On the Origin of the Aharonov–Bohm Effect

.t is now generally accepted that the Aharonov–Bohm effect originates in the interaction between an electron and an external gauge-potential **A** whose **B**-field vanishes locally. Here it is shown that the effect can equally well be regarded as originating in the interaction of the magnetic field of the electron with the distant **B** field. From this point of view the effect is seen to have a natural classical origin and loses much of its mystery.

1. INTRODUCTION

Although the Aharonov–Bohm (AB) effect[1] has been discussed in the literature for almost thirty years it is still regarded as a matter of interest and, indeed, as something of a mystery. For example, the question as to whether it implies that gauge potentials are necessary in quantum mechanics is still a matter of controversy, Wightman and Strocchi[2] maintaining that they are not, and Yang[3,4] maintaining that at least the phases $\exp(i \int A_\mu dx^\mu)$ are necessary. There is even a school[4] which maintains that the present interpretation of the effect is incorrect.

Recently, to settle at least the question of the existence of a real effect, Peshkin[5] wrote a very interesting review in which he pointed out that unless the effect existed all our ideas about angular momentum would have to be revised. Because of his emphasis on angular momentum, however, Peshkin did not consider the parallel question of energy, and in this Comment we wish to point out that the translation of Peshkin's analysis into an *energy* rather than an angular-momentum analysis has distinct advantages. First, the analysis is technically simpler and more intuitive simply because energy is a scalar, not a three-vector. But, more importantly, the analysis in terms of energy shows quite clearly and simply where the effect originates, namely not in the interaction of the external magnetic field with the electron (from which it is shielded) but with the magnetic field of the electron, from which it is not shielded. As a by-product one sees that the disagreement between the Strocchi–Wightman (SW) and Yang (Y) points of view is spurious, the true situation being that Yang's phases are indeed necessary if one requires a local description in terms of the electron but are not necessary if one drops this requirement. A further by-product is that the AB-effect is seen as a classical effect that can only be measured by quantum-mechanical means, and thus provides a counter-example to the conventional statements concerning measurement[6] in quantum mechanics.

2. THE SETTING

The traditional setting for the AB-effect is a cylinder of radius a (say) in the z-direction, with a magnetic field **B** confined inside and an electron confined (and scattered) outside. Following Peshkin we wish to generalize this setting slightly by adding a larger concentric cylinder of radius b (say) and confining the electron to the region $a < r < b$ between the two cylinders (hereafter referred to as the shell; see Fig. 1). The **B** field is allowed to be non-zero also for $r \geq b$, a situation that allows for the "return" of the **B**-field flux and removes any controversy that there might be on that core, but the main reason for the adaption of the more general setting is that since the electron is totally confined, it converts the scattering phase shift into a shift of discrete energy levels (from which the phase shifts are automatically recovered in the limit $b \to \infty$).[7] This simplifies the problem from both the technical and intuitive points of view and allows the origin of the effect to be traced in a straightforward manner. For simplicity we also assume that the magnetic field is cylindrically symmetric.

It is well-known, indeed is a simple case of Stokes' theorem,

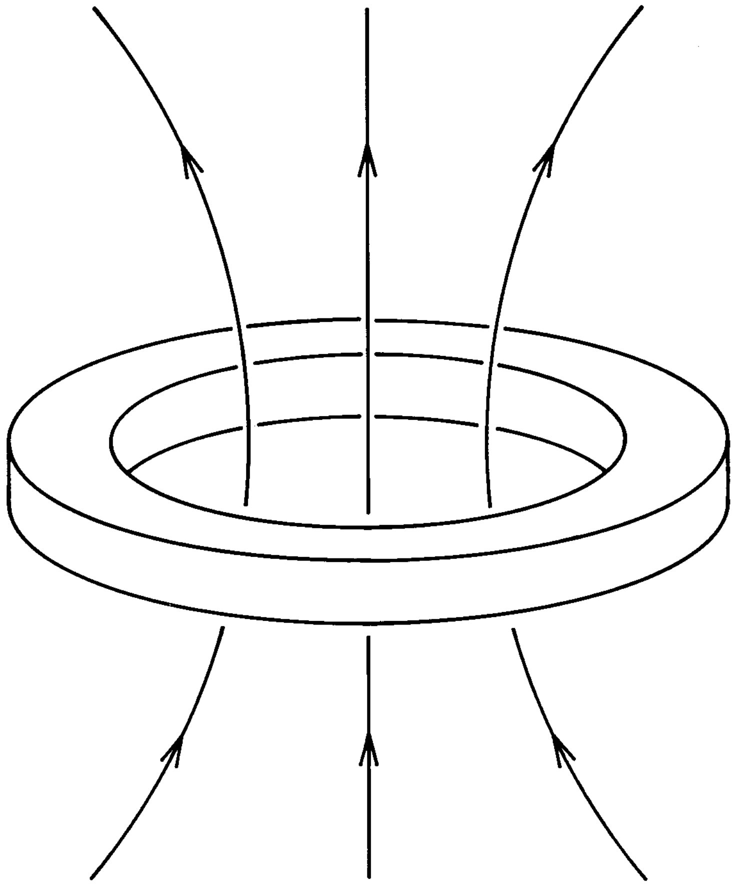

FIGURE 1 Confining torus region for the electron. The external magnetic field is shielded from this domain.

that for a magnetic field of the kind described (i.e., zero in the shell) the gauge potential in the shell is not zero, and in the cylindrical gauge $A_r = 0$ can be chosen to be of the form

$$rA_\phi = \frac{1}{2\pi}\int_{r=a} A_\phi r d\phi = \frac{1}{2\pi}\int_{r\le a} B_z d^2x = \frac{\Phi}{2\pi}. \qquad (1)$$

Thus there is no question that there is a nontrivial gauge potential

within the shell. It cannot be measured classically within the shell by means of the electron, of course, since according to the Lorentz force law, a classical electron experiences the local magnetic field, which is zero, but nevertheless it is there and its presence could be measured even classically by simply measuring the magnetic flux in the core $r \le a$ and using Stokes' theorem. What makes the AB-effect interesting, however, is that for a QM electron the potential A_ϕ can be measured (modulo 2π) even within the shell, either by means of the phase shift for $b \to \infty$, or by means of the energy shift for $b < \infty$, as follows:

In the shell, the Schrödinger equation $((1/i)\nabla + e\mathbf{A})^2\psi(r, \phi) = 2mE\psi(r, \phi)$ (we suppress the trivial dependence on z) reduces to $(F = \Phi/\Phi_0,\ \Phi_0 = 2\pi/e)$

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{(l+F)^2}{r^2}\right)\psi(r) = 2mE\psi(r),$$

$$l = 0, \pm 1, \ldots, \qquad (2)$$

where the boundary conditions $\psi(a) = \psi(b) = 0$ take the place of the confining potential. Since $rA_\phi = F/e$ is constant in the shell the solutions of (2) are just the Bessel functions of order $l + F$ and the energy levels are then determined from the boundary conditions

$$J(ka) = J(kb) = 0, \quad J(kr) = \alpha j_{l+F}(kr) + \beta j_{-(l+F)}(kr), \qquad (3)$$

where $k = \sqrt{2mE}$ and α and β are constants. From this and the standard properties of Besel functions[8] it is easy to see that the energy levels will depend strongly on $l + F$ and thus will be different* for different A_ϕ, provided the difference is not an integer.

* It is assumed here, of course, that l is an integer even in the presence of A_ϕ. The argument for this is that the integral value of l follows from the single-valuedness of the complete wave-function $\psi(r)\exp(il\phi)$ when considered as a function of x, y. The single-valuedness has sometimes been questioned, but the argument of SW to the effect that no potential barrier is infinitely high in practice so that $\psi(r, \phi)$ has an exponential small tail inside $r \le a$ and thus is really a function on R_3, not just within the shell, seems to us adequate to dispose of this objection.

The question then is: What is the source of this mysterious energy that can only be measured quantum-mechanically and not classically. We wish to show that this mystery has nothing to do with quantum mechanics since the same question (and answer) appears in classical mechanics. The only role of quantum mechanics is to provide a means of *measuring* it through the existence of phase shifts, or equivalently, discrete energy levels.

3. ORIGIN OF THE ENERGY SHIFT

To see the origin of the energy shift (or phase shift) let us return to our Peshkin scenario for the magnetic field **B**, but with the difference that we now allow **B** to be gradually switched on from the value zero everywhere to the final time-independent value (with $\mathbf{B} = 0$ inside the shell $a < r < b$ at all times) and we consider *any* electric current **j** which is confined to the shell, either classical or quantum-mechanical. Then from ordinary EM-theory the adiabatic work δW done by the induced electric field (which is needed to increase the flux Φ to $\Phi + \delta\Phi$) on the electron is given by the integral[9]

$$\frac{\delta W}{\delta t} = \frac{1}{c} \int \mathbf{j}(\mathbf{x})\mathbf{E}(\mathbf{x}) d^3x = -\frac{1}{c} \int \mathbf{j}(\mathbf{x}) \frac{\delta}{\delta t} \mathbf{A}(\mathbf{x}) d^3x. \qquad (4)$$

In the case when the current $\mathbf{j}(\mathbf{x})$ is due to the angular velocity of a classical electron localized at **x**, say, where $a < |\mathbf{x}| < b$, then the expression (4) reduces to

$$\delta W = -\frac{e}{c} v_\Phi \delta A_\Phi \qquad (5)$$

from which we see that it produces an increase in the kinetic energy of the electron, namely

$$\delta \left(\frac{m}{2} v_\Phi^2 \right) = -e v_\Phi \delta A_\Phi. \qquad (6)$$

(Note that this does not contradict the earlier statement concerning the Lorentz force law because during the time of switch-on the fields are time-dependent and hence there is an electric field $\dot{A}_\Phi$ inside the shell). From (6) we see at once that the angular velocity of the electron increases according to

$$\delta v_\Phi = -\frac{e}{m} \delta A_\Phi, \qquad (7)$$

and that if we integrate this equation the electron ends up with a final angular velocity $v_\Phi = v_\Phi(0) - e\Phi/2\pi mr$, and hence a final angular momentum $L = L(0) - e\Phi/2\pi c$, where $L(0)$ is its initial angular momentum. Thus it ends up with a final energy

$$E = E_{\text{radial}} + \frac{(L - e\Phi/2\pi c)^2}{2mr^2}. \qquad (8)$$

The Schrödinger Hamiltonian (2) is obviously just the quantum mechanical analogue of (8). So the energy term is already of the form (8) in the classical theory. So where is the mystery? Insofar as there is a mystery it lies in Eq. (4) which states that the work done at the current when increasing the flux *outside* the shell $a < r < b$ is related to the integral of $\mathbf{j}\delta\mathbf{A}$ which is completely *inside* the shell. But this is no more than an expression of the fact that although the right-hand side (current side) of Maxwell's equations (or indeed any other local field equations) may be local, the solutions are certainly not. Thus the equation

$$\nabla \times \mathbf{h}(\mathbf{x}) = 4\pi \mathbf{j}(\mathbf{x}) \qquad (9)$$

for the magnetic field produced by the electron is the non-local quantity

$$\mathbf{h}(\mathbf{x}) = \int \mathbf{j}(\mathbf{x}') \times \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x', \qquad (10)$$

and expression (4) may also be expressed in the form

$$\delta W = -\frac{1}{4\pi} \int (\nabla \times \mathbf{h}) \cdot \delta \mathbf{A}. \qquad (11)$$

From (11) we see (using partial integration) that it can also be expressed as

$$\delta W = -\frac{1}{4\pi}\int \mathbf{h}\cdot(\nabla\times\delta\mathbf{A}) = -\frac{1}{4\pi}\int \mathbf{h}\cdot\delta\mathbf{B} \qquad (12)$$

which integrates to

$$W = -\frac{1}{4\pi}\int \mathbf{h}\cdot\mathbf{B}, \qquad (13)$$

and shows that the energy originates in the interaction energy between the magnetic field **h** of the electron (which, unlike the electron itself, *must* penetrate the regions $r \leq a$ and $r \geq b$ because of Ampère's law) and the external magnetic field **B**.

In sum, therefore, we see that the situation is that there is always a physical energy difference associated with the presence or absence of a magnetic field outside a shell within which there is an electric current, be it classical or quantum mechanical. The only difference is that the energy difference, and hence the potential A_ϕ, cannot be measured *within* the shell at the classical level (of course it can be measured classically by simply measuring **B** directly outside the shell), whereas in QM it can be measured even inside the shell, i.e., locally. We also see that the resolution of the apparent SW-Y discrepancy is simply a matter of whether one requires a local description in terms of the electron or not. If one requires a local description, i.e., a description in terms of the electron current **j(x)**, either classically or quantum-mechanically, then one must use the expression (4) for the energy shift, so for this purpose the potential is necessary. But if one permits a non-local description then the expression (13) is permitted and no gauge-potential is needed. Of course, SW worked in the equivalent of the shell region so their description looks local. But it is easy to see that the non-locality resides in their boundary conditions. Similarly, Yang only claimed that the phases $\exp(i \int A_\mu dx^\mu)$ were necessary, but again this corresponds to his localizing the shell region to the ideal limit where the electron had no tail outside, thus changing radically the topology of the problem. The phases

are then indeed needed to compensate for this radical change in topology.

One final remark: Since we claim that the effect is due to the field of the electron, rather than the electron itself, having a local interaction with the external **B**-field, one might ask whether the AB-effect disappears when the **h**-field, as well as the electron, is confined to the shell. One case in which this can happen is when the regions $r \leq a$ and $r \geq b$ are superconducting, and in this case the AB-effect does disappear on account of the London flux quantization condition.[10]

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