Chapter 5

2-dimensional Gauge Theories

The response of physical systems to a change of external conditions is of eminent importance in physics. In particular the dependence of expectation values on temperature, the particle density, the space region, the imposed boundary conditions or external fields has been widely studied [18]. Despite all these efforts we are still unable to understand, for example, the mechanism leading to the spontaneous symmetry breaking of the $SU_A(N)$ in low temperature QCD [43]. Clearly such subtle effects require a better understanding of the non-perturbative effects and in particular non-perturbative the vacuum sector of gauge theories. From our experience with 2-dimensional gauge theories [41] which we suppose to mimic one-flavor QCD [35], we are lead to believe that gauge fields with windings are responsible for the non-vanishing chiral condensate and in particular its temperature dependence. A related problem is how quantum systems behave in a hot and dense environment as it exists or existed in heavy ion collision, neutron stars or the early epochs of the universe [43].

On another front there has been much effort to quantize self-interacting field theories in a background gravitational field [5]. For example, one is interested whether a black hole still emits thermal radiation when self-interaction is included. Due to general arguments by Gibbons and Perry [25] this question is intimately connected with universality of the second law of thermodynamics.

Rather than seeking new partial results for more general and realistic 4-dimensional systems we have chosen an idealized 2-dimensional model with self-interaction to investigate the questions mentioned and others. It is a theory containing photons [1], charged mass-less fermions, scalars and pseudo-scalars in interaction with themselves and a gravitational background field. The model has the action

$$
\begin{aligned}
S &= \int \sqrt{-g}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\nabla_{\mu} - ig_1\partial_{\mu}\lambda + ig_2\eta_{\mu}{}^{\nu}\partial_{\nu}\phi)\psi \\
&\qquad + g^{\mu\nu}(\partial_{\mu}\phi\partial_{\nu}\phi + \partial_{\mu}\lambda\partial_{\nu}\lambda) - g_3\mathcal{R}\lambda\Big],
\end{aligned}
\tag{5.1}
$$

where $F_{\mu\nu}$ is the electromagnetic field strength, the gamma-matrices in curved space are related to the flat ones as $\gamma^{\mu} = e^{\mu}_{a}\hat{\gamma}^{a}$, $\nabla_{\mu} = \partial_{\mu} + i\omega_{\mu} - ieA_{\mu}$ is the generally and gauge covariant derivative containing the $U(1)$ gauge potential and spin connection, $\eta_{\mu\nu} = \sqrt{-g}\,\epsilon_{\mu\nu}$ denotes the totally antisymmetric tensor and $\mathcal{R}$ the Ricci scalar. The gravitational field $g_{\mu\nu}$ (or rather the 2-bein e^{a}_{μ}, since the theory contains fermions) is treated as classical background field, whereas the 'photons' A_{μ}, 'electrons' ψ, scalars λ and pseudo-scalars ϕ are fully quantized. The classical theory is invariant under $U(1)$ gauge- and axial transformations and correspondingly possesses conserved vector and axial-vector currents. Despite its complexity the general model (5.1) is solvable for arbitrary classical backgrounds $g_{\mu\nu}$ and allows for an analytical treatment.

We have chosen this model since it allows to address the above raised questions and since it relates to known soluble models for certain values of the coupling constants. For example it contains the *gauged Thirring model*, the *Schwinger model in curved space time* and the *minimal models* in conformal field theory as particular limits. For finite volumes the theory possesses instantons which minimize the Euclidean action in a given topological sector. These instantons lead to a non-trivial vacuum structure, i.e. to θ-vacua [10], and to chirality violating amplitudes. For example, a non-zero chiral condensate develops which vanishes exponentially for temperature and curvature bigger than the induced 'photon' mass $m_{\gamma}^2 = e^2/(\pi + \frac{1}{2}g_2^2)$. This mass is generated via the Schwinger mechanism and it the analog of $m_{\eta'}^2$ in QCD [23].

In two dimensions the electric charge e has the dimension of a mass. The other 3 couplings are dimensionless. The physical role of the coupling constants is the following: The coupling of ϕ to the transversal current decreases the effective electromagnetic interaction between fermions. For example, the electric charge becomes renormalized to

[1] Although photons in $1+1$ dimensions possess no transversal degrees of freedom we still call them photons. However, through their interaction with charged fermions they may become dynamical fields as exemplified by the Schwinger mechanism.

$$e_{ren} = \frac{e}{\sqrt{1+g_2^2/2\pi}} \tag{5.2}$$

the chiral condensate decreases as $\sim (2\pi + g_2^2)^{-\frac{1}{2}}$. The mass in the bosonised theory depends on g_2.

For $e = 0$ all coupling constants are dimensionless, the model has a trivial vacuum structure and becomes conformally invariant. It possesses the Virasoro algebra extended by left-right $U(1)$ Kac-Moody algebras as symmetry algebra. The central extensions, conformal weights and $U(1)$ charges all depend on g_2. The coupling constant g_3 amplifies the Hawking radiation which remains thermal for the interacting model. It is $(3 + 24\pi g_3^2)$ times as strong as that of a free mass-less scalar field. The central charge and conformal weights depend also on g_3. Actually, the weights of the fermionic fields become complex for $g_3 \neq 0$. However, g_3 does not enter in the finite size effects. The coupling constant g_1 to the longitudinal current weakens the long range gauge invariant electron-electron correlators in the one-instanton sector (see 5.104). In the un-gauged sector it enters in expectation values of local operators and in particular in the short distance expansions of the fermionic fields and energy momentum tensor. It does not influence the thermodynamics of the model.

Since for particular choices of the coupling constants the model reduces to well-known and well-studied exactly soluble models there are many earlier works which are related to ours. Some of them concentrated more on the gauge sector and investigated the renormalization of the electric charge in the gauged Thirring model by the four-Fermi interaction [30] or the non-trivial vacuum structure in the Schwinger model [41, 29]. Others concentrated on the un-gauged conformal sector. Freedman and Pilch calculated the partition function of the un-gauged Thirring model on arbitrary Riemann surfaces [21]. We do not agree with their result and in particular show that there is no holomorphic factorization for general fermionic boundary conditions. Also we deviate from Destri and deVega [16] which investigated the un-gauged model on the cylinder with twisted boundary conditions. We shall comment on the discrepancies in sections 5.2 and 5.4.1. Other papers which are relevant and are dealing with different aspects of certain limiting cases of (5.1) are [50], where the thermodynamics of the Thirring model has been studied or [5] in which the Hawking radiation has been derived.

This chapter is organized as follows: In section 5.1 we analyze the classical model to prepare the ground for the quantization. In particular we derive the general solution of the field equations, discuss the conservation laws and investigate the limiting theories. By employing the graded structure we derive the classical Poisson (anti) commutators of the fundamental fields with the energy momentum tensor. In the following section we quantize the finite temperature model. To avoid infrared problems we assume space to be finite. Together with the finite temperature boundary conditions we are lead to considering the theory on the 2-dimensional *Euclidean torus*. Due to the twists in the fermionic boundary conditions, the non-trivial vacuum structure and the associated instantons and fermionic zero-modes the quantization is rather subtle. Actually we show that some of the results in the literature are incorrect. In subsection 5.2.1 the general results are applied to derive the partition function of the gauged model. Its dependence on the spatial size, temperature and gravitational field is explicitly found. In subsection 5.2.2 we show that the gauged model on curved spacetime can be bosonised. It turns out that only the non-constant parts of the currents can be bosonised and that the well-known bosonization rules of the Thirring model are modified. In the following section the chiral symmetry breaking is studied. The exact form of the chiral condensate is found. On the flat torus the formula simplifies to (5.96). Various limits, e.g. $L \to \infty, T \to 0, T \to \infty$ or $g_2 \to \infty$ are investigated. By comparing the temperature and curvature dependence of the condensate we derive an effective curvature induced temperature. In section 5.4.1 the thermodynamics of the un-gauged model is studied. We derive the ground state energy and its dependence on the coupling constants, size of the system and boundary conditions. We compute the equation of state and our result does not agree with [50]. In subsection 5.4.2 we investigate the conformal sector of (5.1), that is the un-gauged model in flat spacetime. Besides the Virasoro algebra the model contains an $U(1)$ Kac-Moody algebra. We calculate the important commutators and in particular determine the conformal weights and $U(1)$-charges of the fundamental fields from first principles. Also we show that the finite size effects are in general not proportional to the central charge as has been conjectured by Cardy [12]. The appendix A contains our conventions and scaling formulas for the various geometrical objects. In appendix B we collected some useful variational formulas which we have used in this chapter.

5.1 Classical theory

Equations of motion: The field equations of the model (5.1) are

$$\begin{aligned} & i\gamma^\mu(\nabla_\mu - ig_1\partial_\mu\lambda + ig_2\eta_{\mu\nu}\partial^\nu\phi)\ \psi \equiv i\gamma^\mu D_\mu = \ 0 \\ & 2\ \nabla^2\lambda = -g_3\mathcal{R} - g_1\nabla_\mu j^\mu \\ & 2\ \nabla^2\phi = -g_2\nabla_\mu j^{5\mu} \\ & \nabla_\nu F^{\mu\nu} = e\, j^\mu\ , \end{aligned} \tag{5.3}$$

which are the Dirac equation for mass-less charged fermions propagating in a curved space-time and interacting with the scalar and pseudoscalar-fields, Klein Gordon type of equation and Maxwell equation. Here $j^{5\mu}$ is the axial vector current which is defined by

$$j^{5\mu} = \bar{\psi}\gamma^{\mu}\gamma_5\psi = \eta^{\mu}{}_{\nu} j^{\nu} . \tag{5.4}$$

When one decomposes the gauge field as

$$A_\mu = \partial_\mu \alpha - \eta_{\mu\rho}\partial^\rho \varphi \qquad \text{so that} \quad F_{01} = \sqrt{-g}\nabla^2\varphi, \tag{5.5}$$

and chooses isothermal coordinates for which $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, then the generalized Dirac operator reads

$$\begin{aligned} D\!\!\!\!/ = \; & e^{iF - i\gamma_5 G - \frac{3}{2}\sigma} \, \partial\!\!\!/ \, e^{-iF - i\gamma_5 G + \frac{1}{2}\sigma}, \quad \text{where} \\ F = \; & g_1\lambda + e\,\alpha \quad , \quad G = g_2\phi + e\,\varphi \, . \end{aligned} \tag{5.6}$$

Hence, if $\psi_0(x)$ solves the free Dirac equation in flat Minkowski space time, then

$$\psi(x) \equiv e^{iF + i\gamma_5 G - \frac{1}{2}\sigma}\psi_0 \tag{5.7}$$

solves the Dirac equation of the interacting theory on curved spacetime. The vector currents are related as

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \bar{\psi}_0\hat{\gamma}^\mu\psi_0 e^{-2\sigma} \equiv \frac{1}{\sqrt{-g}}\, j_0^\mu ; .$$

The same relation holds for the axial vector current. From $\sqrt{-g}\nabla_\mu j^\mu = \partial_\mu\sqrt{-g}j^\mu$ the conservation of the vector and axial currents follow at once,

$$\nabla_\mu j^\mu = \nabla_\mu j^{5\mu} = 0 \, ,$$

expressing the classical $U(1) \times U_A(1)$ invariance of the model. Using these conservation laws in (5.1) we conclude that

$$2\nabla^2\lambda = -g_3\mathcal{R} \quad \text{and} \quad \nabla^2\phi = 0 \tag{5.8}$$

or that there is no back-reaction from fermions onto scalars. Finally the conservation laws imply that the currents are free fields

$$\nabla^2 j^\mu = \nabla^2 j^{5\mu} = 0 \, , \tag{5.9}$$

which is the reason which accounts for the solubility of the model [33], even in the presence of photons and an external gravitational field. As is well-known, for any gauge invariant regularization the axial current possesses an anomalous divergence in the quantized model and (5.9) is modified. Thus the normal $U_A(1)$ Ward identities in the un-gauged Thirring model [30] become anomalous when the fermions couple to a gauge field.

Solution to the equations of motion: In isothermal coordinates the general solution of the field equations can be expressed in terms of 6 chiral functions as follows: Introducing light cone coordinates $x^{\pm}=x^0\pm x^1$ so that $ds^2=e^{2\sigma}dx^+dx^-$, the solutions of (5.8) read

$$\lambda = g_3\sigma + \lambda_+(x^+) + \lambda_-(x^-), \quad \text{and} \quad \phi = \phi_+(x^+) + \phi_-(x^-) \tag{5.10}$$

and depend on 4 chiral functions which are fixed by the initial data on some space-like hypersurface. The solutions of the free Dirac equations depend on 2 chiral functions as

$$\psi_0 = \begin{pmatrix} \psi_-(x^-) \\ \psi_+(x^+) \end{pmatrix}.$$

In these coordinate system the Maxwell equations (5.3) can easily be integrated and one finds

$$\partial_+\partial_-\phi = F_{01} = 2e^{2\sigma}\Big[\int^{x^-} \psi_-^\dagger(\xi)\psi_-(\xi)d\xi - \int^{x^+} \psi_+^\dagger(\xi)\psi_+(\xi)d\xi\Big]. \tag{5.11}$$

To go further we must fix the gauge. Conveniently one chooses the Lorentz gauge such that $\alpha=0$ in (5.5) and thus ϕ in (5.11) determines A_μ. We see that in isothermal coordinates and this gauge the general solution of (5.3) is given by (5.10), (5.11) and (5.7), that is in terms of 6 chiral functions.

Energy-momentum tensor: Besides the currents the symmetric energy momentum tensor of the matter fields

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta\mathcal{S}}{\delta g_{\mu\nu}} \tag{5.12}$$

plays an important role in any theory in curved space time. Applying the variational identities in Appendix B one obtains after a lengthy but straightforward computation

$$\begin{aligned} T^{\mu\nu} \;=\; & \frac{1}{4}g^{\mu\nu}F^{\sigma\rho}F_{\sigma\rho} - F^{\sigma\nu}F_\sigma{}^\mu + \frac{i}{2}[\bar\psi\gamma^{(\mu}D^{\nu)}\psi - (D^{(\mu}\bar\psi)\gamma^{\nu)}\psi] \\ & +2\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi \quad + \quad (\phi\leftrightarrow\lambda) \\ & -2g_3(g^{\mu\nu}\nabla^2 - \nabla^\mu\nabla^\nu)\lambda \\ & +\frac{1}{2}j^\mu\,(g_1\nabla^\nu\lambda - g_2\eta^{\nu\alpha}\nabla_\alpha\phi) \quad + \quad (\mu\leftrightarrow\nu) \\ & +g_2g^{\mu\nu}j^\alpha\eta_{\alpha\beta}\nabla^\beta\phi - 2g_2j^\alpha\eta_\alpha{}^{(\mu}\nabla^{\nu)}\phi\,, \end{aligned} \tag{5.13}$$

where we have introduced the symmetrization $A^{(\mu}B^{\nu)}=\frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu)$. The first two lines are just the energy momentum of the electromagnetic field, charged fermions and free neutral (pseudo-) scalars. The terms containing second derivatives of λ are the improvement terms [9] which are always present when one couples scalars non-minimally to a background curvature. The remaining terms reflect the interaction between the fermionic and auxiliary fields.

On shell $T^{\mu\nu}$ is conserved as required by general covariance. Using the field equations for ψ and λ its trace reads

$$T^{\mu}_{\mu} = g_3^2 \mathcal{R} - \frac{1}{2} F^{\sigma\rho} F_{\sigma\rho} \; . \tag{5.14}$$

In particular for $g_3 = 0$ and $A_\mu = 0$ it vanishes, and the theory becomes Weyl-invariant. As a consequence it reduces to a conformal field theory in the flat spacetime limit [22]. It is remarkable that it can be made Weyl invariant even when $g_3 \neq 0$. Indeed, without changing the flat spacetime limit we may add a nonlocal Wess-Zumino-type term to the action, namely

$$S \longrightarrow S' = S - \frac{g_3^2}{4} S_p \quad \text{where} \quad S_p = \int \sqrt{-g} \mathcal{R} \frac{1}{\nabla^2} \mathcal{R}$$

the variation of which is

$$\begin{aligned} \delta S_p = \int \Big\{ 4[g^{\mu\nu} \mathcal{R} - \nabla^\mu \nabla^\nu \frac{1}{\nabla^2} \mathcal{R}] + 2 \nabla^\mu \left(\frac{1}{\nabla^2} \mathcal{R} \right) \nabla^\nu \left(\frac{1}{\nabla^2} \mathcal{R} \right) \\ - g^{\mu\nu} \nabla_\alpha \left(\frac{1}{\nabla^2} \mathcal{R} \right) \nabla^\alpha \left(\frac{1}{\nabla^2} \mathcal{R} \right) \Big\} \sqrt{-g} \delta g_{\mu\nu} \end{aligned} \tag{5.15}$$

The trace of the modified energy momentum tensor is now zero, and for $g_{\mu\nu} \to \eta_{\mu\nu}$ the Lagrangian corresponds to a conformal field theory in Minkowski spacetime.

Choosing the coupling constants appropriately, the model reduces to various well known exactly solvable models:

- For $g_3 = 0$ and $g_1^2 = -g_2^2 = g^2$ the fermionic sector reduces to the gauged version of the *Thirring model* [47] in curved space time. To see that we solve the Klein Gordon equations in (5.3) for the $U(1)$ current which yields

  $$j_\mu = -\frac{2}{g_1} \partial_\mu \lambda - \frac{2}{g_2} \eta_\mu{}^\nu \partial_\nu \phi.$$

  Inserting this into the Dirac equation we find

  $$i\gamma^\mu \nabla_\mu \psi - \frac{g^2}{2} j^\mu \gamma_\mu \psi = 0 \; ,$$

which is the field equation of the gauged Thirring model in curved spacetime with Lagrangian

$$\boxed{\mathcal{L}_{\text{Thir}}[A_\mu,\bar{\psi},\psi] = \bar{\psi} i\gamma^\mu \nabla_\mu \psi - \frac{g^2}{4} j^\mu j_\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\ .} \tag{5.16}$$

If we further specialize to $g = 0$ we recover the *Schwinger model in curved spacetime* [29].

- For the special choice $g_1 = g_2 = e = 0$ and for vanishing gauge field the λ- dependent part of (5.1) is just the Lagrangian of scalar fields coupled to a background charge and and for imaginary g_3 describes the *minimal models* of conformal field theory [4].

Hamiltonian formalism and classical conformal structure: In this subsection we investigate the Hamiltonian structure of the model (5.1) in the conformal limit, i.e. in flat Minkowski space and for vanishing gauge field. In the presence of both fermions and bosons it is convenient to exploit the graded Poisson structure [11]. We recall, that the equal time Poisson brackets are

$$\{A(x), B(y)\} \equiv \sum_O \int dz^1 \Big(\frac{A(x)\overleftarrow{\delta}}{\delta O(z)} \frac{\overrightarrow{\delta} B(y)}{\delta \pi_O(z)} \mp \frac{A(x)\overleftarrow{\delta}}{\delta \pi_O(z)} \frac{\overrightarrow{\delta} B(y)}{\delta O(z)} \Big)\Big|_{x^0=y^0}\ .$$

The sum is over all fundamental fields $O(x)$ in the theory . The sign is minus if one or both of the fields A and B are bosonic (even) and it is plus if both are fermionic (odd) fields. The momentum densities $\pi_O(x)$ conjugate to the O-fields are given by functional left-derivatives

$$\pi_O(x) = \frac{\overrightarrow{\delta} S}{\delta \partial_0 O(x)}.$$

A simple calculation yields the following momenta

$$\pi_\psi = -i\psi^\dagger, \quad \pi_\phi = g_2 j_0^5 + 2\partial_0\phi \quad \text{and} \quad \pi_\lambda = g_1 j_0 + 2\partial_0 \lambda$$

which form the fundamental Poisson brackets with the fields

$$\begin{aligned} \{\psi^\dagger_\alpha(x), \psi_\beta(y)\} &= i\delta_{\alpha\beta}\delta(x^1 - y^1), \\ \{\lambda(x), \pi_\lambda(y)\} &= \delta(x^1 - y^1), \\ \{\phi(x), \pi_\phi(y)\} &= \delta(x^1 - y^1). \end{aligned} \tag{5.17}$$

For the Hamiltonian we obtain

$$
\begin{aligned}
H &= \int dx^1 \Big[\partial_0\psi\pi_\psi + \partial_0\lambda\pi_\lambda + \partial_0\phi\pi_\phi - \mathcal{L}\Big] \\
&= \int dx^1 \Big[\pi_\psi\gamma_5\partial_1\psi - ig_1\partial_1\lambda\pi_\psi\gamma_5\psi - ig_2\partial_1\phi\pi_\psi\psi + (\partial_1\lambda)^2 \\
&\qquad +(\partial_1\phi)^2 + \frac{1}{4}(\pi_\lambda - ig_1\pi_\psi\psi)^2 + \frac{1}{4}(\pi_\phi - ig_2\pi_\psi\gamma_5\psi)^2\Big].
\end{aligned} \tag{5.18}
$$

It can be checked that the corresponding Hamiltonian equations are just the field equations (5.3) with flat metric and vanishing gauge potential, as required. Since $T^\mu{}_\mu = 0$ (see 5.14) the only non-zero components of $T^{\mu\nu}$ are the light-cone components T_{++} and T_{--}. To continue it is convenient to introduce adapted light cone coordinates $x^\pm = x^0 \pm x^1$ and the chiral components of the Dirac spinor $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi$. Then T_{--} in (5.13) simplifies to

$$
\begin{aligned}
T_{--} &= -\frac{1}{2}(\pi_{\psi_+}\partial_-\psi_+ - \partial_-\pi_{\psi_+}\psi_+) + 2(\partial_-\lambda)^2 + 2(\partial_-\phi)^2 \\
&\qquad +g_3\partial_-^2\lambda + i\partial_-(g_1\lambda + g_2\phi)\pi_{\psi_+}\psi_+.
\end{aligned} \tag{5.19}
$$

Using the equations of motion one shows explicitly that it is a chiral field, i.e. depends only on x^-. With (5.19) we can now find the conformal weights of the fundamental fields which determine their transformations under infinitesimal conformal symmetry transformations. For that we must calculate the commutator of the symmetry generators $T_f = \int dx^- f(x^-)T_{--}$ with the fields. The result is

$$
\begin{aligned}
\delta_f\phi &= \{\phi, T_f\} \equiv f\partial_-\phi \\
\delta_f\lambda &= f\partial_-\lambda - \frac{g_3}{2}\partial_- f \\
\delta_f\psi_+ &= f\partial_-\psi_+ + \frac{1}{2}(1 - ig_1g_2)\psi_+\partial_- f \\
\delta_f\psi_+^\dagger &= f\partial_-\psi_+^\dagger + \frac{1}{2}(1 + ig_1g_2)\psi_+^\dagger\partial_- f.
\end{aligned} \tag{5.20}
$$

Whereas ϕ and ψ_+ are primary fields, λ is not. Actually, the non-primary character of λ is very much linked with the g_3-dependent term in the transformation of the Dirac field. To see that more clearly we note that under an infinitesimal left conformal transformation generated by $\bar{T}_f = \int dx^+ f(x^+)T_{++}$ the scalar and fermi field transform as

$$
\bar{\delta}_f\lambda = f\partial_-\lambda - \frac{g_3}{2}\partial_- f \quad \text{and} \quad \bar{\delta}\psi_+ = f\partial_-\psi_+ - ig_1g_2\psi_+\partial_- f.
$$

Since ψ_+ is not any longer a scalar under left transformation the term

$$
\int dx^+ dx^- \Big(2i\psi_+^\dagger(\partial_+ - ig_1\partial_+\lambda)\psi_+\Big)
$$

appearing in the action is only conformally invariant because λ transforms inhomogenously like a spin connection. It maybe surprising that the symmetry transformations depend on the coupling constant g_3 which is not present in the flat space time Lagrangian. Actually, the same happened for the gauged WZNW models considered in the previous. Indeed, the g_3-dependent term in the energy momentum tensor (5.19) contains second derivatives of the field λ and is the analog of the improvement term Tr HJ' in (4.59) in the constrained WZNW theory.

The current transforms as

$$\delta_f j_= f\partial_- j_- + j_- \partial_- f \tag{5.21}$$

and the energy momentum tensor as

$$\delta_f T_{--} = f\partial_- T_{--} + 2T_{--}\partial_- f - g_3^2 \partial_-^3 f. \tag{5.22}$$

Recalling that a primary field O with weight h transforms as

$$\delta_f O = \{O, T_f\} = f\partial_- O + hO\partial_- f$$

and comparing with the above results we have found the following structure:

- The pseudoscalar field ϕ is primary with $h_\phi = 0$. The scalar field λ is only primary for $g_3 = 0$ in which case $h_\lambda = 0$.
- The Dirac field ψ_+ is primary with $h_{\psi_+} = \frac{1}{2}(1 - ig_1 g_3)$. The conformal weight is real for imaginary g_3.
- The current is primary with weight 1.
- Already on the classical level the energy momentum tensor is only quasi-primary. The corresponding Virasoro algebra (5.22) has central charge $c = 24\pi g_3^2$.

In the following sections we are lead to consider the *Euclidean version* of the model. Then one must replace the Lorentzian $\gamma^\mu, g_{\mu\nu}$ and ω_μ by there Euclidean counterparts. For example, with our conventions (see appendix A) the relation (5.4) becomes

$$j^{5\mu} = -i\eta^\mu{}_\nu j^\nu$$

and as a consequence the generalized Dirac operator in Euclidean spacetime becomes

$$\not{D} = e^{iF + \gamma_5 G - \frac{3}{2}\sigma} \not{\partial}\, e^{-iF + \gamma_5 G + \frac{1}{2}\sigma}$$

instead of (5.6). Also, to recover the Euclidean Thirring model as particular limit of (5.1) we must set $g_3 = 0$ and $g_1^2 = g_2^2 = g^2$.

5.2 Quantization of the generalized gauged Thirring model

In this section we quantize the general model (5.1) in curved space-times. The results are then applied in the following sections, where we calculate the *partition function, ground state energy, equation of state* and certain *correlators* of interest and their dependence on the chemical potential, volume of space, temperature and background metric. To do that we couple the conserved U(1)-charge to a chemical potential μ. We enclose the system in a box with length L to avoid infrared divergences. To investigate the temperature dependence the time is taken to be purely imaginary in the functional approach [19]. The imaginary time x^0 varies then from zero to the inverse temperature β and we must impose periodic- and anti-periodic boundary conditions for the bosonic- and fermionic fields, respectively. Thus to study the finite temperature model we must assume that space-time is an Euclidean torus $[0,\beta]\times[0,L]$.

To see how the partition function and correlators depend on the gravitational field we assume that the torus is equipped with an arbitrary metric with Euclidean signature or equivalently with a 2-bein $e_{\mu a}$. The curved gamma matrices are $\gamma_\mu = e_{\mu a}\hat\gamma^a$ and in particular $\gamma_5 = -\frac{i}{2}\eta_{\mu\nu}\gamma^\mu\gamma^\nu = \sigma_3$ is constant (see appendix A for our conventions). We can always choose (quasi) isothermal coordinates and a Lorentz frame such that

$$\begin{aligned} e_{\mu a} &= e^\sigma\,\hat e_{\mu a} \equiv e^\sigma \begin{pmatrix} \tau_0 & \tau_1 \\ 0 & 1 \end{pmatrix} \\ g_{\mu\nu} &= e^{2\sigma}\,\hat g_{\mu\nu} \equiv e^{2\sigma}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \end{aligned} \tag{5.23}$$

where $\tau = \tau_1 + i\tau_0$ is the Teichmueller parameter and σ the gravitational Liouville field. Space-time is then a square of length L and has volume$V = \int_0^L d^2x\sqrt{g}$. We allow for the general twisted boundary conditions for the fermions

$$\begin{aligned} \psi(x^0+L,x^1) &= -e^{2\pi i(\alpha_0+\beta_0\gamma_5)}\psi(x^0,x^1) \\ \psi(x^0,x^1+L) &= -e^{2\pi i(\alpha_1+\beta_1\gamma_5)}\psi(x^0,x^1). \end{aligned} \tag{5.24}$$

The parameters α_i and β_i represent vectorial and chiral twists, respectively. We could allow for twisted boundary conditions for the (pseudo) scalars as well, e.g. $\phi(x^0+nL,x^1+mL) = \phi(x^1,x^0) + 2\pi(m+n)$. However, to recover the Thirring model for certain values of the couplings we assume that these fields are periodic. For $\sigma = 0$, $\tau = i\beta/L$ and $\alpha_0 = \beta_0 = 0$ the partition function has then the usual thermodynamical interpretation. Its logarithm is proportional to the free energy at temperature $T=1/\beta$.

Fermionic path integral: Twisted boundary conditions as in (5.24) require some care in the fermionic path integral. Indeed the fermionic determinant is not uniquely defined when one allows for such twists. The ambiguities are not related to the unavoidable ultra-violet divergences but to the transition from Minkowski- to Euclidean space-time. To see that more clearly let $\mathcal{S}^{\pm}$ denote the set of fermionic fields in *Minkowski space-time* with chirality ± 1. Since both the commutation relations and the action do not connect $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ we can consistently impose different boundary conditions on $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$. On the other hand, in the *Euclidean path-integral* for the generating functional

$$Z_F[\eta,\bar{\eta}] = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{\int \sqrt{g}\,\psi^\dagger i \not{D}\psi + \int \sqrt{g}\,(\bar{\eta}\psi + \psi^\dagger \eta)}, \tag{5.25}$$

the Dirac operator

$$\not{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

exchanges the two chiral components of ψ, i.e. $\not{D} : \mathcal{S}^{\pm} \to \mathcal{S}^{\mp}$. Thus, in contrast to the situation in Minkowski space the two chiral sectors are related in the action. Of course, the eigenvalue problem for $i\not{D}$ is then not well defined. This is the origin of the ambiguity in the definition of the determinant. It is related to the ambiguities one encounters when one quantizes chiral fermions [2]. To solve this problem we shall analytically continue the well-defined determinants in the untwisted sector $\beta = 0$ to $\beta \neq 0$. The resulting determinants do not factorize into (anti-) holomorphic pieces and differ from previous ones in the literature [21].

Let us now study the generating functional for fermions in an external gravitational and gauge field and coupled to the auxiliary fields. For that we observe that on the torus the decomposition (5.4) of the gauge potential generalizes to

$$A_\mu = A^I_\mu + \frac{2\pi}{L} t_\mu + \partial_\mu \alpha - \eta_{\mu\nu}\partial^\nu \varphi, \tag{5.26}$$

where the last 3 terms are recognized as Hodge decomposition of the single valued part of A in a given topological sector, that is the harmonic-, exact- and co-exact pieces. In arbitrary coordinates the toron field t_μ obeys the harmoniticity conditions $\nabla_\mu t^\mu = t_{[\mu,\nu]} = 0$. It follows then that in isothermal coordinates t_μ must be constant. The role of the toron fields has recently been emphasized within the canonical approach [34]. In the Hamiltonian formulation they are quantum mechanical degrees of freedom which are needed for an understanding of the infrared sector in gauge theories. Also, in [45] it has been demonstrated that the Z_N-phases of hot pure Yang-Mills theories

[46] should correspond to the same physical state if one takes care of the toron fields.

The first term in (5.26) is an *instanton potential* which gives rise to a non-vanishing quantized flux Φ or integer-valued instanton number k:

$$\Phi = e\int F_{01} \equiv e\int E = e\int E^I = 2\pi\, k.$$

As representative in the k-instanton sector we choose the, up to gauge transformations, *unique absolute minimum* of the Maxwell action in (5.1). It has field strength $e\,E^I = \sqrt{g}\,\Phi/V$. As instanton potential we choose

$$eA^I_\mu = e\hat{A}^I_\mu - \Phi\,\eta_\mu{}^\nu\partial_\nu\chi, \quad \text{where} \quad e\hat{A}^I = -\frac{\sqrt{\hat{g}}}{\hat{V}}(x^1,0) \tag{5.27}$$

is the instanton potential on the flat torus with the same flux but field strength $\sqrt{\hat{g}}\,\Phi/\hat{V}$. The function χ is then determined (up to a constant) by

$$\sqrt{g}\frac{\Phi}{V} - \sqrt{\hat{g}}\frac{\Phi}{\hat{V}} = \sqrt{g}\triangle\chi. \tag{5.28}$$

The solution of this equation is given by

$$\chi(x) = -\frac{1}{\hat{V}}(\frac{1}{\triangle}e^{-2\sigma})(x) = \frac{1}{\hat{V}}\int d^2y\sqrt{g(y)}G_0(x,y)\,e^{-2\sigma(y)}, \tag{5.29}$$

where

$$G_0(x,y) = \langle x|\frac{1}{-\triangle}|y\rangle = \sum_{\lambda_n>0}\frac{\phi_n(x)\phi^\dagger(y)}{\lambda_n} \tag{5.30}$$

is the Green-function belonging to $-\triangle$. In deriving (5.29) we have used that $\frac{1}{\triangle}(\Phi/V)=0$ which follows from the spectral resolution (5.30) for the Green function in which the constant zero mode $\phi_0=1/\sqrt{V}$ of $\triangle$ is missing.

Note that 2-dimensional gauge theories are not scale or Weyl invariant as 4-dimensional ones are. For that reason the instantons on conformally flat spacetimes are not just the 'flat' instantons.

To be more explicit we relate G_0 to the Green-function $\hat{G}_0$ on the flat torus with the hatted metric [28]

$$\hat{G}_0(x,y) = -\frac{1}{4\pi}\log|\frac{1}{\eta(\tau)}\Theta\begin{bmatrix}\frac{1}{2}+\frac{\xi^0}{L}\\ \frac{1}{2}+\frac{\xi^1}{L}\end{bmatrix}(0,\tau)|^2, \quad \text{where} \quad \xi = x-y. \tag{5.31}$$

For that we note that due to the missing zero-mode in (5.30) the usual flat spacetime equations for the Green-functions are modified to

$$-\triangle_x G_0(x,y)=\frac{\delta(x-y)}{\sqrt{g}}-\frac{1}{V}\quad,\quad -\hat{\triangle}_x\hat{G}_0(x,y)=\frac{\delta(x-y)}{\sqrt{\hat{g}}}-\frac{1}{\hat{V}}.$$

Furthermore one sees at once that both Green functions annihilate the corresponding constant zero modes

$$\int d^2y\sqrt{g(y)}G_0(x,y)=\int d^2y\sqrt{\hat{g}}\hat{G}_0(x,y)=0. \tag{5.32}$$

From these two equations one concludes that Green-function on the curved torus is related to the flat one (5.31) as

$$\begin{aligned} G_0(x,y) &= \hat{G}_0(x,y)+\frac{1}{V^2}\int d^2u d^2v\sqrt{g(u)g(v)}\hat{G}_0(u,v)\\ &- \frac{1}{V}\int \hat{G}_0(x,u)\sqrt{g(u)}d^2u-\frac{1}{V}\int d^2u\sqrt{g(u)}\hat{G}_0(u,y) \end{aligned} \tag{5.33}$$

and this replaces the infinite space relations $G_0=\hat{G}_0$ [6].

Our choice for the instanton potential (5.26,5.27) corresponds to a particular trivialization of the $U(1)$-bundle over the torus [41]. In other words, the gauge potentials and fermion fields at (x^0,x^1) and (x^0,x^1+L) are necessarily related by a *nontrivial gauge transformation* with windings

$$\begin{aligned} &A_\mu(x^0,x^1+L)-A_\mu(x^0,x^1)=\partial_\mu\alpha(x)\\ &\psi(x^0,x^1+L)=-e^{ie\alpha(x)}\,e^{2\pi i(\alpha_1+\beta_1\gamma_5)}\,\psi(x^0,x^1). \end{aligned} \tag{5.34}$$

For the choice (5.27) we find

$$e\alpha(x)=-\frac{\Phi}{L}x^0.$$

Note that A is still periodic in x^0 with period L and ψ still obeys the first boundary condition in (5.24). Our trivialization differs from the one chosen in [31] and so do our instantons and fermionic zero modes.

Similarly as for the gauge potential we must add a harmonic piece to the auxiliary vector field B_μ to which the fermions couple in (5.1), so that

$$B_\mu=\frac{2\pi}{L}g_0h_\mu+g_1\partial_\mu\lambda-g_2\eta_{\mu\nu}\partial^\nu\phi \tag{5.35}$$

appears in the Dirac operator in (5.1) on the torus. λ and ϕ couple to the divergence of the vector and axial vector currents. The harmonic fields h_μ couple to the harmonic part of the current and are needed to recover the Thirring model in the limit $g_0^2=g_1^2=g_2^2$. Also, we shall see that t_μ

and h_μ are essential to obtain the correct answer for the thermodynamic potential. Note that B_μ contains no instanton part since it couples to the gauge invariant fermionic current.

Finally we introduce a *chemical potential* for the conserved $U(1)$ charge. In the Euclidean functional approach this is equivalent to coupling the fermions to a constant imaginary gauge potential A_0 [1].

Inserting the above decompositions and the chemical potential into the Dirac operator finally yields in isothermal coordinates

$$\begin{aligned} \not{D} &= \gamma^\nu D_\nu = e^{iF+\gamma_5(G+\Phi\chi)-\frac{3}{2}\sigma}\,\hat{\not{D}}\,e^{-iF+\gamma_5(G+\Phi\chi)+\frac{1}{2}\sigma}, \quad \text{where} \\ &\qquad \hat{\not{D}} = \gamma^\mu(\partial_\mu + i\hat{\omega}_\mu - ie\hat{A}^I_\mu - \frac{2\pi i}{L}[H_\mu + \mu_\mu]), \\ &\qquad H_\mu = e\,t_\mu + g_0 h_\mu \quad \text{and} \quad \mu_\mu = -i\frac{\tau_0 L}{2\pi}\mu\,\delta_{\mu 0}. \end{aligned} \tag{5.36}$$

Here $\hat{\omega}$ is the spin connection belonging to $\hat{e}_{\mu a}$. It vanishes for our choice of the reference zweibein. $\hat{A}^I$ is the instanton potential (5.27) on the flat torus. The scalar and pseudo scalar functions F, G and χ have been introduced in (5.6,5.29). In the chosen coordinates t and h and hence H are all constant. In [41] it has been shown that $\not{D}$ possesses $|k|$ zero-modes of definite chirality and their chirality is given by the sign of k. They are crucial in any correct quantization. For example, if one would leave out instanton sectors in which $i\not{D}$ has zero-modes then the cluster property would be violated.

In a first step we quantize the fermions in the flat instanton and harmonic background and reference metric $\hat{g}_{\mu\nu}$, that is we assume $\not{D} \to \hat{\not{D}}$ in (5.25). The dependence on the remaining fields F, G, χ and σ, that is the relation between Z_F and $\hat{Z}_F$, is then found by integrating the chiral and trace anomalies [7] and exploiting the relation (5.36) between $\not{D}$ and $\hat{\not{D}}$.

We expand the fermionic field in a orthonormal basis of the Hilbert space

$$\begin{aligned} \psi(x) &= \sum_n a_n\psi_{n+}(x) + \sum_n b_n\psi_{n-}(x) \\ \psi^\dagger(x) &= \sum_n \bar{a}_n\chi^\dagger_{n+}(x) + \sum_n \bar{b}_n\chi^\dagger_{n-}(x), \end{aligned} \tag{5.37}$$

where $a_n, b_n, \bar{a}_n, \bar{b}_n$ are independent Grassmann variables.

Topologically trivial sector: For $k = 0$ or vanishing instanton potential we can immediately write down a basis

$$\psi_{n\pm}(x) = \frac{1}{\sqrt{V}}\,e^{i(p^\pm_n,x)}\,e_\pm, \quad (p^\pm_n)_i = \frac{2\pi}{L}(\frac{1}{2} + \alpha_i \pm \beta_i + n_i), \tag{5.38}$$

and $e_\pm$ are the eigenvectors of γ_5. The ψ_{n+} and ψ_{n-} must obey the $\mathcal{S}^+$ and $\mathcal{S}^-$ boundary conditions, respectively. These boundary conditions fix the admissible momenta $p_n^\pm$ in (5.38). Since the Dirac operator maps $\mathcal{S}^\pm$ into $\mathcal{S}^\mp$ the $\chi_{n\pm}$ must then obey the same boundary conditions as the $\psi_{n\mp}$. Thus $\chi_{n\pm}(x)$ is obtained from $\psi_{n\pm}(x)$ by exchanging p_n^+ and p_n^-. It follows then that

$$i\hat{\not{D}}\psi_{n\pm} = \lambda_n^\pm \chi_{n\mp} \tag{5.39}$$

with

$$\begin{aligned} \lambda_n^+ &= \frac{2\pi}{\tau_0 L}[\bar{\tau}(\frac{1}{2}+a_1+\beta_1+n_1)-(\frac{1}{2}+a_0+\beta_0+n_0)] \\ \lambda_n^- &= \frac{2\pi}{\tau_0 L}[\tau(\frac{1}{2}+a_1-\beta_1+n_1)-(\frac{1}{2}+a_0-\beta_0+n_0)]. \end{aligned} \tag{5.40}$$

Here we have introduced $a_\mu \equiv \alpha_\mu - H_\mu - \mu_\mu$. Substituting (5.37,5.39,5.40) into the generating functional (5.25) and applying the standard Grassmann integration rules we arrive at

$$\begin{aligned} \hat{Z}_F[\eta,\bar{\eta}] &= \det i\hat{\not{D}}\; e^{-\int \bar{\eta}(x)\hat{S}(x,y)\eta(y)}, \quad \det i\hat{\not{D}} = \prod_n \lambda_n^+\lambda_n^-, \\ \hat{S}(x,y) &= \sum_n (\frac{\psi_{n+}(x)\chi_{n-}^\dagger(y)}{\lambda_n^+} + \frac{\psi_{n-}(x)\chi_{n+}^\dagger(y)}{\lambda_n^-}). \end{aligned} \tag{5.41}$$

$\hat{S}$ is the fermionic Green function in the 0-instanton sector. Note that both the 'eigenvalues' and the Green function depend on the Teichmueller parameter, harmonic potentials, twists and chemical potential.

We proceed to calculate the infinite product or generalized determinant in (5.41). This is one of the central points of this section and for non-zero chiral twists and chemical potential our result deviates from previous ones [21]. Actually the twists and chemical potential are related as one can see from (5.39,5.40).

One may be tempted so identify

$$\det(D_+D_-) \sim \prod \lambda_n^+\lambda_n^- \quad \text{and} \quad \det D_+ \det D_- \sim \prod \lambda_n^+ \prod \lambda_m^- \tag{5.42}$$

and thus conclude that the determinant is a product, $f(\tau)\bar{f}(\tau)$, that is factorizes into holomorphic and anti-holomorphic pieces (the overall factor $\sim 1/\tau_0 L$ in the eigenvalues (5.40)) drops in the infinite product, since the torus has vanishing Euler number). However, the infinite product in (5.41) must be regularized and the two expressions in (5.42) may differ. In conformal field theory [28] one is naturally lead to consider the individual chiral sectors and thus finds holomorphic factorization. For Dirac fermions one uses $\not{D}^2$ to regularize the product and this leads to the determinant of the product D_+D_-.

To continue we recast the infinite product in the form

$$\prod^{\infty}\lambda_n^+\lambda_n^- = \prod_{\vec{n}\in Z^2}\Big(\frac{2\pi}{L}\Big)^2\hat{g}^{\mu\nu}(\frac{1}{2}+c_\mu+n_\mu)(\frac{1}{2}+c_\nu+n_\nu)$$

where $\hat{g}^{\mu\nu}$ is the inverse of the reference metric (5.23) and

$$c_\mu = a_\mu + i\hat{\eta}_\mu{}^\nu\beta_\nu, \quad \text{where} \quad (\hat{\eta}_\mu{}^\nu) = -\frac{1}{\tau_0}\begin{pmatrix}\tau_1 & -|\tau|^2 \\ 1 & -\tau_1\end{pmatrix}. \tag{5.43}$$

The point is that for real c_μ, that is for vanishing chiral twists β_μ and chemical potential (see the definitions of a_μ below (5.40) and μ_μ in (5.36)) the zeta function defined by

$$\zeta(s) = \sum_n (\lambda_n^+\lambda_n^-)^{-s} \tag{5.44}$$

has a well defined analytic continuation to $s<1$ via a Poisson resummation. An explicit calculation yields [41, 48, 8]

$$\begin{aligned}\det(i\hat{\not{D}}) &\equiv (\prod_n \lambda_n^+\lambda_n^-)_{reg} = e^{-\zeta'(s)|_{s=0}}, \quad \text{where} \\ \zeta'(s)|_{s=0} &= -\log\Big[\frac{1}{|\eta(\tau)|^2}\Theta\Big[\begin{matrix}-c_1\\ c_0\end{matrix}\Big](0,\tau)\ \bar{\Theta}\Big[\begin{matrix}-c_1\\ c_0\end{matrix}\Big](0,\tau)\Big].\end{aligned} \tag{5.45}$$

However, for complex c_μ the Poisson resummation is not applicable and $\zeta'(0)$ cannot be calculated by direct means. To circumvent these difficulties we note that the infinite product (5.44) defining the ζ-function for $s>1$ is a meromorphic function in c. Thus we may first continue to $s<1$ for real c_μ and then continue the result to complex values. Using the transformation properties of theta functions the resulting determinant can be written as

$$\begin{aligned}\det(i\hat{\not{D}}) &= e^{2\pi(\sqrt{\hat{g}}\hat{g}^{\mu\nu}\beta_\mu\beta_\nu - 2i\beta_1 a_0)} \\ &\cdot \frac{1}{|\eta(\tau)|^2}\Theta\Big[\begin{matrix}-a_1+\beta_1\\ a_0-\beta_0\end{matrix}\Big](0,\tau)\bar{\Theta}\Big[\begin{matrix}-\bar{a}_1-\beta_1\\ \bar{a}_0+\beta_0\end{matrix}\Big](0,\tau).\end{aligned} \tag{5.46}$$

It can be shown that this determinant is *gauge invariant*, i.e. invariant under $\alpha_\mu \to \alpha_\mu{+}1$, but not invariant under chiral transformations, $\beta_\mu \to \beta_\mu{+}1$, as expected. Furthermore it transforms covariantly under modular transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$. In other words, $\det i\hat{\not{D}}$ is invariant under modular transformations if at the same time the boundary conditions are transformed accordingly. The exponential prefactor is needed for modular covariance and is not present in the literature [21]. It correlates the two chiral sectors and will have important consequences.

Topologically nontrivial sectors: Before deriving chirality violating amplitudes one comment is in order. Due to the integrated Gauss law the expectation value of the electric charge must vanish in the fully quantized theory, although it may be nonzero in the intermediate step where one treats the gauge field as external field. This then implies that the partition function and expectation values must be independent of the chemical potential coupled to the electric charge. For example, if the partition function would depend on μ then the expectation value of the charge would not vanish as can easily be seen by differentiating the effective action with respect to μ. Now we note that a chiral twists is equivalent to a chemical potential, and a non-chiral twist to a harmonic gauge potential. Thus we conclude that the partition function can not depend on the twists. This can be checked by explicit calculation. For example, the normal twists are wiped out by the toron integration. Thus we shall set the twists to zero for the *gauged model* so that we have the same boundary conditions in the left and right handed sectors and the Dirac operator becomes selfadjoint. In particular we may use eigenfunctions of the Dirac operator to perform the path integral. The twist will only be relevant for the un-gauged model which we considered later.

Let us now, for definiteness, assume that the instanton number is positive, $k>0$. Then $i\hat{\not{D}}$ possesses k zero-modes $\hat{\psi}_0^p$, $p=1,\dots,k$ with positive chirality. They must be included in an expansion of ψ in (5.37). The Grassmann integral over the variables belonging to the excited modes is performed as in the trivial sector. Also, the integration over the Grassmann variables accompanying the zero-modes can easily be done (see [41] for a careful discussion) and one obtains

$$\begin{aligned}\hat{Z}_F[\eta,\bar{\eta}] &= \prod_{p=1}^{|k|}(\bar{\eta},\hat{\psi}_0^p)(\hat{\psi}_0^p,\eta)\det{}'i\hat{\not{D}}e^{-\int\bar{\eta}(x)\hat{S}_e(x,y)\eta(y)},\\ \det{}'i\hat{\not{D}} &= \prod_{\lambda_n\neq 0}\lambda_n,\qquad \hat{S}_e(x,y)=\sum_{\lambda_n\neq 0}\frac{\psi_n(x)\psi_n^\dagger(y)}{\lambda_n}.\end{aligned}\tag{5.47}$$

Note that the excited Green function S_e anticommutes with γ_5.

To calculate the determinant we observe that

$$\not{D}^2=\begin{pmatrix}D_-D_+ & 0\\ 0 & D_+D_-\end{pmatrix}=\frac{1}{\sqrt{g}}D_\mu\sqrt{g}g^{\mu\nu}D_\nu-\frac{1}{4}\mathcal{R}+\frac{e}{2}\eta^{\mu\nu}F_{\mu\nu}\gamma_5$$

simplifies in the instanton background $\hat{A}^I$ and on the flat torus to

$$-\hat{\not{D}}^2=-\hat{g}^{\mu\nu}\hat{D}_\mu\hat{D}_\nu-\frac{\Phi}{\hat{V}}\gamma_5.\tag{5.48}$$

In other words, it is the same in the two chiral sectors, up to the constant

$2\Phi/\hat{V}$. This observation allows one to reconstruct the spectrum of $-\hat{\not{D}}^2$ completely. For that we observe that the excited eigenmodes of the Dirac operator come in pairs with opposite eigenvalues, since γ_5 anticommutes with $\not{D}$. Since γ_5 commutes with the squared Dirac operator the chiral projections $P_\pm\psi_n$ of these modes are eigenmodes of $\hat{\not{D}}^2$. Thus the excited modes of the squared Dirac operator come also in pairs and two partners have the same energies but opposite chiralities. Earlier we have seen that there are exactly k-zero modes with chirality $+1$ (we assumed $k>0$). Because of (5.48) they are at the same time excited modes of $-\hat{\not{D}}^2$ with energy $2\Phi/\hat{V}$ and chirality -1. Due to the pairing there are then k excited modes with the same energies $2\Phi/\hat{V}$ but chirality $+1$. This procedure may now be iterated and one ends up with the following spectrum of $-\hat{\not{D}}^2$:

$$\lambda_n^2=\begin{cases}0 & \text{degeneracy}=k\\ 2n\Phi/\hat{V} & \text{degeneracy}=2k.\end{cases}$$

With the explicit spectrum at hand we can compute the zero-mode truncated determinant with zeta-function methods and find [41]

$$\det{}'(i\hat{\not{D}})=\Big(\frac{\pi\hat{V}}{\Phi}\Big)^{\Phi/4\pi}.$$

We proceed with computing the *zero modes* of $\hat{\not{D}}^2$. For that we note that the operator commutes with the time translations which leads to the ansatz

$$\tilde{\chi}_p=e^{2\pi i c_p x^0/L}\,e^{2\pi i H_1 x^1/L}\,\xi_p(x^1)\,e_+,\quad c_p=\frac{1}{2}+p,$$

where we have assumed $k>0$. The choice of c_p is dictated by the time-like boundary conditions in (5.24). Inserting this ansatz into the zero mode equation $\tilde{\not{D}}^2\tilde{\chi}_p=0$ yields

$$\begin{aligned}&(|\tau|^2\frac{d^2}{dy^2}-\frac{\Phi^2}{L^4}y^2-2i\tau_1\frac{\Phi}{L^2}y\frac{d}{dy}-i\tau\frac{\Phi}{L^2})\xi_p=0,\\ &\text{where}\quad y=x^1+\frac{L}{k}(c_p-H_0).\end{aligned}\tag{5.49}$$

This is just the differential equation for the ground state of a generalized harmonic oscillator to which it reduces for $\tau=i\tau_0$. The solution is given by

$$\xi_p=\exp\Big[-\frac{\Phi}{2i\bar{\tau}L^2}\{x^1+\frac{L}{k}(c_p-H_0)\}^2\Big].$$

These functions do not obey the boundary condition (5.34), but the correct eigenmodes can be constructed as superpositions of them. For that we observe that

$$\tilde{\chi}_p(x^0,x^1+L)=e^{-i\Phi x^0/\beta}\,e^{2i\pi H_1}\,\tilde{\chi}_{p+k}(x^0,x^1)$$

so that the sums

$$\hat{\psi}_0^p \;=\; \frac{(2k\tau_0)^{\frac{1}{4}}}{\sqrt{|\tau|\hat{V}}} \cdot \sum_{n\in Z} e^{-2i\pi(n+p/k)(\frac{1}{2}-H_1)}\tilde{\chi}_{p+nk}\, e_+, \tag{5.50}$$

where $p=1,\ldots,k$, obey the boundary conditions and thus are the k required zero-modes. The overall factor normalizes these functions to one. Modes with different p are orthogonal to each other, so that the system (5.50) forms an orthonormal basis of the zero-mode subspace. For $k<0$ the zero-modes are the same if one replaces e_+ by e_-.

Integrating the chiral and trace anomalies: To relate the determinants of $i\hat{\not{D}}$ and $i\not{D}$ we introduce the one-parameter family of Dirac operators

$$\not{D}_\tau = e^{\tau[iF+\gamma_5(G+\Phi\chi)-\frac{3}{2}\sigma]}\,\hat{\not{D}}\,e^{\tau[-iF+\gamma_5(G+\Phi\chi)+\frac{1}{2}\sigma]} \tag{5.51}$$

which interpolates between $\hat{\not{D}}$ and $\not{D}$ [39]. The τ-derivative of the corresponding determinants is determined by the chiral and trace anomaly. An explicit calculation yields

$$\log\frac{\det' i\not{D}}{\det' i\hat{\not{D}}} = \int\limits_0^1 \frac{d\tau}{4\pi}\int\sqrt{g^\tau}\,\mathrm{tr}\, a_1^\tau\Big(2\gamma_5[G+\Phi\chi]-\sigma\Big) + \log\det\frac{\mathcal{N}_\psi}{\hat{\mathcal{N}}_\psi}. \tag{5.52}$$

Here g^τ is the determinant of the deformed metric $g^\tau_{\mu\nu}=e^{2\tau\sigma}\hat{g}_{\mu\nu}$, and

$$a_1^\tau = -\frac{1}{12}R^\tau + \gamma_5\tau\triangle^\tau G + \frac{1}{\sqrt{g^\tau}}\Big[(1-\tau)\sqrt{\hat{g}}\,\frac{\Phi}{\hat{V}} + \tau\sqrt{g}\,\frac{\Phi}{V}\Big]\gamma_5 \tag{5.53}$$

is the relevant Seeley-deWitt coefficient of $\not{D}_\tau^2$. Furthermore, $\hat{\mathcal{N}}_\psi$ is the norm-matrix of the zero-modes $\hat{\psi}_0^p$ in (5.50). Since those are orthonormal it is just the k-dimensional identity matrix. $\mathcal{N}_\psi$ is the norm-matrix of the zero-modes of $i\not{D}$ which are related to the $\hat{\psi}_0^p$ as

$$\psi_0^p = e^{iF-\gamma_5(G+\Phi\chi)-\frac{1}{2}\sigma}\hat{\psi}_0^p \tag{5.54}$$

as follows from (5.36). Inserting (5.53) into (5.52) one finds the following formula for the determinant in arbitrary background gravitational and gauge fields:

$$
\begin{aligned}
\det{}' i\not{D} = \det \frac{\mathcal{N}_\psi}{\hat{\mathcal{N}}_\psi} \det{}'(i\hat{\not{D}}) \exp\Big(\frac{S_L}{24\pi} + \frac{1}{2\pi}\int \sqrt{\hat{g}} G\hat{\triangle} G\Big) \\
\cdot \exp\Big(\frac{2k}{V}\int \sqrt{g} G + \frac{\Phi^2}{2\pi\hat{V}}\int \sqrt{\hat{g}}\chi\Big),
\end{aligned} \tag{5.55}
$$

where

$$
S_L = \int \sqrt{\hat{g}}[\hat{\mathcal{R}}\sigma - \sigma\hat{\triangle}\sigma] \tag{5.56}
$$

is the *Liouville action.* In deriving this result we used that $\int \sqrt{g}\chi = 0$. Actually, for our reference metric the Ricci scalar $\hat{\mathcal{R}}$ vanishes and the Liouville action simplifies to $-\int \sqrt{\hat{g}}\sigma\hat{\triangle}\sigma$. However, as it stands the formula (5.55) holds for arbitrary reference metrics and arbitrary Riemannian surfaces.

As expected for a gauge-invariant regularization, the function F and thus the pure gauge part of the vector potential does not appear in the determinant.

For later use we also give the analogous formula for the zero-mode truncated scalar determinant [44]

$$
\det{}'^{\frac{1}{2}}(-\triangle) = \det{}'^{\frac{1}{2}}(-\hat{\triangle})(\frac{V}{\hat{V}})^{\frac{1}{2}} \exp\Big(-\frac{1}{24\pi}S_L\Big). \tag{5.57}
$$

This completes the computations of the determinants.

The *generating functional for the full theory* is then obtained as follows: First one notes that the formulas (5.41) and (5.47) for the fermionic functionals still hold without hats. Thus to calculate the functionals in arbitrary gauge-, auxiliary- and gauge fields we need to know the Green-functions, determinants and zero-modes in these backgrounds.
To relate the fermionic Green-functions S in the different topological sectors to the hatted ones we define

$$
S_1(x,y) = e^{-g(x)}\,\hat{S}(x,y)\,e^{-\bar{g}(y)}, \quad g = -iF + \gamma_5(G + \Phi\chi) + \frac{1}{2}\sigma.
$$

On the infinite space we would have $S = S_1$ [6]. However, if the Dirac operator possesses zero modes this simple relation is modified to to

$$
\begin{aligned}
S(x,y) = S_1(x,y) + \int P_0(x,u)S_1(u,v)P_0(v,y)\sqrt{g(u)g(v)}d^2u d^2v \\
- \int S_1(x,u)P_0(u,y)\sqrt{g(u)}d^2u - \int P_0(x,u)S_1(u,y)\sqrt{g(u)}d^2u,
\end{aligned} \tag{5.58}
$$

and this formula should be compared with the analogues one for scalars (5.33). Here P_0 is the orthonormal projector onto the zero modes. For gauge fields with vanishing flux $S = S_1$. Together with the relation (5.55) between

the full and hatted determinant and the explicit form (5.45,5.46) for $\det i\hat{\not{D}}$ this yields the fermionic generating functional in the various topological sectors.
In the *trivial sector* one finds explicitly

$$\begin{aligned} Z_F[\eta,\bar\eta] = \frac{1}{|\eta(\tau)|^2}\Theta\Big[\begin{matrix}-c_1\\ c_0\end{matrix}\Big](0,\tau)\ \bar\Theta\Big[\begin{matrix}-\bar c_1\\ \bar c_0\end{matrix}\Big](0,\tau) \\ e^{-\int\bar\eta(x)S(x,y)\eta(y)}\cdot\exp\Big(\frac{1}{24\pi}S_L+\frac{1}{2\pi}\int\sqrt{g}G\triangle G]\Big). \end{aligned} \tag{5.59}$$

By using the scaling properties of the Ricci-scalar and Laplacian (see appendix B) the exponent can be rewritten as

$$-\frac{1}{96\pi}\int\sqrt{g}\mathcal{R}\frac{1}{\triangle}\mathcal{R}+\frac{1}{2\pi}\int\sqrt{g}G\triangle G,$$

which makes clear that the resulting functional is diffeomorphism invariant. Here we used that $\mathcal{R}$ integrates to zero or that the Euler number of the torus vanishes. On the sphere or higher genus surfaces the last formula is modified.
To relate the hatted and full functionals in the *non-trivial sectors* one recalls that in the formula (5.47) for the full partition function (without hats) one must use orthonormal zero-modes. These can be expanded in terms of the un-normalized modes ψ_0^p defined in (5.54). Inserting these expansions into (5.47) yields the inverse square roots of the determinants of the corresponding norm matrices $\mathcal{N}_\psi$ and $\mathcal{N}_\chi$ which partly chancel $\det\mathcal{N}_\psi$ in (5.47). Thus one ends up with

$$\begin{aligned} Z_F[\eta,\bar\eta] = \Big(\frac{\pi\hat V}{\Phi}\Big)^{\frac{\Phi}{4\pi}}\,e^{\Phi^2/2\pi\hat V\cdot\int\sqrt{\hat g}\chi}\prod_{p=1}^{|k|}(\bar\eta,\psi_0^p)(\psi_0^p,\eta) \\ \cdot e^{-\int\bar\eta(x)S_e(x,y)\eta(y)}\exp\Big(\frac{S_L}{24\pi}+\frac{1}{2\pi}\int\sqrt{\hat g}G\hat\triangle G+\frac{2k}{V}\int\sqrt{g}G\Big), \end{aligned} \tag{5.60}$$

where the ψ_{0+}^p are the un-normalized zero-modes (5.50).

Bosonic path integral: To arrive at the generating functional for the complete theory we must finally quantize the photon and auxiliary fields A_μ and B_μ (see (5.35)). For that we insert the decomposition (5.26) into the bosonic part of the (Euclidean) action (5.1). This results in

$$\begin{aligned} S_B &= \frac{\Phi^2}{2e^2V}+(2\pi)^2\sqrt{\hat g}\hat g^{\mu\nu}h_\mu h_\nu \\ &+\int\sqrt{g}\Big(\frac{1}{2}\varphi\triangle^2\varphi-\lambda\triangle\lambda-\phi\triangle\phi-g_3R\lambda\Big). \end{aligned} \tag{5.61}$$

The term quadratic in the h field is not present in the action (5.1) on Minkowski space-time. But on the torus h is part of the Hodge decomposition of B_μ and thus on the same footing as $\partial\lambda$ and $\eta\partial\phi$. Since S_B and the fermionic determinants are both gauge invariant and thus independent of the pure gauge mode α in (5.26), it is natural to change variables from A_μ to $(\varphi, \alpha, t_\mu, \Phi)$ in each topological sector. One can show [41] that this transformation is one to one, provided

$$\int \sqrt{g}\varphi = \int \sqrt{g}\alpha = 0 \quad \text{and} \quad et_\mu \in [0,1]. \tag{5.62}$$

The measures are related as

$$\mathcal{D}A_\mu = J\sum_k dt_0 dt_1 \mathcal{D}\varphi\mathcal{D}\alpha, \quad \text{where} \quad J = (2\pi)^2 \det{}'(-\triangle). \tag{5.63}$$

The Jacobian J is independent of the dynamical fields. In expectation values of gauge invariant and thus -independent operators the α-integration cancels against the normalization. This is of course related to the fact that in QED the ghosts decouple in the Lorentz gauge.

Finally observe that via the derivative couplings to the fermionic current [24] we introduced artificial degrees of freedom. The relation between B_μ in (5.35) and the fields (ϕ, λ, h_μ) is only one to one if we impose the conditions similar to (5.62), namely

$$\bar{\phi} \equiv \frac{1}{V}\int \sqrt{g}\phi = 0, \qquad \bar{\lambda} = 0 \quad \text{and} \quad h_\mu \in [-\infty, \infty]. \tag{5.64}$$

There is no restriction on the harmonic part of the auxiliary field, since B_μ is not a gauge field. The constraints are imposed in the functional integral as

$$\int dh_0 dh_1 \mathcal{D}\phi\mathcal{D}\lambda\delta(\bar{\phi})\delta(\bar{\lambda})\cdots. \tag{5.65}$$

The normalization by the volume in (5.64) is needed such that the constraints and hence the partition function are both dimensionless. For example, expanding ϕ in eigenmodes of the Laplacian as

$$\phi = a_0\phi_0 + \sum_{n>0} a_n\phi_n\ , \quad \text{where} \quad \phi_0 = \frac{1}{\sqrt{V}}$$

is the zero mode, one finds the dimensionless partition function

$$\int \mathcal{D}\phi\ \delta(\bar{\phi})\ e^{\phi\triangle\phi} = \sqrt{V}\frac{1}{\det{}'^{\frac{1}{2}}(-\triangle)} \tag{5.66}$$

for free bosons.
Constraining the mean field to zero as in (5.66) is equivalent to fixing the field at an arbitrary point ξ on the torus to zero [49]

$$\int \mathcal{D}\phi\ \delta(\bar{\phi})\cdots = \int \mathcal{D}\phi\ \delta(\phi(\xi))$$

This can be seen as follows:

$$\int \mathcal{D}\phi\ \delta(\phi(\xi))\cdots = \int du\delta(\bar{\phi}-u)\mathcal{D}\phi\ \delta(\phi(\xi))\cdots.$$

Now one shifts the field as $\phi \to \phi + u$. Using that the action is left invariant by this shift, the measure becomes

$$\int du\mathcal{D}\phi\ \delta(\bar{\phi})\delta(\phi(\xi)+u)\cdots = \int \mathcal{D}\phi\ \delta(\bar{\phi})\cdots$$

which shows that the two constraints are the same. When integrating over the auxiliary fields it is always understood that the divergent zero modes are suppressed as in (5.65).

5.2.1 Partition function

As a first application of our general results we calculate the partition function of the theory (5.1). To compute it we must put the sources η and $\bar{\eta}$ in (5.25) to zero. Then it is evident from (5.60) that the non-trivial sectors do not contribute and hence we may assume $\Phi=0$. Thus the partition function is given by

$$Z_0 = J \int d^2t d^2h \mathcal{D}\varphi \mathcal{D}\phi \mathcal{D}\lambda\ Z_F[0,0]\ e^{-S_B[\Phi=0]}, \tag{5.67}$$

where J is the Jacobian of the transformation (5.63). Z_F the fermionic partition function (5.59) in the trivial sector and the integration is over fields obeying the conditions (5.62,reft51). Now we perform the various integrals in turn.

integration over the harmonics: By using the series representation of the theta functions one computes

$$\int_0^1 d^2(et)\Theta\begin{bmatrix}-c_1\\ c_0\end{bmatrix}(0,\tau)\ \bar{\Theta}\begin{bmatrix}-c_1\\ c_0\end{bmatrix}(0,\tau) = \frac{1}{\sqrt{2\tau_0}} \tag{5.68}$$

Since the result appears always together with the η-function factor in (5.59) it is convenient to introduce

$$\kappa := \frac{1}{\sqrt{2\tau_0}}\frac{1}{|\eta(\tau)|^2}$$

in the following expressions. The result (5.68) does not depend on the h-field and hence the h-integration in (5.67) becomes Gaussian. It yields a factor $1/4\pi$ so that

$$Z_0 = \pi\kappa \det{}'(-\triangle)\, e^{S_L/24\pi} \int \mathcal{D}_\delta(\varphi\phi\lambda)\; e^{\frac{1}{2\pi}\int\sqrt{g}G\triangle G - S_B[h=0]}, \tag{5.69}$$

where G has been defined in (5.6).We inserted the explicit expression (5.63) for the Jacobian. If we would have kept the chemical potential and twists then already the toron-integration in (5.68) would have washes out the dependence on the boundary conditions and chemical potential.

Integration over λ and ϕ: The integral over λ, subject to the condition (5.64), modifies the Liouville factor and yields one inverse square-root of the determinant of $-2\triangle$ in (5.69). To continue we recall the scaling formula for the determinant of $\triangle$ [14]:

$$\log\frac{\det{}'(-a\triangle)}{\det{}'(-\triangle)} = \log a\cdot\zeta(0) = \log a\cdot[\frac{1}{4\pi}\int a_1 - p],$$

where p is the number of zero modes of the operator. On the torus $\int a_1 = 0$ and we find

$$\det{}'(\,-a\triangle) = \frac{1}{a}\det{}'(-\triangle). \tag{5.70}$$

Using this scaling property the λ-*integral* together with (5.66) we obtain

$$\begin{aligned} Z_0 \;&=\; \kappa\pi\sqrt{2V\det{}'(-\triangle)}\; e^{(g_3^2+1/24\pi)S_L} \\ &\qquad\int \mathcal{D}_\delta(\varphi\phi)\; e^{\frac{1}{2\pi}\int\sqrt{g}G\triangle G - S_B[h=\lambda=0]}, \end{aligned} \tag{5.71}$$

To quantize the ϕ field we need to recall that $G = e\varphi + g_2\phi$. Since $\varphi\triangle\varphi \sim (A^T, A^T)$, the anomalous term $\sim \int G\triangle G$ in the exponent contains an explicit photon mass term with bare-mass $e/\sqrt{\pi}$. However, when quantizing the ϕ field this mass is renormalized. This can be seen explicitly in the resulting expression for the partition function after the ϕ-*integration* has been performed

$$Z_0 = \frac{2\sqrt{\pi}\kappa eV}{m_\gamma}\; e^{(g_3^2+1/24\pi)S_L} \int \mathcal{D}\varphi e^{-\frac{1}{2}\int\sqrt{g}\varphi(\triangle^2 - m_\gamma^2\triangle)\varphi}, \tag{5.72}$$

where the renormalized photon mass is

$$m_\gamma^2 = \frac{e^2}{\pi}\frac{2\pi}{2\pi + g_2^2}.$$

Integration over φ: The zeta-function regulated determinant which one obtains when performing the integral (5.72) factorizes

$$\det'(\triangle^2 - m_\gamma^2 \triangle) = \det'(-\triangle) \cdot \det'(-\triangle + m_\gamma^2).$$

This factorization property is not obvious since all determinants must be regulated. But it holds for commuting operators and in the zeta-function scheme. Then the partition function simplifies to

$$Z_0 = \frac{2\sqrt{\pi}\kappa e V}{m_\gamma} \big(\det'(-\triangle)\det'(-\triangle + m_\gamma^2)\big)^{-\frac{1}{2}} \exp\Big((g_3^2 + \frac{1}{24\pi})S_L\Big).$$

We can go further by using (5.57) and the known result for the determinant of $\hat{\triangle}$ [28] which together yield

$$\det'^{\frac{1}{2}}(-\triangle) = \tau_0 L|\eta(\tau)|^2 \sqrt{\frac{V}{\hat{V}}} \exp\Big(-\frac{1}{24\pi}S_L\Big) \tag{5.73}$$

which finally leads to

$$Z_0 = \sqrt{2\pi V}\, \frac{e}{m_\gamma} \frac{1}{\tau_0|\eta(\tau)|^4} \frac{1}{\det'^{\frac{1}{2}}(-\triangle + m_\gamma^2)} \exp\Big((\frac{1}{12\pi} + g_3^2)\, S_L\Big) \tag{5.74}$$

for the partition function of the general model (5.1) on curved spaces. It shows explicitly that in the topologically trivial sector the theory should be equivalent to a theory of free mass-less and massive bosons with mass m_γ. It is interesting to follow the various contributions to the explicit dependence on the gravitational field since they contribute to the *Hawking radiation*. For that we recall that when one quantizes a conformal field theory with central charge c in an external gravitational fields one ends up with the Liouville term, $Z \sim \exp[c\, S_L/24\pi]$ [44]. Thus the fermions contribute with $c = 1$, as expected. The ϕ and λ field contribute with 1 and $1 + 24\pi g_3^2$, respectively. However, the Jacobian combined with the conformal part of the gauge sector contribute with $c = -1$ and we are left with a total central charge $c = 2 + 24\pi g_3^2$. Of course, the gauged model is not conformally invariant and the breaking is manifest in the massive determinant in (5.74). The partition function of the *un-gauged theory* is (5.72) multiplied by an inverse determinant (the missing Jacobian) and without φ-integration. In this limit one obtains a conformal theory with central charge $c = 3 + 24\pi g_3^2$.

By using an elegant result of Christensen and Fulling [42], that relates the conformal anomaly to the asymptotic Hawking flux, one concludes that the Hawking radiation of the un-gauged model is $3 + 24\pi g_3^2$ times that of free mass-less scalars. For the gauged model the Hawking radiation is still thermal and consists of mass-less and massive particles.

The appearance of m_γ in (5.72) should be interpreted as *renormalization of the electric charge* induced by the interaction of the auxiliary fields with the fermions. After summing over all fermion-loops this leads to an effective coupling between the photons and the ϕ-field and in turn to a modified effective mass for the photons in (5.72). In the limit $g_2 \to 0$ this mass tends to the well-known Schwinger model result, $m_\gamma \to e/\sqrt{\pi}$ [8].
We conclude this subsection with deriving an explicit formula for the partition function on the flat torus. Applying the results in [3] one obtains for the massive determinant

$$\det{}'(-\hat{\triangle} + m_\gamma^2)^{\frac{1}{2}} = \frac{1}{m_\gamma} e^{-\frac{1}{2}\zeta'(0)},$$

with

$$\zeta'(0) = \sum_{n\neq 0} \frac{1}{\pi L} \frac{\hat{V} m_\gamma}{\sqrt{(n,n)}} K_1(m_\gamma L \sqrt{(n,n)}) - \frac{\hat{V} m_\gamma^2}{4\pi}, \tag{5.75}$$

where $(n,n) = \hat{g}_{ij} n^i n^j$ is the inner product taken with the reference metric, and the sum is over all $(n^i) \in Z_2$ with the origin excluded. For $g_{\mu\nu} = \delta_{\mu\nu}$, in which case the partition function has the usual thermodynamical interpretation, the result reduces to one derived previously by Ambjorn [27]. In addition, if L approaches infinity we recover a result in [1]. The free energy for $\tau_1 = 0$ and on flat space simplifies then to

$$F = -\frac{1}{\beta} \log Z = \frac{1}{2\beta}\, \zeta'(0).$$

with $\zeta'(0)$ from (5.75) and the particular choice for the parameters.

5.2.2 Bosonisation

In the classical analysis we have already seen that in the limiting case $g_3 = 0$ and $g_1 = g_2 = g$ the general model reduces to the gauged Thirring model. Now we show that the same is true for the quantized theory on the torus if in addition we set $g_0 = g$. More precisely, the Hubbard-Stratonovich transform of the Thirring model is just the derivative coupling model (5.1) with identical couplings. In the process of showing that we shall arrive at the Bosonisation formulas for the gauged Thirring model on the curved torus. We shall see that only the non-harmonic part of the fermion current can naively be bosonised and that for this part the rules of the un-gauged model on flat space time [15] need just be covariantized.

For that we calculate the partition function (5.67) in a different order. First we integrate out the auxiliary fields. In order to understand the role of λ and ϕ we introduce sources for them. Thus we study the generating functional for the correlators of the auxiliary fields

$$Z[\xi,\zeta]=\int\mathcal{D}(\lambda\phi h\psi A_\mu)e^{-S+\int\sqrt{g}[\xi\lambda+\zeta\phi]}.$$

Here

$$S=-i\int\sqrt{g}\psi^\dagger \not{D}\psi+S_B[g_3\!=\!0]$$

is the action of the full theory. $\not{D}$ is the Dirac operator in (5.36) with all couplings set equal and S_B the bosonic action (5.61). Since λ and ϕ integrate to zero (see 5.64) we may assume the same property to hold for the sources. Also, since there are no fermionic sources only configurations in the trivial sector contribute, so that there is not instanton potential in (5.36) and hence $\Phi=0$ in (5.61). The integration over the auxiliary fields is Gaussian and yields

$$Z=\mathcal{N}_0\int\mathcal{D}(\psi A_\mu)\,e^{-S_T}\,\exp\int\sqrt{g}\Big[\;-\;\frac{1}{4}(\xi\frac{1}{\triangle}\xi+\zeta\frac{1}{\triangle}\zeta)\;+\;\frac{g}{2}(\xi\frac{1}{\triangle}j^\mu_{;\mu}+\zeta\frac{1}{\triangle}j^\mu_{5;\mu})\Big] \tag{5.76}$$

where

$$S_T=-\frac{1}{4}\int\sqrt{g}\Big(F_{\mu\nu}F^{\mu\nu}-i\psi^\dagger\not{D}\psi-\frac{g^2}{4}j^\mu j_\mu\Big) \tag{5.77}$$

is the action of the *gauged Thirring model* on curved space-time and

$$\mathcal{N}_0=\frac{V}{2\pi\det'(-\triangle)} \tag{5.78}$$

comes from the integration over the auxiliary fields.
Let us first consider the partition function, that is set the sources to zero. Comparing (5.76) with (5.72) and using (5.73) we easily find

$$\boxed{\int\mathcal{D}(\psi t)e^{-S_T}=\sqrt{\frac{1}{2}+\frac{g^2}{4\pi}}\;e^{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}\int\mathcal{D}\gamma\,\delta(\bar\gamma)\;e^{-S_\gamma},} \tag{5.79}$$

where $\bar\gamma$ is the mean field (see 5.64) and we used (5.63) and (5.70). The action for the neutral scalar field γ is found to be

$$S_\gamma=\frac{1}{2}\int\sqrt{g}\partial_\mu\gamma\partial^\mu\gamma-\frac{ie}{\sqrt{\pi}}\frac{1}{\sqrt{1+g^2/2\pi}}\int\sqrt{g}\gamma\triangle\varphi.$$

Since (5.79) holds for any φ (and thus for the non-harmonic part of any A_μ, because of gauge-invariance) we read off the following *Bosonisation rules*:

$$
\begin{aligned}
j'^{\mu} &\longrightarrow \frac{i}{\sqrt{\pi}}\frac{1}{\sqrt{1+g^2/2\pi}}\eta^{\mu\nu}\partial_\nu\gamma \\
j_5'^{\mu} &\longrightarrow -\frac{i}{\sqrt{\pi}}\frac{1}{\sqrt{1+g^2/2\pi}}\partial^\mu\gamma,
\end{aligned}
\tag{5.80}
$$

where prime denotes the non-harmonic part of the currents. Thus, only the non-harmonic parts of the currents can be bosonised in terms of a single valued scalar field. To bosonise their harmonic parts one would have to allow for a scalar field γ with windings as ϕ below . On the infinite plane the harmonic part is not present and we may leave out the primes in (5.80). If we further assume space time to be flat we recover the well-known Bosonisation rules in [15]. What we have shown then, is that for the gauged model on curved space time the Bosonisation rules are just the flat ones properly covariantized and with the omission of the zero-modes.

Since (5.79) holds for any gauge field the current correlators in the Thirring model are correctly reproduced by the Bosonisation rules (5.80). To see that more clearly we calculate the two-point functions of the auxiliary fields in the Thirring model (5.76-5.78). For that we differentiate (5.76) (φ is treated as external field) with respect to the sources and find

$$
\begin{aligned}
\langle\lambda(x)\lambda(y)\rangle &= \frac{1}{2}G_0(x,y) + \frac{g^2}{4}\int\langle G_0(x,u)j^\mu_{;\mu}(u)G_0(y,v)j^\nu_{;\nu}(v)\rangle_T \\
\langle\phi(x)\phi(y)\rangle &= \frac{1}{2}G_0(x,y) + \frac{g^2}{4}\int\langle G_0(x,u)j^\mu_{5;\mu}(u)G_0(y,v)j^\nu_{5;\nu}(v)\rangle_T,
\end{aligned}
\tag{5.81}
$$

where G_0 is the free mass-less Green-function (5.30,5.33) in curved space-time and the integrations are over the variables u and v with the invariant measure on the curved torus. Here $\langle\ldots\rangle_T$ are vacuum expectation values of the Thirring model (5.77). Alternatively we can calculate these expectation values from (5.69) and (5.71), where the fermionic integration has been performed and find

$$
\begin{aligned}
\langle\lambda(x)\lambda(y)\rangle &= \frac{1}{2}G_0(x,y) \\
\langle\phi(x)\phi(y)\rangle &= \frac{\pi m_\gamma^2}{2e^2}G_0(x,y) + \frac{m_\gamma^2}{2}(1-\frac{\pi m_\gamma^2}{e^2})\varphi(x)\varphi(y).
\end{aligned}
\tag{5.82}
$$

Comparing this with the result (5.81) we see at once that

$$
\begin{aligned}
\int\langle G_0(x,u)j^\mu_{;\mu}(u)G_0(y,v)j^\nu_{;\nu}(v)\rangle_T &= 0 \\
\int\langle G_0(x,u)j^\mu_{5;\mu}(u)G_0(y,v)j^\nu_{5;\nu}(v)\rangle_T &= \frac{m_\gamma^2}{e^2}(m_\gamma^2\varphi(x)\varphi(y) - G_0(x,y)).
\end{aligned}
\tag{5.83}
$$

These correlators express the gauge invariance and the axial anomaly $\langle j^\mu_{5;\mu}\rangle = -m_\gamma\triangle\varphi$ in the gauged Thirring model. They can be correctly reproduced with the bosonization rules (5.80). They are not reproduced with the ones derived for the un-gauged model [15].

5.3 Chiral condensate

Recalling that S_e in (5.60) anti-commutes with γ_5 one sees at once that only configuration supporting one fermionic zero-mode with positive chirality contribute to the chiral condensate

$$\langle\psi^\dagger P_+\psi\rangle = -\frac{J}{Z_0}\frac{\delta^2}{\delta\eta_+(x)\delta\bar\eta_+(x)}\int\mathcal{D}(\dots)Z_F[\eta,\bar\eta]|_{\eta=\bar\eta=0}\,e^{-S_B},$$

where $\eta_+ = P_+\eta$. Earlier we have seen that these are the gauge fields with flux $\Phi = 2\pi$ or instanton number $k=1$. Thus the condensate becomes

$$\langle\psi^\dagger P_+\psi\rangle = -\frac{J}{Z_0}\sqrt{\frac{\hat V}{2}}\int\mathcal{D}(.)\,\psi_0^\dagger(x)\psi_0(x)\exp(.)\,e^{-S_B[k=1]}, \tag{5.84}$$

where $\exp(\dots)$ is the last exponential factor in (5.60). First we integrate over the toron field t. The t dependence enters only through the zero mode and more specifically $\hat\psi_0$ in (5.54) and (5.50) with $p=1$. Using the series representation for the theta functions one finds

$$\int d^2t\,\hat\psi_0^\dagger(x)\hat\psi_0(x) = \frac{1}{\hat V}. \tag{5.85}$$

Note that the result does not depend on the chemical potential similarly as in our calculation of the partition function.

To continue we observe that the term $\int\sqrt{g}G$ in $\exp(\dots)$ vanishes because of our conditions (5.62) and (5.64) on the fields φ and ϕ. Also note, that the fermionic Green function does not enter in the expression for the chiral condensate. It follows that the fermionic functional (5.59) in the trivial sector and (5.60) in the one-instanton sector are the same, up to the factors in the fist lines. From (5.85) and (5.69) we see that the toron integral of the fist line in (5.60) is $|\eta|^2\sqrt{\tau_0/\hat V}\exp(2\pi\int\sqrt{g}\chi/\hat V)$ times the toron integral over the factor in (5.59). Also, since

$$S_B[k=1] = S_B[k=0] + \frac{2\pi^2}{e^2V}$$

the functional integral and normalizing partition function in (5.84) are the same, up to these factor and the field-dependent factors which relate the hatted and un-hatted zero-modes in (5.54). Finally note that the λ integrals in (5.84) and in the normalizing partition function cancel so that we end up with the following formula for the condensate

$$\langle\psi^\dagger P_+\psi\rangle = \sqrt{\frac{\tau_0}{\hat{V}}}\,|\eta(\tau)|^2 e^{-2\pi^2/e^2V+2\pi/\hat{V}\int\sqrt{\hat{g}}\chi}\Big\langle e^{-2(g\phi+e\varphi)(x)-\sigma(x)}\Big\rangle_{\phi\varphi} \tag{5.86}$$

The expectation value is evaluated with

$$S_{eff} = \int\sqrt{g}\Big[\frac{1}{2}\varphi(\triangle^2 - \frac{e^2}{\pi}\triangle)\varphi - \frac{e^2}{\pi m_\gamma^2}\phi\triangle\phi - \frac{eg_2}{\pi}\phi\triangle\varphi\Big].$$

A formal calculation of the resulting Gaussian integrals yield

$$\begin{aligned}\langle\psi^\dagger P_+\psi\rangle &= \sqrt{\frac{\tau_0}{\hat{V}}}|\eta(\tau)|^2 e^{-2\pi^2/e^2V+2\pi/\hat{V}\int\sqrt{\hat{g}}\chi}\, e^{-\sigma(x)-2\Phi\chi(x)}\\ &\cdot\quad \exp\,[\frac{2\pi^2m_\gamma^4}{e^2}\,K(x,x)]\exp\,[\frac{2\pi g_2^2}{2\pi+g_2^2}\,G_0(x,x)],\end{aligned} \tag{5.87}$$

where

$$K(x,y) = \langle x|\frac{1}{\triangle^2 - m_\gamma^2\triangle}|y\rangle = \frac{1}{m_\gamma^2}(G_0(x,y) - G_{m_\gamma}(x,y)) \tag{5.88}$$

and G_m, G_0 are the massive and mass-less Green-functions.

Here we encounter ultra-violet divergences since $G_0(x,y)$ is logarithmically divergent when x tends to y. To extract a finite answer we need to renormalize the operator $\exp(\alpha\phi)$. This wave function renormalization is equivalent to the renormalization of the fermion field in the Thirring model and thus is very much expected [44, 15]. In order to do that we first determine the short distance behavior of the mass-less Green function (5.31). Using the identity

$$|\Theta\Big[\begin{matrix}\frac{1}{2}+\frac{\xi^0}{L}\\ \frac{1}{2}+\frac{\xi^1}{L}\end{matrix}\Big](0,\tau)|^2 = |e^{i\pi\tau(\xi^0/L)^2}\Theta_1(\frac{\tau\xi^0+\xi^1}{L},\tau)|^2$$

and the small z expansion

$$\Theta_1(z,\tau) = 2\pi\eta(\tau)^3 z + O(z^2),$$

we see that $\hat{G}_0$ possesses the expected logarithmic short distance singularity

$$\hat{G}_0(x,y) = -\frac{1}{4\pi}\log\frac{\hat{g}_{\mu\nu}\xi^\mu\xi^\nu}{\hat{V}} - \frac{1}{4\pi}\log\,(4\pi^2\tau_0|\eta(\tau)|^4) + O(\xi). \tag{5.89}$$

From the relation (5.33) between the full and hatted Green function and the definition of χ in (5.28) it follows that G_0 possesses the short distance expansion

$$G_0(x,y) \sim \hat{G}_0(x,y) + 2\chi(x) - \frac{1}{\hat{V}}\int \sqrt{\hat{g}}\chi + O(\xi)$$

To continue we need to regularize the composite operator $\exp(\alpha\phi)$ appearing in (5.86). The normal ordering prescription

$$: e^{\alpha\phi(x)} := \frac{e^{\alpha\phi(x)}}{\langle e^{\alpha\phi(x)}\rangle}. \tag{5.90}$$

works well on the whole plane [44, 15]. On the curved torus we must be more careful when renormalizing this operator. The required wave function renormalization is not unique but it is very much restricted by the following requirements: First we take as reference system (the denominator in 5.90) one with a minimal number of dynamical degrees of freedom since we do not want to loose information by our regularization. Second, the renormalized operator should have a well-defined infinite volume limit and its expectation values should cluster. Finally, the regularization should respect general covariance. These requirements then force us to take as reference system the infinite plane with metric $g_{\mu\nu}$. The flat metric $\delta_{\mu\nu}$ is not permitted since it leads to a ill-defined expression for $\langle\exp(\alpha\phi)\rangle$. With these choice the normal ordering in (5.90) is equivalent to replacing the mass-less Green function in (5.87) by

$$G_0^{reg}(x,y) := G_0(x,y) + \frac{1}{4\pi}\log\,[\mu^2 s^2(x,y)]. \tag{5.91}$$

Here $s(x,y)$ denotes the geodesic distance between x and y. The occurrence of the arbitrary mass scale μ comes from the ambiguities in the required ultra-violet regularization. On the flat torus $\hat{G}_0^{reg}$ has now the finite coincidence limit

$$\hat{G}_0^{reg}(x,x) = -\frac{1}{4\pi}\log\Big(\frac{4\pi^2\tau_0|\eta(\tau)|^4}{\mu^2\hat{V}}\Big). \tag{5.92}$$

To determine the chiral condensate we also need to determine $K(x,y)$ on the diagonal. In a first step we shall obtain it for the flat torus. Its σ-dependence is then determined in a second step. For $\sigma=0$ and $\tau=i\tau_0$ the Green function $\hat{K}$ has been computed in [41]. The generalization to arbitrary τ is found to be

$$
\begin{aligned}
m_\gamma^2 \hat{K}(x,x) &= -\frac{1}{2m_\gamma L\tau_0}\coth\left(\frac{\pi\tau_0 a}{|\tau|^2}\right)+\frac{1}{m_\gamma^2\hat{V}} \\
&+ \frac{1}{2\pi}\Big(-\log|\eta(\frac{-1}{\tau})|^2+F(L,\tau)-H(L,\tau)\Big),
\end{aligned}
\tag{5.93}
$$

where we introduced the dimensionless constant $a = Lm_\gamma|\tau|/2\pi$ and the functions

$$
\begin{aligned}
F(L,\tau) &= \sum_{n>0}\Big[\frac{1}{n}-\frac{1}{\sqrt{n^2+a^2}}\Big] \\
H(L,\tau) &= \sum_{n>0}\frac{1}{\sqrt{n^2+a^2}}\Big[\frac{1}{e^{-2\pi i z_+(n)}-1}+\frac{1}{e^{2\pi i z_-(n)}-1}\Big].
\end{aligned}
\tag{5.94}
$$

We used the abbreviations

$$
z_\pm = \frac{1}{|\tau|^2}(n\tau_1 \pm i\tau_0\sqrt{n^2+a^2}). \tag{5.95}
$$

Substituting (5.93) and (5.92) into (5.87) with $\sigma=0$ we obtain the following *exact formula for the chiral condensate* on the torus with flat metric $\hat{g}_{\mu\nu}$:

$$
\begin{aligned}
\langle\psi^\dagger P_+\psi\rangle_{\hat{g}} &= \frac{1}{L|\tau|}\Big(\frac{m_\gamma L|\tau|}{2\pi}\Big)^{\frac{g_2^2}{2\pi+g_2^2}}\exp\Big(\frac{\pi^2 m_\gamma}{e^2 L\tau_0}\coth\frac{Lm_\gamma\tau_0}{2|\tau|}\Big) \\
&\cdot\ \exp\Big[\frac{\pi m_\gamma^2}{e^2}\Big(F(L,\tau)-H(L,\tau)\Big)\Big],
\end{aligned}
\tag{5.96}
$$

where we used that on the flat torus $\chi=0$ and $V=\hat{V}$. Furthermore, we identified μ with the natural mass scale m_γ of the theory.

To study the finite temperature behavior of the chiral condensate we must assume that $\tau = i\beta/L$ and then $\beta = 1/T$ is just the inverse temperature. Furthermore we perform the thermodynamic limit $L\to\infty$. Then $\coth(\ldots)\to 1$, $H\to 0$ and the expression for the chiral condensate simplifies to

$$
\boxed{\langle\psi^\dagger P_+\psi\rangle_\beta = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}}\exp\Big[-\frac{\pi^2 m_\gamma}{e^2}T+\frac{2\pi}{2\pi+g_2^2}F\Big].} \tag{5.97}
$$

Let us now investigate the low and high temperature limits in turn. To study the *low temperature limit* we use that

$$
F(\beta)\to\gamma+\log\frac{a}{2}+\frac{1}{2a}\qquad\text{for}\quad a\to\infty,
$$

where $\gamma = 0.57721\ldots$ is the Euler number. Inserting this expansion into (5.97) yields

$$\langle\psi^\dagger P_+\psi\rangle = -\frac{m_\gamma}{4\pi}\, 2^{g_2^2/(2\pi+g_2^2)}\, \exp\Big(\frac{2\pi}{2\pi+g_2^2}\gamma\Big) \qquad \text{for} \quad T\to 0. \tag{5.98}$$

For temperatures large compared to the induced photon mass F vanishes. Thus we obtain the *high temperature behavior*

$$\langle\psi^\dagger P_+\psi\rangle_T = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}} \exp\Big(-\frac{\pi^2 m_\gamma}{e^2}T\Big) \quad \text{for} \quad T\to\infty \tag{5.99}$$

It is instructive to discuss the various limiting cases. For all $g_i=0$, i.e. the Schwinger model limit, the exact result (5.97) simplifies to

$$\langle\psi^\dagger P_+\psi\rangle_T = -T\, e^{-\frac{\pi}{m_\gamma}T+F(\beta)} \longrightarrow \begin{cases} -\frac{m_\gamma}{4\pi}e^\gamma & T\to 0 \\ -T\, e^{-\pi T/m_\gamma} & T\to\infty, \end{cases} \tag{5.100}$$

where now $m_\gamma^2 = e^2/\pi$ is the induced photon mass in the Schwinger model. This formula for the temperature dependence of the chiral condensate in QED_2 agrees with the earlier results in [41].

Next we wish to investigate how the self-interaction of the fermions affect the breaking. For large coupling g_2 and fixed temperature the exponent in (5.97) vanishes so that

$$\langle\psi^\dagger P_+\psi\rangle_T \sim \frac{1}{\sqrt{2\pi+g_2^2}} \qquad \text{for} \quad T \quad \text{fixed}, \quad g_2\to\infty.$$

Hence, for very large current-current coupling the chiral condensate vanishes. Or in other words, the electromagnetic interaction which is responsible for the chiral condensate, is shielded by the pseudoscalar-fermion interaction. For intermediate temperature and coupling g_2 we must retreat to numerical evaluations of the sums defining the chiral condensate in (5.97). The numerical results are depicted in figure 1.

The study of the influence of the gravitational field is complicated by the presence of the massive Green function G_{m_γ} in (5.87,5.88). This Green function is known only for very particular curved spaces. Fortunately we only need the coincidence limit for which we can use its short distance expansion [37]. For simplicity we assume infinite volume and zero temperature. Then [13]

$$G_m(x,y) \sim \frac{1}{4i}\sum_{j=0}^\infty a_j(x,y)\Big(-\frac{\partial}{\partial m^2}\Big)^j H_0^{(2)}(ms), \tag{5.101}$$

for small geodesic distances $s=s(x,y)$. Here $H_0^{(2)}$ denotes the Hankel function of the second kind and order zero. In particular

$$H_0^{(2)}(z) \to \frac{2}{i\pi}[\log\frac{z}{2} + \gamma] \quad \text{for} \quad z \to 0.$$

Inserting that into (5.101) we find with $G_0 = \hat{G}_0$ from (5.89) the following short distance expansion

$$\begin{aligned} G_0(x,y) - G_m(x,y) \sim \quad & - \quad \frac{1}{2\pi}\Big[\log\big(\frac{2\pi|\eta(\tau)|^2}{m_\gamma L e^{\sigma(x)}}\big) - \gamma\Big] \\ & + \quad \frac{1}{4\pi}\sum_{j=1}^{\infty} a_j(x)\big(-\frac{\partial}{\partial m^2}\big)^j \log(m^2). \end{aligned} \tag{5.102}$$

We have used that $a_0(x) = 1$ and $s \sim e^{\sigma(x)}\hat{s}$, where $\hat{s}$ is the geodesic distance on the flat spacetime with hatted metric, $\hat{s}^2 = \hat{g}_{\mu\nu}(x-y)^\mu(x-y)^\nu$. Finally, substituting (5.102) into (5.87) we end up with

$$\langle\psi^\dagger P_+\psi\rangle_\sigma = \langle\psi^\dagger P_+\psi\rangle_{\sigma=0}\cdot\exp\Big[-\frac{1}{2}\big(\frac{\pi m_\gamma}{e}\big)^2\sum_1^\infty a_j(x)\frac{(j-1)!}{m^{2j}}\Big].$$

The Seeley-deWitt coefficients a_j have been computed up to $j=5$ [26]. They are of order j in the curvature and its derivatives. The first two are

$$a_0(x) = 1 \qquad \text{and} \qquad a_1(x) = \frac{1}{6}\mathcal{R}.$$

For $\mathcal{R} << m^2$ and slowly varying $\mathcal{R}$ we conclude that the chiral condensate decreases with increasing curvature as

$$\langle\psi^\dagger P_+\psi\rangle \sim \exp\big[-\frac{\pi^2}{12e^2}\mathcal{R}\big].$$

If we compare this with the temperature dependence (5.99) we are lead to define a curvature induced effective temperature

$$T_{eff} = \frac{\mathcal{R}}{12 m_\gamma}.$$

For this identification of curvature with temperature no horizon is needed as in black hole physics where the temperature is related to the surface gravity at the horizon. Note that contrary to the temperature the curvature may become negative. Then the condensate is amplified and the identification of $\mathcal{R}$ with T is only a formal one.

Finally we consider the *chiral two point function* for non-coinciding points. The gauge invariant form reads

$$S_+(x,y) \equiv \langle\psi^\dagger(x)\; e^{ie\int_y^x A_\mu dx^\mu} P_+\psi(y)\rangle.$$

It is related to a bound state between a static external charge and a dynamical fermion [20].
The integration over the various fields is similar as in the calculation of the condensate. The result takes a simple form in the infinite volume and zero temperature limit:

$$S_+(x,y) = S_+(x)^{\frac{1}{4}} S_+(y)^{\frac{1}{4}} \; \exp\Big[\frac{1}{2}(\frac{\pi m_\gamma}{e})^2 (K(x,y)+K(y,x))\Big]$$
$$\cdot \exp\big[(\frac{\pi g_2^2}{2\pi + g_2^2} + \frac{g_1^2}{2}) G_0(x,y) - \frac{g_1^2}{4}(G_0(x,x)+G_0(y,y))\Big], \quad (5.103)$$

where $S_+(x) \equiv S_+(x,x) = \langle \psi^\dagger(x)\psi(x)\rangle$ denotes the chiral condensate. Again the mass-less propagator must be regularized. We do this using the prescription (5.91). Then

$$S_+(x,y) = S_+(x)^{\frac{1}{4}} S_+(y)^{\frac{1}{4}} \; \frac{\exp\Big[\frac{1}{2}(\frac{\pi m_\gamma}{e})^2 (G_{m_\gamma}(x,y)+G_{m_\gamma}(y,x))\Big]}{\sqrt{2\pi} m_\gamma^{\frac{g_1^2}{4\pi}} (g(x)^{\frac{1}{8}} g(y)^{\frac{1}{8}} \hat{s})^{\frac{1}{2}(1+\frac{g_1^2}{2\pi})}}.$$

Note that the coupling strength g_1 to the longitudinal current enters the scaling exponent. On *flat space* G_m reduces to $\frac{1}{2\pi}K_0(m\hat{s})$ which decays exponentially for large separations. Hence we find

$$\hat{S}_+(x,y) \sim \frac{\hat{S}_+(x)^{\frac{1}{2}}}{\sqrt{2\pi\hat{s}} (m_\gamma \hat{s})^{\frac{g_1^2}{4\pi}}} \quad (5.104)$$

for large separations of x and y. We have used that the chiral condensate $\hat{S}_+(x)$ in (5.98) is constant, due to translational invariance. For $g_1 = 0$ this simplifies to the Schwinger model result [41]

$$\hat{S}_+(x,y) \sim \sqrt{\frac{m_\gamma e^\gamma}{2}} \frac{1}{2\pi\sqrt{|x-y|}}.$$

Unlike the correlators of fields which in the bosonised version are local in the massive boson field, this two-point function does not decay exponentially. However the long range correlations are suppressed by the coupling to the longitudinal current.

5.4 The un-gauged sector

5.4.1 Thermodynamics

In this section we derive the grand canonical potential, equation of state and ground state energy for $A_\mu = 0$. For the un-gauged model there is no

Gauss constraint and the charge of the vacuum need not vanish. Indeed, for $A_\mu = 0$ the partition function depends on the chemical potential and on the fermionic boundary conditions. Technically this is due to the absence of the toron integration which for the gauged model wiped out any dependence on μ, α and β.

The partition function of the un-gauged model is given by

$$Z = \int d^2h \mathcal{D}\phi \mathcal{D}\lambda \; Z_F[\eta = \bar{\eta} = A = 0] \; e^{-S_B[A=0]}, \tag{5.105}$$

where Z_F is the fermionic generating functional (5.59) and S_B the bosonic action (5.61). The integration over the harmonic fields is Gaussian and yields

$$\int\limits_{-\infty}^{\infty} d^2h \Theta\begin{bmatrix} -c_1 \\ c_0 \end{bmatrix} \bar{\Theta}\begin{bmatrix} -\bar{c}_1 \\ \bar{c}_0 \end{bmatrix} e^{-(2\pi)^2 \sqrt{\hat{g}} \hat{g}^{\mu\nu} h_\mu h_\nu} = \frac{\Theta\left[{u \atop w}\right](\Lambda)}{4\pi\sqrt{1 + g_0^2/2\pi}}$$

where

$$\Theta\begin{bmatrix} u \\ w \end{bmatrix}(\Lambda) = \sum_{n \in Z^2} e^{i\pi(n+u)\Lambda(n+u) + 2\pi i(n+u)w} \tag{5.106}$$

is the theta function with characteristics

$$u = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} (\alpha_1 + i\eta_1{}^\nu \beta_\nu) \quad \text{and} \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\alpha_0 + i\eta_0{}^\nu \beta_\nu - \mu_0) \tag{5.107}$$

and covariance

$$\Lambda = \begin{pmatrix} \tau & 0 \\ 0 & -\bar{\tau} \end{pmatrix} + i \frac{\pi g_0^2 \tau_0}{2\pi + g_0^2} \begin{pmatrix} g_0^2 & -4\pi - g_0^2 \\ -4\pi - g_0^2 & g_0^2 \end{pmatrix}. \tag{5.108}$$

The remaining functional integrals in (5.105) are performed as in the calculation of the condensate. To obtain the partition function of the Thirring model in the limit $g_i = g$ we divide Z by the corresponding partition function $\mathcal{N}_0$ of the free bosons (see 5.78). Using (5.70) and (5.59) we obtain

$$\frac{Z}{\mathcal{N}_0} = \frac{1}{|\eta(\tau)|^2} \sqrt{\frac{2\pi + g_2^2}{2\pi + g_0^2}} \; \Theta\begin{bmatrix} u \\ w \end{bmatrix}(\Lambda) \, e^{(1/24\pi + g_3^2) S_L}. \tag{5.109}$$

In the Thirring model limit $g_2 = g_0$ and the square-root in this formula disappears.

Zero-temperature limit: To investigate the thermodynamics of the model we assume spacetime to be flat and that $\tau = i\beta/L$. Then

$$\Omega = -\frac{1}{\beta}\log\frac{Z}{\mathcal{N}_0}$$

is the *grand canonical potential.* Let us now investigate the low temperature limit of Ω. For $\mu = 0$ this yields the ground state energy.
To study this limit we observe that for $\tau = i\beta/L$ the covariance matrix Λ in (5.108) simplifies to

$$i\pi\Lambda = -\frac{\pi\beta}{L}\Big[\mathrm{Id} + \frac{g_0^2}{4\pi}\frac{1}{2\pi + g_0^2}\begin{pmatrix} g_0^2 & -4\pi - g_0^2 \\ -4\pi - g_0^2 & g_0^2 \end{pmatrix}\Big] \tag{5.110}$$

and has eigenvalues

$$\lambda_1 = -\frac{\pi\beta}{L}\frac{2\pi + g_0^2}{2\pi} \quad \text{and} \quad \lambda_2 = -\frac{\pi\beta}{L}\frac{2\pi}{2\pi + g_0^2} \tag{5.111}$$

with corresponding eigenvectors

$$v_1 = (-1, 1) \quad \text{and} \quad v_2 = (1, 1). \tag{5.112}$$

Also the η tensor (see 5.43) and μ_0 (see 5.36) in (5.107) simplify to

$$\eta_\mu{}^\nu = \begin{pmatrix} 0 & \beta/L \\ -L/\beta & 0 \end{pmatrix} \quad \text{and} \quad \mu_0 = -i\frac{\beta}{2\pi}\mu.$$

Now we can determine the low temperature limit of the grand potential from (5.109) (with $S_L = 0$) and (5.110-5.112]. For that we note that the saddle point approximation to the Gaussian sum (5.106) defining the theta-function becomes exact when $\beta \to \infty$. Also, using that

$$\log|\eta(\tau)|^2 \longrightarrow -\frac{\pi\beta}{6L} \qquad \text{for} \quad \beta \to \infty$$

we end up with

$$\begin{aligned}\Omega(\beta\to\infty) = &-\frac{\pi}{6L} - \frac{4\pi}{2\pi + g_0^2}\frac{\pi}{L}(\beta_1 + \frac{\mu L}{2\pi})^2 \\ &+\frac{\pi}{2L}\min_{n\in Z^2}\Big[\frac{2\pi + g_0^2}{2\pi}\{n_2 - n_1 - \frac{4\pi}{2\pi + g_0^2}(\beta_1 + \frac{\mu L}{2\pi})\}^2 \\ &\qquad + \frac{2\pi}{2\pi + g_0^2}\{n_1 + n_2 - 2\alpha_1\}^2\Big]\end{aligned} \tag{5.113}$$

for the zero-temperature grand potential of the un-gauged model. The chemical potential and chiral twist enter only through the combination $\beta_1 + \mu L/2\pi$.

Up to the second term the potential is invariant under

$$\alpha_1 \longrightarrow \alpha_1 + 1 \quad \text{and} \quad \beta_1 + \frac{\mu L}{2\pi} \longrightarrow \beta_1 + \frac{\mu L}{2\pi} + 1 + g_0^2/2\pi.$$

Let us now discuss the potential in the various *limiting cases.*

No chiral twist, $\beta_1 = 0$, and vanishing chemical potential: Then $\Omega(\beta \to \infty)$ coincides with the *ground state energy.* The minimum in (5.113) is attained for $n_1 = n_2 = [\frac{1}{2} + \alpha_1]$ and we find

$$E_0(L, \alpha_1, \beta_1 = 0) = -\frac{\pi}{6L} + \frac{2\pi}{L}\frac{2\pi}{2\pi + g_0^2}(\alpha_1 - [\frac{1}{2} + \alpha_1])^2. \tag{5.114}$$

Only for anti-periodic boundary conditions, that is for $\alpha_1 = 0$, does this Casimir energy coincide with the corresponding result for free fermions. For $g_0^2 \geq 4\pi$ the Casimir force is always attractive whereas for $g_0^2 < 4\pi$ it can be attractive or repulsive, depending on the value of α_1. The result (5.114) is in agreement with the literature [16]. For example, it coincides with De Vegas and Destri's result if we make the identification $\omega_{DD} = 2\pi\alpha_1$ and $1/\beta_{DD} = 1 + g_0^2/2\pi$ in formula (42) of that paper.

Small twists and chemical potential: For small β_1 and μ the minimum is assumed for $n_i = 0$ and the potential simplifies to

$$\Omega(\beta \to \infty) = -\frac{\pi}{6L} + \frac{2\pi}{L}\frac{2\pi}{2\pi + g_0^2}\alpha_1^2$$

and does not dependent on the chemical potential.
For *vanishing* g_0, that is for *free fermions*, the minimum of (5.113) is attained for

$$n_1 = [\frac{1}{2} + \alpha_1 - \beta_1 - \frac{\mu L}{2\pi}] \quad \text{and} \quad n_2 = [\frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi}],$$

where $[x]$ denotes the biggest integer which is smaller or equal to x. This then leads to the following zero temperature potential

$$\begin{aligned} \Omega = & - \frac{\pi}{6L} - \frac{2\pi}{L}(\beta_1 + \frac{\mu L}{2\pi})^2 \\ & + \frac{\pi}{L}\Big\{\alpha_1 - \beta_1 - \frac{\mu L}{2\pi} - [\frac{1}{2} + \alpha_1 - \beta_1 - \frac{\mu L}{2\pi}]\Big\}^2 \\ & + \frac{\pi}{L}\Big\{\alpha_1 + \beta_1 + \frac{\mu L}{2\pi} - [\frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi}]\Big\}^2. \end{aligned} \tag{5.115}$$

For $\mu = \beta_1 = 0$ this reduces to the Casimir energy for free fermions with left-right symmetric twists and agrees with the results in [32].

Note, however, that for $\beta_1 \neq 0$ we disagree with [16]. The difference is due to the second term on the right in (5.113). Let us give two arguments in favor of our result:

The discrepancy arises from the prefactor appearing in the fermionic determinant (5.46). As discussed earlier this prefactor implies the breakdown of holomorphic factorization, a property which has been presupposed in [16]. One can show that our results can be reproduced by starting with massive fermions and taking the limit $m \to 0$.

The second argument goes as follows: Suppose that $\beta_1 = \alpha_1 = 0$. Then (5.115) simplifies to

$$\Omega(\beta \to \infty) = -\frac{\pi}{6L} - \frac{2\pi}{L}(\frac{\mu L}{2\pi})^2 + \frac{2\pi}{L}(\frac{\mu L}{2\pi} - [\frac{1}{2} + \frac{\mu L}{2\pi}])^2. \tag{5.116}$$

For mass-less fermions the fermi energy is just μ and at $T=0$ all electron states with energies less then μ and all positron states with energies less then $-\mu$ are filled. The other states are empty. Since $d\Omega/d\mu$ is the expectation value of the electric charge in the presence of μ we see that it must jump if μ crosses an eigenvalue of the first quantized Dirac Hamiltonian h. For vanishing twists the eigenvalues of h are just $E_n = (n - \frac{1}{2})\pi/L$. Indeed, from (5.116) one finds that the electric charge

$$\langle Q \rangle = \frac{d\Omega}{d\mu} = 2[\frac{1}{2} + \frac{\mu L}{2\pi}] = 2n \qquad \text{for} \quad E_n \leq \mu < E_{n+1}$$

jumps at these values for μ. Further observe, that in the *thermodynamic limit* $L \to \infty$ the density

$$\frac{\Omega}{L} \to -\frac{2\pi}{2\pi + g_0^2}\frac{\mu^2}{2\pi},$$

reduces for $g_0 = 0$ to the standard result for free electrons.

Equation of state: We wish to derive the equation of state for finite T in the infinite volume limit $L \to \infty$. This may be achieved by interchanging the roles played by L and β. More precisely, using that

$$\Theta\begin{bmatrix} u \\ w \end{bmatrix}(\Lambda) = \sqrt{\det(i\Lambda^{-1})}\, e^{2\pi i w\cdot u}\, \Theta\begin{bmatrix} -w \\ u \end{bmatrix}(i\Lambda^{-1})$$

we find in analogy with the low temperature limit that for $L \to \infty$ the pressure is given by

$$\begin{aligned}
\beta p &= \lim_{L\to\infty}\frac{1}{L}\log\frac{Z}{\mathcal{N}_0}=\frac{\pi}{6\beta}+\frac{2\pi}{\beta}\frac{2\pi+g_0^2}{2\pi}\beta_0^2 \\
&- \frac{\pi}{2\beta}\min_{n\in Z^2}\Big[\frac{2\pi+g_0^2}{2\pi}\{n_1+n_2+2\beta_0\}^2 \\
&\qquad +\frac{2\pi}{2\pi+g_0^2}\{n_2-n_1+2\alpha_0+2i\frac{\beta\mu}{2\pi}\}^2\Big]
\end{aligned} \tag{5.117}$$

Here the minimum of the real part has to be taken. Again the minimization arises from the saddle point approximation to the theta function which becomes exact when $L\to\infty$. For *small twists* the minimum is assumed for $n_i=0$ and then

$$\beta p=\frac{\pi}{6\beta}-\frac{2\pi}{\beta}\frac{2\pi}{2\pi+g_0^2}(\alpha_0+i\frac{\beta\mu}{2\pi})^2$$

becomes independent on the chiral twist β_0. As we have interchanged the roles of the temporal and spatial twists this is consistent with the earlier result that for small twists Ω is independent of β_0. In particular, for $\alpha_1=0$, we are lead to the following equation of state

$$p(\beta,\mu,\alpha_0=0)=\frac{\pi}{6\beta^2}+\frac{\mu^2}{2\pi}\frac{2\pi}{2\pi+g_0^2},$$

which for small β_0 relates the pressure to the chemical potential and temperature. This result is consistent with the renormalization of the electric charge which is conjugate to the chemical potential. It shows in particular that the thermodynamic behavior of the Thirring model is not just the one of free fermions as has been claimed in [50]. Indeed, the zero point pressure is multiplied by a factor $2\pi/(2\pi+g_0^2)$. This modification arises from the coupling of the current to the harmonic fields. It can not be seen if only the local part of the auxiliary field is considered, which is the case if one quantizes the model on the infinite Euclidean space. Furthermore, we see that the 'pressure' p is real only for $\alpha_0=0$. This phenomenon occurs also in the Hamiltonian formalism [38]. However, finite temperature physics dictates anti-periodic boundary conditions, i.e $\alpha_0=0$, and then p becomes real.

5.4.2 Conformal structure

When we discussed the properties of the classical model (5.1) we have noticed that for $A_\mu=0$ it reduces to a conformal field theory on flat Minkowski spacetime. We have found the results listed at the end of section 2.
We determine the quantum corrections to these classical results. As in the previous sections we do that within the Euclidean functional approach. Thus we start from first principles and need not postulate the emerging

Kac-Moody and *Virasoro algebras* in advance [30, 22]. When comparing the classical with the quantum results one should keep in mind that roles of $\psi_0^\dagger$ and $\psi_1^\dagger$ are interchanged when one switches from Minkowski to Euclidean spacetime. For further changes the reader is referred to appendix A.

In what follows it is convenient to exploit the holomorphic structure of the model. On the torus with flat metric $\hat{g}_{\mu\nu}$ the Cauchy-Riemann equations read

$$(\eta_\mu{}^\nu \partial_\nu - i\partial_\mu) f = 0. \tag{5.118}$$

Then one chooses coordinates $x'^a = e^a{}_\mu x^\mu$ and the corresponding complex coordinates $x = x'^0 + ix'^1$ such that (5.118) takes the standard form. More explicitly we chose

$$x = i\bar{\tau}x^0 + ix^1 \quad \text{so that} \quad \partial_x = \frac{1}{2\tau_0}(\partial_{x^0} - \tau\partial_{x^1}).$$

In this section x and $\bar{x}$ always denote the complex coordinates belonging to x^μ. In these coordinates the free Dirac operator and the corresponding Green function are simple

$$i\partial\!\!\!/ = 2i\begin{pmatrix} 0 & \partial_x \\ \partial_{\bar{x}} & 0 \end{pmatrix} \quad \text{and} \quad S(x^\alpha, y^\beta) = \frac{1}{2\pi i}\begin{pmatrix} 0 & 1/\xi \\ 1/\bar{\xi} & 0 \end{pmatrix} + O(1),$$

where $\xi = x - y$. The chiral components of the energy momentum tensor and current are then given by

$$T_{xx} = \frac{\tau_0}{2i}(\tau T^{00} + T^{01}) = \frac{\tau_0}{2i}\frac{d\hat{g}\mu\nu}{d\bar{\tau}}T^{\mu\nu} \quad \text{and} \quad j_x = \frac{1}{2i}(\tau j^0 - j^1).$$

Using that the energy momentum tensor is conserved and traceless and that the vector and axial-vector currents are conserved it is easy to check that these chiral components only depend on x and not on $\bar{x}$.

Virasoro and Kac-Moody algebras First we determine the central charge from the short distance expansion of the T_{xx} correlators. As in the classical theory (see (5.12)) the symmetric energy momentum tensor measures the change of the effective action $\Gamma = \log Z$ under arbitrary variations of the metric. For the torus there are two independent contributions. One being due to variations of the modular parameter τ and its conjugate $\bar{\tau}$ which depend implicitly on the metric. The other is due to the variations of terms which depend explicitly on the metric. Since the chiral component T_{xx} is gotten by contracting $T^{\mu\nu}$ with $d\hat{g}_{\mu\nu}/d\bar{\tau}$ it follows that

$$\langle T_{xx}\rangle = \frac{i\tau_0}{\sqrt{g(x^\alpha)}}\Big(\frac{1}{L^2}\frac{\partial}{\partial\bar{\tau}} + \frac{d\hat{g}_{\mu\nu}}{d\bar{\tau}}\frac{\delta}{\delta g_{\mu\nu}(x^\alpha)}\Big)\,\Gamma[g,\tau,\bar{\tau}] \equiv \delta_x\Gamma[g,\tau,\bar{\tau}].$$

It is always understood when doing metric variations, that we take the flat spacetime limit afterward. The $\bar{\tau}$ variation is constant and may be skipped in the short distance expansion.
Taking several metric variations of the curvature dependent part of $\log Z$ with Z from (5.109), (5.78) and (5.73) we find the following short distance expansions for the three point correlation function

$$\langle T_{uu}\, T_{vv}\, T_{zz}\rangle \sim \; -\frac{3+24\pi g_3^2}{(2\pi)^3}\frac{1}{(u-v)^2(u-z)^2(v-z)^2}.$$

Comparing with the general expression [22] we read off the *central charge* and the conformal weight of the energy momentum tensor

$$c = 3 + 24 g_3^2 \pi \quad \text{and} \quad h_{T_{xx}} \;=\; 2\,. \tag{5.119}$$

The first contribution is that of three free fields. The g_3-dependent term we have already met in our classical analysis and comes from the coupling to the background curvature. It is well known from the minimal conformal series. Note that the couplings g_1 and g_2 do not affect the central charge. In particular, if we subtract the central charge of the auxiliary fields and set $g_3\!=\!0$ then c is the same as for the Thirring model, namely $c\!=\!1$ [22].

Next we determine the *Kac-Moody algebra* of the $U(1)$ currents. To derive the correlation functions with current insertions we couple the fermions to a gauge field, that is consider the 'gauged' model without Maxwell term. For example,

$$< j^\mu(x^\alpha)\; j^\nu(y^\beta) > = \frac{1}{e^2\sqrt{g(x^\alpha)g(y^\beta)}}\,\frac{\delta^2\Gamma[g,A]}{\delta A_\nu(x^\alpha)\delta A_\mu(y^\beta)}|_{A=0}.$$

Using (5.72) on flat spacetime and without Maxwell term, together with

$$\partial_\mu\phi = \eta_\mu{}^\nu A^T_\nu, \quad \text{where} \quad A^T_\mu = A_\mu - \frac{2\pi}{L}t_\mu - \nabla_\mu\frac{1}{\triangle}\nabla^\nu A_\nu$$

is the transversal part of A_μ, one obtains the following short distance expansion

$$\langle j_x\; j_y\rangle \;\sim -\frac{1}{2\pi}\frac{1}{2\pi+g_2^2}\frac{1}{(x-y)^2}\;.$$

We read off the value k of the *central extension* in the $U(1)$-Kac-Moody algebra to be

$$k = \frac{2\pi}{2\pi + g_2^2}. \tag{5.120}$$

Finally we need to determine the conformal weight of the current. From

$$\langle j_x\; j_y\; T_{zz}\rangle \;\sim -\frac{1}{4\pi^2}\frac{1}{2\pi+g_2^2}\frac{1}{(x-z)^2(y-z)^2}$$

we obtain $h_j = 1$. To summarize, the symmetry algebra is the semi-direct product of a Virasoro algebra with central charge (5.119) and a $U(1)$ Kac-Moody current algebra with central extension (5.120).

Conformal weights: To unravel the possible representations of the Virasoro algebra realized in the model we must determine the conformal weights of the fundamental fields. The short distance expansions of the *fermionic two-point function* with T_{zz} follows from the metric variation of the Green function

$$\langle\psi_0(x)\ \psi_1^\dagger(y)\rangle = S_{ij}(x,y)\cdot \exp\left[ig_1g_3\sigma(x) + \alpha G_R(x,x)\right] - [x\to y] - 2\alpha G(x,y)$$

where

$$\alpha = \frac{1}{4}\Big(g_1^2 - \frac{2\pi g_2^2}{2\pi + g_2^2}\Big).$$

and S_{ij} is the fermionic Green function in the external gravitational field and harmonic gauge field but with ϕ and λ set to zero. More precisely,

$$\langle\psi_0(x)\ \psi_1^\dagger(y)\ T_{zz}\rangle = \frac{1}{Z}\delta_z\Big(Z\langle\psi_0(x)\ \psi_1^\dagger(y)\rangle\Big).$$

However, since $Z\sim \exp[F(\mathcal{R}^2)]$, its metric variation vanishes after the flat spacetime limit has been taken. We refer to appendix B for the variation of S_{ij} and $G(x,y)$. Collecting the most singular terms, we arrive at

$$\begin{aligned}\langle\psi_0(x)\ \psi_1^\dagger(y)\ T_{zz}\rangle \ \sim\ & \frac{1}{2\pi i}\frac{1}{4\pi}\Big[\frac{1}{(z-x)(z-y)}(\frac{1}{z-x}-\frac{1}{z-y}) \\ -\frac{ig_1g_3}{x-y}(\frac{1}{(z-x)^2}-\frac{1}{(z-y)^2}) &+ \frac{\alpha}{2\pi}(\frac{1}{z-x}-\frac{1}{z-y})^2\Big]\ e^{2\alpha G(x,y)}.\end{aligned} \tag{5.121}$$

Using that

$$\partial_x e^{2\alpha G(x,y)} = -\partial_y e^{2\alpha G(x,y)} = -\frac{\alpha}{2\pi}\frac{1}{x-y}\ e^{2\alpha G(x,y)},$$

we find that the 2-point function varies under a infinitesimal conformal transformation, parametrized by $f(z)$, as

$$\begin{aligned}&\frac{1}{i}\oint dz f(z)\langle\psi_0(x)\ \psi_1^\dagger(y)\ T_{zz}\rangle = \Big\{f(x)\partial_x + f(y)\partial_y \\ &+\frac{1}{2}(1+\frac{\alpha}{2\pi})[f'(x)+f'(y)] - \frac{ig_1g_3}{2}[f'(x)-f'(y)]\Big\}\ \langle\psi_0(x)\psi_1^\dagger(y)\rangle.\end{aligned}$$

Note that the exponential factor has been absorbed to recover the correlation function $\langle\psi_0(x)\psi_1^\dagger(y)\rangle$. The short distance expansion with $T_{\bar z\bar z}$ is calculated similarly. Then one reads off the conformal weights

$$
\begin{aligned}
h_{\psi_0} &= \frac{1}{2} + \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi} \frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{i g_1 g_3}{2} \\
h_{\psi_1^\dagger} &= (h_{\psi_0})^\dagger \\
\bar{h}_{\psi_0} &= \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi} \frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{i g_1 g_3}{2} \;.
\end{aligned}
\tag{5.122}
$$

Thus we have reproduced the classical results supplemented by additional g_1 and g_2 dependent quantum corrections. In the Thirring model limit $g_3 = 0$ and $g_1 = g_2 = g$, these terms add up to give the known anomalous dimension appearing in the Thirring model [22]. The last classical term is a peculiar feature of the solution. For the conformal weight to be real we are obliged to choose g_3 imaginary.

Let us now turn to the *auxiliary fields*. It is straightforward to compute the correlators

$$
\begin{aligned}
\langle \lambda_x \, T_{zz} \rangle &\sim \frac{1}{4\pi} g_3 \frac{1}{(x-z)^2} \\
\langle \lambda_x \, \lambda_y T_{zz} \rangle &\sim -\frac{1}{32\pi^2} \frac{1}{(x-z)(y-z)} \\
\langle \phi_x \, \phi_y \, T_{zz} \rangle &\sim -\frac{1}{16\pi} \frac{1}{(x-z)(y-z)} .
\end{aligned}
\tag{5.123}
$$

We see that the classical results are unchanged, that is for $g_3 \neq 0$ the scalar field λ is not primary and for $g_3 = 0$ we find the conformal weights $h_\lambda = h_\phi = 0$.

Finally we turn to *vertex operators* or exponentials of the auxiliary fields. In contrast to λ and ϕ those are well defined even on the extended plane. Recalling the regularization prescription (5.91) we find

$$
\begin{aligned}
\langle : e^{\alpha_1 \phi(x)} :: e^{\alpha_2 \phi(y)} : T_{zz} \rangle \;\sim\; & -\frac{1}{16\pi} \frac{1}{2\pi + g_2^2} \Big[\frac{\alpha_1}{z-x} + \frac{\alpha_2}{z-y} \Big]^2 \\
& \cdot \langle : e^{\alpha_1 \phi(x)} :: e^{\alpha_2 \phi(y)} : \rangle
\end{aligned}
\tag{5.124}
$$

and hence

$$
\begin{aligned}
\frac{1}{i} \int_C f(z) \langle : e^{\alpha_1 \phi(x)} :: e^{\alpha_2 \phi(y)} : T_{zz} \rangle \sim & \Big[f(x) \partial_x + f(y) \partial_y \\
- \frac{1}{8(2\pi + g_2^2)} \big(\alpha_1^2 f'(x) + \alpha_2^2 f'(y) \big) \Big] & \langle : e^{\alpha_1 \phi(x)} :: e^{\alpha_2 \phi(y)} : \rangle \; .
\end{aligned}
\tag{5.125}
$$

From this we read off the conformal weights of the vertex operators

$$h_i = \bar{h}_i = -\frac{\alpha_i^2}{8(2\pi + g_2^2)} \ . \tag{5.126}$$

Note that α_i must be imaginary to get a positive weight. A similar analysis for the λ-field yields

$$\begin{aligned}\frac{1}{i}\int_C f(z)\langle : e^{\alpha_1\lambda(x)} :: e^{\alpha_2\lambda(y)} : T_{zz}\rangle \ \sim \Big[f(x)\partial_x + f(y)\partial_y \\ -\frac{\alpha_1}{2}(\frac{\alpha_1}{8\pi}+g_3)f'(x) - \frac{\alpha_2}{2}(\frac{\alpha_2}{8\pi}+g_3)f'(y)\Big]\langle : e^{\alpha_1\lambda(x)} :: e^{\alpha_2\lambda(y)} :\rangle\end{aligned} \tag{5.127}$$

and hence

$$h_i = -\frac{1}{2}\alpha_i\,(\frac{\alpha_i}{8\pi} + g_3) \ . \tag{5.128}$$

Here both α_i and g_3 must be imaginary for the weights to be positive. Note that contrary to λ the fields $: e^{\alpha\lambda(x)} :$ remain primary when the $\lambda\mathcal{R}$ coupling is switched on. This coupling results only in a shift of the conformal weights.

$U(1)$**-charges:** To see how the left and right Kac Moody currents act on the fermionic fields we notice that after the integration over the auxiliary fields the A-dependence of the fermionic Green function factorizes as

$$\langle\psi_0(x)\psi_1^\dagger(y)\rangle_A = e^{\frac{1}{2}m_\gamma\int\varphi\triangle\varphi}\cdot e^{-eg(x)}\ \langle\psi_0(x)\psi_1^\dagger(y)\rangle_{A=0}\ e^{-e\bar{g}(y)},$$

where

$$g(x) = -i\alpha(x) + \gamma_5\beta\varphi(x), \qquad \beta = \frac{2\pi}{2\pi + g_2^2}.$$

Also, using that on flat spacetime

$$\begin{aligned}\phi(x) &= -i\int\partial_z G(x,z)A^z + i\int\partial_{\bar{z}}G(x,z)A^{\bar{z}} \\ \alpha(x) &= \int\partial_z G(x,z)A^z + \int\partial_{\bar{z}}G(x,z)A^{\bar{z}},\end{aligned} \tag{5.129}$$

one ends up with

$$\begin{aligned}\langle\psi_0(x)\psi_1(y)^\dagger j_z\rangle &= \frac{1}{4\pi i}\Big[\frac{4\pi + g_2^2}{2\pi + g_2^2}\,\frac{1}{z-x} + \frac{g_2^2}{2\pi + g_2^2}\,\frac{1}{z-y}\Big]\langle\psi_0(x)\psi_1(y)^\dagger\rangle \\ \langle\psi_0(x)\psi_1(y)^\dagger j_{\bar{z}}\rangle &= \frac{1}{4\pi i}\Big[\frac{g_2^2}{2\pi + g_2^2}\,\frac{1}{\bar{z}-\bar{x}} + \frac{4\pi + g_2^2}{2\pi + g_2^2}\,\frac{1}{\bar{z}-\bar{y}}\Big]\langle\psi_0(x)\psi_1(y)^\dagger\rangle\end{aligned}$$

and thus obtains the following the $U(1)$ charges

$$q_{\psi_0} = \frac{1}{2}(1 + \frac{2\pi}{2\pi + g_2^2}) \quad , \quad \bar{q}_{\psi_0} = \frac{1}{2}(1 - \frac{2\pi}{2\pi + g_2^2}). \tag{5.130}$$

We have used the convention where the electric charge $q+\bar{q}$ is unity. In the Thirring model limit we can compare (5.130) with the results obtained in [22]. For that we need to rescale the currents such that the central extension (5.120) of the Kac-Moody algebra becomes unity

$$j_z \to \sqrt{1 + g_2^2/2\pi}\, j_z \ .$$

Now it is easy to see that we agree with [22] if we make the identification

$$\bar{g}_{Fu} = \frac{g_2^2}{4\pi} \frac{1}{\sqrt{1 + g_2^2/2\pi}} \ .$$

To summarize, what we have found is that the classical conformal and axial transformations of all fields besides ϕ and λ are deformed. The longitudinal part of the current-current interaction in (5.1) changes the conformal weights of the fermion field only. The transversal part affects all weights and $U(1)$-charges. The background charge changes the conformal weight of the vertex operators belonging to the scalar field.
Of course, the same structure is found in the other chiral sector.

5.4.3 Finite size effects

When quantizing a conformal field theory on a spacetime with finite volume one introduces a length scale. The presence of this length scale in turn breaks the conformal invariance and gives rise to finite size effects. It has been conjectured [12] that the finite size effects are proportional to the central charge. For example when one stretches space time, $x^\alpha \to a x^\alpha$, then the change of the effective action is proportional to c:

$$\Gamma_{ax} - \Gamma_x = -\frac{c}{6} \log a \cdot \chi, \tag{5.131}$$

where χ is the Euler number of the Euclidean space time. In [17] this conjecture has been proven for a class of conformal field theories on spaces with boundaries. The only important assumption has been that the regularization respects general covariance. In this subsection we shall show that the conjecture does not hold for the model (5.1) on Riemannian surfaces.

Unfortunately, the only global conformal transformations on the torus are translations which do not give rise to finite size effects. Also, the Euler number vanishes and according to (5.131) the finite size effects are insensitive to the value of c. For that reason we quantize the ungauged model (5.1) on the sphere where the global conformal group is the Moebius group.

An effective method to compute finite size effects has been developed in [17]. It is based on the following observation: Any conformal transformation $z \to w(z)$ is a composition of a diffeomorphism (defined by the same w) and a compensating Weyl transformation $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$ with

$$e^{2\sigma} = \frac{dw(z)}{dz}\frac{d\bar{w}(\bar{z})}{d\bar{z}}, \qquad z = x^0 + ix^1.$$

Therefore, choosing a diffeomorphism invariant regularization one has

$$0 = \delta\Gamma_{Diff} = \delta\Gamma_{Conf} - \delta\Gamma_{Weyl}.$$

Now we apply the techniques of the previous sections to derive the change $\delta\Gamma_{Weyl}$ of the effective action on the sphere under Weyl transformations. This change is given by the trace anomaly.
The change of the effective action under Weyl rescaling is

$$\delta\Gamma_{Weyl} = -\log \frac{\int \mathcal{D}(\lambda\phi)\det(i\not{D})\exp(-S_B[A=0,g])}{\int \mathcal{D}(\lambda\phi)\det(i\hat{\not{D}})\exp(-S_B[A=0,\hat{g}])},$$

where S_B is the bosonic action (5.61) with vanishing gauge field. Also, since on the sphere there are no harmonic vector fields the term $\sim h^2$ in S_B is not present. Thus the calculation on the sphere is actually simpler as on the torus (see 5.105) since there is no integration over the harmonic fields. As on the torus we must impose the conditions (5.64) in order to eliminate the additional degrees of freedom we introduced in the derivative coupling representation. Thus we obtain

$$\delta\Gamma_{Weyl} = \log\frac{\hat{V}}{V} - \frac{S_L}{24\pi} + +\frac{g_3^2}{4}\int \mathcal{R}\frac{1}{\triangle}\mathcal{R} + \log\frac{\det'\triangle}{\det'\hat{\triangle}}. \tag{5.132}$$

Here we used that (5.55) in the trivial sector still holds on the sphere. Also we used the scaling law (5.70). S_L is the Liouville action (5.56) in which we can not put $\hat{\mathcal{R}}$ to zero, since

$$\int \sqrt{g}\mathcal{R} = 8\pi = 4\pi\chi$$

for any curvature and thus in particular for $\hat{\mathcal{R}}$. As for the fermions (see 5.51) one introduces the 1-parametric family of Laplacians

$$\triangle_\tau = e^{-2\tau\sigma}\hat{\triangle}$$

interpolating between $\hat{\triangle}$ and $\triangle$. The τ derivative of the corresponding determinant is given by the trace anomaly [39, 17]. The explicit calculation yields

$$\log \det{}' \frac{\triangle}{\hat{\triangle}} = 2\int_0^1 d\tau \int \sqrt{g^\tau}\Big(-\frac{1}{4\pi}a_1^\tau - P^\tau\Big)\sigma, \tag{5.133}$$

Again g^τ is the determinant and $a_1^\tau = \frac{1}{6}\mathcal{R}^\tau$ the relevant Seeley-deWitt coefficient of the deformed metric $g^\tau_{\mu\nu} = e^{2\tau\sigma}\hat{g}_{\mu\nu}$. P^τ is the projection onto the zero-mode of $\triangle^\tau$. Using that the normalized zeromode is constant and $\sim 1/\sqrt{V^\tau}$, one finds

$$\log \det{}' \frac{\triangle}{\hat{\triangle}} = \log\frac{V}{\hat{V}} + \frac{1}{12\pi}S_L.$$

The $\sim \log V$ chancel against the same term in (5.132) and we end up with

$$\delta\Gamma = \frac{g_3^2}{4}\int \sqrt{g}\mathcal{R}\frac{1}{\triangle}(\mathcal{R} - \frac{8\pi}{V}) - \frac{3}{24\pi}\int \sqrt{\hat{g}}\hat{\mathcal{R}}\sigma + \frac{3}{24\pi}\int\sqrt{\hat{g}}\sigma\hat{\triangle}\sigma \tag{5.134}$$

which depends only on g_3. Now we can see why the finite size conjecture generally fails to be true, although it holds for theories without background charge on domains with boundaries [17]. Take the simple case of a dilatation $w(z) = az$. Then, the conformal angle is a constant $\sigma = \log a$ and $(\mathcal{R} - 8\pi/V) = 0$. Then the first term in (5.134) vanishes and the finite size effect does not depend on g_3^2. It is given by

$$\delta\Gamma = -\frac{3}{24\pi}\log a \int \sqrt{\hat{g}}\hat{\mathcal{R}} = -\log a$$

and does not agrees with (5.131) since c in (5.119) depends on g_3. Thus we have disproved the conjecture. On other Riemannian surfaces one would find the same result: the effective action scales as in (5.119) where c is the central charge of the model without background charge. It is evident that the finite size scaling comes from the middle term $\sim \log a \int \sqrt{\hat{g}}\hat{R}$ in (5.134).

It is interesting to compare the finite size scaling on Riemannian surfaces with the one on domains with boundaries. In the presence of boundaries (5.133) is modified to

$$\log \det{}' \frac{\triangle}{\hat{\triangle}} = -\frac{1}{2\pi}\int_0^1 d\tau \Big(\int \sqrt{g^\tau} a_1^\tau\,\sigma + \oint \sqrt{\tilde{g}^\tau} b_1^\tau\,\sigma\Big), \tag{5.135}$$

where the second integral is over the boundary of spacetime and $\tilde{g}_{\mu\nu}$ the induced metric on this boundary. On a domain we can always put $\hat{R}$ to zero and the middle term in (5.135) does not contribute to the scaling. The scaling comes from the surface term in (5.135). Diffeomorphism invariance implies that the bulk term determines the surface term (up to diffeomorphism invariant surface terms). This is how the central charge, defined by the short distance expansion of the T_{zz}-correlators and thus by the bulk term, re-emerges in the scaling law (5.131), which is determined by the surface term.

Appendix A

Conventions

In this appendix we set up our notation and give a list of useful formulas. Let $g_{\mu\nu}$ be the metric of spacetime. The sign convention for the curvature tensors is such that

$$\mathcal{R}^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\delta\beta,\gamma} - \Gamma^{\alpha}_{\gamma\beta,\delta} + \Gamma^{\sigma}_{\delta\beta}\Gamma^{\alpha}_{\gamma\sigma} - \Gamma^{\sigma}_{\gamma\beta}\Gamma^{\alpha}_{\delta\sigma} \quad \text{and} \quad \mathcal{R}_{\beta\delta} = \mathcal{R}^{\alpha}_{\beta\alpha\delta}. \tag{A.1}$$

In 2 dimensions the only independent component is $\mathcal{R}_{0101}$. In order to couple fermions to gravity we must introduce a local Lorentz frame (or tetrad), $e_{\mu a}$, relating the Lorentz and spacetime indices:

$$e_{\alpha a} e_{\beta}{}^{a} = g_{\alpha\beta} \quad , \quad e_{\alpha a} e^{\alpha}{}_{b} = \eta_{ab} \quad , \quad \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The Latin and Greek indices are Lorentz and spacetime indices, respectively. All physical laws should be general- and Lorentz covariant. If $g_{\alpha\beta}$ has Euclidean signature then η_{ab} in (A.2) is changed to δ_{ab}.

The 'curved' gamma matrices are related to the flat ones as

$$\gamma^{\mu} = e^{\mu}{}_{a}\tilde{\gamma}^{a}. \tag{A.3}$$

We us the following chiral representation for the flat γ's:

$$\hat{\gamma}^{0}_{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad , \qquad \hat{\gamma}^{1}_{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.4}$$

and in Euclidean spacetime we may choose

$$\hat{\gamma}^{0}_{E} = \hat{\gamma}^{0}_{M} \qquad , \qquad \hat{\gamma}^{1}_{E} = i\hat{\gamma}^{1}_{M}. \tag{A.5}$$

We may also define

$$\tilde{\gamma}_5 = \hat{\gamma}^0_M \hat{\gamma}^1_M = -i\hat{\gamma}^0_E \hat{\gamma}^1_E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.6}$$

The relations

$$\tilde{\gamma}^a_M \tilde{\gamma}_5 = \epsilon^a{}_b \tilde{\gamma}^b \quad , \quad \tilde{\gamma}^a_E \tilde{\gamma}_5 = -i\epsilon^a{}_b \tilde{\gamma}^b, \quad \text{where} \quad \epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.7}$$

are particular to 2 dimensions and play an important role in this chapter. Note that depending whether one is in Minkowskian or Euclidean spacetime the Lorentz index a is raised with η^{ab} or δ^{ab}. The curved space analogue of (A.6) reads

$$\gamma_5 = \frac{1}{2}\eta_{\mu\nu}\gamma^\mu_M \gamma^\nu_M = \frac{1}{2i}\eta_{\mu\nu}\gamma^\mu_E \gamma^\nu_E = \tilde{\gamma}_5, \tag{A.8}$$

where $\eta_{\mu\nu} = \sqrt{|g|}\epsilon_{\mu\nu}$ is the antisymmetric tensor (whereas the flat metric has Lorentz- indices, the antisymmetric tensor has space-time indices). To implement local Lorentz invariance one needs to introduce a connection $\omega_{\mu ab}$. For example, in the Lagrangian the Lorentz-covariant derivative acting on the spinors read

$$D_\mu = \partial_\mu + i\omega_\mu, \tag{A.9}$$

where the spin connection ω_μ is defined by

$$\begin{aligned} D_\mu &\equiv \partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \omega_{\mu ab} e_\nu{}^b = 0, \\ \omega_\mu &= \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}, \quad \Sigma^{ab} = \frac{1}{4i}[\tilde{\gamma}^a, \tilde{\gamma}^b]. \end{aligned} \tag{A.10}$$

In 2 dimensions this reduces to

$$\omega^M_\mu = \frac{1}{2i}\omega_{\mu 01}\gamma_5 \qquad \text{or} \qquad \omega^E_\mu = \frac{1}{2}\omega_{\mu 01}\gamma_5. \tag{A.11}$$

Finally we list some useful scaling relations. If the 2-bein scales as $e_\mu{}^a = e^\sigma \hat{e}_\mu{}^a$ then the above introduced quantities scale as

$$\begin{aligned} g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu} \quad , \quad \sqrt{g} = e^{2\sigma}\sqrt{\hat{g}} \quad , \quad \mathcal{R} = e^{-2\sigma}(\hat{\mathcal{R}} - 2\hat{\triangle}\sigma) \\ \omega_{\mu ab} = \hat{\omega}_{\mu ab} - \partial_a\sigma\hat{e}_{\mu b} + \partial_b\sigma\hat{e}_{\mu a}, \\ \Gamma^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu} + \left(\partial_\mu\sigma\delta^\alpha_\nu + \partial_\nu\sigma\delta^\alpha_\mu - \partial_\beta\sigma\hat{g}^{\beta\alpha}\hat{g}_{\mu\nu}\right), \\ \triangle = e^{-2\sigma}\hat{\triangle} \quad , \quad \not{\partial} + i\not{\omega} = e^{-\frac{3}{2}\sigma}(\hat{\not{\partial}} + i\hat{\not{\omega}})e^{\frac{1}{2}\sigma}. \end{aligned} \tag{A.12}$$

Appendix B

Variational formulas

Here we collect some useful variational formulas. In the following D_μ denotes the spacetime and Lorentz covariant derivative. How it acts on spacetime and Lorentz tensors follows from the first formula in (A.10).
Using the definition of the Christoffel symbol and (A.2) it is straightforward to show that

$$\begin{gathered}
\delta g_{\mu\nu} = \delta e_\mu{}^a e_{\nu a} + e_\mu{}^a \delta e_{\nu a} \quad , \quad \delta\sqrt{g} = \frac{1}{2}\sqrt{g} g^{\mu\nu}\delta g_{\mu\nu} \\
\delta\gamma^\mu = -\gamma^\nu e^\mu{}_a \delta e_\nu{}^a \quad , \quad \delta\eta_\mu{}^\nu = \frac{1}{2}(\eta^{\alpha\nu}\delta g_{\mu\alpha} - \eta_\mu{}^\sigma g^{\nu\rho}\delta g_{\sigma\rho}) \\
\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(D_\nu \delta g_{\beta\mu} + D_\mu \delta g_{\beta\nu} - D_\beta \delta g_{\mu\nu}).
\end{gathered} \tag{B.1}$$

For some formulas related to the variation of the tetrad let us refer to [36]

$$\begin{gathered}
\delta e^\mu{}_a = \frac{1}{2} e_{\nu a}\delta g^{\mu\nu} - t_a{}^b e^\mu{}_b \quad , \quad \delta e_\mu{}^a = \frac{1}{2} e^{\nu a}\delta g_{\mu\nu} - t^a{}_b e_\mu{}^b \\
\text{where} \quad t^a{}_b = \frac{1}{2}(e^{\nu a}\delta e_{\nu b} - e^\nu{}_b \delta e_\nu{}^a).
\end{gathered} \tag{B.2}$$

Then using (A.10) it is easy to see that

$$\delta\omega_{\mu ab} = D_\mu t_{ab} - \alpha_{\mu ab} \qquad \alpha_{\mu ab} = \frac{1}{2} e^\alpha{}_a e^\beta{}_b (D_\alpha \delta g_{\beta\mu} - D_\beta \delta g_{\alpha\mu}). \tag{B.3}$$

When performing the variation of curvature dependent expressions we have used the identities

$$\begin{gathered}
g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \omega^\alpha_{;\alpha} \, , \quad \text{where} \quad \omega^\alpha = g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\alpha\nu}\delta\Gamma^\mu_{\mu\nu} \\
\text{and} \quad \int \sqrt{g}\,\omega^\alpha A_\alpha = \int \sqrt{g}\{g^{\alpha\beta}\nabla_\mu A^\mu - \nabla^\alpha A^\beta\}\delta g_{\alpha\beta} \; .
\end{gathered} \tag{B.4}$$

Depending on the topology of spacetime, the induced curvature $\hat{\mathcal{R}}$ appearing in (A.12) may be different from zero. In this case it is not possible to express the conformal angle σ in terms of the curvature scalar. Nevertheless, to perform variations of σ-dependent expressions, the identity

$$\delta(\sqrt{g}\mathcal{R}) = -2\delta(\sqrt{g}\triangle\sigma) \tag{B.5}$$

proves to be useful.
Taking the variations of the equations

$$\sqrt{g}\Delta G(x,y) = -\delta(x-y) \quad \text{and} \quad \sqrt{g}\, i\not{D}S(x,y) = \delta^2(x-y) \tag{B.6}$$

for the scalar and fermionic Green functions we may derive (up to contact terms) the following variational formulas

$$\delta G = \int \,(-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} + g^{\alpha\mu}g^{\beta\nu})\partial_\alpha G(x,u)\,\partial_\beta G(u,y)\sqrt{g}\delta g_{\mu\nu}$$

$$\delta S = \frac{i}{4}\int \,\Big(2S(x,u)\gamma^\mu D^\nu S(u,y) - D_\alpha[S(x,u)\gamma^\delta \eta_\delta{}^\mu \eta^{\nu\alpha} S(u,y)]\Big)\sqrt{g}\delta g_{\mu\nu},$$

here all arguments and derivatives which are not made explicit in the integral refer to the coordinate u over which is integrated. Finally, we need the following formula for the variation of the inverse Laplacian

$$\delta\left(\frac{1}{\triangle}f\right) = \frac{1}{\triangle}\left(\delta f - \delta(\triangle)\frac{1}{\triangle}f\right) - \frac{1}{2V}\int \sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}\frac{1}{\triangle}f, \tag{B.7}$$

where V is the volume of spacetime and f an arbitrary function. To prove this identity we note that for $f \in (\mathrm{Kern}\triangle)^\perp$ we have

$$\triangle\frac{1}{\triangle}f = f.$$

Varying this equation yields

$$\triangle(\delta\frac{1}{\triangle}f) = \delta f - (\delta\triangle)\frac{1}{\triangle}f$$

which may be inverted to give

$$\delta\left(\frac{1}{\triangle}f\right) = \frac{1}{\triangle}\left(\delta f - \delta(\triangle)\frac{1}{\triangle}f\right) + \frac{1}{V}\int \sqrt{g}\delta\left(\frac{1}{\triangle}f\right)\,. \tag{B.8}$$

Varying the identity

$$\frac{1}{V}\int \sqrt{g}\frac{1}{\triangle}f = 0$$

allows to replace the last term of (B.8) to obtain the required result (B.7).

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