nu+1+2n}$ related to the system $\hat{H}_\nu$ produces the energy eigenstates of the AFF model,

$$\hat{S}_\nu(\tfrac{1}{\sqrt{2}}x)^{\nu+1+2n} = 2^{\frac{1}{4}}(-1)^n n! \psi_{\nu,n}^{\mathrm{AFF}}(x) , \tag{5.8}$$

where

$$\psi_{\nu,n}^{\mathrm{AFF}}(x) = x^{\nu+1}\mathcal{L}_n^{(\nu+1/2)}(x^2)e^{-x^2/2} , \qquad E_{\nu,n} = 2n + \nu + 3/2 , \tag{5.9}$$
$$\tag{5.10}$$

are the non-normalized eigenstates of the AFF model and their corresponding energy values, and

$$\mathcal{L}_n^{(\alpha)}(\eta) = \sum_{j=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(j+\alpha+1)} \frac{(-\eta)^j}{j!(n-j)!} \tag{5.11}$$

are the generalized Laguerre polynomials.

On the other hand, application of the operator $\hat{S}_\nu$ to the eigenstates (5.2) of the system $\hat{H}_\nu$ gives

$$\hat{S}_\nu \psi_{\kappa,\nu}(\tfrac{1}{\sqrt{2}}x) = 2^{\frac{1}{4}} e^{-\frac{1}{2}x^2 + \frac{1}{4}\kappa^2}\sqrt{x}\mathcal{J}_{\nu+1/2}(\kappa x) := \phi_\nu(x,\kappa) . \tag{5.12}$$

According to the first relation in (5.6), these are the coherent states of the AFF model [31],

$$\hat{J}_-\phi_\nu(x,\kappa) = -\frac{1}{4}\kappa^2\phi_\nu(x,\kappa) . \tag{5.13}$$

[2]One can work, instead, with the analog of the operator $\hat{\mathfrak{S}}_0$, but such a change is not essential.

By allowing the $\kappa > 0$ to become a complex parameter z, coherent states can be constructed with complex eigenvalues of the operator $\hat{J}_-$. Application of the evolution operator $\exp\left\{-it\hat{H}_\nu^{\mathrm{AFF}}\right\}$ to these states gives the time-dependent coherent states

$$\phi_\nu(x,z,t) = 2^{1/4}\sqrt{x}\mathcal{J}_{\nu+1/2}(z(t)x)e^{-x^2/2+z^2(t)/4-it}\,, \tag{5.14}$$

where $z(t) = ze^{-it}$. In the case of $\nu = 0$, these time-dependent coherent states of the AFF model are the odd Schrödinger cat states of the quantum harmonic oscillator [35],

$$\phi_0(x,z,t) \propto e^{-\frac{x^2}{2}+\frac{z^2(t)}{4}-\frac{it}{2}}\sin(z(t)x)\,. \tag{5.15}$$

Similarly to the analysis presented in the previous Section, one can consider the classical picture underlying the quantum non-unitary similarity transformation associated with the constructed "conformal bridge" between the systems (iii) and (iv). We will not discuss this picture in detail here, and only note that instead of (4.6), we will have the phase space function of a similar form but with H_0 changed for the Hamiltonian of the classical two-particle Calogero system, $H_\nu = \frac{1}{2}p^2 + \Delta_\nu$. The phase space functions H_ν, D and K satisfy the classical algebra of the form (4.1), and the Casimir element $C_\nu = J^\mu J_\mu = -H_\nu^{\mathrm{AFF}}K + D^2$ takes here the value defined by the coupling constant, $C_\nu = -\frac{1}{4}\nu(\nu+1)$. Analogously to the pair discussed in Section 4, classical relations $(x^2)' := \{x^2, 2iD\} = 2ix^2$, $(p^2)' := \{p^2, 2iD\} = -2ip^2$ for the two-particle Calogero system correspond here to the dynamics of the classical AFF model given by the relations $\frac{d}{dt}J_\pm = \{J^\pm, H_\nu^{\mathrm{AFF}}\} = \pm 2iJ_\pm$, where $J_\pm$ are the classical analogs of the operators $\hat{J}_\pm$ from (5.6).

On the Hilbert space of the AFF system, the infinite-dimensional unitary irreducible representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra of the discrete type series $\mathcal{D}_\alpha^+$ with $\alpha = \frac{1}{2}\nu + 3/4$ is realized, in which the states $\psi_{\nu,n}^{\mathrm{AFF}}$ from (5.9) are eigenstates of the compact generator $\hat{J}_0$ with eigenvalues $j_{0,n} = \alpha + n$, $n = 0, 1, \ldots$, and the Casimir operator takes the value $\hat{C}_\nu = \hat{J}^\mu \hat{J}_\mu = -\alpha(\alpha-1) = \frac{3}{16} - \frac{1}{4}\nu(\nu+1)$ [36, 37, 38].

6 Two-dimensional examples

When applying the transformation $\hat{\mathfrak{S}}$ to generators of the conformal algebra, one (formally) need not care about the concrete realization of the $\hat{J}_\mu$, since only the algebraic relations presented by equations (3.4) and (3.5) are used. In particular, one may consider to higher dimensional examples, where the range of physical systems of interest is greater. In this section we generalize our construction to relate the two-dimensional free particle system with a planar isotropic harmonic oscillator and the Landau problem.

To begin with, consider the non-unitary operator

$$\hat{\mathfrak{S}}(x,y) = \hat{\mathfrak{S}}(x)\hat{\mathfrak{S}}(y) \tag{6.1}$$

with $\hat{\mathfrak{S}}(x)$ and $\hat{\mathfrak{S}}(y)$ of the form (3.1). Via a similarity transformation, it produces a map

$$\hat{\mathfrak{S}}(x,y) : (x, \hat{p}_x, y, \hat{p}_y) \to (\hat{a}_x^+, -i\hat{a}_x^-, \hat{a}_y^+, -i\hat{a}_y^-)\,. \tag{6.2}$$

Then the two-dimensional free particle characterized by the dynamical conformal symmetry with generators

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2)\,, \qquad \hat{D} = \frac{1}{2}(x\hat{p}_x + y\hat{p}_y + 1)\,, \qquad \hat{K} = \frac{1}{2}(x^2 + y^2) \tag{6.3}$$

is related with the planar isotropic harmonic oscillator and generators of its Newton-Hooke symmetry as follows,

$$\hat{\mathfrak{S}}(x,y)\hat{H}\hat{\mathfrak{S}}^{-1}(x,y) = -\frac{1}{2}((\hat{a}_x^-)^2 + (\hat{a}_y^-)^2) = -\hat{\mathcal{J}}_-\,, \tag{6.4}$$

$$\hat{\mathfrak{S}}(x,y)2i\hat{D}\hat{\mathfrak{S}}^{-1}(x,y) = \hat{a}_x^+\hat{a}_x^- + \hat{a}_y^+\hat{a}_y^- + 1 = \hat{H}_{iso} = 2\hat{\mathcal{J}}_0\,, \tag{6.5}$$

$$\hat{\mathfrak{S}}(x,y)\hat{K}\hat{\mathfrak{S}}^{-1}(x,y) = \frac{1}{2}((\hat{a}_x^+)^2 + (\hat{a}_y^+)^2) = \hat{\mathcal{J}}_+\,, \tag{6.6}$$

where the operators $\hat{\mathcal{J}}_0$ and $\hat{\mathcal{J}}_\pm$ satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra (A.3). Analogously to the one-dimensional case, the stationary and coherent states of $\hat{H}_{\rm iso}$ are produced by $\hat{\mathfrak{S}}(x,y)$ from the corresponding states of the two-dimensional free particle system,

$$\hat{\mathfrak{S}}(x,y)\left(\tfrac{x}{\sqrt{2}}\right)^n\left(\tfrac{y}{\sqrt{2}}\right)^m = 2^{n+m+\frac{1}{2}}e^{-\frac{(x^2+y^2)}{2}}H_n(x)H_m(y)\,, \tag{6.7}$$

$$\hat{\mathfrak{S}}(x,y)e^{\frac{i}{\sqrt{2}}(k_xx+k_yy)} = \sqrt{2}e^{-\frac{(x^2+y^2)}{2}}e^{\frac{k_x^2+k_y^2}{4}}e^{i(k_xx+k_yy)}\,. \tag{6.8}$$

The angular momentum operator,

$$\hat{M} = x\hat{p}_y - y\hat{p}_x = -i(\hat{a}_x^\dagger\hat{a}_y^- - \hat{a}_y^\dagger\hat{a}_x^-)\,, \tag{6.9}$$

is an integral of the planar free particle and of the isotropic harmonic oscillator systems. It is invariant under the similarity transformation produced by $\hat{\mathfrak{S}}$, or, equivalently,

$$\hat{\mathfrak{S}}(x,y)\hat{M} = \hat{M}\hat{\mathfrak{S}}(x,y)\,. \tag{6.10}$$

Consider now the Landau problem for a scalar particle on $\mathbb{R}^2$. In the symmetric gauge $\vec{A} = \frac{1}{2}B(-q_2, q_1)$, the Hamiltonian operator (in units $c = m = \hbar = 1$),

$$\hat{H}_{\rm L} = \frac{1}{2}\hat{\vec{\Pi}}^2, \qquad \hat{\Pi}_j = -i\frac{\partial}{\partial q_j} - eA_j\,, \qquad [\hat{\Pi}_1, \hat{\Pi}_2] = ieB\,, \tag{6.11}$$

has an explicitly rotational invariant form. Assuming $\omega_c = eB > 0$, it can be factorized,

$$\hat{H}_{\rm L} = \omega_c(\hat{\mathcal{A}}^+\hat{\mathcal{A}}^- + \tfrac{1}{2})\,, \tag{6.12}$$

in terms of the Hermitian conjugate operators

$$\hat{\mathcal{A}}^\pm = \frac{1}{\sqrt{2\omega_c}}(\hat{\Pi}_1 \mp i\hat{\Pi}_2)\,, \qquad [\hat{\mathcal{A}}^-, \hat{\mathcal{A}}^+] = 1\,. \tag{6.13}$$

Setting $\omega_c = 2$, we can identify q_i with dimensionless variables $q_1 = x$, $q_2 = y$. Then we present $\hat{\mathcal{A}}^{\pm}$ as linear combinations of the usually defined ladder operators $\hat{a}^{\pm}_x$ and $\hat{a}^{\pm}_y$, in terms of which we also define the operators $\hat{\mathcal{B}}^{\pm}$,

$$\hat{\mathcal{A}}^{\pm} = \frac{1}{\sqrt{2}}(\hat{a}^{\pm}_y \pm i\hat{a}^{\pm}_x)\,, \qquad \hat{\mathcal{B}}^{\pm} = \frac{1}{\sqrt{2}}(\hat{a}^{\pm}_y \mp i\hat{a}^{\pm}_x)\,. \tag{6.14}$$

The operators $\hat{\mathcal{B}}^{\pm}$ satisfy relation $[\hat{\mathcal{B}}^{-}, \hat{\mathcal{B}}^{+}] = 1$, and commute with $\hat{\mathcal{A}}^{\pm}$. They are integrals of motion, and their non-commuting Hermitian linear combinations $\hat{\mathcal{B}}^{+} + \hat{\mathcal{B}}^{-}$ and $i(\hat{\mathcal{B}}^{+} - \hat{\mathcal{B}}^{-})$ correspond to coordinates of the center of the cyclotron motion. In terms of the ladder operators $\hat{a}^{\pm}_x$, $\hat{a}^{\pm}_y$ the Hamiltonian $\hat{H}_{\mathrm{L}}$ takes the form of a linear combination of the Hamiltonian of the isotropic oscillator and angular momentum operator,

$$\hat{H}_{\mathrm{L}} = \hat{H}_{\mathrm{iso}} - \hat{M}\,. \tag{6.15}$$

On the other hand, the angular momentum operator and isotropic oscillator's Hamiltonian are presented in terms of $\hat{\mathcal{A}}^{\pm}$ and $\hat{\mathcal{B}}^{\pm}$ as follows,

$$\hat{M} = \hat{\mathcal{B}}^{+}\hat{\mathcal{B}}^{-} - \hat{\mathcal{A}}^{+}\hat{\mathcal{A}}^{-}\,, \qquad \hat{H}_{\mathrm{iso}} = \hat{\mathcal{B}}^{+}\hat{\mathcal{B}}^{-} + \hat{\mathcal{A}}^{+}\hat{\mathcal{A}}^{-} + 1\,, \tag{6.16}$$

and we have the commutation relations $[\hat{M}, \hat{\mathcal{B}}^{\pm}] = \pm\hat{\mathcal{B}}^{\pm}$, $[\hat{M}, \hat{\mathcal{A}}^{\pm}] = \mp\hat{\mathcal{A}}^{\pm}$. By taking into account the invariance of the angular momentum under similarity transformation, we find that its linear combination with the dilatation operator is transformed into the Hamiltonian of the Landau problem,

$$\hat{\mathfrak{S}}(x,y)(2i\hat{D} - \hat{M})\hat{\mathfrak{S}}^{-1}(x,y) = \hat{H}_{\mathrm{L}}\,. \tag{6.17}$$

Let us now introduce complex coordinate in the plane,

$$w = \frac{1}{\sqrt{2}}(y + ix)\,, \quad \text{and} \qquad \bar{w}\,. \tag{6.18}$$

The elements of conformal algebra and angular momentum operator take then the form

$$\hat{H} = -\frac{\partial^2}{\partial w \partial \bar{w}}\,, \quad \hat{D} = \frac{1}{2i}\left(w\frac{\partial}{\partial w} + \bar{w}\frac{\partial}{\partial \bar{w}} + 1\right)\,, \quad \hat{K} = w\bar{w}\,, \quad \hat{M} = \bar{w}\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial w}\,, \tag{6.19}$$

and we find that the operator (6.1) generates the similarity transformations

$$\hat{\mathfrak{S}}(x,y)w\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{A}}^{+}\,, \qquad \hat{\mathfrak{S}}(x,y)\left(\tfrac{\partial}{\partial w}\right)\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{A}}^{-}\,, \tag{6.20}$$

$$\hat{\mathfrak{S}}(x,y)\bar{w}\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{B}}^{+}\,, \qquad \hat{\mathfrak{S}}(x,y)\left(\tfrac{\partial}{\partial \bar{w}}\right)\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{B}}^{-}\,, \tag{6.21}$$

$$\hat{\mathfrak{S}}(x,y)\left(w\tfrac{\partial}{\partial w}\right)\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{A}}^{+}\hat{\mathcal{A}}^{-}\,, \qquad \hat{\mathfrak{S}}(x,y)\left(\bar{w}\tfrac{\partial}{\partial \bar{w}}\right)\hat{\mathfrak{S}}^{-1}(x,y) = \hat{\mathcal{B}}^{+}\hat{\mathcal{B}}^{-}\,. \tag{6.22}$$

Observe that each pair of relations in (6.20) and (6.21) has a form similar as the one-dimensional transformation (3.2), where, however, the coordinate and momentum are Hermitian operators.

Simultaneous eigenstates of the operators $w\frac{\partial}{\partial w}$ and $\bar{w}\frac{\partial}{\partial \bar{w}}$, which satisfy the relations $w\frac{\partial}{\partial w}\phi_{n,m} = n\phi_{n,m}$ and $\bar{w}\frac{\partial}{\partial \bar{w}}\phi_{n,m} = m\phi_{n,m}$ with $n, m = 0, 1, \ldots$, are

$$\phi_{n,m}(x,y) = w^n(\bar{w})^m = 2^{-(n+m)/2}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}(i)^{n-m+l-k}y^{k+l}x^{n+m-k-l}\,, \tag{6.23}$$

where the binomial theorem has been used. Employing Eq. (6.19) we find that

$$\hat{M}\phi_{n,m} = (m-n)\phi_{n,m}\,, \qquad 2i\hat{D}\phi_{n,m} = (n+m+1)\phi_{n,m}\,, \tag{6.24}$$

$$\hat{K}\phi_{n,m} = \phi_{n+1,m+1}\,, \qquad \hat{H}\phi_{n,m} = -nm\phi_{n-1,m-1}\,. \tag{6.25}$$

The last equality shows that $\phi_{0,m}$ and $\phi_{n,0}$ are the zero energy eigenstates of the two-dimensional free particle, while the $\phi_{n,m}$ with $n, m > 0$ are the Jordan states corresponding to the same zero energy value. Application of the operator $\hat{\mathfrak{S}}(x,y)$ to these functions yields

$$\hat{\mathfrak{S}}(x,y)\phi_{n,m}(x,y) = 2^{2(n+m)+\frac{1}{2}}e^{-\frac{(x^2+y^2)}{2}}H_{n,m}(y,x) = \psi_{n,m}(x,y)\,, \tag{6.26}$$

where

$$H_{n,m}(y,x) = 2^{-(n+m)}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}(i)^{n-m+l-k}H_{k+l}(y)H_{n+m-k-l}(x)\,, \tag{6.27}$$

are the complex Hermite polynomials, see [39]. These functions are eigenstates of the operators $\hat{H}_{\rm L}$, $\hat{M}$ and $\hat{H}_{\rm iso}$,

$$\hat{H}_{\rm L}\psi_{n,m} = (n+\tfrac{1}{2})\psi_{n,m}\,, \qquad \hat{M}\psi_{n,m} = (m-n)\psi_{n,m}\,, \tag{6.28}$$

$$\hat{H}_{\rm iso}\psi_{n,m} = (n+m+1)\psi_{n,m}\,, \tag{6.29}$$

and we note that $\psi_{n,n}$ is rotational invariant.

Eqs. (6.20), (6.21), and (6.25) show that the operators $\hat{\mathcal{A}}^{\pm}$ and $\hat{\mathcal{B}}^{\pm}$ act as the ladder operators for the indexes n and m, respectively, while the operators $\hat{\mathcal{J}}_{\pm}$ given by Eqs. (6.4) and (6.6) increase or decrease simutaneousy n and m by one.

Application of the operator $\hat{\mathfrak{S}}(x,y)$ to exponential functions of the most general form $e^{\alpha w+\beta\bar{w}}$ with $\alpha, \beta \in \mathbb{C}$ gives here, similarly to the one-dimensional case, the coherent states of the Landau problem as well of the isotropic harmonic oscillator,

$$\begin{aligned}\psi_{\rm L}(x,y,\alpha,\beta) &= \hat{\mathfrak{S}}(x,y)e^{\frac{1}{\sqrt{2}}((\alpha+\beta)y+i(\alpha-\beta)x)} = \sqrt{2}e^{-\frac{(x^2+y^2)}{2}+(\alpha+\beta)y+i(\alpha-\beta)x-\alpha\beta} \\ &= \textstyle\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{1}{n!}\binom{n}{l}\alpha^l\beta^{n-l}\psi_{l,n-l}(x,y)\,.\end{aligned} \tag{6.30}$$

Applying to them, in particular, the evolution operator $e^{-it\hat{H}_{\rm L}}$, we obtain the time dependent solution to the Landau problem,

$$\psi_{\rm L}(x,y,\alpha,\beta,t) = e^{-\frac{it}{2}}\psi_{\rm L}(x,y,\alpha e^{-it},\beta)\,, \tag{6.31}$$

whereas under rotations these states transform as

$$e^{i\varphi\hat{M}}\psi_{\rm L}(x,y,\alpha,\beta) = \psi_{\rm L}(x,y,\alpha e^{-i\varphi},\beta e^{i\varphi})\,. \tag{6.32}$$

As the function $e^{\alpha w+\beta\bar{w}}$ is a common eigenstate of the differential operators $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \bar{w}}$ with eigenvalues α and β, respectively, then our transformation yields

$$\hat{\mathcal{A}}^{-}\psi_{\rm L}(x,y,\alpha,\beta)=\alpha\psi_{\rm L}(x,y,\alpha,\beta)\,,\qquad \hat{\mathcal{B}}^{-}\psi_{\rm L}(x,y,\alpha,\beta)=\beta\psi_{\rm L}(x,y,\alpha,\beta)\,,\tag{6.33}$$

that provides another explanation why the wave functions (6.30) are the coherent states for the planar harmonic oscillator as well as for the Landau problem.

7 Discussion and outlook

The intertwining operators of the Darboux transformations allow to construct eigenstates of a Hamiltonian of one system in terms of eigenstates of a Hamiltonian of another, related system. There, the intertwining operators are differential operators of finite order, and so, the relation between corresponding eigenstates is local. Those operators also factorize the corresponding Hamiltonians or polynomials thereof. In our "conformal bridge" constructions, the generator of the similarity transformation is a nonlocal operator, formally given by an infinite series in the momentum operator. It has a nature of the fourth order root of the identity operator from the point of view of its action on generators of the conformal algebra, where it acts as a non-unitary automorphism. Its peculiarity in comparison with generators of the Darboux transformations is that it intertwines the generators of the $\mathfrak{sl}(2,\mathbb{R})$ having different topological nature corresponding to a change of the dynamics in the sense of Dirac. In fact, our transformation changes the conformally invariant asymptotically free dynamics described by the Hamiltonian being a non-compact generator of the $\mathfrak{sl}(2,\mathbb{R})$ to the conformal Newton-Hooke dynamics generated by a compact generator.

In comparison with the well known Niederer's canonical transformation [11, 40],

$$\zeta=\frac{x}{\sqrt{1+(\omega t)^2}}\,,\qquad d\tau=\frac{dt}{1+(\omega t)^2}\,,\tag{7.1}$$

employed to transform the one-dimensional conformal mechanics model into the regularized AFF model, our quantum non-unitary similarity transformations and corresponding complex classical canonical transformations are time-independent: we work within the stationary framework. As a result, instead of the map between solutions of the time-dependent Schrödinger equation of the free particle and harmonic oscillator, or of the conformal mechanics corresponding to the two-particle Calogero model and the AFF model, our transformation acts in an essentially different way. It maps formal eigenstates of the dilatation operator with imaginary eigenvalues being also Jordan states corresponding to the zero energy value of the conformal mechanics model Hamiltonian $\hat{H}_\nu$ (or of the free particle Hamiltonian $\hat{H}_0$) into physical bound eigenstates of the regularized AFF model $\hat{H}_\nu^{\rm AFF}$ (or of the harmonic oscillator $\hat{H}_{\rm osc}$) of the real energy values. Besides, the stationary eigenstates of $\hat{H}_\nu$ (or, of $\hat{H}_0$) are transformed into the coherent states of $\hat{H}_\nu^{\rm AFF}$ ($\hat{H}_{\rm osc}$).

The interesting question is then what is a possible geometrical interpretation of our complex canonical transformation, being an eighth order root of the identity transformation. One could try to characterize it within the framework of the Eisenhart lift [41] or by using the Dirac trick which corresponds to a generation of the time-dependent Schrödinger equation via the presentation of the classical action in the reparametrization invariant form. This way one may expect to establish a possible relation (if it exists) of our "conformal bridge" constructions with the transformations based on the Niederer's time-dependent canonical transformation (7.1). Investigation of the indicated problem may be of interest in the light of the AdS/CFT correspondence [1, 2, 42].

Our canonical transformations correspond to the Hamiltonian vector flows produced by generators of the conformal symmetry with particular complex values of the parameters. The interesting question is whether such transformations having the nature of the fourth order root of the spatial reflection have any interpretation in the context of $\mathcal{PT}$-symmetry [43, 44, 45]. This question is intriguing especially having in mind that perfectly invisible zero-gap systems and the related extreme wave solutions (multi-solitons) to the complexified KdV equation can be obtained from the free particle system by application to it of the $\mathcal{PT}$-regularized Darboux transformations intimately related to conformal symmetry [19, 22, 46].

It is not difficult to find the analog of our canonical transformation but with certain real values of the parameters that transforms the dilatation generator of the free particle into the Hamiltonian of the inverted (repulsive) harmonic oscillator, which has a continuous spectrum at the quantum level [47]. In this way, Gamow states [48] enter naturally into the construction. However, at the quantum level there appears a problem because the transformation, which now will be unitary, acting on the eigenstates of the dilatation operator with real eigenvalues will produce divergent series. This implies the necessity of a more deep analysis of the corresponding unitary transformation.

Yet another interesting question is whether by using a conformal bridge transformation, new solutions to some integrable systems in partial derivatives can be constructed. We have here in mind the generation of the multi-soliton solutions to the KdV equation, including rational solutions of the Calogero type, by application of the Darboux transformations to the quantum free particle system on the one hand, and the "conformal bridge" relation of the free particle with the quantum harmonic oscillator that we have discussed. Such a hypothetical possibility still remains a mystery for us.

As our transformation does not depend on a concrete realization of the generators of the conformal algebra, as we showed, one can study higher dimensional systems. In this way we found an interesting relation between the 2D free particle system and the Landau problem, where the two-dimensional plane waves were mapped into the coherent states of the planar harmonic oscillator and of the Landau problem. It would be interesting to investigate what happens in the case of the higher-dimensional non-separable conformal invariant systems such as multi-particle Calogero model[3], or in the case of Dirac magnetic

[3]Based on a spectrum similarity, Calogero conjectured [13] that there should exist a nontrivial relation between his model and the system of decoupled oscillators. This conjecture was proved, including the case without quadratic interactions, by constructing some (non)-unitary transformations that mutually map Hamiltonians of the corresponding systems of the coupled and decoupled particles [49, 50, 51, 52] and give rise to a non-local $\mathfrak{sl}(2,\mathbb{R})$ generator [52]. The conformal transformations we discussed here, in spite of being of a somewhat similar form to those in [50, 51, 52], are considered from an essentially different perspective of the change of the form of dynamics a la Dirac [4, 5, 1, 2, 10].

monopole system.

As a final interesting problem which seems to be rather natural for further investigation we indicate a generalization of our constructions to the superconformal case [53, 54], bearing, particularly, in mind, the presence of the hidden superconformal symmetry in the quantum harmonic oscillator [26].

We plan to present some results on the listed problems in a near future [55].

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Appendix

A Conformal symmetry

The $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$ algebra is given by the commutation relations

$$[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda\,, \tag{A.1}$$

where $\epsilon^{\mu\nu\lambda}$ is a totally antisymmetric tensor, $\epsilon^{012} = 1$, and metric is $\eta^{\mu\nu} = \text{diag}\,(-1,1,1)$. The algebra (A.1) can be generated, in particular, by $J_\mu = -\epsilon_{\mu\nu\lambda}x^\nu p^\lambda$ realized in terms of the $(2+1)$-dimensional coordinate and momenta operators, $[x_\mu, p_\nu] = i\eta_{\mu\nu}$.

With

$$J_\pm := J_1 \pm iJ_2\,, \tag{A.2}$$

the algebra (A.1) takes the form of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[J_-, J_+] = 2J_0\,, \qquad [J_0, J_\pm] = \pm J_\pm\,. \tag{A.3}$$

The transformation

$$J_0 \to -J_0\,, \qquad J_\pm \to -J_\mp \tag{A.4}$$

defines an outer automorphism of (A.3), and the Casimir element of the algebra is

$$C = J_\mu J^\mu = -(J_0)^2 + (J_1)^2 + (J_2)^2 = -(J_0)^2 + \frac{1}{2}(J_+J_- + J_-J_+)\,. \tag{A.5}$$

The realization

$$J_- = \frac{\partial}{\partial t}\,, \qquad J_0 = t\frac{\partial}{\partial t}\,, \qquad J_+ = t^2\frac{\partial}{\partial t} \tag{A.6}$$

generates the conformal transformations

$$t \to t + \alpha\,, \qquad t \to e^\beta t\,, \qquad t \to \frac{t}{1-\gamma t} \tag{A.7}$$

of a time variable. The transformations (A.7), together with

$$x \to e^{\frac{1}{2}\beta} x\,, \qquad x \to \frac{x}{1-\gamma t} \tag{A.8}$$

represent the conformal symmetry of the free particle action $\mathcal{A} = \frac{m}{2}\int \dot{x}^2 dt$, whose generators are

$$J_- = \frac{\partial}{\partial t}\,, \qquad J_0 = t\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\,, \qquad J_+ = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x}\,. \tag{A.9}$$

The formal change $t \to z$, $z \in \mathbb{C}$, in (A.6) accompanied by the automorphism (A.4) with subsequent identification $L_0 = J_0$, $L_{+1} = J_-$, $L_{-1} = J_+$ gives generators of the $\mathfrak{su}(1,1)$ subalgebra of the Virasoro algebra,

$$L_0 = -z\frac{\partial}{\partial z}\,, \qquad L_{+1} = -z^2\frac{\partial}{\partial z}\,, \qquad L_{-1} = -\frac{\partial}{\partial z}\,. \tag{A.10}$$

This can be realized as

$$J_0 = a^+a^-\,, \qquad J_+ = (a^+)^2a^-\,, \qquad J_- = a^- \tag{A.11}$$

with ladder operators $a^{\mp} = \frac{1}{\sqrt{2}}(x \pm \frac{d}{dx})$ of the quantum harmonic oscillator. Another realization of (A.3) is given by

$$J_0 = \frac{1}{4}(a^+a^- + a^-a^+)\,, \qquad J_\pm = \frac{1}{2}(a^\pm)^2\,, \tag{A.12}$$

which corresponds to the conformal Newton-Hooke symmetry of the quantum harmonic oscillator.

B Hamiltonian vector flows as canonical transformations

A Hamiltonian vector flow generated by a function F on a phase space $\mathcal{M}$ is given by

$$f(\alpha) = \exp(\alpha F) \star f(q,p) := f(q,p) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}\{F, \{\ldots, \{F, f\underbrace{\}\ldots\}\}}_{n} =: T_F(\alpha)(f)\,. \tag{B.1}$$

The parameter α is usually assumed to be real, but we allow for complex values. Transformations (B.1) correspond to the action of a one-parameric Lie group on $\mathcal{M}$,

$$T_F(\alpha) \circ T_F(\beta) = T_F(\alpha+\beta), \qquad T_F(0) = I, \qquad (T_F(\alpha))^{-1} = T_F(-\alpha).$$

The composition of the Hamiltonian flows generated by functions F and G with $\{F, G\} \neq 0$ is non-commutative, and

$$(T_F(\alpha) \circ T_G(\beta))^{-1} = T_G(-\beta) \circ T_F(-\alpha).$$

For functions f and g on phase space, the following relation holds

$$T_F(\alpha) \circ T_G(\beta)(f \cdot g) = (T_F(\alpha) \circ T_G(\beta)(f)) \cdot (T_F(\alpha) \circ T_G(\beta)(g)). \tag{B.2}$$

A flow of a Hamiltonian vector field is a canonical transformation: $\{f(\alpha), g(\alpha)\} = \{f, g\}$. In the general case of $\alpha \in \mathbb{C}$, the transformation (B.1) corresponds to the quantum similarity transformation $\hat{f}(\alpha) = \exp\Big(-i\alpha\hat{F}\Big)\hat{f}\exp\Big(i\alpha\hat{F}\Big)$.

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