Gauge Theories in a Bag

A. Wipf
Institute for Theoretical Physics, ETH-Hönggerberg,
CH-8093 Zürich, Switzerland

S. Duerr
Institute for Theoretical Physics, University of Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

ZU-TH 30/94 and ETH-TH/94-36
15.11.1994

Abstract

We investigate multi-flavour gauge theories confined in $d = 2n$-dimensional Euclidean bags. The boundary conditions for the 'quarks' break the axial flavour symmetry and depend on a parameter θ. We determine the θ-dependence of the fermionic correlators and determinants and find that a CP-breaking θ-term is generated dynamically. As an application we calculate the chiral condensate in multi-flavour QED_2 and the abelian projection of QCD_2. In the second model a condensate is generated in the limit where the number of colours, N_c, tends to infinity. We prove that the condensate in QCD_2 decreases with increasing bag radius R at least as $\sim R^{-1/N_cN_f}$. Finally we determine the correlators of mesonic currents in QCD_2.

1 Introduction

Possible mechanisms for the spontaneous breaking of the chiral symmetry in QCD have repeatedly been discussed in the literature [1], but a derivation from first principles remains to be found. The broken phases can be probed by coupling the fields to a symmetry breaking trigger source which is removed after the infinite volume limit has been taken. Alternatively one may put the system in a finite box, imposes symmetry breaking boundary conditions and then performs the thermodynamic limit $V \to \infty$. This is wellknown from spin models [2]. For example, when coupling the Ising spins to a constant magnetic field a mean magnetization remains at low temperature even when the trigger has been switched off. Such a magnetization can only arise if the ground state is Z_2-asymmetric or in other words if the Z_2-symmetry is spontaneously broken. Instead of switching on a magnetic field one may impose Z_2-breaking, say spin-up, boundary conditions and again a magnetization remains after the infinite volume limit has been taken.

In QCD a great deal of efforts have been undertaken to study the quark condensates in the limit of vanishing current quark masses [1]. These condensates would signal a spontaneous breaking of the axial flavour symmetry $SU_A(N_f)$ as it is required by the low energy phenomenology. Here one runs into the following paradox: In the chiral limit the generating functional for the fermionic Green's functions on a compact spacetime without boundary,

$$\begin{aligned} Z[\eta,\bar\eta] &= \int \mathcal{D}(A,\psi)\, e^{-S_{YM}+\int \bar\psi i\not{D}\psi + \int \bar\eta\psi+\bar\psi\eta} \\ &= \sum_N \int \mathcal{D}A^N\, e^{-S_{YM}} \prod_{k=1}^{N} (\bar\eta,\psi_k)(\bar\psi_k,\eta)\mathrm{det}' i\not{D}\; e^{\int \bar\eta S'\eta}, \end{aligned} \tag{1}$$

where the gauge fields A^N support N zero modes $\psi_1,\ldots,\psi_N$ of $i\not{D}$, gets contributions from sectors with non-zero instanton numbers [3]

$$q = \frac{1}{32\pi^2}\int d^4x F^a_{\mu\nu}\ {}^*F^a_{\mu\nu}. \tag{2}$$

The primes in (1) indicate the suppression of zero modes. If we only allow for smooth configurations on S^4 or $S^3\times R$ then q is an integer [4] and the number of zero modes [5]

$$N = \begin{cases} N_f q & \text{for } N_f\text{-flavour } QCD \\ N_c q & \text{for supersymmetric } QCD \end{cases} \tag{3}$$

is an integer multiple of N_f or N_c. Thus neither the topologically trivial sector contributes to the chiral condensate

$$\langle\bar\psi\psi\rangle = \frac{1}{Z}\frac{\delta^2}{\delta\eta\delta\bar\eta} Z|_{\eta=\bar\eta=0}, \tag{4}$$

since S' in (1) is chirality conserving, nor the nontrivial sectors since there are too many zero modes. Hence the condensate vanishes[1]. This conclusion is certainly in conflict with low energy strong interaction phenomenology or Ward-identities which predict a nonvanishing condensate for susy QCD [6].

Possible ways out (which work if the center of the gauge group is big enough) have been suggested by t'Hooft [7], who introduced twisted instantons, socalled torons, on the 4-dimensional torus, and by Zhitnitsky [8], who considered singular gauge fields on S^4. Both constructions produce configurations with fractional instanton numbers and may resolve the above mentioned paradox. However, for $O(N>4)$ susy-YM-theories, which give rise to a nonvanishing chiral condensate [9], the center is too small and these constructions do not work. Recently Shifman and Smilga have introduced another type of configuration, they called them fractons, which may generate a chiral condensate [10]. By allowing for flavour-dependent twisted boundary conditions they could introduce fractionally charged instantons and those generated a non-vanishing condensate in multi-flavour QED_2. It remains to be seen whether these fractons solve the puzzle posed by the chiral condensate in $O(N)$-susy theories.

Instead of quantizing gauge theories on a sphere or on a torus we propose to quantize them in an even-dimensional ($d=2n$) Euclidean bag $\mathcal{M}$ [11] and to impose $SU_A(N_f)$-breaking boundary conditions to trigger a chiral symmetry breaking. In a bag the instanton number is not quantized and the system itself is allowed to decide which are the dominant configurations. We investigate how the various correlators depend on the parameter θ characterizing the boundary conditions and shall see that in the models we studied the bag boundary conditions are a substitute for small quark masses and also reproduce the fracton results.

In the chiral limit of massless 'quarks' in the fundamental representation of $SU(N_c)$ the Euclidean action

$$
\begin{aligned}
S[A,\psi] = S_{YM}[A] \quad &+ \quad S_D[A,\psi], \quad \text{where} \\
S_{YM} = \frac{1}{4g^2}\int\limits_{\mathcal{M}} \operatorname{tr} F_{\mu\nu}F^{\mu\nu} \quad , &\quad S_D = \sum_{p=1}^{N_f}\int\limits_{\mathcal{M}} \psi_p^\dagger i\not{D}\psi_p,
\end{aligned}
\tag{5}
$$

is invariant under global $SU_V(N_f)\times SU_A(N_f)$ rotations[2] of the fermions since the Dirac operator

[1] When switching on a small quark mass one arrives at the same conclusion on a compact spacetime without boundary, since $\det(i\not{D}+m)\sim m^N$.

[2] Actually, for $N_c=2$ the symmetry group is $SU(2N_f)$ [12].

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - iA_\mu) \tag{6}$$

is the same for all N_f flavours. We shall impose the following boundary conditions, which relate the different spin components on the bag boundary,

$$(B(\theta) \times I_f \times I_c)\psi = \psi \quad \text{on} \quad \partial\mathcal{M}. \tag{7}$$

They break the $SU_A(N_f)$-symmetry but are vector-flavour and colour neutral so that the gauge invariant fermionic determinant is the same for all flavours. This approach has various advantages. First, the configuration space of gauge potentials in a bag is topologically trivial and hence there are no disconnected instanton sectors. Related to that is the absence of fermionic zero modes which would complicate the quantization of gauge theories considerably [13, 14, 15]. Second, the θ-dependence of the fermionic determinant, which appears in the measure of functional integration over the gauge field configurations after the fermions have been integrated out,

$$\langle O\rangle = \int d\mu_\theta(A)\,\langle O\rangle_A \quad , \quad d\mu_\theta(A) = \frac{1}{Z}e^{-S_{YM}[A]}\mathrm{det}_\theta(i\not{D})\,\mathcal{D}A \tag{8}$$

can be calculated explicitly, contrary to its mass dependence. Here $\langle O\rangle_A$ denotes the expectation value of O in a fixed background gauge field A,

$$\langle O\rangle_A = \frac{1}{\mathrm{det}_\theta\, i\not{D}}\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\; O\, e^{\int \psi^\dagger i\not{D}\psi}. \tag{9}$$

In writing (8) we anticipated that in a bag $\not{D}$ possesses no zero modes and absorbed the gauge fixing factor with corresponding Fadeev-Popov determinant in $\mathcal{D}A$.

The results of our investigations are presented as follows: In section 2 we introduce the bag boundary conditions for the 'quarks'. Some simple consequences for the spectrum of the Dirac operator are then discussed in section 3. We show that $\not{D}$ possesses no zero modes, discuss the (modified) parity transformation and derive a boundary Hellmann-Feynman formula. In section 4 we determine the θ-depencence of the fermionic Green's functions in a (spherical) bag and find their explicit forms when the gauge field is switched off. In the following section we derive the θ-dependence of the fermionic determinants for arbitrary $2n$-dimensional bags. We shall prove that through the interaction of the 'quarks' with the boundary an effective CP-breaking θ-term is generated. In the remaining part of the paper we investigate 2-dimensional gauge theories in the chiral limit. We start in section 6 with applying the deformation technique to evaluate the exact fermionic determinant in a bag. We prove that for $U(N_c)$-theories the measure of functional integration $d\mu_\theta(A)$ factorizes into the $U(1)$ and $SU(N_c)$ measures. Then we gain further insight into the spectrum of these models by calculating all mesonic current correlators in section 7. For $U(N_c)$ gauge theories with N_f flavours we find that the spectrum contains 1 massive and N_f^2-1 massless bosons, similarly as in the multi-flavour Schwinger model, and that they decouple from the remaining degrees of freedom. In the last section we investigate the chiral symmetry breaking in 2-dimensional gauge theories. First we derive the exact form of the chiral condensate for multi-flavour QED_2 in a spherical bag. A comparison with the perturbation by small 'quark'-masses [16] shows that the bag-boundary conditions serve as trigger similarly as small 'quark'-masses do. However, in a bag we need not worry about instantons, torons or fractons. Then we derive an upper bound on the chiral condensate in nonabelian gauge theories as a function of the bag-radius. As a particular application we prove that for 2-dimensional $SU(N_c)$ gauge theories with arbitrary $N_c<\infty$ the condensate vanishes in the thermodynamic limit. Finally we calculate the condensate in the abelian projected gauge theories and discuss the large N_c-limit. We shall see that for 1 flavour and $N_c\to\infty$ a condensate is generated. In the discussion we show that for multi-flavour QED_2, confined in a *spatial* bag and at finite temperature, the chiral condensate agrees with that generated by fractons on the torus [10]. In the appendix we derive the boundary-Seeley-deWitt coefficient which is needed in the main body of the paper.

2 Bag Boundary Conditions

For Dirac fermions propagating in an Euclidean bag $i\not{D}$ should be selfadjoint (or at least normal) for the partition function Z to be real. A necessary condition for selfadjointness is that

$$(\chi,i\not{D}\psi)-(i\not{D}\chi,\psi)\equiv\int\limits_{\mathcal{M}}\chi^\dagger i\not{D}\psi-\int\limits_{\mathcal{M}}(i\not{D}\chi)^\dagger\psi=i\oint\limits_{\partial\mathcal{M}}\chi^\dagger\gamma_n\psi \tag{10}$$

vanishes. Here $\gamma_n=n^\mu\gamma_\mu$ is the projection of the hermitean γ-matrices on the outward oriented normal vectorfield $n^\mu(x)$ on the bag-boundary $\partial\mathcal{M}$.

We will impose *local* linear boundary conditions

$$\psi(x) = B(x)\psi(x) \quad \text{on} \quad \partial\mathcal{M}, \tag{11}$$

since nonlocal spectral boundary condition, as introduced and discussed in [17], respect the axial flavour symmetry and probably would lead to a vanishing condensate in the multiflavour case. The local boundary conditions must be compatible with both gauge- and vector-flavour symmetry which means that B must be a singlett under the corresponding transformations. Hence it ought to be in the center of these transformations.

The surface integral in (10) vanishes if

$$B^\dagger\gamma_n B = -\gamma_n \quad \text{and we may assume} \quad B^2 = Id. \tag{12}$$

We shall choose the following one-parametric solution[3] [19]

$$\boxed{\psi = B_\theta\psi \quad \text{on} \quad \partial\mathcal{M} \qquad \text{with} \quad B_\theta = i\bar{\gamma}e^{\theta\bar{\gamma}}\gamma_n \times I_f \times I_c.} \tag{13}$$

Here $\bar{\gamma} = (-i)^n\gamma_0\gamma_1\cdots\gamma_{d-1}$ is the generalization of γ_5 which always exists in even dimensions. We shall choose $\bar{\gamma} = \text{diag}(1_n, -1_n)$, i.e. a chiral representations in which the hermitean γ_μ are off-diagonal. In the following we shall not spell out the trivial action of B_θ in flavour and colour space as we did in (13).

When the 'quarks' are reflected from the bag boundary they may change their chirality [18] which means that the boundary conditions break the axial-flavour symmetry

$$\psi \longrightarrow e^{\bar{\gamma}A}\psi, \quad \text{where} \quad e^{iA} \in SU(N_f).$$

We shall see that θ in (13) plays a similar role as the θ-parameter in QCD. Let us now derive some properties of the Dirac operator in an arbitrary external gauge field. The results will be used later on.

[3]Expanding B in a basis of the Clifford algebra the general solution in even $d = 2n$ is found to be

$$B_{\theta,\xi} = i\bar{\gamma}\gamma_n \exp\Big(-\theta\bar{\gamma}e^{i\xi\gamma_n}\Big)\exp\big(-i\xi\gamma_n\big) \times \mathcal{C}_f \times \mathcal{C}_c,$$

with center elements $\mathcal{C}$, and depends on two real parameters θ and ξ.

3 On the Spectrum of the Dirac Operator in a Bag

The Dirac equation for fermions confined to a bag and subject to the bag boundary conditions,

$$i\not{D}\psi_m(\theta) = \lambda_m(\theta)\psi_m(\theta), \qquad B_\theta\psi_m(\theta) = \psi_m(\theta)|_{\partial\mathcal{M}}, \tag{14}$$

possesses a discrete spectrum $\{\lambda_n\}$. Unlike the non-zero eigenvalues on a sphere or torus the eigenvalues do not come in pairs $\lambda_n, -\lambda_n$. The reason is that ψ_n and $\gamma_5\psi_n$ can not both obey the bag boundary conditions. Below we prove that $i\not{D}$ possesses no zero modes, display how the eigenvalues and -modes transform under the parity operation and derive a boundary Hellmann-Feynman formula for the θ-variation of the eigenvalues.

3.1 Absence of fermionic zero modes.

By explicit mode-analysis Balog and Hrasko have shown [19] that in a 2-dimensional spherical bag $i\not{D}$ possesses no zero modes which obey the bag boundary conditions (13). Here we shall extend their result to arbitrarily shaped even-dimensional bags. Indeed, if there would be a zero mode ψ then we would arrive at the contradiction

$$0 = (\bar{\gamma}\psi, i\not{D}\psi) - (i\not{D}\bar{\gamma}\psi, \psi) = i\oint \psi^\dagger\bar{\gamma}\gamma_n\psi = \oint \psi^\dagger e^{-\theta\bar{\gamma}}\psi > 0.$$

Here we used that as elements of the Clifford algebra $\bar{\gamma}$ and $\not{D}$ anticommute[4], the identity (10) and the boundary conditions (13) which a possible zero mode would have to obey.

3.2 Parity transformations.

Here we study how the eigenvalues $\lambda_m(A,\theta)$ in (14) change under parity transformations of the gauge field

$$\begin{array}{lcll} A_0(x) & \longrightarrow & \tilde{A}_0(x) = A_0(\tilde{x}) & , \quad \tilde{x} = (x^0, -x^i) \\ A_i(x) & \longrightarrow & \tilde{A}_i(x) = -A_i(\tilde{x}). & \end{array} \tag{15}$$

First we notice that the transformed modes

[4] This is not true on the Hilbertspace defined by (13) since $\bar{\gamma}$ does not commute with the boundary conditions. But this is not needed to arrive at the contradiction.

$$\tilde{\psi}_m(x) = \bar{\gamma}\gamma_0\psi_m(\tilde{x}) \tag{16}$$

solve the Dirac equation with potential $\tilde{A}$ and eigenvalues $-\lambda_m$. Second, if ψ_m obeys the boundary condition (13) then $\tilde{\psi}_m$ does, but with θ replaced by $-\theta$. In other words

$$\lambda_m(\tilde{A}, \theta) = -\lambda_m(A, -\theta) \tag{17}$$

and this property will constrain the fermionic determinants and Green's functions.

3.3 A boundary Hellmann-Feynman formula.

The Hellmann-Feynman theorem [20] relates the infinitesimal variation of an eigenvalue with the expectation value of the infinitesimal variation of the operator in the corresponding normalized eigenstate. Here we derive a similar formula for the variation of the eigenvalues λ_m when the parameter θ entering the boundary conditions is varied.

To continue we choose the eigenfunctions $\psi_m(\theta)$ in (14) to be orthornormal for all values of θ. The θ-variation of the eigenvalues is then simply

$$\frac{d}{d\theta}\lambda_m \equiv \lambda'_m = (\psi'_m, i\not{D}\psi_m) + (\psi_m, i\not{D}\psi'_m) = i \oint_{\partial\mathcal{M}} \psi^\dagger_m \gamma_n \psi'_m, \tag{18}$$

where we made use of (10). The last expression depends only on the eigenmodes restricted to the bag boundary and there the boundary conditions (13) imply $\psi'_m = \bar{\gamma}\psi_m + B\psi'_m$. Using the boundary conditions once more, together with the first formula in (12), we arrive at

$$i\psi^\dagger_m\gamma_n\psi'_m = i\psi^\dagger_m\gamma_n\bar{\gamma}\psi_m + i\psi^\dagger_m\gamma_n B_\theta\psi'_m = i\psi^\dagger_m\gamma_n\bar{\gamma}\psi_m - i\psi^\dagger_m\gamma_n\psi'_m$$

and this can be solved for $i\psi^\dagger_m\gamma_n\psi'_m$. Inserting the resulting expression into (18) finally yields

$$\boxed{\frac{d}{d\theta}\lambda_m = \frac{i}{2}\oint \psi^\dagger_m\gamma_n\bar{\gamma}\psi_m = -\lambda_m(\psi_m, \bar{\gamma}\psi_m),} \tag{19}$$

where once again we made use of (10). Eq. (19) is the analog of the Hellmann-Feynman formula and exhibits how λ_m changes if the boundary conditions are varied.

4 The Fermionic Green's Functions

When calculating correlators of 'quark' fields in a bag one needs in an intermediate step the Green's function S^θ of the Dirac operator in an arbitrary background field. This Green's function must obey

$$i\not{D}S^\theta(x,y;A)=\delta(x,y) \quad , \quad B_\theta(x)S^\theta(x,y;A)=S^\theta(x,y;A)|_{x\in\partial\mathcal{M}}, \qquad (20)$$

and the adjoint relations with respect to y. Since

$$\not{D}e^{\frac{1}{2}\theta\bar{\gamma}}=e^{-\frac{1}{2}\theta\bar{\gamma}}\not{D} \quad \text{and} \quad B_\theta e^{\frac{1}{2}\theta\bar{\gamma}}=e^{\frac{1}{2}\theta\bar{\gamma}}B_0$$

its θ-dependence is easily found to be

$$S^\theta=e^{\frac{1}{2}\theta\bar{\gamma}}\,S^0\,e^{\frac{1}{2}\theta\bar{\gamma}}=\begin{pmatrix} e^\theta S^0_{++} & S^0_{+-} \\ S^0_{-+} & e^{-\theta}S^0_{--}\end{pmatrix}, \qquad (21)$$

where the subscripts indicate the chiral projections, for example $S_{++}=P_+SP_+, P_\pm=\frac{1}{2}(1\pm\bar{\gamma})$. Note that the θ-dependent diagonal entries $S_{\pm\pm}$ lead to chirality violating amplitudes and may therefore trigger a chiral symmetry breaking. Also note that when we parity-transform the eigenvalues and eigenmodes of the Dirac operator in the spectral resolution of the Green's function according to (15-17) we conclude that

$$S^\theta(x,y;A)=-\gamma_0\bar{\gamma}S^{-\theta}(\tilde{x},\tilde{y};\tilde{A})\bar{\gamma}\gamma_0. \qquad (22)$$

This property will relate different correlators in the fully quantized theories.

Next we derive some explicit expressions for S^θ in spherical bags when the gauge field is switched off. These free Green's functions are needed in perturbative expansions for small couplings and/or small bags. For the explicit calculation it is useful to observe that in a spherical bag B_θ commutes with the total angular momentum,

$$J_{\mu\nu}=\frac{1}{i}(x_\mu\partial_\nu-x_\nu\partial_\mu)+\frac{1}{4i}[\gamma_\mu,\gamma_\nu]\equiv M_{\mu\nu}+\Sigma_{\mu\nu},$$

so that the free Green's functions are rotationally invariant,

$$US^\theta(Rx,Ry;0)U^{-1}=S^\theta(x,y;0), \quad \text{where} \quad U\gamma^\mu U^{-1}=R(U)^\mu_{\ \nu}\gamma^\nu,$$

and only depend on the rotationally invariant quantities $(\gamma,x),(\gamma,y),x^2,y^2$, (x,y) and the bag-radius R. If we continue to Minkowski spacetime then

$\mathcal{M}$ becomes the interior of a hyperboloid and all non-vanishing correlators would be Lorentz invariant.

We may compute the free Green's functions either by angular-momentum decomposition or by applying the mirror charge method. We found that their chirality conserving off-diagonal terms are just those on the infinite spacetime[5] but also that they contain chirality violating diagonal terms. The final result in $d=2n$ dimensions reads

$$\boxed{S^\theta(x,y;0)=S_0(x,y)+\frac{\Gamma(n)}{2R\pi^n}\bar\gamma\, e^{\theta\bar\gamma}\frac{R^2-(x,\gamma)(y,\gamma)}{(R^2-2xy+\frac{x^2y^2}{R^2})^n},} \tag{23}$$

where

$$S_0(x,y)=\frac{\Gamma(n)}{2i\pi^n}\frac{(x-y,\gamma)}{|x-y|^d} \tag{24}$$

is the free Green's function in d-dimendional Euclidean spacetime. Beeing the Green's functions of a selfadjoint operator they fulfill the reality condition $S^{\theta\dagger}(x,y)=S^\theta(y,x)$. In 2 dimensions (23) has been derived earlier in [19].

Note that the θ-dependent chirality violating entries $S^\theta_{\pm\pm}$ are regular at all interior points and vanish if the bag size tends to infinity. For example, at the center of the bag

$$S^\theta_{\pm\pm}(0,0;0)=\pm\frac{e^{\pm\theta}}{2\pi^n}\Gamma(n)R^{1-d}\longrightarrow 0\quad\text{for}\quad R\to\infty. \tag{25}$$

They become singular only if x and y both approach the boundary and each other since then the mirror charge comes close to $\partial\mathcal{M}$,

$$S^\theta_{\pm\pm}(|x|=R,y=(1-\epsilon)x;0)\sim\epsilon^{1-d}. \tag{26}$$

The Green's functions of the squared Dirac operator,

$$G^\theta(x,y;A)=\langle x|\frac{1}{-\not{D}^2}|y\rangle \tag{27}$$

obey the same boundary conditions as S^θ and in addition

[5] This is a particular property of the spherical geometry. For instance, on the torus the off-diagonal terms are modified.

$$i\not{D}G^\theta(x,y;A)=S^\theta(x,y;A). \tag{28}$$

They transform under the parity operation as

$$G^\theta(x,y;A)=\gamma_0\bar{\gamma}G^{-\theta}(\tilde{x},\tilde{y};\tilde{A})\bar{\gamma}\gamma_0. \tag{29}$$

After some manipulations we arrived at the following explicit formulae:

$$G^\theta(x,y;0)=G^D(x,y)-C_\theta(x)F(x,y)C^\dagger_\theta(y), \tag{30}$$

where the Dirichlet Green's functions G^D are constructed from the infinite spacetime Green's functions

$$G_0(x,y)=-\frac{1}{2\pi}\log\mu|x-y| \quad \text{resp.} \quad G_0(x,y)=\frac{\Gamma(n-1)}{4\pi^n}|x-y|^{2-d} \tag{31}$$

in 2 and more than 2 dimensions, respectively, by the mirror charge method and are found to be

$$G^D(x,y) = G_0(x,y)-(\frac{R^2}{x^2})^{n-1}G_0(x',y) \qquad \text{for } d>2 \tag{32}$$

$$G^D(x,y) = -\frac{1}{2\pi}\log\Big(\frac{R}{|x|}\frac{|x-y|}{|x'-y|}\Big) \qquad \text{for } d=2. \tag{33}$$

Here $x'=R^2x/x^2$ denotes the mirror point of x. We have introduced the functions

$$C_\theta(x)=1+iR\bar{\gamma}e^{\theta\bar{\gamma}}\frac{(\gamma,x)}{x^2}.$$

and

$$F(x,y)=\frac{i(\gamma,x)}{R}\oint\limits_{\partial\mathcal{M}}S_0(x,z)G_0(z,y)d\Omega(z)=\oint\limits_{\partial\mathcal{M}}G_0(x,z)S_0(z,y)d\Omega(z)\frac{(\gamma,y)}{iR},$$

where the z-integration extends over the bag-boundary. That G^θ in (30) obeys the boundary conditions is easily verified. To check (28) one needs to use the identity

$$\oint S_0(x,z)S_0(z,y)d\omega(z)=\frac{\Gamma(n)}{2R\pi^n}\frac{R^2-(\gamma,x)(\gamma,y)}{\left(R^2-2(x,y)+\frac{x^2y^2}{R^2}\right)^n}.$$

We have calculated $F(x,y)$ in 2 and 4 dimensions explicitly. In 2 dimensions it reads

$$F(x,y) = \frac{1}{4\pi} \log \Big(1 - \frac{\gamma x \gamma y}{R^2}\Big), \tag{34}$$

and in 4 dimensions

$$\begin{aligned} F(x,y) &= \frac{1}{8\pi^2}\Bigg\{ \frac{x^2y^2 - (x,y)(\gamma,x)(\gamma,y)}{\Delta^{3/2}} \arctan \frac{\sqrt{\Delta}}{R^2-(x,y)} \\ &- \frac{1}{\Delta}\frac{[R^2-(x,y)]x^2y^2 - [R^2(x,y)-x^2y^2](\gamma,x)(\gamma,y)}{R^4 - 2R^2(x,y) + x^2y^2}\Bigg\}, \end{aligned} \tag{35}$$

where $\Delta = x^2y^2 - (x,y)^2$.

5 The Fermionic Determinant in a Bag

In this section we shall compute the θ-dependence of the fermionic determinants. We shall see that the interaction of the fermions with the bag-boundary induces a CP-violating θ term in the effective action for the gauge bosons.

The Dirac operator and boundary conditions are both flavour neutral and hence the determinants are the same for all flavours and is suffices to study the 1-flavour models. For the explicit calculations we employ the gauge invariant ζ-function definition of the determinants [21]

$$\log \det_\theta (i \not{D}) \equiv \frac{1}{2} \log \det_\theta (-\not{D}^2) = -\frac{1}{2}\frac{d}{ds}\zeta_\theta(s)|_{s=0} \tag{36}$$

and calculate their θ-dependence with the help of the boundary Hellmann-Feynman formula (19). Denoting the eigenvalues of $-\not{D}^2$ by μ_m, the ζ-function is defined by

$$\zeta_\theta(s) = \sum_m \mu_m^{-s}(\theta) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{tr}_{\,\theta}\, e^{t\not{D}^2} \quad , \quad \Re(s) > \frac{d}{2} \tag{37}$$

and its analytic continuation to $\Re(s) \le d/2$. Using (19) and the fact that $i\not{D}$ possesses no zero modes, so that a partial integration with respect to t is justified, the θ-variation of ζ_θ is found to be

$$\frac{d}{d\theta}\zeta_\theta(s) = \frac{2s}{\Gamma(s)}\int t^{s-1}\mathrm{tr}_{\,\theta}\, e^{t\not{D}^2}\bar{\gamma}. \tag{38}$$

Now we can insert the asymptotic small-t expansion of the heat kernel of $-\not{D}^2$ [25] to arrive at the general result [22, 23, 24]

$$\frac{d}{d\theta}\log\det{}_\theta(i\not{D}) = -\frac{1}{(4\pi)^n}\int\limits_{\mathcal{M}} \mathrm{tr}\, a_n(\bar{\gamma}) - \frac{1}{(4\pi)^n}\oint\limits_{\partial\mathcal{M}} \mathrm{tr}\, b_n(\bar{\gamma}) \tag{39}$$

which holds in an arbitrary $2n$-dimensional bag. Here the n'th $(n = d/2)$ Seeley-deWitt coefficients in the small t-expansion of the heat kernel,

$$\mathrm{tr}_{\,\theta} e^{t\not{D}^2}\phi \sim \frac{1}{(4\pi t)^n}\sum_m t^{m/2}\ \mathrm{tr}\Big\{\int a_{m/2}(\phi) + \oint b_{m/2}(\phi)\Big\} \tag{40}$$

showed up. Unlike the a_n the coefficients b_n depend on the boundary conditions and thus on θ.

For the squared Dirac operator, $\not{D}^2 = D^2 + \Sigma^{\mu\nu}F_{\mu\nu}$, that part of the ($\theta$-independent) a_n which leads to a non-vanishing $\bar{\gamma}$-trace is known in any dimension [25] and inserting it we obtain

$$\boxed{\log\frac{\det_\theta i\not{D}}{\det_0 i\not{D}} = \frac{-\theta}{n!(4\pi)^n}\int\limits_{\mathcal{M}}\epsilon_{\mu_1\ldots\mu_d}F_{\mu_1\mu_2}\ldots F_{\mu_{d-1}\mu_d} - \int\limits_0^\theta d\theta'\oint\limits_{\partial\mathcal{M}}\mathrm{tr}\, b_n(\bar{\gamma}),} \tag{41}$$

and this formulae are the main results of this section. We see that the θ variation is proportional to the parity-odd instanton number q which is not quantized in a bag. Our result is in agreement with

$$\det i\not{D}(A,\theta) = \det i\not{D}(\tilde{A},-\theta) \tag{42}$$

which immediately follows from (17) and the fact that the determinant of $i\not{D}$ is defined via the spectrum of $-\not{D}^2$. This relation means that parity odd (even) factors in the determinant are multiplied by functions that are odd (even) in θ so that the last surface integral in (41) must be parity odd. Since the Yang-Mills action is parity even we immediately see that the measure of functional integration (8) satisfies

$$\boxed{d\mu_\theta(A) = d\mu_{-\theta}(\tilde{A})} \tag{43}$$

which implies that expectation values of parity even (odd) operators are even (odd) functions of θ.

In particular in 2-dimensions we have

$$\log \frac{\det_\theta i\not{D}}{\det_0 i\not{D}} = -\frac{\theta}{2\pi}\int \operatorname{tr} F_{01}, \tag{44}$$

where we have already anticipated that $\oint \operatorname{tr} b_1(\bar{\gamma}) = 0$, a fact that is proven by the heat kernel method in the appendix. In 4 dimensions we find

$$\log \frac{\det_\theta i\not{D}}{\det_0 i\not{D}} = -\frac{\theta}{2(4\pi)^2}\int \epsilon_{\mu\nu\alpha\beta} \operatorname{tr} F_{\mu\nu}F_{\alpha\beta} + \oint f_4(\theta, A). \tag{45}$$

An explicit calculation of surface coefficients like b_2 (which leads to the last surface integral) is not an easy task [22]. Contrary to b_1 we did not compute it explicitly. However, there seems to exist no local polynomial which is parity odd, gauge invariant and has dimension -3 and thus may contribute to b_2. Thus we believe that this surface term is absent as it is in 2 dimensions.

Since the Dirac operator in a bag is hermitean its determinant is real and positive. Thus, to make contact with the θ-worlds in QCD [26] we would have to continue θ in (45) to $i\theta$. However, when doing this replacement naively in (44,45) then then one runs into the following apparant paradox: the boundary conditions and thus the eigenvalues and Green's function of $i\not{D}$ are unchanged if we replace θ by $\theta + 2\pi i n$, $n \in Z$. On the other hand, the determinant seems not to be periodic since the instanton number is not quantized. The solution of this apparant paradox is simply that θ in (44,45) should read $\log(e^\theta)$ as is shown in the appendix.

6 Effective Action in 2-dimensional Bags

It has been realized by Polyakov and Wiegmann [28] and Alvarez [29] that the fermionic determinant on the 2-dimensional plane may be computed exactly using the chiral anomaly. Here we shall extend their result to fermions confined in a 2-dimensional bag.

We shall employ the deformation technique developped in [13, 22, 24] to find the various contributions to the fermionic determinant. For that we recall that an arbitrary gauge potential in a two-dimensional bag (without holes) can always be written as [24, 27]

$$A_z \equiv A_0 - iA_1 = ig^{-1}(\partial_0 - i\partial_1)g \equiv ig^{-1}\partial_z g \tag{46}$$

with g from the complexified gauge group G^c, e.g. $g \in GL(n,C)$ for $U(n)$-gauge theories[6]. Now it is easy to see that

$$\not{D} = G^\dagger \not{\partial} G, \quad \text{where} \quad G = \begin{pmatrix} g^{-1\dagger} & 0 \\ 0 & g \end{pmatrix} \quad , \quad \not{\partial} = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \tag{47}$$

and we made the matrix-forms in spinor space explicit. Note that if we replace g by gU, where U lies in the gauge group G, then

$$G \longrightarrow GU \quad \text{and} \quad \not{D} \longrightarrow U^{-1}\not{D}U \tag{48}$$

and hence the corresponding gauge potential is just the gauge-transformed one. The field strength is

$$F_{01} = -\frac{1}{2}g^\dagger \bar{\partial}(J^{-1}\partial J)g^{-1\dagger} = -\frac{1}{2}g^{-1}\partial(\bar{\partial}JJ^{-1})g, \tag{49}$$

where the gauge invariant field

$$J = gg^\dagger \tag{50}$$

with values in the coset space G^c/G appeared. J will play an important role since all gauge invariant Green's functions depend on the gauge field only via this gauge invariant field. The Yang-Mills action reads

$$S_{YM} = \frac{1}{8g^2}\int \operatorname{tr} \bar{\partial}(J^{-1}\partial J)\bar{\partial}(J^{-1}\partial J). \tag{51}$$

Let us now introduce a τ-dependent family $g(x,\tau)$ which interpolates between the identity and the field $g(x)$ as

$$g(x,0) = I \quad , \quad g(x,1) = g \quad \text{and} \quad \frac{d}{d\tau}g(\tau) \equiv \dot{g}(\tau) = -g(\tau)a(\tau). \tag{52}$$

With (47) it follows at once that

[6] On compact spacetimes without boundaries (46) needs some modifications, see [24].

$$\dot{\lambda}_m = \lambda_m(\psi_m, (A + A^\dagger)\psi_m) + i\oint \psi_m^\dagger \gamma_n A \psi_m, \quad A = \begin{pmatrix} a^\dagger & 0 \\ 0 & -a \end{pmatrix}. \tag{53}$$

To get rid of the annoying surface term we observe that the gauge potential in (46) is unaffected by the replacement

$$g \longrightarrow \alpha^{-1}(\bar{z})g \tag{54}$$

and we can use this freedom to get rid of this term. Indeed, we can always find a unique α such that $\alpha(\bar{z})\alpha(z)^\dagger = gg^\dagger$ on the bag-boundary. The equivalent g obeys then

$$gg^\dagger|_{\partial\mathcal{M}} \equiv J|_{\partial\mathcal{M}} = I \Longleftrightarrow G^{-1}B_\theta G = B_\theta \quad \text{on} \quad \partial\mathcal{M}. \tag{55}$$

Imposing the first condition for all τ implies that on the bag boundary $a + a^\dagger = 0$ or that A is the identity in spinor space. Then the surface term in (53) vanishes on account of the bag boundary conditions. The second condition is just the statement that the G-transformation (47) is compatible with the bag boundary conditions so that the Green's function is related to the free one[7], (23), as

$$S^\theta(x, y; A) = G^{-1}(x)S^\theta(x, y; 0)G^{-1\dagger}(y). \tag{56}$$

In the following we assume (55) to hold for all τ so that the whole deformation (52) is compatible with the boundary conditions.

Now we can apply the wellknown deformation techniques for the ζ-function defined determinant [22, 24] and obtain

$$\frac{d}{d\tau}\log\det i\not{D} = \frac{1}{4\pi}\int\limits_{\mathcal{M}} \mathrm{tr}\, a_1(A + A^\dagger) + \frac{1}{4\pi}\oint\limits_{\partial\mathcal{M}} \mathrm{tr}\, b_1(A + A^\dagger). \tag{57}$$

Here A and the Seeley-deWitt coefficients a_1, b_1 of the τ-deformed Dirac operator are to be inserted. The volume coefficient a_1 is wellknown [25],

$$\int a_1(\phi) = \int F_{01}\bar{\gamma}\phi \tag{58}$$

[7] we use the same symbol $S^\theta(x, y; 0)$ independently on whether the free Green's function (23) is tensored with the identities in flavour- and/or colour space or not. The local meanings should be clear from the context.

contrary to the surface coefficient b_1. We have calculated b_1 via the heat-kernel in the appendix and up to purely geometric terms, which cancel in expectation values, the result is

$$\oint b_1(\phi) = \frac{1}{2}\oint \Big\{1 - \frac{\log e^\theta}{\sinh(\theta)}\begin{pmatrix} e^\theta & -1 \\ -1 & e^{-\theta}\end{pmatrix}\Big\}\partial_n\phi. \tag{59}$$

Note that for a constant function ϕ the surface Seeley-deWitt coefficient $b_1(\phi)$ vanishes, and we have used this fact earlier in deriving (44). Note, however, that although $A + A^\dagger = 0$ on $\partial\mathcal{M}$ the last surface integral in (57) does not vanish, since $\mathrm{tr}\, b_1(\phi)$ contains the normal derivatives of ϕ at the boundary.

Inserting (59) into (57) we end up with the exact formula

$$\log\frac{\det_\theta i\not{D}}{\det_\theta i\not{\partial}} = \frac{1}{2\pi}\int\limits_0^1 d\tau\Big\{\int\limits_{\mathcal{M}} \mathrm{tr}\, F_{01}(a+a^\dagger) - \frac{\theta}{2}\oint\limits_{\partial\mathcal{M}} \mathrm{tr}\,\partial_n(a+a^\dagger)\Big\}. \tag{60}$$

To continue we express a and F_{01} in terms of g and its derivatives and find

$$\log\frac{\det_\theta i\not{D}}{\det_\theta i\not{\partial}} = -\frac{1}{4\pi}\int\limits_0^1 d\tau\Big\{\int\limits_{\mathcal{M}} \mathrm{tr}\,\Big(J^{-1}\partial J\bar\partial(J^{-1}\dot J)\Big) - \theta\oint\limits_{\partial\mathcal{M}} \mathrm{tr}\,\partial_n(J^{-1}\dot J)\Big\}.$$

The τ-integral of the volume term can be calculated in the same way as on the infinite plane[8] and leads to the Wess-Zumino action [24, 30]. That of the surface term is easily found since $\partial_\tau \mathrm{tr}\,\bar\partial(J^{-1}\partial J) = \triangle(J^{-1}\dot J)$. Hence we arrive at the following explicit answer for the fermionic determinant in a bag:

$$\begin{aligned}\log\frac{\det_\theta i\not{D}}{\det_\theta i\not{\partial}} \;=\; & -\frac{1}{8\pi}\int\limits_{\mathcal{M}} \mathrm{tr}\,\Big(J^{-1}\partial J J^{-1}\bar\partial J\Big) + \frac{i}{12\pi}\int\limits_{\mathcal{Z}} \mathrm{tr}\,(J^{-1}d_3 J)^3 \\ & + \frac{\theta}{4\pi}\int\limits_{\mathcal{M}} \mathrm{tr}\,\bar\partial(J^{-1}\partial J).\end{aligned} \tag{61}$$

In the Wess-Zumino term in the middle on the right hand side $J = J(x,\tau)$ and thus $\mathcal{Z} = \mathcal{M}\times[0,1]$ is the finite cylinder over the bag. We recall that the deformation is subject to the boundary-, initial- and final conditions

[8] the various partial integrations needed to arrive at the result are allowed if one takes into account that J is the identity on the bag-boundary

$$J(x\in\partial\mathcal{M},\tau)=I \quad , \quad J(x,0)=I \quad \text{and} \quad J(x,1)=J(x). \tag{62}$$

As for the last surface term in (61) we see immediately that for semisimple gauge groups it vanishes, since $J^{-1}\partial J$ lies in the complexified gauge algebra. Also note that this term is equal to $-\theta/2\pi\int\mathrm{tr}\,F_{01}$ so that our result is indeed compatible with (44). Also, for $J=J_1J_2$ it becomes the sum of such terms for the individual fields J_i. This means that the wellknown Polyakov-Wiegman identity [28], which relates the determinant belonging to $J=J_1J_2$ with those of J_1 and J_2,

$$\begin{aligned}\log\frac{\det_\theta i\not{D}(J_1J_2)}{\det_\theta i\not{\partial}} &= \log\frac{\det_\theta i\not{D}(J_1)}{\det_\theta i\not{\partial}}+\log\frac{\det_\theta i\not{D}(J_2)}{\det_\theta i\not{\partial}}\\ &- \frac{1}{4\pi}\int\limits_{\mathcal{M}}\mathrm{tr}\left(J_1^{-1}\partial J_1\bar{\partial}J_2J_2^{-1}\right).\end{aligned} \tag{63}$$

still holds in a bag.

Let us now suppose that $G=U(1)\times SU(N_c)$. The results for this particular case will be important when we calculate mesonic current correlators and chiral condensates. We represent the gauge potential $A=\tilde{A}+\hat{A}$ as in (46) and factorize the $U(1)$ field, that is we set $g=\tilde{g}\hat{g}$. We parametrize the $U(1)$-part as $\tilde{g}=e^{-e\varphi-ie\lambda}$, where e is the electric charge, and then

$$A_\mu=\tilde{A}_\mu+\hat{A}_\mu=-e\epsilon_{\mu\nu}\partial_\nu\varphi+e\partial_\mu\lambda+\hat{A}_\mu \quad \text{and} \quad F_{01}=e\triangle\varphi+\hat{F}_{01}. \tag{64}$$

Repeating the above analysis for the deformation

$$J(x,\tau)=e^{-2e\varphi(x)\tau}\hat{J}(x,\tau) \quad \text{with} \quad \varphi|_{\partial\mathcal{M}}=0 \quad \text{and} \quad \hat{J}(\tau)|_{\partial\mathcal{M}}=I,$$

or equivalently applying the Polyakov-Wiegman identity to $J=e^{-2e\varphi}\hat{J}$, shows that the determinant (61) factorizes,

$$\boxed{\det_\theta i\not{D}=e^{-\frac{N_c}{2\pi}[e^2\int\partial\varphi\bar{\partial}\varphi+\theta e\oint\partial_n\varphi]}\det i\hat{\not{D}},} \tag{65}$$

where the last determinant is θ-independent. The same happens then for the functional measure for the Euclidean gauge fields

$$d\mu_\theta(A)=d\mu_\theta(\tilde{A})\;d\mu(\hat{A})=\frac{e^{-\Gamma_\theta[\varphi]}}{\tilde{Z}_\theta}\mathcal{D}\tilde{A}\;\;\frac{e^{-\Gamma[\hat{A}]}}{\hat{Z}}\mathcal{D}\hat{A}. \tag{66}$$

Here we introduced the θ-dependent effective action for the $U(1)$-gauge potential $\tilde{A}$ and the θ-independent one for the $\hat{G}$-gauge potential $\hat{A}$. For the N_f-flavour model with flavour-independent $U(1)$-charge e and $\hat{G}$-coupling constant g they read

$$\begin{aligned}
\Gamma_\theta[\varphi] &= \frac{N_c}{2}\left\{\int(\triangle\varphi)^2 - m_\eta^2\int\varphi\triangle\varphi + \frac{e\theta N_f}{\pi}\oint\partial_n\varphi\right\} \\
\Gamma[\hat{A}] &= S_{YM}[\hat{A}] + \frac{N_f}{8\pi}\int\limits_{\mathcal{M}}\mathrm{tr}\,(\hat{J}^{-1}\partial\hat{J}\hat{J}^{-1}\bar{\partial}\hat{J}) - \frac{iN_f}{12\pi}\int\limits_{\mathcal{Z}}\mathrm{tr}\,(\hat{J}^{-1}d_3\hat{J})^3.
\end{aligned} \tag{67}$$

Note that due to the wellknown Schwinger mechanism the mass

$$m_\eta^2 = N_f\frac{e^2}{\pi}, \tag{68}$$

which is the analog of the η'-mass in QCD, has been induced in the abelian subsector of the theory.

7 Correlation Functions of Mesonic Currents

Fermionic correlation functions are gotten from the generating functional (1), which in a bag simplifies to

$$Z[\eta,\bar{\eta}] = \int d\mu_\theta(A)\; e^{\int\eta^\dagger(x)S^\theta(x,y;A)\eta(y)}, \tag{69}$$

by functional differentiation with respect to the grassmann valued sources. Here $d\mu_\theta$ is the measure of functional integration (8) and we recall that the fermionic Green's function S^θ is the identity in flavour space. Let $\mathcal{C} = \mathcal{S}\otimes\mathcal{F}\otimes I_c$ be a numerical matrix which acts trivial in colour space. Then we obtain for the gauge invariant connected two- and four-point functions in a fixed background field

$$\begin{aligned}
&\langle\psi^\dagger(x)\mathcal{C}\psi(x)\rangle_A = -\mathrm{tr}\,\mathcal{F}\;\mathrm{tr}\,\mathcal{S}\,S^\theta(x,x;A) \\
&\langle\psi^\dagger(x)\mathcal{C}_1\psi(x)\psi^\dagger(y)\mathcal{C}_2\psi(y)\rangle_{A,c} \\
&\qquad = -\mathrm{tr}\,\mathcal{F}_1\mathcal{F}_2\;\mathrm{tr}\,[\mathcal{S}_1S^\theta(x,y;A)\mathcal{S}_2S^\theta(y,x;A)],
\end{aligned} \tag{70}$$

where it is understood that the first traces are in flavour space and the second ones in spinor- and colour space.

Vector currents The 2-point functions of the mesonic vector- and pseudovector currents

$$j^\mu_{\mathcal{F}} = \psi^\dagger\gamma^\mu\mathcal{F}\,\psi \quad\text{and}\quad j^{5\mu}_{\mathcal{F}} = \psi^\dagger\,\bar{\gamma}\gamma^\mu\mathcal{F}\,\psi = i\epsilon_{\mu\nu}j^\nu_{\mathcal{F}} \tag{71}$$

will already shed some light on the particle spectrum of 2-dimensional gauge theories. We obtain the following formal expressions for the connected 1 and 2-point functions

$$\begin{aligned}\langle j^\mu_{\mathcal{F}}(x)\rangle_A &= -\mathrm{tr}\,\mathcal{F}\ \mathrm{tr}\,\gamma^\mu S^\theta(x,x;A)\\ \langle j^\mu_{\mathcal{F}_1}(x)j^\nu_{\mathcal{F}_2}(y)\rangle_{A,c} &= -\mathrm{tr}\,\mathcal{F}_1\mathcal{F}_2\ \mathrm{tr}\,\gamma^\mu S^\theta(x,y;A)\gamma^\nu S^\theta(y,x;A).\end{aligned} \tag{72}$$

In 2 spacetime dimensions the Green function S^θ is given by (56) and (23). When inserting the explicit form (56,23) of S^θ one notices that the gauge field and θ-parameter both drop in these expectation values. In principle one would have to regularize the currents, e.g. by a gauge invariant point splitting prescription and this may reintroduce a gauge field and θ-dependence. However, by noticing that the mesonic currents couple to the abelian gauge potential $\tilde{A}_\mu$ in (64) we can calculate the regularized connected correlators in a fixed background as

$$\begin{aligned}\langle j^\mu_{\mathcal{F}}(x)\rangle_A &= \frac{\mathrm{tr}\,\mathcal{F}}{e}\frac{\delta\log\det i\not{D}}{\delta\tilde{A}_\mu(x)} = \frac{N_c\mathrm{tr}\,\mathcal{F}}{\pi}\epsilon_{\mu\nu}\Big(e\partial^\nu\varphi + \frac{\theta}{2}\delta(r-R)n^\nu\Big),\\ \langle j^\mu_{\mathcal{F}}(x)j^\nu_{\mathcal{F}}(y)\rangle_{A,c} &= \frac{\mathrm{tr}\,\mathcal{F}^2}{e^2}\frac{\delta^2\log\det i\not{D}}{\delta\tilde{A}_\mu(x)\delta\tilde{A}_\nu(y)} = -\frac{N_c\mathrm{tr}\,\mathcal{F}^2}{\pi}\mathcal{P}^{\mu\nu}(x,y).\end{aligned} \tag{73}$$

All higher connected correlators vanish. In deriving (73) we have factorized the flavour dependence by diagonalizing $\mathcal{F}$ so that the determinants are those of the one-flavour model. The last equalities follow from the explicit dependence of $\det i\not{D}$ in (65) on the field φ and the decomposition of $\tilde{A}_\mu$ in (64). $\mathcal{P}^{\mu\nu}$ projects onto the transversal degrees of freedom and is consistent with the boundary conditions,

$$\mathcal{P}^{\mu\nu}(x,y) = \pi\mathrm{tr}\,\gamma^\mu S^\theta(x,y;0)\gamma^\nu S^\theta(y,x;0) = \epsilon^{\mu\alpha}\epsilon^{\nu\beta}\partial_{x^\alpha}\partial_{y^\beta}G^D(x,y). \tag{74}$$

Here $G^D(x,y)$ is the Dirichlet Green's function of $-\triangle$, see (33). Since $\varphi=0$ on $\partial\mathcal{M}$ the current normal to $\partial\mathcal{M}$ vanishes and no $U(1)$-charge is leaking through the boundary as required by the boundary conditions on the 'quark' fields. Furthermore, our result is compatible with vector flavour symmetry and the axial vector anomaly,

$$\partial_\mu \langle j^\mu_{\mathcal{F}} \rangle_A = 0 \quad \text{and} \quad \partial_\mu \langle j^{5\mu}_{\mathcal{F}} \rangle_A = \frac{\mathrm{tr}\,\mathcal{F}}{i\pi} \,\mathrm{tr}\,_c \Big\{ eF_{01} + \frac{\theta}{2}\delta'(r-R) \Big\}. \tag{75}$$

Note that the nonabelian part $\hat{A}$ of the gauge potential has completely disappeared in the above formulae. Since we know all correlators in an arbitrary gauge field and since those only depend on the abelian part of the gauge potential the averaging over the gauge fields reduces to that in the multi-flavour Schwinger model. Here we may use the results in [31], up to some modification due to the presence of the bag boundary. Let us choose a trace-orthonormal basis T_a, $a = 2, 3, \ldots, N_f^2$ of $SU(N_f)$, together with the identity in flavour space which we denote by T_1. The correlators of the associated currents $j^\mu_a = \bar{\psi}\gamma^\mu T_a \psi$ are reproduced by the generating functional

$$\begin{aligned} \langle \exp \Big(\int j^\mu_a b^a_\mu \Big) \rangle = \exp \Bigg\{ &- \frac{N_c}{2} \Big[m^2_\eta \int b^1_\mu(x) \mathcal{P}^{\mu\nu}_{m_\eta}(x,y) b^1_\nu(y) \\ &+ \frac{m^2_\eta}{N_f} \sum_2^{N_f^2} b^a_\mu(x) \mathcal{P}^{\mu\nu}(x,y) b^a_\nu(y) + \frac{e\theta N_f}{\pi} \int I_e(r,R) \epsilon^{\mu\nu} \partial_\mu b^1_\nu \Big] \Bigg\}, \end{aligned} \tag{76}$$

where we introduced the function

$$I_e(r,R) = \frac{I_0(m_\eta r)}{I_0(m_\eta R)}. \tag{77}$$

The projector $\mathcal{P}^{\mu\nu}_m$ onto the transverse massive vector-particles is derived from the massive Green function G^D_m in (83) in the same way as $\mathcal{P}^{\mu\nu}$ was derived from G^D in (74). Actually, the generating functional for the currents in the Cartan subalgebra can be calculated directly since the associated fermionic determinant is calculable. The identities needed to prove that the generating functional (76) yields the correct current correlators are derived in the next section, see for example (84).

Now it is easy to bosonize the mesonic currents, since the bosonization is identical to that of the multi-flavour Schwinger model [31], up to boundary terms. One finds that the generating functional for all currents can be rewritten as

$$\langle \exp \Big(\int j^\mu_a b^a_\mu \Big) \rangle = \langle \exp \Big(i \int \epsilon^{\mu\nu} \partial_\nu \, \varphi_a b^a_\mu \Big) \rangle_B, \tag{78}$$

where the Gaussian measure for the N_f^2-bonsonic fields φ_a has the action

$$B[\varphi]=\frac{1}{2N_c m_\eta^2}\Big[\int\varphi_1(-\triangle+m_\eta^2)\varphi_1-N_f\sum_2^{N_f^2}\int\varphi_a\triangle\varphi_a\Big]+\frac{i\theta}{e}\oint\partial_n\varphi_1.$$

We recovered the wellknown bosonization rule $j_a^\mu\to i\epsilon^{\mu\nu}\partial_\nu\varphi_a$, where the field φ_1 belonging to the $U(1)$-current $\bar\psi\gamma^\mu\psi$ has mass m_η and the remaining N_f^2-1 pseudo-scalar fields are massless. What we have shown is that 2-dimensional multi-flavour $U(N_c)$ gauge theories contain one massive and N_f^2-1 massless pseudoscalar 'mesons'. For $G=SU(N_c)$ the massive 'meson' is absent.

8 Chiral Symmetry Breaking in $2d$-Gauge Theories

We begin with calculating the chiral condensate of the N_f-flavour Schwinger model [32, 31] enclosed in a spherical bag. As an application we derive an upper bound for the condensate in $SU(N_c)$ gauge theories and prove that for $N_c<\infty$ it vanishes in the thermodynamic limit. On the other hand, for the abelian projected non-abelian theories we calculate the R-dependence of the condensate explicitly and show that in the limit $N_c\to\infty$ a 'quark' condensate is generated which remains when $R\to\infty$.

The $u=\psi_1$-'quark' condensate is the particular 2-point-function (70) with $\mathcal{S}=P_+$ and $\mathcal{F}_{ab}=\delta_{a1}\delta_{b1}$. Inserting the explicit form of the Green function S^θ we arrive at

$$\langle\bar uP_+u\rangle(x)=-\frac{e^\theta}{2\pi R}\frac{1}{1-r^2/R^2}\int d\mu_\theta(A)\ \mathrm{tr}\,J(x)\tag{79}$$

and it remains to calculate the average of the colour trace of the gauge invariant field J. For 2-dimensional $SU(N_c)$-gauge theories the measure $d\mu$ does not depend on θ and the condensate is proportional to e^θ. On the other hand, we shall see that for $U(N_c)$-theories the 'quark' condensates become θ-independent, up to exponentially small (in R) finite size corrections.

8.1 Multi-flavour QED_2

When one quantizes multi-flavour QED_2 with massless fermions on S^2 [15] or the torus [13, 33] or some other Riemann surface one finds $\langle\bar uP_+u\rangle=$

0. The same result is found in the geometric Schwinger model [14] which is equivalent to QED_2 with 2-flavours. The condensate vanishes for the same reason as it does in QCD if one only allows for gauge fields with integer instanton number. Only for nonzero 'quark'-masses or if one allows for flavour dependent twisted boundary conditions does one find a nonzero condensate in finite volumes. Here we shall show that the $U_A(N_f)$-breaking bag-boundary conditions also trigger a chiral condensate. No fermionic zero modes are needed to generate it and actually there are none of them. The condensate decreases with increasing bag-radius unless $N_f=1$ or the number of colours is infinite.

As earlier we choose the parametrization $g=e^{-e\varphi-ie\lambda}$ (we skip the tilde in this subsection) for the abelian field so that the functional integral representation for the u-'quark' condensate reads

$$\langle\bar{u}P_+u\rangle=-\frac{e^\theta}{2\pi R}\,\frac{1}{1-r^2/R^2}\,\frac{\int\mathcal{D}A_\mu\,e^{-2e\varphi(x)-\Gamma_\theta[\varphi]}}{\int\mathcal{D}A_\mu\,e^{-\Gamma_\theta[\varphi]}},\tag{80}$$

where $\Gamma_\theta[\varphi]$ is the effective action (67) for one colour. The Jacobian of the transformation (64) from the potential A_μ (there it was denoted by $\tilde{A}_\mu$) to the new fields λ,φ is field independent and we can replace $\mathcal{D}A_\mu$ by $\mathcal{D}\varphi$ in expectation values of gauge invariant operators. Also recall that we integrate over those fields φ which vanish on the bag-boundary.

The integral (80) is Gaussian with source

$$j(y)=-2e\delta(x-y)+\frac{e\theta N_f}{2\pi}\frac{1}{r_y}\partial_{r_y}\Big(r_y\delta(r_y-R)\Big)$$

and thus is found to be

$$\langle\bar{u}P_+u\rangle=\frac{-e^\theta}{2\pi R}\frac{1}{1-r^2/R^2}\exp\Big\{\frac{2\pi}{N_f}K(x,x)+\theta\int d^2y\triangle_yK(x,y)\Big\}.\tag{81}$$

Here we introduced

$$K(x,y)=\langle x|\frac{1}{-\triangle}|y\rangle-\langle x|\frac{1}{-\triangle+m_\eta^2}|y\rangle\equiv G^D(x,y)-G^D_{m_\eta}(x,y),\tag{82}$$

i.e. the difference between the massless and massive Green's functions with respect to Dirichlet boundary conditions. In a spherical bag with radius R G^D has been given in (33) and

$$
\begin{aligned}
G_m^D(x,y) &= \frac{1}{2\pi}\Big\{K_0(m|x-y|) \\
&- \sum_0^\infty \epsilon_n \frac{K_n(mR)}{I_n(mR)} I_n(mr_x) I_n(mr_y)\cos n(\varphi_x-\varphi_y)\Big\},
\end{aligned}
\tag{83}
$$

where $\epsilon_0 = 1$, $\epsilon_{n>0} = 2$ and I_n, K_n are the modified Bessel functions. Using the explicit form of the Green's functions one calculates

$$
\int_{\mathcal{M}} d^2y \, \triangle_y G^D(x,y) = -1 \quad \text{and} \quad \int_{\mathcal{M}} d^2y \, \triangle_y G_m^D(x,y) = -\frac{I_0(mr)}{I_0(mR)}, \tag{84}
$$

so that

$$
\langle \bar{u}P_+u \rangle = -\frac{1}{2\pi R}\,\frac{1}{1-r^2/R^2}\exp\Big\{\theta I_e(r,R) + \frac{2\pi}{N_f}K(x,x)\Big\}.
$$

The function I_e in the exponent has been defined in (77). Inserting the expansion of K_0 for small arguments we obtain

$$
2\pi K(x,x) = \gamma + \log\Big(\frac{m_\eta R}{2}[1-\frac{r^2}{R^2}]\Big) + F_e(r,R),
$$

where $\gamma = 0.577\ldots$ is Euler's constant and we have introduced the function

$$
F_e(r,R) = \sum \epsilon_n \frac{K_n(m_\eta R)}{I_n(m_\eta R)} I_n^2(m_\eta r), \tag{85}
$$

Inserting all that we get the following exact formula for the chiral condensate in multi-flavour QED_2 confined in a bag with radius R:

$$
\boxed{\langle \bar{u}P_+u \rangle(x) = -\frac{m_\eta e^\gamma}{4\pi}\Big(\frac{m_\eta R e^\gamma}{2}[1-\frac{r^2}{R^2}]\Big)^{-1+1/N_f} e^{\theta I_e + F_e/N_f}.} \tag{86}
$$

The function F_e has the asymptotic expansions

$$
F_e(r,R) \sim \begin{cases} e^{-m_\eta R} & \text{for } 1 \ll m_\eta R \gg m_\eta r \\ -\log \frac{1}{2} m_\eta R e^\gamma [1-\frac{r^2}{R^2}] & \text{for } m_\eta R \ll 1. \end{cases} \tag{87}
$$

Thus for large and small bags or equivalently for strong and weak coupling constant e the condensate simplifies to

$$\langle \bar{u}P_+u\rangle \sim \begin{cases} -\frac{m_\eta e^\gamma}{4\pi}\Big(\frac{1}{2}m_\eta R e^\gamma\Big)^{-1+1/N_f} & \text{for } 1 \ll m_\eta R \gg m_\eta r \\ -\frac{e^\theta}{2\pi R}(1-r^2/R^2)^{-1} & \text{for } m_\eta R \ll 1. \end{cases} \tag{88}$$

As expected, for weak couplings and/or small bags the condensate tends to the chirality violating entry $-S^\theta_{++}(x,x;0)$ of the free Green's function (23).

For *one flavour* and large bags we recover the wellknown value for the condensate in the Schwinger model [34]

$$\langle \bar{u}P_+u\rangle = -\frac{m_\eta}{4\pi}e^\gamma. \tag{89}$$

We stress that this result has been obtained without doing any instanton physics. The calculations in a bag are actually much simpler as compared with those on a torus [13, 14, 33] or sphere [15], where a careful treatment of the different instanton sectors is required to find the result (89).

For *several flavours* the condensate inside the bag, e.g. at the center of a large bag,

$$\langle \bar{u}P_+u\rangle(0) = -\frac{1}{2\pi R}\Big(\frac{m_\eta R e^\gamma}{2}\Big)^{1/N_f} \tag{90}$$

decreases with increasing bag radius and vanishes in the thermodynamic limit.

The cluster property holds since the 4-point function

$$\begin{aligned} &\langle \bar{u}(x)P_+u(x)\bar{u}(y)P_-u(y)\rangle \\ &= \langle S^\theta_{++}(x,x;A)S^\theta_{--}(y,y;A) - S^\theta_{-+}(x,y;A)S^\theta_{+-}(y,x;A)\rangle \\ &\longrightarrow -(\frac{m_\eta}{4\pi})^2\Big(\frac{m_\eta}{2}|x-y|\Big)^{-2+2/N_f} e^{2\gamma/N_f} \quad \text{for} \quad R\to\infty \end{aligned} \tag{91}$$

tends to the product of the left- and righthanded condensates for large separations $|x-y|$.

Let us finally prove that in the thermodynamic limit all fermionic correlators in multi-flavour QED_2 become θ-independent. This follows from the explicit form of the fermionic Green's function (for $\lambda=0$)

$$S^\theta(x,y;A) = e^{-\bar{\gamma}[e\varphi(x)-\frac{1}{2}\theta]}S^0(x,y;0)e^{-\bar{\gamma}[e\varphi(y)-\frac{1}{2}\theta]},$$

which implies that all correlators are proportional to

$$e^{-\theta\sum\alpha_i}\,\langle e^{2e\sum\alpha_i\varphi(x_i)}\rangle,$$

and from the formula

$$\langle e^{2e\sum\alpha_i\varphi(x_i)}\rangle = e^{\theta\sum\alpha_i[1-I_\epsilon(r_i,R)]}\, e^{2\pi/N_f\sum\alpha_i K(x_i,x_j)\alpha_j}.$$

Thus, up to exponentially small finite size corrections $\sim \exp(\theta I_\epsilon)$ the θ-dependence cancels in all fermionic correlators.

Let us compare our result with that of Smilga [16] who calculated the condensate in multiflavour QED_2 for small 'quark' masses. Using bosonization techniques he found that the mass μ of the lightest particle and the 'quark' condensate depend on the electric charge e and small current quark masses m as

$$\mu \sim (m_\eta m^{N_f})^{\frac{1}{N_f+1}} \quad \text{and} \quad \langle\bar\psi\psi\rangle \sim (m_\eta^2 \mu^{N_f-1})^{\frac{1}{N_f+1}}$$

so that

$$\langle\bar\psi\psi\rangle \sim \mu\Big(\frac{m_\eta}{\mu}\Big)^{1/N_f}. \tag{92}$$

Comparing with (90) we see that the bag- and small quark mass calculations yield the same result if we identify the mass of the lightest particle in the spectrum with the inverse radius of the bag. In other words, small quark masses and bag boundary conditions both trigger the same condensate if μ is identified with $1/R$.

In passing we note that the left- and right-handed condensates are related as

$$\langle\bar u P_- u\rangle_\theta = -\langle\bar u P_+ u\rangle_{-\theta}. \tag{93}$$

This follows from the transformations (43) and (22) under the parity operation. Since the function I_e in (86) vanishes exponentially with increasing bag radius R (assuming that $r \ll R$) we conclude that

$$\langle\bar u u\rangle = \langle\bar u P_+ u\rangle + \langle\bar u P_- u\rangle = O\Big(\sinh(\theta e^{-m_\eta R})\Big) \tag{94}$$

for large bags or in the strong coupling limit.

To summarize, up to a phase the thermodynamic limits of the left- and right-handed condensates in a bag are identical to the instanton induced condensates in the 1-flavour model on the torus or sphere and to the condensates in the multi-flavour models obtained via perturbative expansion in the small quark masses. The same is true for the condensate $\langle\bar u u\rangle$ only for particular values of the parameter θ in the θ-world.

8.2 Multi-flavour nonabelian gauge theories.

Due to the factorization of the measure for the gauge bosons, (66), the chiral condensate (79) in $U(N_c)$ gauge theories factorizes as

$$\begin{aligned}\langle\bar{u}P_+u\rangle_{U(N_c)} &= -\frac{e^\theta}{2\pi R}\frac{1}{1-r^2/R^2}\int d\mu_\theta(\tilde{A})e^{-2e\varphi}\int d\mu(\hat{A})\,\mathrm{tr}\,\hat{J} \\ &= -\frac{2\pi R}{e^\theta}(1-\frac{r^2}{R^2})\,\langle\bar{u}P_+u\rangle_{U(1)}\,\langle\bar{u}P_+u\rangle_{SU(N_c)},\end{aligned} \tag{95}$$

and thus is proportional to the Schwinger model result times the $SU(N_c)$ condensate. When calculating the $U(1)$-condensate one should remember that $\Gamma_\theta[\varphi]$ in (66,67) is N_c times that of the multi-flavour Schwinger model, so that (86) is modified to

$$\langle\bar{u}P_+u\rangle_{U(1)} = -\frac{e^{\theta I_e}}{2\pi R}\frac{1}{1-r^2/R^2}\Big(\frac{m_\eta R\,e^{\gamma+F_e}}{2}[1-\frac{r^2}{R^2}]\Big)^{1/N_cN_f}, \tag{96}$$

where the functions I_e and F_e have been defined in (77) and (85), respectively. Inserting all that into (95) we find the following exact relation between the $U(N_c)$ and $SU(N_c)$ condensates:

$$\langle\bar{u}P_+u\rangle_{U(N_c)} = e^{\theta(I_e-1)}\,\Big(\frac{m_\eta R\,e^{\gamma+F_e}}{2}[1-\frac{r^2}{R^2}]\Big)^{1/N_cN_f}\langle\bar{u}P_+u\rangle_{SU(N_c)}. \tag{97}$$

Using the asymptotic expansion of F_e for small arguments, (87), we see that for $e\to 0$ the $U(N_c)$ result reduces to the $SU(N_c)$ one, as expected.

For the condensates at the center of large bags (97) simplifies to

$$\langle\bar{u}P_+u\rangle_{U(N_c)} = e^{-\theta}[\frac{m_\eta Re^\gamma}{2}]^{1/N_cN_f}\langle\bar{u}P_+u\rangle_{SU(N_c)}. \tag{98}$$

Assuming that the $U(N_c)$ condensate has a smooth thermodynamic limit we conclude at once that for a finite number of colours the quark condensate in $SU(N_c)$ gauge theories tends to zero as the bag increases at least as

$$\boxed{\langle\bar{u}P_+u\rangle_{SU(N_c)} \le \mathrm{const}\cdot R^{-1/N_cN_f}.} \tag{99}$$

Only when we take the limit in which the number of colours tends to infinity *before* we perform the thermodynamic limit $R\to\infty$ can a quark condensate survive.

It would be interesting to see how (99) is modified for two-dimensional QCD with adjoint Majorana fermions. Arguments based on the bosonized representation of the theory imply that a nonvanishing condensate is generated, even for $N_c \geq 3$ in which case the instantons fail to generate a condensate [10, 35].

8.3 Baby-QCD_2

For doing explicit calculations it is useful to parametrize the g-field in (46). We take a parametrization for which the fermionic determinant becomes local and simple. The price we pay for the locality is that the Yang-Mills action is not quadratic as it would be in a gauge like $A_r = 0$. For simplicity we assume that $G = SU(2)$, that is we consider the baby-version of QCD_2 [36]. For baby-QCD the field g lies in $SL(2,C)$ and in a bag without holes any such g can globally be decomposed as [37]

$$g = hU, \quad \text{where} \quad h = \begin{pmatrix} e^{\frac{1}{2}\varphi} & ve^{\frac{1}{2}\varphi} \\ 0 & e^{-\frac{1}{2}\varphi} \end{pmatrix} \quad \text{and} \quad U \in SU(2). \tag{100}$$

Here U contains the pure gauge part of the potential and cancels in expectation values of gauge invariant operators[9]. The condition (55) means that φ and v both vanish on the bag boundary.

Now we can apply the Polyakov-Wiegman identity (63) with $J = hh^\dagger$ and this yields

$$\boxed{\log\det\frac{i\not{D}}{i\not{\partial}} = -\frac{1}{4\pi}\int\Big((\nabla\varphi,\nabla\varphi) + \alpha\bar{\alpha}\Big).} \tag{101}$$

The 3-dimensional integral in (61) converted into an ordinary 2-dimensional spacetime integral because we have chosen a triangular h in the decomposition (100). The property that the Wess-Zumino term becomes local for a triangular h has been exploited in a different context in [38].

At this point we wish to comment on the θ-independence of the fermionic determinant. For $v = 0$ this fact is easily understood as follows:
In this case $A_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu\varphi\sigma_3$ and $i\not{D}$ is just the tensor product of two $U(1)$ Dirac operators, one with $\varphi \to \frac{1}{2}\varphi$ and the other with $\varphi \to -\frac{1}{2}\varphi$. This means that the log det is just the sum of the two abelian results with the corresponding replacements and in this sum the θ-dependent terms cancel.

[9] The gauge field measure is discussed below

In section 6 we have shown that this cancellation between the various colour degrees of freedom takes actually place for arbitrary gauge potentials and semi-simple gauge groups.

With the parametrization (100) there is actually a much quicker way to arrive at (101). When we replace φ, v in (100) and in

$$F_{01} = -\frac{1}{2}\begin{pmatrix} \triangle\varphi + \alpha\bar{\alpha} & \bar{\partial}\alpha - \alpha\bar{\partial}\varphi \\ \partial\bar{\alpha} - \bar{\alpha}\partial\varphi & -\triangle\varphi - \alpha\bar{\alpha} \end{pmatrix}, \qquad \text{where} \quad \alpha = \partial v + v\partial\varphi$$

by the deformed fields $\tau\varphi, \tau v$, then F_{01} and

$$a + a^\dagger = -\begin{pmatrix} \varphi & v(1+\tau\varphi) \\ \bar{v}(1+\tau\varphi) & -\varphi \end{pmatrix}$$

in (60) both become polynomial in τ and the τ-integral can easily be performed.

Similar as the fermionic determinant the Yang-Mills action

$$S_{YM} = \frac{1}{2g^2}\int\limits_{\mathcal{M}} \mathrm{tr}\, F_{01}^2 = \frac{1}{4g^2}\int \Big\{ (\partial\bar{\partial}\varphi + \alpha\bar{\alpha})^2 + |\bar{\partial}\alpha - \alpha\bar{\partial}\varphi|^2 \Big\} \tag{102}$$

depends on v only via the α-field and this suggests that we should change variables $A_\mu^a \to (\varphi, \alpha, \bar{\alpha}, U)$. To find the Jacobian of this transformation we note that, up to a gauge transformation,

$$A_z = i\begin{pmatrix} \frac{1}{2}\partial\varphi & \alpha \\ 0 & -\frac{1}{2}\partial\varphi \end{pmatrix} + i\partial U U^{-1}$$

and parametrize the gauge transformations as

$$U = U(\xi) \Rightarrow \partial_\mu U U^{-1} = \mathcal{N}_{ab}\tau_a\partial_\mu\xi^b, \quad \text{where} \quad \mathcal{N}_{ab} = 2\mathrm{tr}\left(\frac{\partial U}{\partial\xi^b}U^{-1}\tau_a\right)$$

and the τ_a are half of the Pauli-Matrices. Then the transformation to the new variables is given by

$$\begin{pmatrix} A_0^a \\ A_1^a \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & \\ 0 & -1 & 0 & \mathcal{N}\partial_0 \\ \partial_1 & 0 & 0 & \\ 0 & -1 & 0 & \\ 0 & 0 & 1 & \mathcal{N}\partial_1 \\ -\partial_0 & 0 & 0 & \end{pmatrix}\begin{pmatrix} \varphi \\ \alpha \\ \bar{\alpha} \\ \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}$$

and we conclude that the Jacobian of this transformation depends only on U,

$$\mathcal{D}A = J(U)\mathcal{D}\varphi\mathcal{D}\alpha\mathcal{D}\bar{\alpha}\mathcal{D}U \quad , \quad J(U)\mathcal{D}U \sim \det \triangle d\mu(U). \tag{103}$$

When calculating expectation values of gauge invariant operators the factor $\det \triangle$ und the integrations over the Haar measure $d\mu(U)$ in the numerator and denominator cancel.

In particular for the chiral condensate (79) in N_f-flavour baby-QCD we find

$$\langle \bar{u}P_+u\rangle(x) = -S^\theta_{++}(x,x;0)\frac{\int \mathcal{D}(\varphi,\alpha,\bar{\alpha})\,\mathrm{tr}\,J\,e^{-S_{YM}}\det^{N_f}(i\not{D})}{\int \mathcal{D}(\varphi,\alpha,\bar{\alpha})\,e^{-S_{YM}}\det^{N_f}(i\not{D})} \tag{104}$$

or after inserting the explicit expressions we are left with the non-Gaussian functional integral

$$\langle \bar{u}P_+u\rangle = -\frac{e^\theta}{2\pi R}\frac{1}{1-r^2/R^2}\frac{\int \mathcal{D}(.)\left\{e^\varphi(1+v\bar{v})+e^{-\varphi}\right\}e^{-\Gamma}}{\int \mathcal{D}(.)\,e^{-\Gamma},} \tag{105}$$

with effective action

$$\Gamma = S_{YM} + \frac{N_f}{4\pi}\int\Big\{(\nabla\varphi,\nabla\varphi)+\alpha\bar{\alpha}\Big\}. \tag{106}$$

Thus we have reduced the task of calculating the 'quark' condensate to computing the functional integral (105) over the gauge invariant variables φ and α. For an evaluation of the integral it maybe relevant to decide on the boundary conditions for the gauge fields. For the abelian models it makes no difference whether we take free boundary conditions or impose the gauge invariant bag boundary conditions [11]

$$n^\mu F_{\mu\nu}|_{\partial\mathcal{M}} = 0,$$

but for the non-abelian theories this choice may affect the final results for correlators.

The formula (106) immediately leads to a gauge invariant perturbation expansion for the condensate and similarly for other expectation values. Note that if we perturb about the quadratic part of the effective action then we obtain an infinite resummation of the ordinary perturbative expansion in the gauge coupling constant. We hope to report on the corresponding results elsewhere. Here we shall truncate the nonabelian theories and shall investigate their abelian projections.

8.4 Abelian projection of $SU(N_c)$ gauge theories.

Here we calculate the condensate in the approximation where the 'gluons' are confined to the Cartan subalgebra of $SU(N_c)$. Hence only N_c-1 gluons propagate around a 'gluon' loop and there are no 3 or 4-gluon vertices in this approximation. In other words, we assume that g in $A_z = ig^{-1}\partial_z g$ lies in the maximal abelian subgroup of $SL(N_c)$, i.e.

$$g = \prod_{i=1}^{N_c-1} e^{-g(\varphi_i+i\lambda_i)H_i} \tag{107}$$

with trace-orthonormal H_i in the Cartan subalgebra of $SU(N_c)$. The Jacobian of the transformation $(A_\mu) \to (\varphi_i, \lambda_i)$, where A lies in the Cartan subalgebra, is field independent and cancels in expectation values of gauge invariant observables. Thus in the abelian projected theory the 'quark' condensate (79) simplifies to

$$\langle \bar{u}P_+u\rangle_{SU(N_c)} = -\frac{e^\theta}{2\pi R}\frac{1}{1-r^2/R^2}\,\mathrm{tr}\prod_{i=1}^{N_c-1}\frac{\int \mathcal{D}\varphi_i\, e^{-2g\varphi_i H_i}\, e^{-\Gamma_0[\varphi_i]}}{\int \mathcal{D}\varphi_i\, e^{-\Gamma_0[\varphi_i]}}, \tag{108}$$

where Γ_0 is the effective action Γ_θ in (67) without boundary term ($\theta=0$), with e replaced by g and with $N_c=1$. The N_c-1 functional integrals can be calculated by using that

$$\frac{\int \mathcal{D}\varphi\, e^{-2g\varphi H} e^{-\Gamma_0}}{\int \mathcal{D}\varphi\, e^{-\Gamma_0}} = \Big(\frac{\tilde{m}Re^{\gamma+F_g}}{2}[1-\frac{r^2}{R^2}]\Big)^{H^2/N_f},$$

where now $\tilde{m}^2=N_f g^2/\pi$ and F_g is the function (85) with the electric charge e replaced by the gauge coupling g or equivalently m_η by $\tilde{m}$. Since $N_c\sum H_i^2=(N_c-1)I_c$ we arrive at the following expression for the chiral condensate in the projected theories

$$\langle \bar{u}P_+u\rangle_{SU(N_c)} = -\frac{e^\theta}{2\pi R}\frac{N_c}{1-r^2/R^2}\Big(\frac{\tilde{m}Re^{\gamma+F_g}}{2}[1-\frac{r^2}{R^2}]\Big)^{(N_c-1)/N_cN_f}. \tag{109}$$

In the one-flavour model the condensate depends on the bag-radius as $\sim R^{-1/N_c}$ and therefore saturates the upper bound (99).

The $U(N_c)$-condensate is related to the one in $SU(N_c)$ gauge theories as in (97) and thus is found to be

$$\begin{aligned} \langle \bar{u}P_+u\rangle_{U(N_c)} &= -\frac{e^{\theta I_e}}{2\pi R}\frac{N_c}{1-r^2/R^2}\Big(\frac{e}{g}\Big)^{1/N_cN_f}e^{(F_e-F_g)/N_cN_f} \\ &\cdot\Big(\frac{\tilde{m}Re^{\gamma+F_g}}{2}[1-\frac{r^2}{R^2}]\Big)^{1/N_f}. \end{aligned} \tag{110}$$

Let us now discuss the various limiting cases in turn.

Large N_c limit. The large N_c limits of the ablian projected theories are different from the same limits in the full theories since there is no suppression of fermionic loops relative to the bosonic ones. But as in the full theories a condensate remains in the thermodynamic limit in the one-flavour models. Indeed, when $N_c\to\infty$ the condensates at the center of a large bag simplify to

$$\langle \bar{u}P_+u\rangle_{SU(N_c)} = e^{\theta}\langle \bar{u}P_+u\rangle_{U(N_c)} = -\frac{e^{\theta}N_c}{2\pi R}\Big(\frac{\tilde{m}Re^{\gamma}}{2}\Big)^{1/N_f}. \tag{111}$$

For $N_f=1$ a condensate remains for infinite volume and its limiting value is just

$$\frac{1}{N_c}\langle \bar{u}P_+u\rangle_{SU(N_c)} = \frac{e^{\theta}}{N_c}\langle \bar{u}P_+u\rangle_{U(N_c)} = -\frac{e^{\theta+\gamma}g}{4\pi^{3/2}}. \tag{112}$$

Weak couplings. For a small electric charge e the function I_e in the first factor in (110) tends to 1 and inserting the asymptotic expansion (87) for small $m_\gamma R$ we see that for $e\to 0$ the $U(N_c)$-condensate converges to the $SU(N_c)$ one, as expected.
When the gauge coupling g is weak the $SU(N_c)$-condensate becomes equal to $-N_c$ times the chirality violating entry S^{θ}_{++} of the free Green's function (23) and thus vanishes in the thermodynamic limit. The $U(N_c)$-condensate simplifies to N_c times the $U(1)$ condensate (96).

Strong couplings. When both couplings e and g become strong, or equivalently the bag very large, then the condensates at the bag center are just

$$
\begin{aligned}
\langle \bar{u}P_+u\rangle_{SU(N_c)} &= -\frac{e^\theta N_c}{2\pi R}\Big(\frac{\tilde{m}Re^\gamma}{2}\Big)^{(N_c-1)/N_cN_f} \\
\langle \bar{u}P_+u\rangle_{U(N_c)} &= -\frac{N_c}{2\pi R}(\frac{e}{g})^{1/N_cN_f}\Big(\frac{\tilde{m}Re^\gamma}{2}\Big)^{1/N_f}.
\end{aligned}
\tag{113}
$$

9 Discussion

In this paper we have investigated Euclidean gauge theories with massless Dirac fermions enclosed in a bag. We have imposed $U_A(N_f)$-breaking boundary conditions to trigger a breaking of the chiral symmetry. In the first part of the paper we considered gauge theories in arbitrary $2n$-dimensional bags. We found the explicit θ-dependence of the fermionic Green's functions and determinants in arbitrary background gauge fields. In contrast to the situation on a sphere or torus the Dirac operator possesses no zero modes in a bag and this property simplifies the quantization considerably. In the second part of the paper we investigated 2-dimensional gauge theories. We found the mesonic current correlators and calculated the chiral condensates both for abelian and non-abelian gauge theories. Our results are in full agreement with earlier instanton-type or small 'quark'-mass calculations. We conclude that the bag boundary conditions are a substitute for introducing small quark masses to drive the breaking of the chiral symmetry. Of course, for several flavours the condensate dissappears when the volume of the bag tends to infinity, in accordance with general theorems. Only when the number of colours is sent to infinity before the thermodynamic limit is performed there remains a 'quark'-condensate.

On a sphere or torus one finds that in the chiral limit only configurations with vanishing topological charge

$$
q=\frac{e}{2\pi}\int d^2x\,F_{01}\quad\text{resp.}\quad q=\frac{g^2}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\,F_{\mu\nu}F_{\alpha\beta}
$$

contribute to the partition functions in 2 resp. 4-dimensions [39]. For $U(N_c)$ gauge theories confined in a 2-dimensional bag we can find the expectation values of arbitrary powers of the topological charge by differentiating the partition function sufficiently often with respect to θ. The correlators are reproduced by the following Gaussian distribution for the topological charge:

$$d\mu(q)=\sqrt{\frac{N_cN_f}{\pi\sigma}}\,e^{-N_cN_f\sigma[q+\theta/2\sigma]^2}\,dq\quad,\quad\sigma=\frac{I_0(m_\eta R)}{m_\eta RI_1(m_\eta R)}. \tag{114}$$

The expectation value of the instanton number vanishes for vanishing θ, but its fluctuation does not. Only for very small volumes and/or weak coupling (for which the semiclassical approximation makes sense) is the instanton number distribution sharply peaked about $q=0$ as can be seen by inspection from (114) or from

$$\langle|q|\rangle=\begin{cases}0 & \text{for } m_\eta R\to 0\\ \sqrt{\frac{eR}{\pi N_c}}(\pi N_f)^{-1/4} & \text{for } m_\eta R\to\infty.\end{cases} \tag{115}$$

For big volumes and/or strong coupling, which would correspond to small quark masses, configurations with $q^2\sim 1/\sqrt{N_f}$ dominate the functional integral.

In this paper we have regarded the bags as mathematical constructs rather than real objects in spacetime. For example, to be a model for a hadron at finite temperature, $\mathcal{M}$ must be a bag in space and hence $[0,\beta]\times\mathcal{M}$ a subspace of the Euclidean spacetime. The gluon (quark) fields must then be periodic (antiperiodic) in the Euclidean time with period $\beta=1/T$. In [40] we have studied multi-flavour QED_2 at finite temperature enclosed in a spatial bag $[0,L]$. Besides the finite temperature boundary conditions we imposed the bag boundary conditions $B_\theta\psi=\psi$ at $x^1=0$ and $x^1=L$. By applying the methods developped in this paper we found for the chiral condensate in the low temperature limit $T\ll 1/L\ll m_\eta$ [40]

$$\langle\bar uP_+u\rangle=-\frac{1}{4L}\,e^{\gamma/N_f}\Big(\frac{m_\eta L}{\pi}\Big)^{1/N_f} \tag{116}$$

In particular, for 2 flavours this reads

$$\langle\bar uP_+u\rangle=-\Big(\frac{e^\gamma m_\eta}{16\pi L}\Big)^{1/2} \tag{117}$$

and this result is identical to that of Shifman and Smilga [10] when they allowed for fracton configurations.

The condensate in an d-dimensional Euclidean bag obeys the scaling relation [41]

$$\langle\bar{\psi}P_+\psi\rangle(\lambda R,\lambda x,g)=\lambda^{1-d}\,Z(\lambda)\langle\bar{\psi}P_+\psi\rangle(R,x,\lambda^{2-d/2}g(\lambda)),\tag{118}$$

where $Z(\lambda)$ and $g(\lambda)$ are the wave-function renormalization of the condensate and running gauge coupling constant, respectively[10]. The relative size λ of the two bags plays the role of the inverse energy scale in the Callan-Symanzik equation. For example, the condensates in the multi-flavour Schwinger models, (86), obey this scaling relation with $g(\lambda)=g$ and $Z(\lambda)=1$ and this agrees with the wellknown fact that the β-function vanishes and that there is no wave function renormalization in these theories. In 4-dimensions $g(\lambda)$ becomes weak in small bags because of asymptotic freedom and the chiral condensate should again tend to the chirality violating entry S^θ_{++} of the free Green's function. The change of the condensate at $x=0$, when the size of the bag is increased, is then determined by the nonperturbative beta-function and anomalous dimension of the condensate. Thus we could extrapolate the QCD-condensate to large volumes if we would know its anomalous dimension and the QCD beta-function. Conversely, we may put bounds on the functions $g(\lambda),Z(\lambda)$ since a condensate must remain in the infinite volume limit.

Acknowledgements: The reported work was partially supported by the Swiss National Science Foundation. We are indebted to A. Abrikosov jr., S.I. Azakov, I. Sachs and C. Wiesendanger for valuable discussions and to J. Fröhlich for pointing our attention to Ref. [2].

A Appendix

In this appendix we fill the gaps in the calculation of the fermionic determinants confined in 2-dimensional bags in section 6. What remains is to calculate the surface Seeley deWitt coefficent b_1 in (59) which enters in (41,57).

First we note that $\oint \mathrm{tr}\, b_n(\phi)$ has the expansion

$$\oint \mathrm{tr}\, b_n(\phi)=\sum_0^{d-1}\oint \mathrm{tr}\, c_p(F_{\mu\nu},\mathcal{R},\chi)\,\partial_n^p\phi,\tag{119}$$

where c_p is a gauge- and Lorentz-invariant local polynomial in the field-

[10] up to possible runnings of the surface coupling constants

strength and its covariant derivatives, the extrinsic and intrinsic curvatures of the bag boundary and their covariant derivatives and has length-dimension $1-d+p$. Here ∂_n^p is the p'th derivative normal to the bag boundary. In particular in two dimensions we need b_1 which is the sum of two terms (again neglecting purely geometric contributions)

$$\oint \mathrm{tr}\, b_1(\phi) = \oint \mathrm{tr}\, f_1(\theta)\chi\,\phi + \oint \mathrm{tr}\, f_2(\theta)\,\partial_n\phi. \tag{120}$$

Here we are not interested in the term containing f_1. In (57) it would not contribute since $A+A^\dagger$ vanishes on the bag boundary and in (41) it would yield an uninteresting constant which cancels in expectation values[11] The invariance of the fermionic determinant under parity, $(\theta, A, x) \to (-\theta, \tilde{A}, \tilde{x})$, restricts the form of the free function f_2. To determine this function it suffices to calculate the heat kernel expansion for free fermions confined to the halfplane $\mathcal{M} = \{x^0, x^1 | x^1 \geq 0\}$ and subject to bag boundary conditions at $x^1 = 0$.

Besided the wellknown properties the heat kernel must obey the boundary conditions

$$\begin{aligned} B_\theta K(t,x,y)|_{x^1=0} &= K(t,x,y)|_{x^1=0} \\ B_\theta \not{\partial}_x K(t,x,y)|_{x^1=0} &= \not{\partial}_x K(t,x,y)|_{x^1=0}. \end{aligned} \tag{121}$$

After some algebra we have found the following explicit formula

$$\begin{aligned} K(t,x,y) &= \frac{1}{4\pi t} e^{-(\xi_0^2+\xi_1^2)/4t} \\ &+ \frac{1}{4\pi t}\begin{pmatrix} e^\theta \sinh\theta & -\cosh\theta \\ -\cosh\theta & -e^{-\theta}\sinh\theta \end{pmatrix} e^{-(\xi_0^2+\eta^2)/4t} \\ &+ \frac{i\mathcal{P}\sinh\theta}{8t\sqrt{\pi t}}\begin{pmatrix} e^\theta & -1 \\ -1 & e^{-\theta} \end{pmatrix} e^{-\mathcal{P}^2/4t}\Big[1+\mathrm{erf}\Big(\frac{i\xi_0\sinh\theta - \eta\cosh\theta}{2\sqrt{t}}\Big)\Big], \end{aligned} \tag{122}$$

where $\xi^\mu = x^\mu - y^\mu$, $\eta = x^1 + y^1$ and $\mathcal{P} = \xi_0\cosh\theta + i\eta\sinh\theta$. To determine the relevant Seeley-deWitt coefficient we need to calculate

$$\int_{\mathcal{M}} K(t,x,x) f(x) \sim \int_{\mathcal{M}} K(t,x,x)\Big(f(x^0,0) - x^1\partial_1 f(x^0,0) + ..\Big), \tag{123}$$

[11]It would contribute to the free energy or to the Casimir effect [42].

where we anticipated that the integrand is sharply peaked at $x_1 = 0$ and thus expanded the test function f about $x^1=0$. On the diagonal $(x=y)$ we have $\xi=0$ and $\eta=2x_1$ and we are left with calculating the integrals

$$\begin{aligned}&\int\limits_{x_1\geq 0} dx_1\, e^{-x_1^2/t}\Big(f(x_0,0)+x_1\partial_1 f(x_0,0)+..\Big)\\ &\int\limits_{x_1\geq 0} dx_1\, x_1\, e^{x_1^2\sinh^2\theta/t}\Big[1-\mathrm{erf}(\frac{x_1\cosh\theta}{\sqrt{t}})\Big]\Big(f(x_0,0)+x_1\partial_1 f(x_0,0)+..\Big).\end{aligned} \tag{124}$$

The first integral is easily evaluated by using that

$$\int\limits_{x\geq 0} dx\, e^{-x^2/t}=\frac{1}{2}\sqrt{\pi t}\quad,\quad \int\limits_{x\geq 0} dx\, x e^{-x^2/t}=\frac{t}{2}.$$

For evaluating the second integral we need the formulae

$$\begin{aligned}\int\limits_0^\infty dx\,[1-\mathrm{erf}(\beta x)]\,e^{\mu x^2}\,x &= -\frac{1}{2\mu}\Big(1-\frac{\beta}{\sqrt{\beta^2-\mu}}\Big)\\ \int\limits_0^\infty dx\,[1-\mathrm{erf}(\beta x)]\,e^{\mu x^2}\,x^2 &= \frac{1}{2\mu\sqrt{\pi}}\Big(\frac{\beta}{\beta^2-\mu}+\frac{1}{2\sqrt{\mu}}\log\frac{\beta-\sqrt{\mu}}{\beta+\sqrt{\mu}}\Big)\end{aligned} \tag{125}$$

which apply if $\mu > 0$ and $\Re(\mu) < \Re(\beta^2)$. Using these results one finds the following small-t expansion for the integral (123)

$$\begin{aligned}\int d^2x K(t,x,x)f(x) &= \frac{1}{4\pi t}\int\limits_{\mathcal{M}} d^2x f(x)\\ &+\frac{1}{8\sqrt{\pi t}}\int dx^0\left\{\begin{pmatrix} e^\theta & -1\\ -1 & e^{-\theta}\end{pmatrix}-I\right\}f(x^0,0)\\ &+\frac{1}{8\pi}\int dx^0\left\{\frac{\log e^\theta}{\sinh\theta}\begin{pmatrix} e^\theta & -1\\ -1 & e^{-\theta}\end{pmatrix}-I\right\}\partial_1 f(x^0,0)+O(t^{1/2}).\end{aligned} \tag{126}$$

The first term on the right yields the wellknown a_0 coefficient, the second term $b_{1/2}$ and the third one is the b_1-coefficient (59) (after noting that $\partial_1 = -\partial_n$) we have been aiming at. We see that the small t-expansion of K is invariant under $\theta\to\theta+i2\pi n$, $n\in Z$, as required.

References

[1] see e.g. E.V. Shuryak, The QCD Vacuum, Hadrons and the Superdense Matter, World Scientific, Singapore, 1988.

[2] J.L. Lebowitz, J. Stat. Phys. **16** (1977) 463.

[3] G. t'Hooft, Phys. Rev. Lett. **37** (1976) 8; Phys. Rev. **D14** (1976) 3432; R.D. Peccei and H.R. Quinn, Nuovo Cimenti **41A** (1977) 309; N.V. Krasnikov, V.A. Rubakov and V.F. Tokarev, Sov. J. Nucl. Phys. **29** (1979) 581; K.D. Rothe and J.A. Swieca, Ann. Phys. **117** (1979) 382; M. Hortacsu, K.D. Rothe and B. Schroer, Phys. Rev. **D20** (1979) 3203;

[4] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Yu.S. Tyupkin, Phys. Lett. **59B** (1975) 85.

[5] A.S. Schwartz, Phys. Lett. **67B** (1977) 172.

[6] V. Novikov, M. Shifman, A. Vainshtain and V. Zakharov, Nucl. Phys. **B229** (1975) 85.

[7] G. t'Hooft, Commun. Math. Phys. **81** (1981) 267; E. Witten, Nucl. Phys. **B202** (1982) 253; E. Cohen and C. Gomez, Phys. Rev. Lett. **52** (1984) 237.

[8] A.R. Zhitnitsky, Nucl. Phys. **B340** (1990) 56; **B374** (1992) 183.

[9] M.A. Shifman and A.I. Vainstein, Nucl. Phys. **B296** (1988) 445.

[10] M.A. Shifman and A.V. Smilga, preprint TPI-MINN-94/24-T, UMN-TH-1262-94.

[11] A. Chodos and C.B. Thorn, Phys. Rev. **D12** (1975) 2733; T. Inowe and T. Maskawa, Prog. Theor. Phys. **54** (1975) 1833; P. Hasenfratz and J. Kuti, Phys. Rep. **40C** (1978) 76; G.E. Brown and M. Rho, Phys. Lett. **82B** (1979) 177.

[12] A. Smilga and J.J.M. Verbaarschot, preprint SUNY-NTG-94/18 and TPI-Minn-94/10-T.

[13] I. Sachs and A. Wipf, Helv. Phys. Acta **65** (1992) 652.

[14] H. Joos, Helv. Phys. Acta **63** (1990) 670; Nucl. Phys. (Proc. Suppl.) **B17** (1990) 704; H. Dilger and H. Joos, Nucl. Phys. (Proc. Suppl.) **B34** (1994) 195; H. Joos and S.I. Azakov, preprint DESY-94-142.

[15] C. Jayewardena, Helv. Phys. Acta **61** (1988) 636.

[16] A.V. Smilga, Phys. Lett. **B278** (1992) 1992.

[17] M.F. Atiyah, V.K. Patodi and I.M. Singer, Proc. Camb. Phil. Soc. **77** (1075) 43; P. Forgacs, L. O'Raifeartaigh and A. Wipf, Nucl. Phys. **B293** (1987) 559.

[18] C.G. Callan, R. Dashen and D.J. Gross, Phys. Rev. **D19** (1979) 1826.

[19] J. Balog and P. Hrasko, Nucl. Phys. **245** (1984) 118.

[20] see e.g. E. Merzbacher, Quantum Mechanics, Wiley, New York, 1961, p.396.

[21] J.S. Dowker and R. Critchley, Phys. Rev. **D13** (1976) 3224; S.W. Hawking, Commun. Math. Phys. **55** (1977) 133.

[22] A. Dettki and A. Wipf, Nucl. Phys. **B377** (1992) 252.

[23] I. Sachs and A. Wipf, Phys. Lett. **B326** (1994) 105; A. Dettki, I. Sachs and A. Wipf, Phys. Lett. **B317** (1993) 545.

[24] S. Blau, M. Visser and A. Wipf, Int. J. Mod. Phys., **A4** (1989) 1467.

[25] P. Gilkey, Invariance theory, the heat equation and the Atiyah Singer index theorem, Publish or Perish, Houston, 1984; A.E.M. van der Ven, Nucl. Phys. **B250** (1985) 593; N. Berline, E. Getzler and M. Verne, Heat Kernels and Dirac Operators, Springer, Berlin, 1991.

[26] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37** (1976) 172; R.D. Peccei and H.R. Quinn, Phys. Rev. **D16** (1977) 1977.

[27] C.N. Yang, Phys. Rev. Lett. **38** (1977) 1377; L.L. Chan, M.K. Prasad and A. Sinha, Phys. Rev. **D24** (1981) 1574.

[28] A.M. Polyakov and P.M. Wiegman, Phys. Lett. **B131** (1983) 121.

[29] O. Alvarez, Nucl. Phys. **B238** (1984) 61.

[30] A. Kupiainen and J. Mickelsson, Phys. Lett. **B185** (1987) 107.

[31] C. Gattringer and E. Seiler, Ann. Phys. **233** (1994) 97; L.V. Belvedere, J.A. Swieca, K.D. Rothe and B. Schroer, Nucl. Phys. **B153** (1979) 112.

[32] J. Schwinger, Phys. Rev. **128** (1977) 244.

[33] A.V. Smilga, Phys. Lett. **B278** (1992) 371; A. Fayyazuddin, T.H. Hansson, M.A. Nowak, J.J.M. Verbaarschot and I. Zahed, preprint USITP-93/19 and SUNY-NGT-93-43; J.V. Steele, J.J.M. Verbaarschot and I. Zahed, preprint SUNY-NTG-94-28.

[34] J.H. Loewenstein and J.A. Swieca, Ann. Phys. **68** (1961) 172; N.K. Nielsen and B. Schroer, Nucl. Phys. **B120** (1977) 62.

[35] A.V. Smilga, preprint TPI-Minn-94/6-T

[36] S. Coleman, Aspects of Symmetry, Cambridge Univ. Press, Cambridge, 188.

[37] E. Moreno and F.A. Shaposnik, Int. J. Mod. Phys. **A4** (1989) 2827; Z. Haba, Int. J. Mod. Phys. **A5** (1990) 4241; I. Sachs, PhD Thesis 1994.

[38] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1.

[39] H. Leutwyler and A.V. Smilga, Phys. Rev. **D46** (1992) 5607.

[40] S. Duerr and A. Wipf, in preparation.

[41] C. Wiesendanger and A. Wipf, Ann. Phys. **233** (1994) 125.

[42] H. Falomir and E.M. Santangelo, Phys. Rev. **D43** (1991) 539.