Chapter 4

Hamiltonian Reduction of WZNW Theories

Two dimensional conformally invariant field theories are based on various extensions of the chiral Virasoro algebras. The best-known extensions is the Kac-Moody (KM) algebra [23], whose most prominent Lagrangian realization is the Wess-Zumino-Witten-Novikov (WZNW) model [50]. Another extension is the so-called $\mathcal{W}$-extension [52], which is a polynomial extension of the Virasoro algebra by higher spin fields. These $\mathcal{W}$-algebras proved very fruitful in analyzing conformal field theories and they have become the subject of intense study (see [13] for a review on these algebras). It has been realized by Gervais and Bilal that Toda theories provide a Lagrangian realization of $\mathcal{W}$-algebras [9]. In [20] we have shown that the exact relationship is that Toda theories may be regarded as WZNW models reduced by conformally invariant constraints. More precisely, Toda theories can be identified as the constrained WZNW models, modulo the left-moving upper triangular and right-moving lower triangular KM transformations, which are the gauge transformations generated by the first class constraints.

The constrained WZNW (KM) setting of the Toda theories ($\mathcal{W}$-algebras) calls for generalizations, some of which have been investigated. For example, in [39] the reduction was generalized to produce a series of conformally invariant integrable theories which interpolate between the WZNW and Toda theories. These theories contain WZNW fields belonging to reducible WZNW groups, with the irreducible pieces in nearest neighbor interaction, thus providing a natural generalization of Toda theories. A remarkable feature of the theories is the emergence of a field which plays the role of the two-dimensional gravitational density $\sqrt{-g}$. Further features are the ease

with which the general solutions of the field equations in these theories can be obtained from the well-known WZNW solutions, and the formula for the centers of the Virasoro algebra in terms of the WZNW center.
Also, it has been realized in [5, 18] that it is possible to associate a generalized $\mathcal{W}$-algebra to every embedding of the Lie algebra $sl(2)$ into the simple Lie algebras. The standard $\mathcal{W}$-algebra occurring in Toda theory, corresponds to the so called principal $sl(2)$. Another interesting development is the W_n^l algebras introduced by Bershadsky [8]. It is known that the simplest non-trivial case W_3^2, which was originally proposed by Polyakov [44], is a special $sl(2)$-based $\mathcal{W}$-algebra. the classification based on $sl(2)$ embeddings
In addition, the whole construction has been supersymmetrized by constraining super-WZNW theories [14]. As in the bosonic case one finds that the reduced supersymmetric theories contain super $\mathcal{W}$-algebras as non-linear symmetry algebras. Here the classification of the $\mathcal{W}$-algebras is based on $OSp(1|2)$ embeddings in a simple superalgebra $\mathcal{G}$. A specially simple example where the algebra closes linearly is the $N = 1$ superconformal algebra made from the stress energy tensor and a conformal spin 3/2 fermionic field.

Here we undertake a systematic study of the Hamiltonian reductions of WZNW theory, aiming at uncovering the general structure of the reduction. We shall derive the effective field theories (some of them will contain fields with half-integer spins) which contain generalized $\mathcal{W}$-algebras as symmetry algebras and investigate the relation between the different $\mathcal{W}$-algebras. We give purely Lie-algebraic conditions for the constraints to be first class, conformally invariant and that they lead to a polynomial extension of the Virasoro algebra. Finally we investigate the quantum reduction of WZNW theories and derive the central charge for the effective reduced theories for arbitrary reductions.

We start with recalling, that WZNW-theories are field theories for group valued fields $g(x) \in G$ with action [1]

$$S_{\mathrm{WZ}}(g) = \frac{\kappa}{2} \int d^2x \, \mathrm{Tr}\, (g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g) - \frac{\kappa}{3} \int_{B_3} \mathrm{Tr}\, (g^{-1}dg)^3. \tag{4.1}$$

We assume that G is a a simple, maximally non-compact, connected real Lie group or in other words that the simple Lie algebra, $\mathcal{G}$, corresponding to G allows for a Cartan decomposition over the field of real numbers. Thus $\mathcal{G}$ is defined as the real span of a Chevalley basis H_i, $E_{\pm\alpha}$ of the corresponding complex Lie algebra $\mathcal{G}_c$, and in the case of the classical series A_n, B_n, C_n and D_n is given by $sl(n+1, R)$, $so(n, n+1, R)$, $sp(2n, R)$ and $so(n, n, R)$, respectively. The Cartan-Killing form of $\mathcal{G}$ is denoted by $\langle .,. \rangle \equiv \mathrm{Tr}\,(..)$.

The field equation of the WZNW theory can be written in the equivalent

[1] The KM level k is $-4\pi\kappa$. The space-time conventions are: $\eta_{00} = -\eta_{11} = 1$ and $x^{\pm} = \frac{1}{2}(x^0 \pm x^1)$. The WZNW field g is periodic in x^1 with period $2\pi r$.

forms

$$\partial_- J = 0 \quad \text{or} \quad \partial_+ \tilde{J} = 0, \tag{4.2}$$

where

$$J = \kappa \partial_+ g \cdot g^{-1} \quad \text{and} \quad \tilde{J} = -\kappa g^{-1} \partial_- g. \tag{4.3}$$

These equations express the conservation of the left- and right KM currents, J and $\tilde{J}$, respectively. The general solution of the field equation have the simple form

$$g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-), \tag{4.4}$$

where g_L and g_R are arbitrary G-valued functions, constrained only by the boundary conditions imposed on g.

In what follows we shall need the remarkable Polyakov-Wiegmann identity [43],

$$\begin{aligned} S_{WZ}(abc^{-1}) &= S_{WZ}(a) + S_{WZ}(b) + S_{WZ}(c^{-1}) \\ &- \kappa \int \mathrm{Tr} \left\{ (a^{-1}\partial_- a) bc^{-1} (\partial_+ c) b^{-1} \right\} \\ &+ \kappa \int \mathrm{Tr} \left\{ (a^{-1}\partial_- a)(\partial_+ b) b^{-1} - (b^{-1}\partial_- b) c^{-1} \partial_+ c \right\}. \end{aligned} \tag{4.5}$$

4.1 Gauging the WZNW theories

For gauging the WZNW theories we couple the fields a and c in (4.5) minimally to gauge potentials, that is replace the ordinary derivatives by covariant ones

$$D_- a = \partial_- a + A a \quad \text{and} \quad D_+ c = \partial_+ c - \tilde{A} c,$$

which transform covariant under the *gauge transformations*

$$\begin{array}{ccc} a \to e^{\alpha} a, \quad c \to e^{\tilde{\alpha}} c & \Longleftrightarrow & g \to e^{\alpha} g e^{-\tilde{\alpha}} \\ A \to e^{\alpha} A e^{-\alpha} + e^{\alpha} \partial_- e^{-\alpha} & ; & \tilde{A} \to e^{\tilde{\alpha}} \tilde{A} e^{-\tilde{\alpha}} + (\partial_+ e^{\tilde{\alpha}}) e^{-\tilde{\alpha}}. \end{array} \tag{4.6}$$

The b field in the decomposition of g is gauge invariant. Clearly, if we replace the derivatives of a, c in (4.5) by covariant ones and if we drop the WZ-action of a and c^{-1}, then the resulting action

$$
\begin{aligned}
I' &= S_{WZ}(b) - \kappa \int d^2x \mathrm{Tr} \left\{ (a^{-1}D_-a)bc^{-1}(D_+c)b^{-1} \right\} \\
&+ \kappa \int d^2x \mathrm{Tr} \left\{ (a^{-1}D_-a)(D_+b)b^{-1} - (b^{-1}D_-b)c^{-1}D_+c \right\}
\end{aligned}
\tag{4.7}
$$

would be gauge invariant. However, in general this action cannot be re-expressed in terms of the original field g. For example, for vanishing gauge potential this is only possible if we would add the WZ-actions of a and c^{-1}. However, such terms are are not gauge invariant. The way out is to assume that S_{WZ} vanish on these fields. This is equivalent to assuming that a and c vary in subgroups of G with Lie-algebras Γ and $\tilde{\Gamma}$, respectively, which have the property $\Gamma \subset \Gamma^{\perp}$ and $\tilde{\Gamma} \subset \tilde{\Gamma}^{\perp}$. Of course, the gauge potentials lie then also in these subalgebras, $A \in \Gamma$ and $\tilde{A} \in \tilde{\Gamma}$.

With these assumptions the gauge-invariant action (4.7) can be written in terms of the original field, up to a term

$$
\kappa \int \mathrm{Tr} \left\{ A(\partial_+a)a^{-1} - \tilde{A}(\partial_-c)c^{-1} \right\}.
$$

But because A and $(\partial_+a)a^{-1}$ are both in Γ and we assumed that $\Gamma \subset \Gamma^{\perp}$ (and similarly for $\tilde{\Gamma}$) this difference vanishes. However, the resulting gauge-theories are still rather uninteresting, they are essentially WZNW-theories for the gauge-invariant field b. To get interesting new theories we couple the gauge field to constant elements M and $\tilde{M}$ and define

$$
I = I' - \kappa \int \mathrm{Tr} \left(AM + \tilde{A}\tilde{M} \right).
$$

which, with our assumption on $\Gamma, \tilde{\Gamma}$ can be rewritten as

$$
\begin{aligned}
I(g, A, \tilde{A}) \equiv S_{\mathrm{WZ}}(g) &+ \kappa \int d^2x \Big(\mathrm{Tr} \ (A(\partial_+gg^{-1} - M) \\
&+ \tilde{A}(g^{-1}\partial_-g - \tilde{M}) + Ag\tilde{A}g^{-1} \Big).
\end{aligned}
\tag{4.8}
$$

Later we shall see that for particular choices of $M, \tilde{M}$ the reduced theories are interesting interacting Toda-type theories. Note that the terms containing M and $\tilde{M}$ are not invariant under the general transformations (4.6). However, they become invariant if we assume that M is orthogonal to the derived algebra $[\Gamma, \Gamma]$. For example, under an infinitesimal gauge transformation belonging to $e^{\alpha} \simeq 1 + \alpha$, the term $\langle A, M \rangle$ changes by

$$
\delta \langle A, M \rangle = -\langle \partial_-\alpha, M \rangle + \langle M, [\alpha, A] \rangle \ ,
$$

which is a total divergence since with our assumption on Γ the second term vanishes, as both A and α are from Γ. This then proves that the action

(4.8) is gauge invariant, provided we impose the following conditions on M and Γ: for $\alpha, \beta \in \Gamma$

$$\boxed{[\alpha, \beta] \in \Gamma, \qquad \langle \alpha, \beta \rangle = 0 \quad \text{and} \quad \omega_M(\alpha\,, \beta) = 0,} \tag{4.9}$$

where I introduced the anti-symmetric Kostant-Kirillov 2-form on $\mathcal{G}$:

$$\omega_M(u, v) \equiv \langle M, [u, v] \rangle \quad \text{for a fixed} \quad M \in \mathcal{G} \quad \text{and} \quad \forall u, v \in \mathcal{G}. \tag{4.10}$$

This means that Γ is a subalgebra on which the Cartan-Killing form and ω_M vanish. Of course, we must impose exactly the same conditions on $\tilde{M}, \tilde{\Gamma}$. It is easy to see that the 3 conditions in (4.9) can be equivalently written as

$$[\Gamma, \Gamma^{\perp}] \subset \Gamma^{\perp}, \qquad \Gamma \subset \Gamma^{\perp} \qquad \text{and} \qquad [M, \Gamma] \subset \Gamma^{\perp}\ , \tag{4.11}$$

respectively. Subalgebras Γ satisfying $\Gamma \subset \Gamma^{\perp}$ exist in every real form of the complex simple Lie algebras except the compact one, since for the compact real form the Cartan-Killing inner product is (negative) definite. Now we have the following

Lemma 3 $\Gamma \subset \Gamma^{\perp} \Longrightarrow \Gamma$ *is a solvable subalgebra of* $\mathcal{G}$.

We recall that Γ is *solvable*, if $\Gamma^{(n)} = 0$ for some n, where the $\Gamma^{(k)}\,, k \geq 0$ are defined iteratively by:

$$\Gamma^{(0)} = \Gamma \quad \text{and} \quad \Gamma^{(k)} = [\Gamma^{(k-1)}, \Gamma^{k-1}].$$

The second condition in (4.11) can be satisfied for example by assuming that every $\gamma \in \Gamma$ is a nilpotent element of $\mathcal{G}$ in which case Γ is actually a nilpotent Lie algebra, by Engel's theorem [28]. We also recall that Γ is called *nilpotent*, if $\Gamma_{(n)} = 0$ for some n, where the $\Gamma_{(k)}\,, k \geq 0$ are defined iteratively by:

$$\Gamma_{(0)} = \Gamma \quad \text{and} \quad \Gamma_{(k)} = [\Gamma_{(k-1)}, \Gamma].$$

Clearly, any nilpotent Γ is solvable. However, *the nilpotency of* Γ *is not necessary* for $\Gamma \subset \Gamma^{\perp}$ to hold. In fact, solvable but not nilpotent Γ's which satisfy (4.11) can be found.

The *Euler-Lagrange equation* derived from (4.8) by varying g can be written equivalently as

$$\begin{aligned} \partial_-(\partial_+ g g^{-1} + g\tilde{A}g^{-1}) \quad &+ \quad [A, \partial_+ g g^{-1} + g\tilde{A}g^{-1}] + \partial_+ A = 0 \\ \partial_+(g^{-1}\partial_- g + g^{-1}Ag) \quad &- \quad [\tilde{A}, g^{-1}\partial_- g + g^{-1}Ag] + \partial_-\tilde{A} = 0 \end{aligned} \tag{4.12}$$

and they determine the evolution of the field g. Since the action contains no time-derivative of the gauge fields the $A, \tilde{A}$-equations are *Lagrangian constraints*

$$
\begin{aligned}
&\langle \gamma, \partial_+ g g^{-1} + g\tilde{A}g^{-1} - M\rangle = 0, && \forall\, \gamma \in \Gamma\,, \\
&\langle \tilde{\gamma}\,,\, g^{-1}\partial_- g + g^{-1}Ag - \tilde{M}\rangle = 0, && \forall\, \tilde{\gamma} \in \tilde{\Gamma}.
\end{aligned} \tag{4.13}
$$

We now note that by making use of the gauge invariance, $\tilde{A}$ and A can be set equal to zero simultaneously. The important point for us is that, as is easy to see, in the $A = \tilde{A} = 0$ gauge one recovers from (4.12) the field equations (4.2) of the WZNW theory and from (4.13) the constraints

$$
\phi_\gamma = \langle \gamma, J - \kappa M\rangle = 0, \quad \text{and} \quad \tilde{\phi}_{\tilde{\gamma}} = \langle \tilde{\gamma}, \tilde{J} + \kappa\tilde{M}\rangle = 0, \tag{4.14}
$$

where the γ and the $\tilde{\gamma}$ form bases of Γ and $\tilde{\Gamma}$, respectively.

Note that setting $A, \tilde{A}$ to zero is not a complete gauge fixing, the residual gauge transformations are exactly the chiral gauge transformations

$$
g(x^+, x^-) \longrightarrow e^{\alpha(x^+)} \cdot g(x^+, x^-) \cdot e^{-\tilde{\alpha}(x^-)}\,, \tag{4.15}
$$

where α and $\tilde{\alpha}$ are arbitrary Γ and $\tilde{\Gamma}$ valued chiral functions, respectively.

4.1.1 Hamiltonian formalism of the gauged theory

To discuss the Hamiltonian formalism for these theories we need to specify the canonical variables. For that purpose we parametrize the group elements in some arbitrary way [11], $g = g(\xi)$. We shall regard the parameters ξ^a, $a = 1, ..., \dim G$, as the canonical coordinates in the theory. To find the canonical momenta, we introduce the 2-form $\mathcal{A} = \frac{1}{2}\mathcal{A}_{ab}(\xi)\, d\xi^a d\xi^b$ to rewrite the Wess-Zumino term as

$$
\frac{1}{3}\mathrm{Tr}\ (dg\, g^{-1})^3 = d\mathcal{A}. \tag{4.16}
$$

The 2-form $\mathcal{A}$ is well-defined only locally on G, since the Wess-Zumino 3-form is closed but not exact. Fortunately we do not need to specify $\mathcal{A}$ explicitly below. Next we express the Maurer-Cartan forms as

$$
dgg^{-1} = N_{ab}(\xi)d\xi^a T^b \quad \text{and} \quad g^{-1}dg = \tilde{N}_{ab}(\xi)d\xi^a T^b, \tag{4.17}
$$

where T^b are some orthonormal generators of $\mathcal{G}$ [2]. These non-singular matrices are related with each other and $\mathcal{A}$ by

$$
\begin{aligned}
N\tilde{N}^{-1} = \tilde{N}^{-1}N = B \quad &, \quad B_{ab} = \langle\, gT_a\, g^{-1}, T_b\rangle \Longrightarrow BB^t = 1 \\
\mathcal{A}_{ab,c} + \mathcal{A}_{ca,b} + \mathcal{A}_{bc,a} \quad &= \quad f^{pqr}\tilde{N}_{ap}\tilde{N}_{bq}\tilde{N}_{cr} = f^{pqr}N_{ap}N_{bq}N_{cr}.
\end{aligned} \tag{4.18}
$$

[2] In real forms of the complex Lie algebra some T_a have norm -1

Furthermore, the Maurer-Cartan relations (or integrability condition on g) take the form

$$N_{bc,a} - N_{ac,b} - f_c^{pq} N_{ap} N_{bq} = \tilde{N}_{bc,a} - \tilde{N}_{ac,b} + f_c^{pq} \tilde{N}_{ap} \tilde{N}_{bq} = 0.$$

In these variables the action reads

$$\begin{aligned} I &= \frac{\kappa}{2} \int \eta^{\mu\nu} (N^t \partial_\mu \xi, N^t \partial_\nu \xi) - \kappa \int \mathrm{Tr}\, \Big(AM + \tilde{A}\tilde{M} \Big) \\ &- \kappa \int \Big((\dot{\xi}, \mathcal{A}\xi') - (\partial_+ \xi, NA) - (\partial_- \xi, \tilde{N}\tilde{A}) - (\tilde{A}, BA) \Big). \end{aligned} \tag{4.19}$$

Since it does not depend on $\dot{A}$, one has the *primary constraints*

$$\Pi_a = \tilde{\Pi}_a = 0.$$

The momenta conjugated to the $\dot{\xi}^a$ are easily found to be

$$\pi_a = \kappa N_a{}^p N_{bp} \dot{\xi}^b - \kappa \mathcal{A}_{ab} \xi'^b + \kappa N_{ab} A^b + \kappa \tilde{N}_{ab} \tilde{A}^b. \tag{4.20}$$

The canonical Hamiltonian can be written as

$$\begin{aligned} \mathcal{H} &= \frac{1}{4\kappa} \langle J, J \rangle + \langle A, \kappa M + \frac{1}{2} \kappa A - J \rangle \\ &+ \frac{1}{4\kappa} \langle \tilde{J}, \tilde{J} \rangle + \langle \tilde{A}, \kappa \tilde{M} + \frac{1}{2} \kappa \tilde{A} + \tilde{J} \rangle, \end{aligned} \tag{4.21}$$

where we have defined the *KM-currents*

$$\begin{aligned} J = J_a T^a \quad &, \quad J_a = (N^{-1})_a{}^b (\pi_b + \mathcal{A}_{bc} \xi'^c) + \kappa N_{ba} \xi'^b \\ \tilde{J} = \tilde{J}_a T^a \quad &, \quad \tilde{J}_a = -(\tilde{N}^{-1})_a{}^b (\pi_b + \mathcal{A}_{bc} \xi'^c) + \kappa \tilde{N}_{ba} \xi'^b. \end{aligned} \tag{4.22}$$

The consistency of the primary constraints lead to the following *secondary constraints*

$$\langle \gamma, J - \kappa M - \kappa A \rangle = 0 \quad \text{and} \quad \langle \tilde{\gamma}, \tilde{J} + \kappa \tilde{M} + \kappa \tilde{A} \rangle = 0. \tag{4.23}$$

For arbitrary subalgebras these constraints do not weakly commute with the primary constraints due to terms linear in the gauge fields. Thus to get FCC we are again lead to impose the second condition in (4.9) or in (4.11). Then the quadratic in $A, \tilde{A}$ terms in (4.21) and the linear in $A, \tilde{A}$ terms in (4.23) vanish and we remain with the secondary constraints

$$\phi_\gamma = \langle \gamma, J - \kappa M \rangle = 0 \quad \text{and} \quad \tilde{\phi}_{\tilde{\gamma}} = \langle \tilde{\gamma}, \tilde{J} + \kappa \tilde{M} \rangle = 0. \tag{4.24}$$

After a lengthy but straightforward calculation, where one uses (4.18,4.19) and identities like

$$g^{-1} \Big(\partial_a (\partial_b g g^{-1}) \Big) g = \partial_b \Big(g^{-1} \partial_a g \Big) \Longrightarrow \partial_a N_b{}^c N^{-1}_{cd} = \partial_b \tilde{N}_a{}^c \tilde{N}^{-1}_{cd}$$

or

$$f_c^{pq}\tilde{N}_{ap}B^t_{bq} + f_b^{pq}N_{ap}B_{cq} = 0,$$

one finds the following equal time Poisson brackets for the KM-currents

$$\begin{aligned}
\{\langle u, J(x)\rangle, \langle v, J(y)\rangle\} &= \langle [u,v], J(x)\rangle\delta(\xi) + 2\kappa\langle u,v\rangle\delta'(\xi) \\
\{\langle u, \tilde{J}(x)\rangle, \langle v, \tilde{J}(y)\rangle\} &= \langle [u,v], \tilde{J}(x)\rangle\delta(\xi) - 2\kappa\langle u,v\rangle\delta'(\xi) \\
\{\langle u, J(x)\rangle, \langle v, \tilde{J}(y)\rangle\} &= 0
\end{aligned} \tag{4.25}$$

for arbitrary $u, v \in \mathcal{G}$. Here I abbreviated $x - y = \xi$. Thus we have

$$\begin{aligned}
\{\phi_{\gamma_1}(x), \phi_{\gamma_2}(y)\} &= \Big(\phi_{[\gamma_1,\gamma_2]} + \kappa\,\omega_M(\gamma_1,\gamma_2) + 2\kappa\langle\gamma_1,\gamma_2\rangle\partial_x\Big)\delta \\
\{\tilde{\phi}_{\tilde{\gamma}_1}(x), \tilde{\phi}_{\tilde{\gamma}_2}(y)\} &= \Big(\tilde{\phi}_{[\tilde{\gamma}_1,\tilde{\gamma}_2]} - \kappa\,\omega_{\tilde{M}}(\tilde{\gamma}_1,\tilde{\gamma}_2) - 2\kappa\langle\gamma_1,\gamma_2\rangle\partial_x\Big)\delta
\end{aligned} \tag{4.26}$$

with the same arguments as in (4.25). Again it is evident that the constraints are first class if, and only if, the conditions (4.9) are fulfilled, that is if $\Gamma, \tilde{\Gamma}$ are solvable subalgebras on which the Kostant-Kirillov forms vanish.

Finally we need to check the consistency of the secondary constraints. Using the second assumption in (4.9) we find for the Poisson brackets of the secondary constraints with the Hamiltonian density (4.21)

$$\begin{aligned}
\{\phi_\gamma(x), \mathcal{H}(y)\} &= (\phi_\gamma(x)\delta(\xi))' - \langle[\gamma, A], J\rangle\delta(\xi) + \langle\gamma, M\rangle\delta'(\xi) \\
\{\tilde{\phi}_{\tilde{\gamma}}(x), \mathcal{H}(y)\} &= -(\tilde{\phi}_{\tilde{\gamma}}(x)\delta(\xi))' + \langle[\tilde{\gamma}, \tilde{A}], \tilde{J}\rangle\delta(\xi) + \langle\tilde{\gamma}, \tilde{M}\rangle\delta'(\xi).
\end{aligned} \tag{4.27}$$

Using the last property in (4.9) and integrating over y we obtain for the smeared constraints

$$\begin{aligned}
\int\{\alpha\phi_\gamma, H\} &= -\int(\alpha'\phi_\gamma + \alpha\phi_{[\gamma,A]}) \\
\int\{\tilde{\alpha}\tilde{\phi}_{\tilde{\gamma}}, H\} &= \int(\tilde{\alpha}'\tilde{\phi}_{\tilde{\gamma}} + \tilde{\alpha}\tilde{\phi}_{[\tilde{\gamma},\tilde{A}]}).
\end{aligned} \tag{4.28}$$

We see, that the primary and secondary constraints form a FC system.

Finally, let us check which off-shell symmetries are generated by these FCC. For that we define a general FCC

$$G = \int dx^1\Big(\alpha^i\phi_i + \beta^i\Pi_i + \tilde{\alpha}^i\tilde{\phi}_i + \tilde{\beta}^i\tilde{\Pi}_i\Big), \quad \text{where} \quad \phi_i = \phi_{\gamma_i},$$

calculate its time-derivative

$$\frac{d}{dt}G = \int\Big(\alpha^i{}_{,t}\,\phi_i + \beta^i{}_{,t}\,\Pi_i + \tilde{\alpha}^i{}_{,t}\,\tilde{\phi}_i + \tilde{\beta}^i{}_{,t}\,\tilde{\Pi}_i\Big) + \{G, H\}$$

and demand that this must be proportional to the primary constraints. One easily finds that this can only be the case if the coefficient functions are related as

$$\beta = -\partial_- \alpha + [\alpha, A] \ \ , \ \ \tilde{\beta} = \partial_+ \tilde{\alpha} + [\tilde{\alpha}, \tilde{A}], \quad \text{where} \quad \alpha = \alpha^i \gamma_i, \ldots$$

With these relations between the parameters one obtains the following off mass-shell symmetry transformations

$$\begin{aligned} \delta g &= \{g, G\} = \alpha g - g\tilde{\alpha}, \\ \delta A &= \gamma_i \{A^i, G\} = [\alpha, A] - \partial_- \alpha, \\ \delta \tilde{A} &= \tilde{\gamma}_i \{\tilde{A}^i, G\} = [\tilde{\alpha}, \tilde{A}] + \partial_+ \tilde{\alpha}, \end{aligned} \tag{4.29}$$

which are the infinitesimal gauge transformations (4.6). Finally note, that if we set the gauge fields to zero, then we find [3]

$$\frac{d}{dt} \int (\alpha^i \phi_i + \tilde{\alpha}^i \tilde{\phi}_i) = \int \Big(\langle \partial_- \alpha, J - \kappa M \rangle + \langle \partial_+ \tilde{\alpha}, \tilde{J} + \kappa \tilde{M} \rangle \Big).$$

The right hand side vanishes if the α and $\tilde{\alpha}$ depend only on x^+ and x^-, respectively. The corresponding smeared constraints generate transformations which leave the surface defined by the constraints and the conditions $A = \tilde{A} = 0$ invariant. In other words, the conditions $A = \tilde{A} = 0$ is only a partial gauge fixing and the constraints

$$G_{ch} \equiv \int dx^1 \Big(\alpha^i(x^+) \, \phi_i + \tilde{\alpha}^i(x^-) \, \tilde{\phi}_i \Big) \tag{4.30}$$

generate chiral off mass-shell symmetries on the surface defined by the partial gauge fixing. From (4.29) we see that these symmetries are just the chiral gauge transformations (4.15), as expected. The currents are transformed as

$$\begin{aligned} \delta J = T_a \{J^a(x), G_{ch}\} = [\alpha(x), J(x)] + 2\kappa \alpha' \\ \delta \tilde{J} = T_a \{\tilde{J}^a(x), G_{ch}\} = [\tilde{\alpha}(x), \tilde{J}(x)] - 2\kappa \tilde{\alpha}' \end{aligned} \tag{4.31}$$

which, since $2\alpha' = \partial_+ \alpha$ and $2\tilde{\alpha}' = -\partial_- \tilde{\alpha}$ for chiral functions, are just the infinitesimal forms of the global gauge transformations

$$\begin{aligned} J \to e^{\alpha} J e^{-\alpha} + (\partial_+ e^{\alpha}) \, e^{-\alpha}, \quad \alpha = \alpha(x^+) \\ \tilde{J} \to e^{\tilde{\alpha}} \tilde{J} e^{-\tilde{\alpha}} - e^{\tilde{\alpha}} \, \partial_- e^{-\tilde{\alpha}}, \quad \tilde{\alpha} = \tilde{\alpha}(x^-). \end{aligned} \tag{4.32}$$

These are just the transformation of the currents (4.2) which follow from (4.15).

- From now on I shall always assume that the canonical pairs A^i, Π_i and $\tilde{A}^i, \tilde{\Pi}_i$ have been eliminated.

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[3] up to surface terms which vanish if we impose periodic boundary conditions

The Dirac bracket on the so partially reduced phase space, are just the ordinary Poisson bracket for the remaining degrees of freedom. The situation is very much like in *Yang-Mills theories*, where one remains with the time-independent gauge transformations after elimination of the primary pair A_0, Π_0 and where these transformations are generated by the secondary Gauss-constraints smeared with time-independent test functions. Also, in the rest of this chapter, the notation $f' = 2\partial_1 f$ is used for every function, including the spatial δ-functions. This has the advantage that for a chiral function $f(x^+)$ one has then $f' = \partial_+ f$.

4.1.2 Effective field theories from left-right dual reductions

The aim of this section is to describe the effective field equations and action functionals for an important class of reduced WZNW theories. This class of theories is obtained by making the assumption that the left and right gauge algebras Γ and $\tilde{\Gamma}$ are *dual to each other* with respect to the Cartan-Killing form, which means that one can choose bases $\gamma_i \in \Gamma$ and $\tilde{\gamma}_j \in \tilde{\Gamma}$ so that

$$\langle \gamma_i, \tilde{\gamma}_j \rangle = \delta_{ij} \ . \tag{4.33}$$

This *technical* assumption allows for having a simple general algorithm for disentangling the constraints (4.24) which define the reduction. It holds if one chooses Γ and $\tilde{\Gamma}$ to be the images of each other under a Cartan involution[4] of the underlying simple Lie algebra. For maximally non-compact, connected real Lie groups the Cartan involution is $(-1)\times$ transpose, operating on the Chevalley basis according to

$$H_i \longrightarrow -H_i \qquad E_{\pm\alpha} \longrightarrow -E_{\mp\alpha} \ .$$

It is obvious that $\langle v \, , \, v^t \rangle > 0$ for any non-zero $v \in \mathcal{G}$ and from this one sees that Γ^t is dual to Γ with respect to the Cartan-Killing form, i.e., (4.33) holds for $\tilde{\Gamma} = \Gamma^t$. It should also be mentioned that there is a Cartan involution for every non-compact real form of the complex simple Lie algebras, as explained in detail in [26].

Equation (4.33) implies that the left and right gauge algebras do not intersect, and thus we can consider a direct sum decomposition of $\mathcal{G}$ of the form

$$\mathcal{G} = \Gamma + \mathcal{B} + \tilde{\Gamma} \ , \tag{4.34}$$

where $\mathcal{B}$ is some linear subspace of $\mathcal{G}$. Here $\mathcal{B}$ is in principle an arbitrary complementary space to $(\Gamma + \tilde{\Gamma})$ in $\mathcal{G}$, but one can always make the choice

[4]A Cartan involution σ of the simple Lie algebra is an automorphism for which $\sigma^2 = 1$ and $\langle v, \sigma(v) \rangle < 0$.

$$\mathcal{B} = (\Gamma + \tilde{\Gamma})^{\perp} \,, \tag{4.35}$$

which is natural in the sense that the Cartan-Killing form is non-degenerate on this $\mathcal{B}$. We note that matters simplify if the space $\mathcal{B}$ is a subalgebra of $\mathcal{G}$, but this is not necessary for our arguments and is not always possible either.

We can associate a 'generalized Gauss decomposition' of the group G to the direct sum decomposition (4.34). By 'Gauss decomposing' an element $g \in G$, we mean writing it in the form

$$g = a \cdot b \cdot c \,, \quad \text{with} \quad a = e^{\gamma}, \quad b = e^{\beta} \quad \text{and} \quad c = e^{\tilde{\gamma}}, \tag{4.36}$$

where γ, β and $\tilde{\gamma}$ are from the respective subspaces in (4.34).

There is a neighborhood of the identity in G consisting of elements which allow a unique decomposition of this sort, and in this neighborhood the pieces a, b and c can be extracted from g by algebraic operations. We make the assumption that every G-valued field we encounter is decomposable as g in (4.36). It is easily seen that in this 'Gauss decomposable sector' the components of $b(x^+, x^-)$ provide a complete set of *gauge invariant local fields.*

Below I explain how to solve the constraints (4.24) in the Gauss decomposable sector of the WZNW theory. For our method to work, we restrict ourselves to fields which vary in such a Gauss decomposable neighborhood of the identity where the matrix

$$V_{ij}(b) = \langle \gamma_i, b\,\tilde{\gamma}_j\, b^{-1} \rangle \tag{4.37}$$

is invertible. Due to the assumptions, the analysis given in the following yields a *local* description of the reduced theories. It is clear that for a global description one should use patches on G obtained by multiplying out the Gauss decomposable neighborhood of the identity, but we do not deal with this issue here.

Field equations of the reduced theories: First I derive the field equations of the reduced theory by implementing the constraints directly in the WZNW field equation $\partial_-(\partial_+ g g^{-1}) = 0$. (This is allowed since the WZNW dynamics leaves the constraint surface invariant.) By inserting the Gauss decomposition of g into (4.14) and making use of the constraints being first class, the constraint equations can be rewritten as

$$\begin{aligned} \langle \gamma_i,\, \partial_+ b b^{-1} + b(\partial_+ c c^{-1}) b^{-1} - M \rangle &= 0\,, \\ \langle \tilde{\gamma}_i,\, b^{-1}\partial_- b + b^{-1}(a^{-1}\partial_- a) b - \tilde{M} \rangle &= 0. \end{aligned} \tag{4.38}$$

With the help of the inverse of V_{ij} in (4.37), one can solve these equations for $\partial_+ cc^{-1}$ and $a^{-1}\partial_- a$ in terms of b,

$$\partial_+ cc^{-1} = b^{-1}T(b)b, \qquad \text{and} \qquad a^{-1}\partial_- a = b\tilde{T}(b)b^{-1}, \tag{4.39}$$

where

$$\begin{aligned} T(b) &= \sum_{ij} V_{ij}^{-1}(b)\,\langle\gamma_j,\, M - \partial_+ bb^{-1}\rangle\, b\tilde{\gamma}_i b^{-1}, \\ \tilde{T}(b) &= \sum_{ij} V_{ij}^{-1}(b)\,\langle\tilde{\gamma}_i,\, \tilde{M} - b^{-1}\partial_- b\rangle\, b^{-1}\gamma_j b. \end{aligned} \tag{4.40}$$

The effective field equation for the field $b(x^+, x^-)$ can be obtained, for instance, by noting that the WZNW field equation can be written in the zero-curvature form $[\partial_+ - J, \partial_- - 0] = 0$ or equivalently after a gauge transformation with a as

$$[\partial_+ - \mathcal{A}_+\,,\, \partial_- - \mathcal{A}_-] = 0, \tag{4.41}$$

where

$$\mathcal{A}_+ = \partial_+ b\, b^{-1} + b(\partial_+ cc^{-1})b^{-1} \quad \text{and} \quad \mathcal{A}_- = -a^{-1}\partial_- a\ . \tag{4.42}$$

Inserting the relations (4.39) we see that the field equation of the reduced theory is the zero curvature condition of the following Lax potential:

$$\mathcal{A}_+(b) = \partial_+ b\, b^{-1} + T(b) \quad \text{and} \quad \mathcal{A}_-(b) = -b\tilde{T}(b)b^{-1}\ . \tag{4.43}$$

More explicitly, the effective field equation reads

$$\boxed{\partial_-(\partial_+ bb^{-1}) + [b\tilde{T}(b)b^{-1},\, T(b)] + \partial_- T(b) + b(\partial_+\tilde{T}(b))b^{-1} = 0.} \tag{4.44}$$

The expression on the left-hand-side of (4.44) in general varies in the full space $\mathcal{G}$, but not all the components represent independent equations. The number of the independent equations is the number of the independent components of the WZNW field equation minus the number of the constraints in (4.24), since the constraints automatically imply the corresponding components of the WZNW equation. Thus there are exactly as many independent equations in (4.44) as the number of the reduced degrees of freedom. In fact, the *independent field equations* can be obtained by taking the Cartan-Killing inner product of (4.44) with a basis of the linear space $\mathcal{B}$. The inner product of with the γ_i and the $\tilde{\gamma}_i$ vanishes as a consequence of the constraints together with the independent field equations.

General solution of field equation: The effective field equation (4.44) is in general a non-linear equation for the field $b(x^+, x^-)$, and we can give a procedure which can in principle be used for producing its *general solution.* We are going to do this by making use of the fact that the space of solutions of the reduced theory is the space of the constrained WZNW solutions factorized by the chiral gauge transformations (4.15). Thus the idea is to find the general solution of the effective field equation by first parameterizing, in terms of arbitrary chiral functions, those WZNW solutions which satisfy the constraints (4.24), and then extracting their b-part by algebraic operations.

To be more concrete, one can start the construction of the general solution by first Gauss-decomposing the chiral factors of the general WZNW solution $g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-)$ as

$$\begin{aligned} g_L(x^+) &= a_L(x^+) \cdot b_L(x^+) \cdot c_L(x^+) \\ g_R(x^-) &= a_R(x^-) \cdot b_R(x^-) \cdot c_R(x^-). \end{aligned} \tag{4.45}$$

Then the constraint equations (4.24) become

$$\partial_+ c_L c_L^{-1} = b_L^{-1} T(b_L) b_L \quad \text{and} \quad a_R^{-1} \partial_- a_R = b_R \tilde{T}(b_R) b_R^{-1}\ . \tag{4.46}$$

In addition to the the purely algebraic problems of computing the quantities T and $\tilde{T}$ and extracting b from $g = g_L \cdot g_R = a \cdot b \cdot c$, these first order systems of ordinary differential equations are all one has to solve to produce the general solution of the effective field equation. If this can be done by quadrature then the effective field equation is also integrable by quadrature. In general, one can proceed by trying to solve (4.46) for the functions $c_L(x^+)$ and $a_R(x^-)$ in terms of the arbitrary 'input functions' $b_L(x^+)$ and $b_R(x^-)$. Clearly, this involves only a *finite number of integrations whenever the gauge algebras* Γ *and* $\tilde{\Gamma}$ *are nilpotent.*

We note that in concrete cases some other choice of input functions, instead of the chiral b's, might prove more convenient for finding the general solutions of the systems of first order equations on g_L and g_R given in (4.46) (see for instance the derivation of the general solution of the Liouville equation given in [20]).

Effective action for gauge invariant fields: It is natural to ask for the action functional underlying the effective field theory obtained by imposing the constraints (4.24) on the WZNW theory. In fact, the effective action is given by the following formula:

$$\boxed{I_{\text{eff}}(b) = S_{\text{WZ}}(b) - \int d^2x\, \langle b\tilde{T}(b)b^{-1}\, ,\, T(b)\rangle.} \tag{4.47}$$

One can derive the following condition for the extremum of this action:

$$\langle \delta b b^{-1}, \partial_-(\partial_+ b b^{-1}) + [b\tilde{T}b^{-1}, T] + \partial_- T + b(\partial_+\tilde{T})b^{-1}\rangle = 0. \tag{4.48}$$

It is straightforward to compute this, the only thing to remember is that the objects $b\tilde{T}b^{-1}$ and $b^{-1}Tb$ introduced in (4.40) vary in the gauge algebras Γ and $\tilde{\Gamma}$. The arbitrary variation of $b(x)$ is determined by the arbitrary variation of $\beta(x) \in \mathcal{B}$, according to $b(x) = e^{\beta(x)}$, and thus we see from (4.48) that the Euler-Lagrange equation of the action (4.47) yields exactly the independent components of the effective field equation (4.44).

The effective action given above can be derived from the gauged WZNW action (4.8), by eliminating the gauge fields $A, \tilde{A}$ by means of their Euler-Lagrange equations (4.13). By using the Gauss decomposition, these Euler-Lagrange equations become equivalent to the relations

$$a^{-1}D_- a = b\tilde{T}(b)b^{-1}\ , \quad \text{and} \quad cD_+c^{-1} = -b^{-1}T(b)b, \tag{4.49}$$

where T and $\tilde{T}$ are given by the expressions in (4.40) and $D_\pm$ denotes the gauge covariant derivatives introduced earlier. Now I show that $I_{\rm eff}(b)$ can indeed be obtained by substituting the solution of (4.49) for $A, \tilde{A}$ back into (4.8) with $g = abc$. To this first we rewrite $I(abc, A, \tilde{A})$ in the form (4.5) (plus the terms containing M and $\tilde{M}$) and use (4.49) by noting, for example, that $\langle \partial_- a a^{-1}\ , M\rangle$ is a total derivative.

Parity operations: Here I point out that the particular left-right related choice (4.33) of the gauge algebras can also be used to ensure the *parity invariance* of the effective field theory. Indeed, for maximally non-compact connected Lie group G $S_{\rm WZ}(g)$ is invariant under any of the following two 'parity transformations' $g \longrightarrow Pg$:

$$(P_1 g)(x^0, x^1) \equiv g^t(x^0, -x^1) \quad , \quad (P_2 g)(x^0, x^1) \equiv g^{-1}(x^0, -x^1). \tag{4.50}$$

If one chooses $\tilde{\Gamma} = \Gamma^t$ and $\tilde{M} = M^t$ then the parity transformation P_1 simply interchanges the left and right constraints, ϕ and $\tilde{\phi}$ in (4.24), and thus the corresponding effective field theory is invariant under the parity P_1. The space $\mathcal{B}$ in (4.35) is invariant under the transpose in this case, and thus the gauge invariant field b transforms in the same way under P_1 as g does in (4.50). Of course, the parity invariance can also be seen on the level of the gauged action. Namely, $I(g, A, \tilde{A})$ is invariant under P_1 if one extends the definition in (4.50) to include the following parity transformation of the gauge fields:

$$(P_1 A)(x^0, x^1) \equiv A^t(x^0, -x^1) \ , \tag{4.51}$$

and similarly for $\tilde{A}$. The P_1-invariant reduction procedure does not preserve the parity symmetry P_2, but it is possible to consider reductions preserving just P_2 instead of P_1. In fact, such *axial reductions* can be obtained by taking $\tilde{\Gamma} = \Gamma$ and $\tilde{M} = M$.

It is obvious that to construct parity invariant WZNW reductions in general, for some arbitrary but non-compact real form $\mathcal{G}$ of the complex simple Lie algebras, one can use $-\sigma$ instead of the transpose, where σ is a Cartan involution of $\mathcal{G}$.

Special cases: Finally I would like to mention certain special cases when the above equations simplify. First we note that if one has

$$[\mathcal{B}\,, \Gamma] \subset \Gamma \qquad \text{and} \qquad [\mathcal{B}\,, \tilde{\Gamma}] \subset \tilde{\Gamma} \ , \tag{4.52}$$

then

$$T(b) = M - \tilde{\pi}(\partial_+ b b^{-1}) \quad \text{and} \quad \tilde{T}(b) = \tilde{M} - \pi(b^{-1}\partial_- b) \ , \tag{4.53}$$

where we introduced the projectors onto the spaces Γ and $\tilde{\Gamma}$,

$$\pi = \sum_i |\gamma_i\rangle\langle\tilde{\gamma}_i| \quad \text{and} \quad \tilde{\pi} = \sum_i |\tilde{\gamma}_i\rangle\langle\gamma_i| \ , \tag{4.54}$$

and, without loss of generality, (see 4.33) assumed that $M \in \tilde{\Gamma}$ and $\tilde{M} \in \Gamma$. One obtains (4.53) from (4.39,4.40) by taking into account that in this case $V_{ij}(b)$ in (4.37) is the matrix of the operator Ad_b acting on $\tilde{\Gamma}$, and thus the inverse is given by $\mathrm{Ad}_{b^{-1}}$.

The nicest possible situation occurs when $\mathcal{B} = (\Gamma + \tilde{\Gamma})^\perp$ is a *subalgebra* of $\mathcal{G}$ and also satisfies (4.52). In this case one simply has $T = M$ and $\tilde{T} = \tilde{M}$ and thus (4.44) simplifies to

$$\boxed{\partial_-(\partial_+ b b^{-1}) + [b\tilde{M}b^{-1}\,, M] = 0\ .} \tag{4.55}$$

The derivative term is now an element of $\mathcal{B}$ and by combining the above assumptions with the first class conditions $[M, \Gamma] \subset \Gamma^\perp$ and $[\tilde{M}, \tilde{\Gamma}] \subset \tilde{\Gamma}^\perp$ one sees that the commutator term in (4.55) also varies in $\mathcal{B}$, which ensures the consistency of this equation. Generalized, or non-Abelian, Toda theories of this type have been first investigated by Leznov and Saveliev [32, 33] , who defined these theories by postulating their Lax potential

$$\mathcal{A}^H_+ = \partial_+ b \cdot b^{-1} + M \quad , \quad \mathcal{A}^H_- = -b\tilde{M}b^{-1}, \tag{4.56}$$

which they obtained by considering the problem that if one requires a $\mathcal{G}$-valued pure-gauge Lax potential to take some special form, then the consistency of the system of equations coming from the zero curvature condition becomes a non-trivial problem. Also, in this particular situation the effective action simplifies to

$$I_{\text{eff}}^{H}(b) = S_{\text{WZ}}(b) - \int d^2x \, \langle b\tilde{M}b^{-1}, M\rangle, \tag{4.57}$$

where the field b varies in the subgroup with Lie-algebra $\mathcal{B}$.

4.2 Conformally invariant reductions

The purpose of this section is to find sufficient conditions for the conformal invariance of the constraints. The residual gauge symmetries on the partially gauge fixed configurations consisting of currents of the form

$$J(x) = \kappa M + j(x) \ , \quad \text{with} \quad j(x) \in \Gamma^{\perp} \tag{4.58}$$

are the chiral transformations (4.15) and (4.32) which are generated by the FCC (4.24) smeared with chiral test functions. The analysis applies to each current J and $\tilde{J}$ separately so we choose one of them, J say, for definiteness.

It is clear from (4.24) that M can be shifted by an arbitrary element from $\Gamma^{\perp}$ without changing the actual content of the constraints. This ambiguity is unessential, since one can fix M, for example, by requiring that it is from some given linear complement of $\Gamma^{\perp}$ in $\mathcal{G}$, which can be chosen by convention. We shall assume that $M \notin \Gamma^{\perp}$ from now on.

Now let us discuss sufficient conditions which ensure conformal invariance. The standard conformal symmetry generated by the *Virasoro density*$L_{\text{KM}}(x)$ is broken by the constraints (4.24), since they set some component of the current, which has spin 1, to a non-zero constant. The idea is to circumvent this apparent violation of conformal invariance by changing the standard action of the conformal group on the KM phase space to one which does leave the constraint surface invariant. One can try to generate the new conformal action by changing the usual KM Virasoro density to the new Virasoro density

$$L_H(x) = L_{\text{KM}}(x) - \langle H, J'(x)\rangle, \quad \text{where} \quad L_{\text{KM}} = \frac{1}{2\kappa}\langle J, J\rangle \tag{4.59}$$

is twice the energy-density (4.21) on the partially gauge fixed fields and H is some constant element of $\mathcal{G}$. The conformal action generated by $L_H(x)$ operates on the KM phase space as

$$\begin{aligned}\delta_{f,H} J(x) &\equiv \int dy^1 \{J(x), L_H(y)\}\, f(y^+) \\ &= f(x^+)J'(x) + f'(x^+)(J(x) + [H, J(x)]) + \kappa f''(x^+)H\end{aligned} \tag{4.60}$$

for any parameter function $f(x^+)$, corresponding to the conformal coordinate transformation $\delta_f x^+ = -f(x^+)$. In particular, $j(x)$ in (4.58) transforms under this new conformal action according to [5]

$$\begin{aligned}\delta_{f,H}\, j(x) &= f(x^+)j'(x) + f''(x^+)H \\ &+ f'(x^+)\Big(j(x) + [H, j(x)] + [H, M] + M\Big),\end{aligned} \tag{4.61}$$

and our condition is that this variation should be in $\Gamma^\perp$, which means that this conformal action preserves the constraint surface. From (4.61), one sees that this is equivalent to having the following relations:

$$\boxed{H \in \Gamma^\perp, \qquad [H, \Gamma^\perp] \subset \Gamma^\perp \qquad \text{and} \qquad ([H, M] + M) \in \Gamma^\perp\ .} \tag{4.62}$$

In conclusion, the existence of an operator H satisfying these relations is a *sufficient condition* for the conformal invariance of the KM reduction obtained by imposing (4.24). The conditions in (4.62) are equivalent to $L_H(x)$ being a gauge invariant quantity, inducing a corresponding conformal action on the reduced phase space. Obviously, the second relation in (4.62) is equivalent to

$$[H, \Gamma] \subset \Gamma\ . \tag{4.63}$$

An element $H \in \mathcal{G}$ is called diagonalizable if the linear operator ad_H possesses a complete set of eigenvectors in $\mathcal{G}$. By the eigenspaces of ad_H, such an element defines a grading of $\mathcal{G}$, and below we shall refer to a diagonalizable element as a *grading operator* of $\mathcal{G}$.

If H is a grading operator satisfying (4.62) then it is always possible to shift M by some element of $\Gamma^\perp$ so that the new M satisfies

$$[H, M] = -M\ , \tag{4.64}$$

instead of the last condition in (4.62). It is also clear that if H is a grading operator then one can take graded bases in Γ and $\Gamma^\perp$. On re-inserting (4.64) into (4.61) it then follows that all components of $j(x)$ are *primary fields* with respect to the conformal action generated by $L_H(x)$, with the exception of the H-component, which also survives the constraints according to the first condition in (4.62).

[5] From now on we set $\kappa = 1$. Only when we compute the central extension in the Virasoro algebra do we reinstall κ.

4.2.1 Gauge invariant polynomials

In the previous sections I derived the conditions for the constraints to be first class and for $L_H(J)$ in (4.59) being a gauge invariant polynomial. It is clear that the KM Poisson brackets of all gauge invariant differential polynomials of the current always close on such polynomials and δ-distributions. The corresponding algebra is of special interest in the conformally invariant case when it is a polynomial extension of the Virasoro algebra, the so-called $\mathcal{W}$-algebra. Here I shall give sufficient conditions on the triple (Γ, M, H) which allows one to construct out of the constrained current a complete set of gauge invariant differential polynomials. Their KM Poisson bracket algebra becomes the Dirac bracket algebra of the current components in the so-called Drinfeld-Sokolov (DS) gauges [15]. Thus we can represent $\mathcal{W}$-algebras as KM Poisson bracket algebras of gauge invariant differential polynomials, which in principle allows for its quantization through the KM representation theory. Also we shall exhibit the primary fields for the $\mathcal{W}$-algebras and describe their structure in detail.

Let us suppose that

- (Γ, M, H) satisfy the previously given conditions, (4.9) and (4.62).
- H is a grading operator and M is chosen so that $[H, M] = -M$, cf. (4.64).

The grade-h subspaces of $\mathcal{G}$ are denoted by $\mathcal{G}_h$ and the direct sum of the $\mathcal{G}_{h'}$ with $h' > h$ by $\mathcal{G}_{>h}$. Also note that in the present situation Γ and $\Gamma^\perp$ are graded by the eigenvalues of ad_H. Now we can prove the following

Theorem 6 *If $\Gamma \cap \mathcal{K}_M = \{0\}$ and $\Gamma^\perp \subset \mathcal{G}_{>-1}$, where $\mathcal{K}_M = \mathrm{Ker}(\mathrm{ad}_M)$, then one can construct out of $J(x)$ in (4.58) a complete set of gauge invariant differential polynomials.*

The condition on $\Gamma^\perp$ plays a technical role in our considerations, but perhaps it can be argued for also physically, on the basis that it ensures that the conformal weights of the primary field components of $j(x)$ in (4.58) are *non-negative* with respect to L_H. Second, let us observe that in our situation M satisfying (4.64) is *uniquely determined*, that is, there is no possibility of shifting it by elements from $\Gamma^\perp$, simply because there are no grade -1 elements in $\Gamma^\perp$. The first condition means that the operator ad_M maps Γ into $\Gamma^\perp$ in an *injective* manner, and for this reason we call it *non-degeneracy condition.* Before proving this result, we discuss some consequences of the conditions, which we shall need later.

Lemma 4 *The conditions in the theorem imply the following conditions on the gauge algebra and the kernel of M: $\mathcal{G}_{\geq 1} \subset \Gamma \subset \mathcal{G}_{>0}$, $\mathcal{G}_{\geq 0} \subset \Gamma^\perp \subset \mathcal{G}_{>-1}$ and $\mathcal{K}_M \subset \mathcal{G}_{<1}$.*

Hence every $\gamma \in \Gamma$ is represented by a nilpotent operator in any finite dimensional representation of $\mathcal{G}$.

To prove the lemma we note that the spaces $\mathcal{G}_h$ and $\mathcal{G}_{-h}$ are dual to each other with respect to the Cartan-Killing form which is a consequence of its non-degeneracy and invariance under ad_H. This implies $\mathcal{G}_{\geq 1} \subset \Gamma$. On the other hand, if Γ would contain an element γ of grade ≤ 0, then $\mathrm{ad}_M \gamma$, which is non-zero according to our non-degeneracy condition and lies in $\Gamma^{\perp}$, would have grade ≤ -1. This would be in conflict with our assumption on $\Gamma^{\perp}$. So we conclude that $\Gamma \subset \mathcal{G}_{>0}$. Using the duality property we also conclude then $\mathcal{G}_{\geq 0} \subset \Gamma^{\perp}$. Finally, since Γ contains all elements with grade ≥ 1, the Kernel of ad_M must be a subset of $\mathcal{G}_{<1}$. This then proves the lemma.

Finally, I wish to establish a certain relationship between the dimensions of $\mathcal{G}$ and $\mathcal{K}_M$. For this purpose we consider an arbitrary complementary space $\mathcal{T}_M$ to $\mathcal{K}_M$, defining a linear direct sum decomposition

$$\mathcal{G} = \mathcal{K}_M + \mathcal{T}_M \ . \tag{4.65}$$

Clearly, $\omega_M(\mathcal{K}_M, \mathcal{G}) = 0$, and the restriction of ω_M to $\mathcal{T}_M$ is a *symplectic* form, in other words:

$$\omega_M(\mathcal{T}_M, \mathcal{T}_M) \qquad \mathrm{is} \quad \mathrm{non-degenerate}\ . \tag{4.66}$$

We note in passing that $\mathcal{T}_M$ can be identified with the tangent space at M to the co-adjoint orbit of G through M, and in this picture ω_M becomes the Kirillov-Kostant symplectic form of the orbit [2]. The non-degeneracy condition says that one can choose the space $\mathcal{T}_M$ in (4.65) in such a way that $\Gamma \subset \mathcal{T}_M$. One then obtains the inequality

$$\dim(\Gamma) \leq \frac{1}{2}\dim(\mathcal{T}_M) = \frac{1}{2}(\dim(\mathcal{G}) - \dim(\mathcal{K}_M)) \ , \tag{4.67}$$

where the factor $\frac{1}{2}$ arises since ω_M is a symplectic form on $\mathcal{T}_M$, which vanishes on the subspace $\Gamma \subset \mathcal{T}_M$.

After the above clarification of the meaning of conditions in the theorem, I now wish to show that they indeed allow for exhibiting a complete set of gauge invariant differential polynomials among the gauge invariant functions. Generalizing the arguments of [15, 4, 40], this will be achieved by demonstrating that an arbitrary current $J(x)$ subject to (4.58) *can be brought to a certain normal form by a unique gauge transformation which depends on $J(x)$ in a differential polynomial way.*

A normal form suitable for this purpose can be associated to any *graded subspace* $\Theta \subset \mathcal{G}$ which is dual to Γ with respect to the 2-form ω_M. Because of the non-degeneracy condition and the lemma such a space must obey

$$\Theta \subset \mathcal{G}_{<1} \quad \text{and} \quad \dim(\,\Theta \cap \mathcal{G}_{1-h}) = \dim \mathcal{G}_h \ , \ h \geq 1.$$

It is possible to choose bases γ_h^i and θ_k^j in Γ and Θ respectively such that

$$\omega_M(\gamma_h^l, \theta_k^i) = \delta_{il}\delta_{hk}, \tag{4.68}$$

where the subscript h on γ_h^l denotes the grade, and the indices i and l denote the additional labels which are necessary to specify the base vectors at fixed grade. The subscript k on elements $\theta_k^j \in \Theta$ does not denote the grade, which is $(1-k)$. The *reduced phase space* corresponding to Θ is given by the following equation:

$$\boxed{J_{\rm red}(x) = M + j_{\rm red}(x) \quad \text{where} \quad j_{\rm red}(x) \in \Gamma^\perp \cap \Theta^\perp \equiv \mathcal{V}\ .} \tag{4.69}$$

In other words, the set of reduced currents is obtained by supplementing the FCC (4.24) by the *gauge fixing condition*

$$\chi_\theta(x) = \langle J(x), \theta\rangle - \langle M, \theta\rangle = 0\ , \qquad \forall \theta \in \Theta. \tag{4.70}$$

We call a gauge which can be obtained in the above manner a *Drinfeld-Sokolov (DS) gauge.* It is not hard to see that the space $\mathcal{V}$ is a graded subspace of $\Gamma^\perp$ which is disjoint from the image of Γ under the operator ad_M and is in fact complementary to the image, i.e., one has

$$\Gamma^\perp = [M, \Gamma] + \mathcal{V}\ . \tag{4.71}$$

It also follows from the non-degeneracy condition that any graded complement $\mathcal{V}$ in (4.71) can be obtained in the above manner, by means of using some Θ. Thus it is possible to define the DS normal form of the current directly in terms of a complementary space $\mathcal{V}$ as well, as has been done in special cases in [15, 4, 18].

As the first step in proving that any current in (4.58) is gauge equivalent to one in the DS gauge, let us consider the gauge transformation by $g_h(x^+) = \exp[\sum_l a_h^l(x^+)\gamma_h^l]$ for some fixed grade h. Suppressing the summation over l, it can be written as [6]

$$j(x) \to j^{g_h}(x) = e^{a_h\cdot\gamma_h}(j(x) + M)e^{-a_h\cdot\gamma_h} + (e^{a_h\cdot\gamma_h})' e^{-a_h\cdot\gamma_h} - M\ .$$

[6] Throughout the chapter, all equations involving gauge transformations, Poisson brackets, etc., are to be evaluated by using a fixed time. They are valid both on the canonical phase space and on the chiral KM phase space belonging to space of solutions of the theory.

Taking the inner product of this equation with the basis vectors θ^i_k in (4.68) for all $k \leq h$, we see that there is no contribution from the derivative term. We also see that the only contribution from

$$e^{a_h \cdot \gamma_h} j(x) e^{-a_h \cdot \gamma_h} = j(x) + [a_h \cdot \gamma_h,\, j(x)] + \dots$$

is the one coming from the first term, since all commutators containing the elements γ^l_h drop out from the inner product in question as a consequence of the following crucial relation:

$$\boxed{[\gamma^l_h, \theta^i_k] \in \Gamma, \quad \text{for} \quad k \leq h,} \tag{4.72}$$

which follows from the lemma by noting that the grade of this commutator is at least 1 for $k \leq h$. Taking these into account, and computing the contribution from those two terms in $j^{g_h}(x)$ which contain M by using (4.68) and $\langle \theta^i_k, M \rangle = 0$, we obtain

$$\langle \theta^i_k, j^{g_h}(x) \rangle = \langle \theta^i_k, j(x) \rangle - a^i_h(x^+) \delta_{hk}, \quad \text{for all} \quad k \leq h.$$

We see from this equation that

$$\langle \theta^i_k, j(x) \rangle = 0 \quad \Longleftrightarrow \quad \langle \theta^i_k, j^{g_h}(x) \rangle = 0\ , \quad \text{for} \quad k < h\ ,$$

and

$$a^i_h(x^+) = \langle \theta^i_h, j(x) \rangle \quad \Rightarrow \quad \langle \theta^i_h, j^{g_h}(x) \rangle = 0\ , \quad \text{for} \quad k = h.$$

The last two equations tell us that if the gauge-fixing condition $\langle \theta^i_k, j(x) \rangle = 0$ is satisfied for all $k < h$ then we can ensure that the same condition holds for $j^{g_h}(x)$ for the *extended range of indices* $k \leq h$, by choosing $a^i_h(x^+)$ to be $\langle \theta^i_h, j(x) \rangle$. From this it is easy to see that the DS gauge (4.70) can be reached by an iterative process of gauge transformations, and the gauge-parameters $a^i_h(x^+)$ are unique polynomials in the current at each stage of the iteration.

In more detail, let us write the general element $g(a(x^+)) \in e^\Gamma$ of the gauge group as a product in order of descending grades, i.e.,as

$$g(a(x^+)) = g_{h_n} \cdot g_{h_{n-1}} \cdots g_{h_1}, \quad \text{with} \quad g_{h_i}(x^+) = e^{a_{h_i}(x^+) \cdot \gamma_{h_i}}\ ,$$

where $h_n > h_{n-1} > \ldots > h_1$ is the list of grades occurring in Γ. Let us then insert this expression into

$$j \to j^g = g(j + M)g^{-1} + g' g^{-1} - M\ , \tag{4.73}$$

and consider the condition

$$j^g(x) = j_{\text{red}}(x)\ , \tag{4.74}$$

with $j_{\text{red}}(x)$ in (4.69), as an equation for the gauge-parameters $a_h(x^+)$. One sees from the above considerations that this equation is uniquely soluble for the components of the $a_h(x^+)$ and the solution is a differential polynomial in $j(x)$. This implies that the components of $j_{\rm red}(x)$ can also be uniquely computed from (4.73,4.74) and *the solution yields a complete set of gauge invariant differential polynomials of* $j(x)$, which establishes the required result. The above iterative procedure is in fact a convenient tool for computing the gauge invariant differential polynomials in practice [40]. Of course, any unique gauge fixing can be used to define gauge invariant quantities, but they are in general not polynomial, not even local in $j(x)$.

4.2.2 The polynomiality of the Dirac bracket

It follows from the polynomiality of the gauge fixing that the components of the gauge fixed current $j_{\rm red}$ in (4.69) generate a differential polynomial algebra *under Dirac bracket.*

Now I wish to give a direct proof for the polynomiality of the Dirac bracket algebra of the SCC, that is the FCC (4.24) and gauge fixings (4.70)

$$c_\tau(x) = \langle \tau \,,\, J(x) - M\rangle = 0 \quad \text{where} \quad \tau \in \{\gamma_h^l\} \cup \{\theta_k^i\}. \tag{4.75}$$

We note that for certain purposes SCC might be more natural to use than FCC since in the second class formalism one directly deals with the physical fields. For example, the $\mathcal{W}_\mathcal{S}^\mathcal{G}$-algebra discussed below is very natural from the second class point of view and can be realized by starting with a number of different first class systems of constraints, as we shall see in the next section.

The Dirac brackets (2.61) of the reduced currents is

$$\begin{aligned}\{j^u_{\rm red}(x), j^v_{\rm red}(y)\}^* &= \{j^u_{\rm red}(x), j^v_{\rm red}(y)\}\\ &- \sum_{\mu\nu}\int dz^1 dw^1 \{j^u_{\rm red}(x), c_\mu(z)\}\Delta^{\mu\nu}(z,w)\{c_\nu(w), j^v_{\rm red}(y)\},\end{aligned} \tag{4.76}$$

where $j^u_{\rm red}(x) = \langle u, j_{\rm red}(x)\rangle$ for any $u \in \mathcal{G}$ and $\Delta^{\mu\nu}(z,w)$ is the inverse of the kernel

$$\Delta_{\mu\nu}(z,w) = \{c_\mu(z), c_\nu(w)\}\ ,$$

in the sense that (on the constraint surface)

$$\sum_\nu \int dx^1 \Delta^{\mu\nu}(z,x)\Delta_{\nu\sigma}(x,w) = \delta_{\mu\sigma}\delta(z^1 - w^1).$$

From the structure of the constraints in (4.75), $c_\tau = (\phi_\gamma, \chi_\theta)$, one sees that $\Delta_{\mu\nu}(z)$ is a first order differential operator possessing the following block structure

$$\Delta_{\mu\nu} = \begin{pmatrix} \{\phi,\phi\} & \{\phi,\chi\} \\ \{\chi,\phi\} & \{\chi,\chi\} \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E^\dagger & F \end{pmatrix}, \tag{4.77}$$

where $E^\dagger$ is the formal Hermitian conjugate of the matrix E. We see that $\Delta_{\mu\nu}$ is invertible if and only if its block E is invertible, and in that case the inverse takes the form

$$(\Delta)^{\mu\nu} = \begin{pmatrix} (E^\dagger)^{-1}FE^{-1} & -(E^\dagger)^{-1} \\ E^{-1} & 0 \end{pmatrix} \tag{4.78}$$

Since $E(z)$ and $F(z)$ are polynomial (even linear) in the current and ∂_z it follows that $\Delta^{\mu\nu}$ is a polynomial differential operator if and only if $E^{-1}(z)$ is a polynomial differential operator.

To show that E^{-1} exists and is a polynomial differential operator we note that in terms of the basis of $(\Gamma + \Theta)$ in (4.75) the matrix E is given explicitly by the following formula:

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{mn} + \langle[\gamma_h^m,\theta_k^n], j_{\rm red}(z)\rangle + \langle\gamma_h^m,\theta_k^n\rangle\partial_z \ .$$

The crucial point is that, by the grading and the property in (4.72), we have

$$E_{\gamma_h^m,\theta_k^n}(z) = \delta_{hk}\delta_{nm} \ , \qquad \text{for} \qquad k \leq h \ . \tag{4.79}$$

The matrix E has a block structure labeled by the (block) row and (block) column indices h and k, respectively, and (4.79) means that the blocks in the diagonal of E are unit matrices and the blocks below the diagonal vanish. In other words, E is of the form $E = 1+\varepsilon$, where ε is a strictly upper triangular matrix. It is clear that such a matrix differential operator is polynomially invertible, namely by a *finite series* of the form

$$E^{-1} = 1 - \varepsilon + \varepsilon^2 + \ldots + (-1)^N\varepsilon^N \ , \qquad (\varepsilon^{N+1} = 0),$$

which finishes our proof of the polynomiality of the Dirac bracket in (4.76). One can use the arguments in the above proof to set up an algorithm for actually computing the Dirac bracket. The proof also shows that the polynomiality of the Dirac bracket is guaranteed whenever E is of the form $(1+\varepsilon)$ with ε being *nilpotent as a matrix.* In our case this was ensured by a special grading assumption, and it appears an interesting question whether polynomial reductions can be obtained at all without using some grading structure.

The zero block occurs in $\Delta^{\mu\nu}$ in (4.78) because the SCC originate from the gauge fixing of FCC. We note that the presence of this zero block implies that the Dirac brackets of the gauge invariant quantities coincide with their original Poisson brackets, namely one sees this from the formula of the

Dirac bracket by keeping in mind that the gauge invariant quantities weakly commute with the FCC.

In our proof of the polynomiality of the gauge fixing and of the algebra we actually only used that the graded subspace Θ of $\mathcal{G}$ which defines the gauge fixing in (4.70) is dual to the graded gauge algebra Γ with respect to ω_M and satisfies the condition

$$([\Theta\,,\,\Gamma])_{\geq 1} \subset \Gamma\ , \tag{4.80}$$

which is equivalent to the existence of the bases γ_h^l and θ_k^i satisfying (4.68) and (4.72). We have seen that this condition follows from the assumption in the theorem, but it should be noted that it is a *more general condition*, since the converse is not true. This is best seen by considering an example. To this let now $\mathcal{G}$ be the maximally non-compact real form of a complex simple Lie algebra. Consider the principal $sl(2)$ embedding in $\mathcal{G}$, with commutation rules as in (4.81) below, and choose the one-dimensional gauge algebra $\Gamma \equiv \{M_+\}$ and take $M \equiv M_-$. The ω_M-dual to M_+ can be taken to be $\theta = M_0$, and then (4.80) holds. To show that conditions in the theorem cannot be satisfied, we prove that a grading operator H for which $[H, M_-] = -M_-$ and $\mathcal{G}^H_{\geq 1} \subset \Gamma$, does not exist. First of all, $[H, M_-] = -M_-$ and $\langle M_-, M_+\rangle \neq 0$ imply $[H, M_+] = M_+$, and thus $\Gamma^H_{\geq 1} = \{M_+\}$. Furthermore, writing $H = (M_0 + \Delta)$, we find from $[H, M_\pm] = \pm M_\pm$ that Δ must be an $sl(2)$ singlet in the adjoint of $\mathcal{G}$. However, in the case of the principal $sl(2)$ embedding, there is no such singlet in the adjoint, and hence $H = M_0$. But then the condition $\mathcal{G}^{M_0}_{\geq 1} \subset \Gamma$ is not fulfilled.

4.3 $\mathcal{W}$-algebras

4.3.1 First class constraints for the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras

Let $\mathcal{S} = \{M_\pm, M_0\}$ be an $sl(2)$ subalgebra of the simple Lie algebra $\mathcal{G}$:

$$[M_0, M_\pm] = \pm M_\pm\ , \qquad [M_+, M_-] = 2M_0\ . \tag{4.81}$$

One can associate an extended conformal algebra, denoted as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, to any such $sl(2)$ embedding [5, 18]. Namely, we defined the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra to be the Dirac bracket algebra generated by the components of the constrained KM current of the the following special form:

$$J_{\rm red}(x) = M_- + j_{\rm red}(x)\ , \qquad \text{with} \qquad j_{\rm red}(x) \in {\rm Ker}({\rm ad}_{M_+})\ , \tag{4.82}$$

which means that $j_{\rm red}(x)$ is a linear combination of the $sl(2)$ highest weight states in the adjoint of $\mathcal{G}$. This definition is indeed natural in the sense that the conformal properties are manifest, since, as we shall see below, with the exception of the M_+-component the spin s component of $j_{\rm red}(x)$ turns out to be a primary field of conformal weight $(s+1)$ with respect to L_{M_0}. Before showing this, we wish to find a gauge algebra Γ for which the triple $(\Gamma, H=M_0, M=M_-)$ satisfies our sufficient conditions for polynomiality and (4.82) represents a DS gauge for the corresponding conformally invariant FCC. The corresponding *first class* KM constraints will then be used in the next section to construct generalized Toda theories which realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras as their chiral algebras.

We start by noticing that the dimension of such a Γ has to satisfy the relation

$$\dim {\rm Ker}({\rm ad}_{M_+}) = \dim \mathcal{W}^{\mathcal{G}}_{\mathcal{S}} = \dim \mathcal{G} - 2\dim\Gamma \ .$$

From this, since the kernels of ${\rm ad}_{M_\pm}$ are of equal dimension, we obtain that

$$\dim\Gamma = \frac{1}{2}\dim\mathcal{G} - \frac{1}{2}\dim {\rm Ker}({\rm ad}_{M_-}) \ , \tag{4.83}$$

which means by (4.67) that we are looking for a Γ of *maximal* dimension. By the representation theory of $sl(2)$, the above equality is equivalent to

$$\boxed{\dim\Gamma = \dim\mathcal{G}_{\geq 1} + \frac{1}{2}\dim\mathcal{G}_{\frac{1}{2}} \ ,} \tag{4.84}$$

where the grading is by the, in general half-integral, eigenvalues of ${\rm ad}_{M_0}$. We also know from our lemma that we should choose the graded Lie subalgebra Γ of $\mathcal{G}$ in such a way that $\mathcal{G}_{\geq 1} \subset \Gamma \subset \mathcal{G}_{>0}$. Observe that the non-degeneracy condition in the theorem is automatically satisfied for any such Γ since in the present case ${\rm Ker}({\rm ad}_{M_-}) \subset \mathcal{G}_{\leq 0}$, and $H = M_0 \in \Gamma^\perp$ is also ensured, which guarantees the conformal invariance, see (4.62).

It is obvious from the above that in the special case of an *integral* $sl(2)$ subalgebra, for which $\mathcal{G}_{\frac{1}{2}}$ is empty, one can simply take

$$\Gamma = \mathcal{G}_{\geq 1} \ .$$

For grading reasons, ω_{M_-} vanishes on this Γ and thus one indeed obtains conformal FCC and polynomiality this way.

One sees from (4.84) that for finding the gauge algebra in the non-trivial case of a *half-integral* $sl(2)$ subalgebra, one should somehow add half of $\mathcal{G}_{\frac{1}{2}}$ to $\mathcal{G}_{\geq 1}$, in order to have the correct dimension. The key observation for defining the required *halving* of $\mathcal{G}_{\frac{1}{2}}$ consists in noticing that the restriction of the 2-form ω_{M_-} to $\mathcal{G}_{\frac{1}{2}}$ is non-degenerate. This can be seen as a consequence of (4.66), but is also easy to verify directly. By the well known Darboux normal form of symplectic forms [2], there exists a (non-unique) direct sum decomposition

$$\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}} \tag{4.85}$$

such that ω_{M_-} vanishes on the subspaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$ separately. The spaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, which are the analogues of the usual momentum and coordinate subspaces of the phase space in analytic mechanics, are of equal dimension and dual to each other with respect to ω_{M_-}. The point is that the first-classness conditions in (4.13) are satisfied if we define the gauge algebra to be

$$\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} \ , \tag{4.86}$$

by using *any symplectic halving* of the above kind. It is obvious from the construction that the FCC (4.58) obtained by using Γ in (4.86) satisfy the sufficient conditions for polynomiality given earlier. With this Γ we have

$$\Gamma^{\perp} = \mathcal{G}_{\geq 0} + \mathcal{Q}_{-\frac{1}{2}}, \quad \text{where} \quad \mathcal{Q}_{-\frac{1}{2}} = [M_-, \mathcal{P}_{\frac{1}{2}}] \subset \mathcal{G}_{-\frac{1}{2}}.$$

By combining these relations with (4.86) one also easily verifies the following direct sum decomposition:

$$\Gamma^{\perp} = [M_-, \Gamma] + \mathrm{Ker}(\mathrm{ad}_{M_+}) \ ,$$

which is just (4.71) with $\mathcal{V} = \mathrm{Ker}(\mathrm{ad}_{M_+})$. This means that (4.82) is indeed nothing but a particular DS gauge for the FCC, and this gauge is called the *highest weight gauge* [4]. There exists therefore a basis of gauge invariant differential polynomials of the current in (4.58) such that the base elements reduce to the components of $j_{\rm red}(x)$ in (4.82) by the gauge fixing. The KM Poisson bracket algebra of these polynomials is clearly identical to the Dirac bracket algebra of the corresponding current components, and we can thus realize the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra as a KM Poisson bracket algebra of gauge invariant differential polynomials.

The SCC defining the highest weight gauge (4.82) are natural in the sense that in this case τ in (4.75) runs over the basis of the space $\mathcal{T}_{M_-} = [M_+ \, , \mathcal{G}]$ which is a natural complement of $\mathcal{K}_{M_-} = \mathrm{Ker}(\mathrm{ad}_{M_-})$ in $\mathcal{G}$, eq. (4.65).

In the second class formalism, the conformal action generated by L_{M_0} on the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra is given by the following formula:

$$\delta^*_{f,M_0}\, j_{\rm red}(x) \equiv -\int dy^1\, f(y^+)\, \{L_{M_0}(y)\, , j_{\rm red}(x)\}^* \ , \tag{4.87}$$

where the parameter function $f(x^+)$ refers to the conformal coordinate transformation, cf. (4.60). To actually evaluate (4.87), we first replace L_{M_0} by the object

$$L_{\rm mod}(x) = L_{M_0}(x) - \frac{1}{2}\langle M_+\,,\, J''(x)\rangle\ ,$$

which is allowed under the Dirac bracket since the difference (the second term) vanishes upon imposing the constraints. The crucial point to notice is that $L_{\rm mod}$ weakly commutes with *all* FCC and gauge fixings the KM Poisson bracket. This implies that with $L_{\rm mod}$ the Dirac bracket in (4.87) is in fact identical to the original KM Poisson bracket and by this observation we easily obtain

$$\delta^*_{f,M_0}\, j_{\rm red}(x) = f(x^+)\, j'_{\rm red} + f'(x^+)(j_{\rm red} + [M_0, j_{\rm red}]) - \frac{1}{2}f'''(x^+)M_+.$$

This proves that, with the exception of the M_+-component, the $sl(2)$ highest weight components of $j_{\rm red}(x)$ in (4.82) transform as conformal primary fields, whereby the conformal content of $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ is determined by the decomposition of the adjoint of $\mathcal{G}$ under $\mathcal{S}$ in the aforementioned manner. We end this discussion by noting that in the highest weight gauge $L_{M_0}(x)$ becomes a linear combination of the M_+-component of $j_{\rm red}(x)$ and a quadratic expression in the components corresponding to the singlets of $\mathcal{S}$ in $\mathcal{G}$. From this we see that $L_{M_0}(x)$ and the primary fields corresponding to the $sl(2)$ highest weight states give a basis for the differential polynomials contained in $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, which is thus indeed a (classical) $\mathcal{W}$-algebra in the sense of the general idea in [52].

In the above we proposed a 'halving procedure' for finding purely FCC for which $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ appears as the algebra of the corresponding gauge invariant differential polynomials. I now wish to clarify the relationship between our method and the construction in a recent paper by Bais *et al* [5], where the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra has been described, in the special case of $\mathcal{G} = sl(n)$, by using a different method. I recall that the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra has been constructed in [5] by adding to the FCC defined by the pair $(\mathcal{G}_{\geq 1}, M_-)$ the SCC

$$\langle u\,,\, J(x)\rangle = 0 \qquad \text{for} \quad \forall u \in \mathcal{G}_{\frac{1}{2}}\ . \tag{4.88}$$

Clearly, we recover these constraints by first imposing our complete set of FCC belonging to (Γ, M_-) with Γ in (4.86), and then partially fixing the gauge by imposing the condition

$$\langle u\,,\, J(x)\rangle = 0\ , \qquad \text{for} \quad \forall u \in \mathcal{Q}_{\frac{1}{2}}\ .$$

One of the advantages of our construction is that by using only first class KM constraints it is easy to construct generalized Toda theories which possess

$\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ as their chiral algebra, for any $sl(2)$ subalgebra, namely by using our general method of WZNW reductions. This will be elaborated in the next section. We note that in [5] the authors were actually also led to replacing the original constraints by a system of FCC, in order to be able to consider the BRST quantization of the theory. For this purpose they introduced unphysical 'auxiliary fields' and thus constructed FCC in an extended phase space. However, in that construction one has to check that the auxiliary fields finally disappear from the physical quantities.

The FCC leading to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ are not unique. For example, arbitrary halving in (4.85) lead to the same $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$. It maybe conjectured that these $\mathcal{W}$-algebras always occur under certain natural assumptions on the constraints. To be more exact, let us suppose that we have conformally invariant first class constraints determined by (Γ, M_-, H) where M_- is a *nilpotent* matrix and the *non-degeneracy* condition in the theorem holds together with equation (4.83). I expect that these assumptions are sufficient for the existence of a complete set of gauge invariant differential polynomials and their algebra is isomorphic to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, where $\mathcal{S} = \{M_\pm, M_0\}$ is an $sl(2)$-extension of the nilpotent M_-. Such an $\mathcal{S}$ can always be found, since we have the

Lemma 5 *Let H be a grading operator and $M_- \in \mathcal{G}_{-1}^H$. Then there exists an $sl(2)$ algebra $\mathcal{S} = \{M_\pm, M_0\}$ such that $M_+ \in \mathcal{G}_1^H$.*

Note that as a consequence the difference $H - M_0$ commutes with $\mathcal{S}$. To prove this theorem one first extends the nilpotent M_- to an $sl(2)$ subalgebra, which always exists by the Jacobson-Morozow theorem. Then one decomposes the generators of this $sl(2)$ in components of definite H-grades. The components with the desired grades form then the $sl(2)$ with the properties in the lemma. To prove this last fact one uses the lemma 7 on page 98 in [28].

I am not able to prove the above conjecture in general, but now I sketch the proof in an important special case which illustrates the idea.

Let us assume that we have conformally invariant FCC described by (Γ, M_-, H) subject to the sufficient conditions for polynomiality. But in addition we assume now that H is an *integral grading operator* of $\mathcal{G}$ so that $\Gamma = \mathcal{G}_{\geq 1}$. Then the non-degeneracy condition says that

$$\mathrm{dim}\mathcal{G}_+^H = \mathrm{dim}[M_-, \mathcal{G}_+^H]. \tag{4.89}$$

Now I show that this condition implies

$$[M_-, \mathcal{G}_0^H + \mathcal{G}_-] = \mathcal{G}_-^H \tag{4.90}$$

Indeed, if it would not, then we would find an $u \in \mathcal{G}_+^H$ such that $\langle u, [M_-, \mathcal{G}_0^H + \mathcal{G}_-^H]\rangle$ would vanish. By the invariance and non-degeneracy of the Cartan-Killing form this in turn is equivalent to $[M_-, u] = 0$ which means that the non-degeneracy condition (4.89) would be violated. Also, since $H - M_0$ commutes with $\mathcal{S}$, the difference $\mathrm{ad}_H - \mathrm{ad}_{M_0}$ is constant in each multiplet in the decomposition of $\mathcal{G}$ under $\mathcal{S}$. Then it follows immediately from the $sl(2)$ structure and (4.89,4.90) that

$$\mathrm{dimKer}(\mathrm{ad}_{M_\pm}) = \mathrm{dim}\mathcal{G}_0^H \quad , \quad \mathrm{Ker}(\mathrm{ad}_{M_+}) \subset \mathcal{G}^H_{\geq 0} \Leftrightarrow \mathrm{Ker}(\mathrm{ad}_{M_-}) \subset \mathcal{G}^H_{\leq 0},$$

We introduce a definition at this point, which will be used in the rest of the chapter. Namely, we call $\mathcal{S}$ an *H--compatible $sl(2)$* if there exists an integral grading operator H such that $[H, M_\pm] = \pm M_\pm$ is satisfied together with the non-degeneracy condition. The non-degeneracy condition can be expressed in various equivalent forms, it can be given for example as the relation in above, and its (equivalent) analogue for M_-.

Turning back to the problem at hand, we now point out that by using the H-compatible $sl(2)$ we have the following direct sum decomposition of $\Gamma^\perp = \mathcal{G}^H_{\geq 0}$:

$$\mathcal{G}^H_{\geq 0} = [M_-, \mathcal{G}^H_{>0}] + \mathrm{Ker}(\mathrm{ad}_{M_+}).$$

This means that the set of currents of the form (4.82) represents a DS gauge for the present FCC. This implies the required result, that is that the $\mathcal{W}$-algebra belonging to the constraints defined by $\Gamma = \mathcal{G}^H_{>0}$ together with a non-degenerate M_- is isomorphic to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ with $M_- \in \mathcal{S}$. In this example both $L_H(x)$ and $L_{M_0}(x)$ are gauge invariant differential polynomials. Although the spectrum of ad_H is *integral* by assumption, in some cases the H-compatible $sl(2)$ is embedded into $\mathcal{G}$ in a *half-integral* manner.

I also would like to mention an interesting general fact about the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebras, which will be used in the next section. Let us consider the decomposition of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$. In general, we shall find singlet states and they span a Lie subalgebra in the Lie subalgebra $\mathrm{Ker}(\mathrm{ad}_{M_+})$ of $\mathcal{G}$. Let us denote this zero spin subalgebra as $\mathcal{Z}$. It is easy to see that we have the semi-direct sum decomposition

$$\mathrm{Ker}(\mathrm{ad}_{M_+}) = \mathcal{Z} + \mathcal{R}, \qquad [\mathcal{Z}, \mathcal{R}] \subset \mathcal{R}, \qquad [\mathcal{Z}, \mathcal{Z}] \subset \mathcal{Z}, \tag{4.91}$$

where $\mathcal{R}$ is the linear space spanned by the rest of the highest weight states, which have non-zero spin. It is not hard to prove that the subalgebra of the original KM algebra which belongs to $\mathcal{Z}$, survives the reduction to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$. In other words, the Dirac brackets of the $\mathcal{Z}$-components of the highest weight gauge current, j_{red} in (4.82), coincide with their original KM Poisson brackets, given by (4.25). Furthermore, this $\mathcal{Z}$ KM subalgebra acts on the $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra by the corresponding original KM transformations, which preserve the highest weight gauge:

$$J_{\mathrm{red}}(x) \to e^{a^i(x^+)\zeta_i}\, J_{\mathrm{red}}(x)\, e^{-a^i(x^+)\zeta_i} + (e^{a^i(x^+)\zeta_i})'\, e^{-a^i(x^+)\zeta_i},$$

where the ζ_i form a basis of $\mathcal{Z}$. In particular, one sees that the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra inherits the semi-direct sum structure given by (4.91) [5]. The point is that it is possible to *further reduce* the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra by applying the general method of conformally invariant KM reductions to the present $\mathcal{Z}$ KM symmetry. In principle, one can generate a huge number of new conformally invariant systems out of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras in this way, i.e., by applying conformally invariant constraints to their singlet KM subalgebras. For example, if one can find a subalgebra of $\mathcal{Z}$ on which the Cartan-Killing form of $\mathcal{G}$ vanishes, then one can consider the obviously conformally invariant reduction obtained by constraining the corresponding components of $j_{\rm red}$ in (4.82) to zero.

Finally, note that for a half-integral $sl(2)$, one can consider (instead of using Γ in (4.86)) also those conformally invariant FCC which are defined by the triple (Γ, M_0, M_-) with any graded Γ for which $\mathcal{G}_{\geq 1} \subset \Gamma \subset (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}})$. The polynomiality conditions are clearly satisfied with any such non-maximal Γ, and the corresponding extended conformal algebras are in a sense between the KM and $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras.

4.3.2 The $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ interpretation of the W_n^l-algebras

The W_n^l-algebras are certain conformally invariant reductions of the $sl(n,R)$ KM algebra introduced by Bershadsky [8] using a mixed set of FCC and SCC. It is known [5] that the simplest non-trivial case W_3^2, originally proposed by Polyakov [44], coincides with the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebra belonging to the highest root $sl(2)$ of $sl(3,R)$. The purpose of this section is to understand whether or not these reduced KM systems fit into our framework and to uncover their possible connection with the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras in the general case [7] In fact, we shall construct here purely first class KM constraints leading to the W_n^l-algebras. We will prove the

Lemma 6 *The W_n^l-algebras can in general be identified as further reductions of particular $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras. The secondary reduction process is obtained by means of the singlet KM subalgebras of the relevant $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$-algebras*

By definition [8], the KM reduction yielding the W_n^l-algebra is obtained by constraining the current to take the following form:

$$J_{\rm B}(x) = M_- + j_{\rm B}(x), \qquad j_{\rm B}(x) \in \Delta^{\perp}, \tag{4.92}$$

where Δ denotes the set of all strictly upper triangular $n \times n$ matrices and

$$M_- = e_{l+1,1} + e_{l+2,2} + \dots + e_{n,n-l}, \tag{4.93}$$

the e's being the standard $sl(n,R)$ generators ($l \leq n-1$), i.e., M_- has 1's all

[7] In this section, $\mathcal{G} = sl(n,R)$.

along the l-th slanted line below the diagonal. Generally, these constraints comprise first and second class parts, where the first class part is the one belonging to the subalgebra $\mathcal{D}$ of Δ defined by the relation $\omega_{M_-}(\mathcal{D},\Delta)=0$, (see 4.26). The second class part belongs to the complementary space, $\mathcal{C}$, of $\mathcal{D}$ in Δ. In fact, for $l=1$ the constraints are the usual first class ones which yield the standard $\mathcal{W}$-algebras, but the second class part is non-empty for $l>1$. The above KM reduction is so constructed that it is conformally invariant, since the constraints weakly commute with the Virasoro density $L_{H_l}(x)$, see (4.59), where $H_l=\frac{1}{l}H_1$ and H_1 is the standard grading operator of $sl(n,R)$, for which $[H_1\,,\,e_{ik}]=(k-i)e_{ik}$.

We start our construction by extending the nilpotent generator M_- in (4.93) to an $sl(2)$ subalgebra $\mathcal{S}$. In fact, parameterizing $n=ml+r$ with $m=[\frac{n}{l}]$ and $0\leq r<l$, we can take

$$M_0=\mathrm{diag}\Big(\overbrace{\frac{m}{2},\cdots}^{r\text{ times}},\overbrace{\frac{m-1}{2},\cdots}^{(l-r)\text{ times}},\cdots,\overbrace{-\frac{m}{2},\cdots}^{r\text{ times}}\Big),\tag{4.94}$$

where the multiplicities, r and $(l-r)$, occur alternately and end with r. The meaning of this formula is that the fundamental of $sl(n,R)$ branches into l irreducible representations under $\mathcal{S}$, r of spin $m/2$ and $l-r$ of spin $(m-1)/2$. The explicit form of M_+ is a certain linear combination of the e_{ik}'s with $(k-i)=l$, which is straightforward to compute.

Next I describe the first and the second class parts of the constraints in (4.92) in more detail by using the grading defined by M_0. We observe first that in terms of this grading the space Δ admits the decomposition

$$\Delta=\Delta_0+\mathcal{G}_{\frac{1}{2}}+\mathcal{G}_1+\mathcal{G}_{>1}\ .\tag{4.95}$$

From this and the definition of ω_{M_-}, the subalgebra $\mathcal{D}$ comprising the first class part can also be decomposed into

$$\mathcal{D}=\mathcal{D}_0+\mathcal{D}_1+\mathcal{G}_{>1}\ ,\quad\text{where}\quad\mathcal{D}_0=\mathrm{Ker}\,(\mathrm{ad}_{M_-})\cap\Delta_0\tag{4.96}$$

is the set of the $sl(2)$ singlets in Δ, and $\mathcal{D}_1$ is a subspace of $\mathcal{G}_1$ which we do not need to specify. By combining (4.95) and (4.96), we see that the complementary space $\mathcal{C}$, to which the second class part belongs, has the structure

$$\mathcal{C}=\mathcal{Q}_0+\mathcal{G}_{\frac{1}{2}}+\mathcal{P}_1\ ,$$

where the subspace $\mathcal{Q}_0$ is complementary to $\mathcal{D}_0$ in Δ_0, and $\mathcal{P}_1$ is complementary to $\mathcal{D}_1$ in $\mathcal{G}_1$. The 2-form ω_{M_-} is non-degenerate on $\mathcal{C}$ by construction, and this implies by the grading that the spaces $\mathcal{Q}_0$and $\mathcal{P}_1$ are symplectically conjugate to each other, which is reflected by the notation.

We shall construct a gauge algebra, Γ, so that Bershadsky's constraints will be recovered by a partial gauge fixing from the first class ones belonging to Γ. As a generalization of the halving procedure of the previous section, we take the following ansatz:

$$\Gamma = \mathcal{D} + \mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1 \ , \tag{4.97}$$

where $\mathcal{P}_{\frac{1}{2}}$ is defined by means of some symplectic halving $\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}}$, like in (4.85). It is important to notice that this equation can be recasted into

$$\Gamma = \mathcal{D}_0 + \mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1} \ , \tag{4.98}$$

which would be just the familiar formula (4.86) if $\mathcal{D}_0$ was not here. By using (4.93) and (4.94), $\mathcal{D}_0$ can be identified as the set of $n \times n$ block-diagonal matrices, σ, of the following form:

$$\sigma = \text{block-diag}\{\Sigma_0, \sigma_0, \Sigma_0, ..., \Sigma_0, \sigma_0, \Sigma_0\},$$

where the Σ_0's and the σ_0's are identical copies of strictly upper triangular $r \times r$ and $(l-r) \times (l-r)$ matrices respectively. This implies that

$$\dim \mathcal{D}_0 = \frac{1}{4}[l(l-2) + (l-2r)^2] \ ,$$

which shows that $\mathcal{D}_0$ is non-empty except when $l = 2$, $r = 1$, which is the case of W_n^2 with $n =$ odd. The fact that $\mathcal{D}_0$ is in general non-empty gives us a trouble at this stage, namely, we have now no guarantee that the above Γ is actually a *subalgebra* of $\mathcal{G}$. By using the grading and the fact that $\mathcal{D}_0$ is a subalgebra, we see that Γ in (4.98) becomes a subalgebra if and only if

$$[\mathcal{D}_0 \, , \mathcal{P}_{\frac{1}{2}}] \subset \mathcal{P}_{\frac{1}{2}}. \tag{4.99}$$

I next show that it is possible to find such a 'good halving' of $\mathcal{G}_{\frac{1}{2}}$ for which $\mathcal{P}_{\frac{1}{2}}$ satisfies (4.99).

For this purpose, we use yet another grading here. This grading is provided by using the particular diagonal matrix, $H \in \mathcal{G}$, which we construct out of M_0 in (4.94) by first adding $\frac{1}{2}$ to its half-integral eigenvalues, and then subtracting a multiple of the unit matrix so as to make the result traceless. In the adjoint representation, we then have $\mathrm{ad}_H = \mathrm{ad}_{M_0}$ on the tensors, and $\mathrm{ad}_H = \mathrm{ad}_{M_0} \pm 1/2$ on the spinors. We notice from this that the H-grading is an integral grading. In fact, the relationship between the two gradings allows us to define a good halving of $\mathcal{G}_{\frac{1}{2}}$ as follows:

$$\mathcal{P}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}_1^H \ , \qquad \text{and} \qquad \mathcal{Q}_{\frac{1}{2}} \equiv \mathcal{G}_{\frac{1}{2}} \cap \mathcal{G}_0^H \ . \tag{4.100}$$

Since M_- is of grade -1 with respect to both gradings, the spaces given by (4.100) clearly yield a symplectic halving of $\mathcal{G}_{\frac{1}{2}}$ with respect to ω_{M_-}. That this ensures the condition (4.99), can also be seen easily by observing that $\mathcal{D}_0$ has grade 0 in the H-grading, too. Thus we obtain the required subalgebra Γ of $\mathcal{G}$ by using this particular $\mathcal{P}_{\frac{1}{2}}$ in (4.98).

Let us consider now the FCC corresponding to the above constructed gauge algebra Γ, $\phi_\gamma(x) = 0$ for $\gamma \in \Gamma$, which bring the current into the form

$$J_\Gamma(x) = M_- + j_\Gamma(x) \ , \qquad j_\Gamma(x) \in \Gamma^\perp \ . \tag{4.101}$$

It is easy to verify that the original constraint surface (4.92) can be recovered from (4.101) by a partial gauge fixing in such a way that the residual gauge transformations are exactly the ones belonging to the space $\mathcal{D}$. In fact,this is achieved by fixing the gauge freedom corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{P}_1)$ of Γ, (4.97), by imposing the partial gauge fixing condition

$$\phi_{q_i}(x) = 0 \ , \qquad q_i \in (\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}}),$$

where the q_i form a basis of the space $(\mathcal{Q}_0 + \mathcal{Q}_{\frac{1}{2}})$ and the ϕ_q's are defined like in (4.24). This implies that the reduced phase space defined by the constraints in(4.101) is the same as the one determined by the original constraints (4.92). In conclusion, our purely FCC, (4.101), have the same physical content as Bershadsky's original mixed set of constraints, (4.92).

Finally, we give the relationship between Bershadsky's W_n^l-algebras and the $sl(2)$ systems. Having seen that the reduced KM phase spaces carrying the W_n^l-algebras can be realized by starting from the FCC in (4.101), it follows from (4.98) that the W_n^l-algebras coincide with particular $\mathcal{W}_\mathcal{S}^\mathcal{G}$-algebras if and only if the space $\mathcal{D}_0$ is empty, i.e., for W_n^2 with $n =$ odd. In order to establish the $\mathcal{W}_\mathcal{S}^\mathcal{G}$ interpretation of W_n^l in the general case, note that the reduced phase space can be reached from (4.101) by means of the following two step process based on the $sl(2)$ structure. Namely, one can proceed by first fixing the gauge freedom corresponding to the piece $(\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1})$ of Γ, and then fixing the rest of the gauge freedom. Clearly, the constraint surface resulting in the first step is the same as the one obtained by putting to zero those components of the highest weight gauge current representing $\mathcal{W}_\mathcal{S}^\mathcal{G}$ which correspond to $\mathcal{D}_0$. The final reduced phase space is obtained in the second step by fixing the gauge freedom generated by the constraints belonging to $\mathcal{D}_0$, which we have seen to be the space of the upper triangular singlets of $\mathcal{S}$. Thus we can conclude that W_n^l can be regarded as a further reduction of the corresponding $\mathcal{W}_\mathcal{S}^\mathcal{G}$, where the 'secondary reduction' is of the type mentioned at the end of the previous subsection.

4.4 Generalized Toda theories

The standard conformal Toda field theories

$$\mathcal{L}_{\mathrm{Toda}} = \frac{\kappa}{2}\left(\sum_{ij=1}^{l} \frac{1}{2|\alpha_i|^2} K_{ij}\partial_\mu \varphi^i \partial_\nu \varphi^j - \sum_{i=1}^{l} m_i^2 \exp\left\{ \frac{1}{2}\sum_{j=1}^{l} K_{ij}\varphi^j \right\} \right) \tag{4.102}$$

where K_{ij} is the Cartan matrix and the α_i the simple roots of $\mathcal{G}$, are the most simple cases of reduced WZNW theories, and as a consequence these theories possess the chiral algebras $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$ as their canonical symmetries, where $\mathcal{S}$ is the principal $sl(2)$ subalgebra of the maximally non-compact real Lie algebra $\mathcal{G}$. It is natural to seek for WZNW reductions leading to effective field theories which would realize $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$ as their chiral algebras *for any $sl(2)$ subalgebra $\mathcal{S}$ of any simple real Lie algebra.* The main purpose of this chapter is to obtain *generalized Toda theories* meeting the above requirement in the non-trivial case of the *half-integral $sl(2)$* subalgebras of the simple Lie algebras. Before turning to describing these new theories, next I briefly recall the main features of those generalized Toda theories, associated to the *integral gradings* of the simple Lie algebras, which have been studied before [33, 46, 39, 40, 5, 49, 18]. The simplicity of the latter theories will motivate some subsequent developments.

4.4.1 Generalized Toda theories with integral gradings

The WZNW reduction leading to the generalized Toda theories in question is set up by considering an integral grading operator H of $\mathcal{G}$, and taking the special case

$$\Gamma = \mathcal{G}_{\geq 1}^{H} \;,\; M \in \mathcal{G}_{-1}^{H} \qquad \text{and} \qquad \tilde{\Gamma} = \mathcal{G}_{\leq -1}^{H} \;,\; \tilde{M} \in \mathcal{G}_{1}^{H}. \tag{4.103}$$

In the present case $\mathcal{B}$ in (4.35) is the subalgebra $\mathcal{G}_0^H$, and, because of the grading structure, the properties expressed by equation (4.52) hold. Thus the effective field equation reads as (4.55) and the corresponding action is given by the simple formula (4.57) where the field b varies in the little group G_0^H of H in G.

It was shown in [33, 46, 5] in the special case when H, M and $\tilde{M}$ are taken to be the standard generators of an integral $sl(2)$ subalgebra of $\mathcal{G}$, that the non-Abelian Toda equation allows for conserved chiral currents underlying its exact integrability. These currents then generate chiral $\mathcal{W}$-algebras of the type $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, for integrally embedded $sl(2)$'s.

By means of the argument given in the previous section, we can establish the structure of the chiral algebras of a wider class of non-Abelian Toda systems [18]. Namely, we see that if M and $\tilde{M}$ in (4.103) satisfy the non-degeneracy conditions

$$\mathrm{Ker}(\mathrm{ad}_M) \cap \Gamma = \{0\} \quad \text{and} \quad \mathrm{Ker}(\mathrm{ad}_{\tilde{M}}) \cap \tilde{\Gamma} = \{0\} \ ,$$

then the left $\times$ right chiral algebra of the corresponding generalized Toda theory is isomorphic to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\tilde{\mathcal{S}}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are $sl(2)$ subalgebras of $\mathcal{G}$ containing the nilpotent generator M and $\tilde{M}$, respectively. The H-compatible $sl(2)$ algebras $\mathcal{S}$ and $\tilde{\mathcal{S}}$ occurring here are *not always integrally embedded* ones. Thus for certain *half-integral* $sl(2)$ algebras $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ can be realized in a generalized Toda theory of the type (4.57). As we would like to have generalized Toda theories which possess$\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$ as their symmetry algebra for an arbitrary $sl(2)$ subalgebra, we have to ask whether the theories given above are already enough for this purpose or not. This leads to the technical question as to whether for every half-integral $sl(2)$ subalgebra $\mathcal{S}$ of $\mathcal{G}$ there exists an integral grading operator H such that $\mathcal{S}$ is an H-compatible $sl(2)$, in the sense introduced earlier. The answer to this question is negative. Thus we have to find new integrable conformal field theories for our purpose.

4.4.2 Generalized Toda theories with half-integral $sl(2)$'s

In the following I exhibit a generalized Toda theory possessing the left $\times$ right chiral algebra $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ for an arbitrarily chosen half-integral $sl(2)$ subalgebra $\mathcal{S}$ of the arbitrary but non-compact simple real Lie algebra $\mathcal{G}$. Clearly, if one imposes FCC of the type described in the previous section on the currents of the WZNW theory then the resulting effective field theory will have the required chiral algebra. We shall choose the left and right gauge algebras in such a way to be dual to each other with respect to the Cartan-Killing form.

Thus we choose a direct sum decomposition of $\mathcal{G}_{\frac{1}{2}}$ of the type in (4.85), and then define the *induced decomposition* $\mathcal{G}_{-\frac{1}{2}} = \mathcal{P}_{-\frac{1}{2}} + \mathcal{Q}_{-\frac{1}{2}}$ to be given by the subspaces

$$\mathcal{Q}_{-\frac{1}{2}} \equiv \mathcal{P}^{\perp}_{\frac{1}{2}} \cap \mathcal{G}_{-\frac{1}{2}} = [M_- \, , \, \mathcal{P}_{\frac{1}{2}}] \quad \text{and} \quad \mathcal{P}_{-\frac{1}{2}} \equiv \mathcal{Q}^{\perp}_{\frac{1}{2}} \cap \mathcal{G}_{-\frac{1}{2}} = [M_- \, , \, \mathcal{Q}_{\frac{1}{2}}] \ .$$

It is easy to see that the 2-form ω_{M_+} vanishes on the above subspaces of $\mathcal{G}_{-\frac{1}{2}}$as a consequence of the vanishing of ω_{M_-} on the corresponding subspaces of $\mathcal{G}_{\frac{1}{2}}$. Thus we can take the left and right gauge algebras to be

$$\Gamma = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}}) \quad \text{and} \quad \tilde{\Gamma} = (\mathcal{G}_{\leq -1} + \mathcal{P}_{-\frac{1}{2}}) \ , \tag{4.104}$$

with the constant matrices M and $\tilde{M}$ entering the constraints given by M_-

and M_+, respectively. The duality hypothesis of section 4.1.2 is obviously satisfied by this construction.

In principle, the action and the Lax potential of the effective theory can be obtained by specializing the general formulas of section 4.1.2 to the present particular case. In our case

$$\mathcal{B} = \mathcal{Q}_{\frac{1}{2}} + \mathcal{G}_0 + \mathcal{Q}_{-\frac{1}{2}} \ ,$$

and the physical modes, which are given by the entries of b in the generalized Gauss decomposition $g = abc$ with $a \in e^{\Gamma}$ and $c \in e^{\tilde{\Gamma}}$, are now conveniently parametrized as

$$b(x) = \exp[q_{\frac{1}{2}}(x)] \cdot g_0(x) \cdot \exp[q_{-\frac{1}{2}}(x)], \tag{4.105}$$

where $q_{\pm\frac{1}{2}}(x) \in \mathcal{Q}_{\pm\frac{1}{2}}$ and $g_0(x) \in G_0$, the little group of M_0 in G. Next I introduce some notation which will be useful for describing the effective theory.

The operator Ad_{g_0} maps $\mathcal{G}_{-\frac{1}{2}}$ to itself and, by writing the general element of $\mathcal{G}_{-\frac{1}{2}}$ as a two-component column vector whose upper and lower components belong to $\mathcal{P}_{-\frac{1}{2}}$ and $\mathcal{Q}_{-\frac{1}{2}}$, respectively, we can write this operator as a 2×2 matrix:

$$\mathrm{Ad}_{g_0|\mathcal{G}_{-\frac{1}{2}}} = \begin{pmatrix} X_{11}(g_0) & X_{12}(g_0) \\ X_{21}(g_0) & X_{22}(g_0) \end{pmatrix} \tag{4.106}$$

Analogously, I introduce the notation

$$\mathrm{Ad}_{g_0^{-1}|\mathcal{G}_{\frac{1}{2}}} = \begin{pmatrix} Y_{11}(g_0) & Y_{12}(g_0) \\ Y_{21}(g_0) & Y_{22}(g_0) \end{pmatrix} \ , \tag{4.107}$$

which corresponds to writing the general element of $\mathcal{G}_{\frac{1}{2}}$ as a column vector, whose upper and lower components belong to $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, respectively.

The action functional of the effective field theory resulting from the WZNW reduction at hand reads as follows:

$$\begin{aligned} I_{\text{eff}}^{\mathcal{S}}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) &= S_{\text{WZ}}(g_0) - \int d^2x \, \langle g_0 M_+ g_0^{-1} \, , \, M_- \rangle \\ &+ \int d^2x \, (\langle \partial_- q_{\frac{1}{2}} \, , \, g_0 \partial_+ q_{-\frac{1}{2}} g_0^{-1} \rangle + \langle \eta_{\frac{1}{2}} \, , \, X_{11}^{-1} \cdot \eta_{-\frac{1}{2}} \rangle), \end{aligned} \tag{4.108}$$

where the objects $\eta_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$ are given by the formulas

$$\eta_{\frac{1}{2}} = [M_+, q_{-\frac{1}{2}}] + Y_{12} \cdot \partial_- q_{\frac{1}{2}} \quad \text{and} \quad \eta_{-\frac{1}{2}} = [M_-, q_{\frac{1}{2}}] - X_{12} \cdot \partial_+ q_{-\frac{1}{2}}.$$

The Euler-Lagrange equation of this action is the zero curvature condition of the following Lax potential:

$$
\begin{aligned}
\mathcal{A}_+^{\mathcal{S}} = & \quad M_- + \partial_+ g_0 \cdot g_0^{-1} + g_0(\partial_+ q_{-\frac{1}{2}} + X_{11}^{-1} \cdot \eta_{-\frac{1}{2}}) g_0^{-1}\,, \\
\mathcal{A}_-^{\mathcal{S}} = & \qquad\qquad -g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1} \cdot \eta_{\frac{1}{2}}\,.
\end{aligned}
\tag{4.109}
$$

The above new (conformally invariant) effective action and Lax potential are among the main results of the present chapter. Clearly, for an integrally embedded $sl(2)$ this action and Lax potential simplify to the ones given by equation (4.57) and (4.56).

The derivation of the above formulas is not completely straightforward, and next I wish to sketch the main steps. First, let us remember that, by (4.39), to specialize the general effective action (4.47) and the Lax potential (4.43) to our situation, we should express the objects $\partial_+ cc^{-1}$ and $a^{-1}\partial_- a$ in terms of b by using the constraints on J and $\tilde{J}$, respectively [8]. For this purpose it turns out to be convenient to parametrize the WZNW field g by using the grading defined by the $sl(2)$, i.e., as

$$
g = g_+ \cdot g_0 \cdot g_- \quad \text{where} \quad g_+ = a \cdot \exp[q_{\frac{1}{2}}], \quad g_- = \exp[q_{-\frac{1}{2}}] \cdot c.
$$

We recall that the fields a, c, g_0 and q have been introduced previously by means of the parametrization $g = abc$, with b in (4.105). Also for later convenience, we write $g_\pm$ as

$$
g_+ = \exp[r_{\geq 1} + p_{\frac{1}{2}} + q_{\frac{1}{2}}] \quad \text{and} \quad g_- = \exp[r_{\leq -1} + p_{-\frac{1}{2}} + q_{-\frac{1}{2}}]\,.
$$

Note that here and below the subscript denotes the grade of the variables, and $p_{\pm\frac{1}{2}} \in \mathcal{P}_{\pm\frac{1}{2}}$. In our case this parametrization of g is advantageous, since, as shown below, the use of the grading structure facilitates solving the constraints.

For example, the left constraint are restrictions on $J_{<0}$, for which we have

$$
J_{<0} = (g_+ g_0 N g_0^{-1} g_+^{-1})_{<0} \quad \text{with} \quad N = \partial_+ g_- \cdot g_-^{-1}.
$$

By considering this equation grade by grade, starting from the lowest grade, it is easy to see that the constraints corresponding to $\mathcal{G}_{\geq 1} \subset \Gamma$ are equivalent to the relation

$$
N_{\leq -1} = g_0^{-1} M_- g_0\ .
$$

The remaining left constraints set the $\mathcal{P}_{-\frac{1}{2}}$ part of $J_{-\frac{1}{2}}$ to zero, and to unfold these constraints first we note that

$$
J_{-\frac{1}{2}} = [p_{\frac{1}{2}} + q_{\frac{1}{2}}, M_-] + g_0 \cdot N_{-\frac{1}{2}} \cdot g_0^{-1}, \quad \text{with} \quad N_{-\frac{1}{2}} = \partial_+ p_{-\frac{1}{2}} + \partial_+ q_{-\frac{1}{2}}.
$$

[8] In the present case it would be tedious to compute the inverse matrix of V_{ij} in (4.37), which would be needed for using directly (4.40).

By using the notation introduced in (4.106), the vanishing of the projection of J to $\mathcal{P}_{-\frac{1}{2}}$ is written as

$$[q_{\frac{1}{2}}\,,\, M_-] + X_{11}\cdot\partial_+ p_{-\frac{1}{2}} + X_{12}\cdot\partial_+ q_{-\frac{1}{2}} = 0,$$

and from this we obtain

$$\partial_+ p_{-\frac{1}{2}} = X_{11}^{-1}\cdot\{[M_-\,,\, q_{\frac{1}{2}}] - X_{12}\cdot\partial_+ q_{-\frac{1}{2}}\}.$$

Combining our previous formulas, finally we obtain that on the constraint surface of the WZNW theory

$$N = g_0^{-1}M_- g_0 + \partial_+ q_{-\frac{1}{2}} + X_{11}^{-1}(g_0)\cdot\{[M_-\,,\, q_{\frac{1}{2}}] - X_{12}(g_0)\cdot\partial_+ q_{-\frac{1}{2}}\}.$$

A similar analysis applied to the right constraints yields that they are equivalent to the following equation:

$$-g_+^{-1}\partial_- g_+ = -g_0 M_+ g_0^{-1} - \partial_- q_{\frac{1}{2}} + Y_{11}^{-1}(g_0)\{[M_+\,,\, q_{-\frac{1}{2}}] + Y_{12}(g_0)\partial_- q_{\frac{1}{2}}\}\,.$$

By using the relations established above, we can at this stage easily compute $b^{-1}Tb = \partial_+ cc^{-1}$ and $b\tilde{T}b^{-1} = a^{-1}\partial_- a$ as well, and substituting these into (4.47), and using the Polyakov-Wiegmann identity to rewrite $S_{\rm WZ}(b)$ for b in (4.105), results in the action in (4.108) indeed. The Lax potential in (4.109) is obtained from the general expression in (2.32) by an additional 'gauge transformation' by the field $\exp[-q_{\frac{1}{2}}]$, which made the final result simpler.

The choice of the constraints leading to the effective theory (4.108) guarantees that the chiral algebra of this theory is the required one, $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}\times\tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$, and thus one should be able to express the $\mathcal{W}$-currents in terms of the local fields in the action. For that recall that in section 4.2.1 an algorithm has been given for constructing the gauge invariant differential polynomials $W(J)$. The point I wish to make is that the expression of the gauge invariant object $W(J)$ in terms of the local fields in (4.108) is simply $W(\partial_+ b\,b^{-1} + T(b))$, where b is given by (4.105). Applying the reasoning of [18] to the present case, this follows since the function W is form-invariant under any gauge transformation of its argument, and the quantity $(\partial_+ b\,b^{-1} + T(b))$ is obtained by a (non-chiral) gauge transformation from J, namely by the gauge transformation defined by the field $a^{-1}\in e^{\Gamma}$, see equations (4.42,4.43). We can in principle compute the object $T(b)$, as explained in the above, and thus we have an algorithm for finding the formulas of the W's in terms of the local fields g_0 and $q_{\pm\frac{1}{2}}$.

The conformal symmetry of the effective theory (4.108) is determined by the left and right Virasoro densities $L_{M_0}(J)$ and $L_{-M_0}(\tilde{J})$, which survive the reduction. To see this conformal symmetry explicitly, it is useful to extract the *Liouville field* ϕ by means of the decomposition $g_0 = e^{\phi M_0}\cdot\hat{g}_0$, where $\hat{g}_0$ contains the generators from $\mathcal{G}_0$ orthogonal to M_0. One can easily rewrite the action in terms of the new variables and then its conformal symmetry becomes manifest since e^{ϕ} is of conformal weight $(1,1)$, $\hat{g}_0$ is conformal scalar, and the fields $q_{\pm\frac{1}{2}}$ have conformal weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively. This assignment of the conformal weights can be established in a number of ways, one can for example derive it from the corresponding conformal symmetry transformation of the WZNW field g in the gauged WZNW theory, see eq. (***). We also note that the action (4.108) can be made generally covariant and thereby our generalized Toda theory can be re-interpreted as a theory of two-dimensional gravity since ϕ becomes the gravitational Liouville mode [39].

There is a certain freedom in constructing a field theory possessing the required chiral algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, for example, one has a freedom of choice in the halving procedure used here to set up the gauge algebra. The theories in (4.108) obtained by using different halvings in equation (4.85) have their chiral algebras in common, but it is not quite obvious if these theories are always completely equivalent local Lagrangian field theories or not.

A special case of this problem arises from the fact that one can expect that in some cases the theory in (4.108) is equivalent to one of the form (4.57). This is certainly so in those cases when for the half-integral $sl(2)$ of M_0 and $M_\pm$ one can find an integral grading operator H such that:

$$\begin{array}{llll} \text{i)} & [H\,,\,M_\pm]=\pm M_\pm \quad , & \text{ii)} & \mathcal{P}_{\frac{1}{2}}+\mathcal{G}_{\geq 1}=\mathcal{G}^H_{\geq 1} \\ \text{iii)} & \mathcal{P}_{-\frac{1}{2}}+\mathcal{G}_{\leq -1}=\mathcal{G}^H_{\leq -1} \quad , & \text{iv)} & \mathcal{Q}_{-\frac{1}{2}}+\mathcal{G}_0+\mathcal{Q}_{\frac{1}{2}}=\mathcal{G}^H_0\,, \end{array} \tag{4.110}$$

where one uses the M_0 grading and the H-grading on the left- and on the right hand sides of these conditions, respectively. By definition, we call the halving $\mathcal{G}_{\frac{1}{2}}=\mathcal{P}_{\frac{1}{2}}+\mathcal{Q}_{\frac{1}{2}}$ an *H-compatible halving* if these conditions are met. Those generalized Toda theories in (4.108) which have been obtained by using H-compatible halvings in the WZNW reduction can be rewritten in the simpler form (4.57) by means of a renaming of the variables, since in this case the relevant FCC are in the overlap of the ones which have been considered for the integral gradings and for the half-integral $sl(2)$'s to derive the respective theories. Since the form of the action in (4.57) is much simpler than the one in (4.108), it appears important to know the list of those $sl(2)$ embeddings which allow for an H-compatible halving, i.e., for which conditions (4.110) can be satisfied with some integral grading operator H and halving. The answer to this group theoretic question for the $sl(2)$ subalgebras of the maximally non-compact real forms of the classical Lie algebras are:

- For $\mathcal{G}=sl(n,R)$ an H-compatible halving can be found for every $sl(2)$ subalgebra. This means that any chiral algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ can be realized in a generalized Toda theory associated to an integral grading.

- For the the symplectic and orthogonal Lie algebras such halvings exist only only for special $sl(2)$-embeddings listed in the appendix.

It is interesting to observe that those theories which can be alternatively written in both forms (4.57) and (4.108) allow for several conformal structures. This is so since in this case at least two different Virasoro densities, namely L_H and L_{M_0}, survive the WZNW reduction.

4.4.3 Two examples of generalized Toda theories

I wish to illustrate here the general construction of the previous section by working out two examples. First I shall describe a generalized Toda theory associated to the highest root $sl(2)$ of $sl(n+2,R)$. This is a half-integral $sl(2)$ embedding, but, as we shall see explicitly, the theory (4.108) can in this case be recasted in the form (4.57), since the corresponding halving is H-compatible. Note that the $\mathcal{W}$-algebras defined by these $sl(2)$ embeddings have been investigated before by using auxiliary fields in [45]. According to the group theoretic analysis in the appendix, the simplest case when a $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$-algebra defined by a half-integral $sl(2)$ embedding cannot be realized in a theory of the type (4.57) is the case of $\mathcal{G} = sp(4,R)$. As our second example, I shall elaborate on the generalized Toda theory in (4.108) which realizes the $\mathcal{W}$-algebra belonging to the highest root $sl(2)$ of $sp(4,R)$.

Highest root $sl(2)$ of $sl(n+2,R)$ In the usual basis where the Cartan subalgebra consists of diagonal matrices, the $sl(2)$ subalgebra $\mathcal{S}$ is generated by the elements

$$M_0 = \frac{1}{2}\begin{pmatrix} 1 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & -1 \end{pmatrix} \qquad \text{and} \qquad M_+ = M_-^t = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}.$$

Note that here and below dots mean 0's in the entries of the various matrices. The adjoint of $sl(n+2)$ decomposes into one triplet, $2n$ doublets and n^2 singlets under this $\mathcal{S}$. It is convenient to parametrize the general element, g_0, of the little group of M_0 as

$$g_0 = e^{\phi M_0} \cdot e^{\psi T} \cdot \begin{pmatrix} 1 & \dots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2+n}\begin{pmatrix} n & \cdots & 0 \\ 0 & -2I_n & 0 \\ 0 & \cdots & n \end{pmatrix}$$

is trace orthogonal to M_0 and $\tilde{g}_0$ is from $sl(n)$. We note that T and M_0 generate the center of the corresponding subalgebra, $\mathcal{G}_0$. We consider the halving of $\mathcal{G}_{\pm\frac{1}{2}}$ which is defined by the subspaces $\mathcal{P}_{\pm\frac{1}{2}}$ and $\mathcal{Q}_{\pm\frac{1}{2}}$ consisting of matrices of the following form:

$$
\begin{aligned}
p_{\frac{1}{2}} &= \begin{pmatrix} 0 & p^t & 0 \\ 0 & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad q_{\frac{1}{2}} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & q \\ 0 & \cdots & 0 \end{pmatrix}, \\
p_{-\frac{1}{2}} &= \begin{pmatrix} 0 & \cdots & 0 \\ \tilde{p} & 0_n & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad q_{-\frac{1}{2}} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & 0_n & 0 \\ 0 & \tilde{q}^t & 0 \end{pmatrix},
\end{aligned} \tag{4.111}
$$

where q and $\tilde{p}$ are n-dimensional column vectors and p^t and $\tilde{q}^{\,t}$ are n-dimensional row vectors, respectively. One sees that the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{\pm\frac{1}{2}}$ are invariant under the adjoint action of g_0, which means that the block-matrices in (4.106) and (4.107) are diagonal, and thus $\eta_{\pm\frac{1}{2}} = [M_\pm, q_{\mp\frac{1}{2}}]$. One can also verify that $X_{11} = e^{-\frac{1}{2}\phi-\psi}\tilde{g}_0$, and that using this the effective action (4.108) can be written as follows:

$$
\begin{aligned}
I_{\text{eff}}(g_0, q_{\frac{1}{2}}, q_{-\frac{1}{2}}) = S_{\text{WZ}}(g_0) \quad &- \int d^2x \Big[e^{\phi} + e^{\frac{1}{2}\phi+\psi}\ \tilde{q}^t \cdot \tilde{g}_0^{-1} \cdot q \\
&- \ e^{-\frac{1}{2}\phi+\psi}\ (\partial_+\tilde{q})^t \cdot \tilde{g}_0^{-1} \cdot (\partial_- q), \Big]
\end{aligned} \tag{4.112}
$$

where dot means usual matrix multiplication. With respect to the conformal structure defined by M_0, e^{ϕ} has weights $(1,1)$, the fields q and $\tilde{q}$ have half-integer weights $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively, ψ and $\tilde{g}_0$ are conformal scalars. In particular, we see that ϕ is the Liouville mode with respect to this conformal structure.

In fact, the halving considered in (4.111) can be written like the one in (4.100), by using the integral grading operator H given explicitly as

$$
H = M_0 + \frac{1}{2}T = \frac{1}{n+2}\begin{pmatrix} n+1 & 0 \\ 0 & -I_{n+1} \end{pmatrix}.
$$

It is an H *-compatible halving* as one can verify that it satisfies the conditions (4.110). It follows that our reduced WZNW theory can also be regarded as a generalized Toda theory associated with the integral grading H. In other words, it is possible to identify the effective action (4.112) as a special case of the one in (4.57). To see this in concrete terms,it is convenient to parametrize the little group of H as

$$
b = \exp(q_{\frac{1}{2}}) \cdot g_0 \cdot \exp(q_{-\frac{1}{2}}), \quad \text{where} \quad g_0 = e^{\Phi H} \cdot e^{\xi S} \cdot \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \tilde{g}_0 & 0 \\ 0 & \cdots & 1 \end{pmatrix},
$$

and $S = M_0 - (\frac{n+2}{2n})T$ is trace orthogonal to H. It is easy to check that by inserting this decomposition into the effective action (4.57) and using the Polyakov-Wiegmann identity one recovers indeed the effective action

(4.112), with

$$\phi = \Phi + \xi \qquad \text{and} \qquad \psi = \frac{1}{2}\Phi - \frac{2+n}{2n}\xi.$$

The conformal structure defined by H is different from the one defined by M_0. In fact, with respect to the former conformal structure Φ is the Liouville mode and all other fields, including q and $\tilde{q}$, are conformal scalars.

Highest root $sl(2)$ of $sp(4,R)$ We use the convention when the symplectic matrices have the form

$$g = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \qquad \text{where} \qquad B = B^t\,,\ C = C^t,$$

and the Cartan subalgebra is diagonal. The $sl(2)$ subalgebra $\mathcal{S}$ corresponding to the highest root of $sp(4,R)$ is generated by the matrices

$$M_0 = \frac{1}{2}(e_{11} - e_{33}), \quad M_+ = e_{13}\,, \quad \text{and} \quad M_- = e_{31}\ ,$$

where e_{ij} denotes the elementary 4×4 matrix containing a single 1 in the ij-position. The adjoint of $sp(4)$ branches into $\underline{3} + 2 \cdot \underline{2} + 3 \cdot \underline{1}$ under $\mathcal{S}$. The three singlets generate an $sl(2)$ subalgebra different from $\mathcal{S}$, so that the little group of M_0 is $GL(1) \times SL(2)$. $GL(1)$ is generated by M_0 itself and the corresponding field is the Liouville mode. Using usual Gauss-parameters for the $SL(2)$, we can parametrize the little group of M_0 as

$$g_0 = e^{\phi M_0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\psi} + \alpha\beta e^{-\psi} & 0 & \alpha e^{-\psi} \\ 0 & 0 & 1 & 0 \\ 0 & \beta e^{-\psi} & 0 & e^{-\psi} \end{pmatrix}.$$

We decompose the $\mathcal{G}_{\pm\frac{1}{2}}$ subspaces (spanned by the two doublets) into their $\mathcal{P}$ and $\mathcal{Q}$ parts as follows

$$p_{\frac{1}{2}} + q_{\frac{1}{2}} = \begin{pmatrix} 0 & p & 0 & q \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -p & 0 \end{pmatrix}, \qquad p_{-\frac{1}{2}} + q_{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tilde{p} & 0 & 0 & 0 \\ 0 & \tilde{q} & 0 & -\tilde{p} \\ \tilde{q} & 0 & 0 & 0 \end{pmatrix}.$$

Now the little group, or more precisely the $SL(2)$ generated by the three singlets, mixes the $\mathcal{P}$ and $\mathcal{Q}$ subspaces of $\mathcal{G}_{-\frac{1}{2}}$ so that the matrices X_{ij} and Y_{ij} in (4.106) and (4.107) possess off-diagonal elements:

$$X_{ij} = e^{-\frac{1}{2}\phi} \begin{pmatrix} e^{\psi} + \alpha\beta e^{-\psi} & \alpha e^{-\psi} \\ \beta e^{-\psi} & e^{-\psi} \end{pmatrix}, \qquad Y_{ij} = X_{ji}.$$

Inserting this into (4.108) yields the following effective action:

$$
\begin{aligned}
I_{\text{eff}}^{\mathcal{S}}(g_0, q, \tilde{q}) &= S_{\text{WZ}}(g_0) - \int d^2x \Bigg[e^{\phi} - 2e^{-\frac{1}{2}\phi-\psi}(\partial_- q)\cdot(\partial_+ \tilde{q}) \\
&+ 2e^{\frac{1}{2}\phi}\, \frac{(\tilde{q} + e^{-\frac{1}{2}\phi-\psi}\beta\partial_- q)\cdot(q + e^{-\frac{1}{2}\phi-\psi}\alpha\partial_+ \tilde{q})}{e^{\psi} + \alpha\beta e^{-\psi}} \Bigg],
\end{aligned}
\tag{4.113}
$$

for the Liouville mode ϕ, the conformal scalars ψ, α, β and the fields q, $\tilde{q}$ with weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively.

It is easy to see directly from its formula that it is impossible to obtain the above action as a special case of (4.57). Indeed, if the expression in (4.113) was obtained from (4.57) then the non-derivative term $\sim \tilde{q}\, q(e^{\psi} + \alpha\beta e^{-\psi})^{-1}$ could only be gotten from the second term in (4.57), but, since g_0 and b are matrices of unit determinant, this term could never produce the denominator in the non-derivative term in (4.113).

4.5 Quantum reduction of WZNW-theories

Here we study the quantum version of the WZNW reduction in the path-integral formalism. We first show that the configuration space path-integral of the constrained WZNW theory can be realized by the gauged WZNW theory. We then point out that the effective action of the reduced theory, (4.47), can be derived by integrating out the gauge fields in a convenient gauge. We shall find that for the generalized Toda theories associated with integral gradings the effective measure takes the form determined from the symplectic structure of the reduced theory. This means that in this case the quantum Hamiltonian reduction results in the quantization of the reduced classical theory; in other words, the two procedures, the reduction and the quantization, commute. We shall also exhibit the $\mathcal{W}$-symmetry of the effective action for this example. By using the gauged WZNW theory, we can construct the BRST formalism for the WZNW reduction in the general case. For conformally invariant reductions, this allows for computing the corresponding Virasoro center explicitly. In particular, we derive a general formula for the Virasoro center of $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ for an arbitrary $sl(2)$ embedding.

4.5.1 Path-integral for constrained WZNW theory

In this section we set up the path-integral formalism for the constrained WZNW theory. For this, we recall that classically the reduced theory has been obtained by imposing a set of FCC in the *Hamiltonian formalism*. Thus what we should do is to write down the path-integral of the WZNW theory first in phase space with the constraints implemented and then find the corresponding configuration space expression. The phase space path-integral can formally be defined once the canonical variables of the theory are specified.

Classically, the constrained WZNW theory has been defined as the usual WZNW theory with its KM phase space reduced by the FCC (4.24). No relationship is assumed here between the two subalgebras, Γ and $\tilde{\Gamma}$. The Hamiltonian is then given by (4.21) with $A = \tilde{A} = 0$, that is has the usual Sugawara form

$$H = \int dx^1 \mathcal{H} = \frac{1}{4\kappa} \int dx^1 \Big(\mathrm{Tr}\ J^2 + \mathrm{Tr}\ \tilde{J}^2 \Big) \tag{4.114}$$

where the KM-currents have been defined in (4.22) and the momenta conjugated to the $\dot{\xi}^a$ simplify to

$$\pi_a = \kappa N_a{}^p N_{bp} \dot{\xi}^b - \kappa \mathcal{A}_{ab} \xi'^b. \tag{4.115}$$

Now we write down the phase space path-integral for the constrained WZNW theory. According to Faddeev's prescription [16] it is defined as

$$\begin{aligned} Z = \int d\pi d\xi\ \delta(\phi)\delta(\tilde{\phi})\delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}| \\ \times \exp\Big(i \int d^2x\, (\pi_a \dot{\xi}^a - \mathcal{H}) \Big), \end{aligned} \tag{4.116}$$

where we implement the FCC by inserting $\delta(\phi)$ and $\delta(\tilde{\phi})$ in the path-integral. The δ-functions of χ and $\tilde{\chi}$ refer to gauge fixing conditions corresponding to the constraints, ϕ and $\tilde{\phi}$, which act as generators of gauge symmetries. By introducing Lagrange-multiplier fields, $A = A^i \gamma_i$ and $\tilde{A} = \tilde{A}^i \tilde{\gamma}_i$, (4.116) can be written as

$$\begin{aligned} Z = \int d\pi d\xi d\tilde{A} dA \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}| \\ \times \exp\Big(i \textstyle\int d^2x\, [\mathrm{Tr}\ (\pi\dot{\xi} + A\phi + \tilde{A}\tilde{\phi}) - \mathcal{H}] \Big). \end{aligned} \tag{4.117}$$

By changing the momentum variable from π_a to

$$P = P^a T_a = T_a (N^{-1})^{ab} (\pi_b + \kappa \mathcal{A}_{bc} \partial_1 \xi^c)$$

the measure acquires a determinant factor, $d\pi = dP \det N$, and the integrand of the exponent in (4.117) becomes

$$
\begin{aligned}
&\mathrm{Tr}\,(\pi\dot{\xi} + A\phi + \tilde{A}\tilde{\phi}) - \mathcal{H} \\
&\quad = \kappa \mathrm{Tr}\,\Big[-\tfrac{1}{2}(\tfrac{1}{\kappa}P)^2 - \tfrac{1}{2}(\partial_1 g\, g^{-1})^2 + \tfrac{1}{\kappa}P(A + g\tilde{A}g^{-1} + \partial_0 g\, g^{-1}) \\
&\qquad\quad + A(\partial_1 g\, g^{-1} - M) - \tilde{A}(g^{-1}\partial_1 g + \tilde{M})\Big] - (\dot{\xi}, \mathcal{A}\xi').
\end{aligned}
\tag{4.118}
$$

Since the matrix $N(\xi)$ is independent of P, we can easily perform the integration over P provided that the remaining δ-functions and the determinant factors are also P-independent.We can choose the gauge fixing conditions, χ and $\tilde{\chi}$, so that this is true. (For example, the physical gauge which we will choose in the next section fulfills this demand.) Then we end up with the following formula of the configuration space path-integral:

$$
Z = \int d\xi \det N\, d\tilde{A} dA \delta(\chi)\delta(\tilde{\chi}) \det|\{\phi,\chi\}| \det|\{\tilde{\phi},\tilde{\chi}\}|\, e^{iI(g,A,\tilde{A})}, \tag{4.119}
$$

where $I(g, A, \tilde{A})$ is the gauged WZNW action (4.8). We note that the measure for the coordinates in this path-integral is the invariant Haar measure,

$$
d\mu(g) = \prod_a d\xi^a \det N = \prod_a (dg\, g^{-1})^a. \tag{4.120}
$$

This is a consequence of the fact that the phase space measure in (4.116) is invariant under canonical transformations to which the group transformations belong.

The above formula for the configuration space path-integral means that the gauged WZNW theory provides the Lagrangian realization of the Hamiltonian reduction, which we have already seen on the basis of a classical argument in section 4.1.1.

4.5.2 Effective theory in the physical gauge

We next discuss the effective theory which arises when we eliminate all the unphysical degrees of freedom in a particularly convenient gauge, the physical gauge. We shall re-derive, in the path-integral formalism, the effective action which appeared in the classical context earlier in this paper. For this purpose, within this section we restrict our attention to the left-right dual reductions considered in section 4.1.2 It, however, should be noted that this restriction is not absolutely necessary to get an effective action by the method given below. In this respect, it is also worth noting that Polyakov's 2-dimensional gravity action in the light-cone gauge can be regarded as an effective action in a non-dual reduction, which is obtained by imposing a constraint only on the left-current for $G = SL(2)$ [1, 20]. We will not pursue the non-dual cases here.

To eliminate all the unphysical gauge degrees of freedom, we simply gauge them away from g, i.e., we gauge fix the Gauss decomposed g in (4.36) into the form

$$g = abc \to b.$$

More specifically, with the parametrization $a(x) = \exp[\sigma_i(x)\gamma_i]$, $c(x) = \exp[\tilde{\sigma}_i(x)\tilde{\gamma}_i]$ we define the *physical gauge* by

$$\chi_i = \sigma_i = 0, \qquad \tilde{\chi}_i = \tilde{\sigma}_i = 0.$$

Note that for this gauge the determinant factors in (4.117) are actually constants. Now the effective action is obtained by performing the $A_\pm$ integrations in (4.119). The integration of A gives rise to the delta-function,

$$\prod_i \delta\Big(\langle \gamma_i, b\tilde{A}b^{-1} + \partial_+ b\, b^{-1} - M\rangle\Big),$$

with $\gamma_i \in \Gamma$ normalized by the duality condition (4.33). One then notices that this delta-function implies exactly the condition (4.39) with $\partial_+ c\, c^{-1}$ replaced by $\tilde{A}$. Hence, with the help of the matrix $V_{ij}(b)$ in (4.37) and $T(b)$ in (4.40), it can be rewritten as

$$(\det V)^{-1}\, \delta\Big(\tilde{A} - b^{-1}T(b)b\Big).$$

Finally, the integration of $\tilde{A}$ yields

$$Z = \int d\mu_{\rm eff}(b)\, e^{I_{\rm eff}(b)}, \tag{4.121}$$

where $I_{\rm eff}(b)$ is the effective action (4.47) [9], and $d\mu_{\rm eff}(b)$ is the effective measure given by

$$d\mu_{\rm eff}(b) = (\det V)^{-1}\, d\mu(g)\delta(\sigma)\delta(\tilde{\sigma}) = (\det V)^{-1}\, \frac{d\mu(g)}{d\sigma d\tilde{\sigma}}\bigg|_{\sigma=\tilde{\sigma}=0}. \tag{4.122}$$

Of course, as far as the effective action is concerned, the path-integral approach should give the same result as the classical one, because the integration of the gauge fields is Gaussian and hence equivalent to the classical elimination of the gauge fields by their field equations. However, a non-trivial feature may arise at the quantum level when the effective path-integral measure (4.122) is taken into account. Let us examine the effective

[9] Actually, the effective action always takes the form (4.47) if one restricts the WZNW field to be of the form $g = abc$ with $a \in e^{\Gamma}$, $c \in e^{\tilde{\Gamma}}$ and b such that $V_{ij}(b)$ is invertible.The duality between Γ and $\tilde{\Gamma}$ is not necessary but can be used to ensure this technical assumption.

measure in the simple case where the space $\mathcal{B}$ in (4.34) forms a subalgebra of $\mathcal{G}$ satisfying (4.52), and thus the effective action in (4.121) simplifies (4.57) In this case, the 1-form appearing in the measure $d\mu(g)$ of (4.120),

$$dg\, g^{-1} = da\, a^{-1} + a(db\, b^{-1})a^{-1} + ab(dc\, c^{-1})b^{-1}a^{-1},$$

turns out, in the physical gauge, to be

$$dg\, g^{-1}|_{\sigma=\tilde{\sigma}=0} = \gamma_i d\sigma_i + db\, b^{-1} + V_{ij}(b)\tilde{\gamma}_i d\tilde{\sigma}_j. \tag{4.123}$$

As a result, the determinant factor in (4.122) is canceled by the one coming from (4.123), and the effective measure admits a simple form:

$$d\mu_{\text{eff}}(b) = db\, b^{-1}. \tag{4.124}$$

The point is that this is exactly the measure which is determined from the symplectic structure of the effective theory (4.45) obtained by the *classical* Hamiltonian reduction. This tells us that in this case the *quantum* Hamiltonian reduction results in the quantization of the reduced classical theory. In particular, since the above assumption for $\mathcal{B}$ is satisfied for the generalized Toda theories associated with integral gradings, we conclude that these generalized Toda theories are equivalent to the corresponding constrained (gauged) WZNW theories even at the quantum level, i.e., including the measure. This result has been established before in the special case of the standard Toda theory (4.102) in [40], where the measure $d\mu_{\text{eff}}(b)$ is simply given by $\prod_i d\varphi^i$.

We end this section by noting that it is not clear whether the measure determined from the symplectic structure of the reduced classical theory is identical to the effective measure (4.122) in general. In the general case both measures in question could become quite involved and thus one would need some geometric argument to see if they are identical or not.

4.5.3 The off-shell $\mathcal{W}$-symmetry of the generalized Toda theory

Because of the WZNW origin of the the generalized Toda theories they possess $\mathcal{W}$-currents. It is thus natural to expect that their effective actions, I^H_{eff} in (4.57) and I^S_{eff} in (4.108), allow for *symmetry transformations yielding the $\mathcal{W}$-currents as the corresponding Noether currents.* We demonstrate below that this is indeed the case for the integral graded theories, when the action takes a simple form. We however believe that there are symmetries of the effective action corresponding to the conserved chiral currents inherited from the KM algebra for any reduced WZNW theory.

Let us consider a gauge invariant differential polynomial $W(J)$ in the constrained WZNW theory giving rise to the effective theory described by the action in (4.57). In terms of the generalized Toda field $b(x)$, this conserved $\mathcal{W}$-current is given by the differential polynomial

$$W_{\text{eff}}(\beta) = W(M+\beta), \qquad \text{where} \qquad \beta \equiv \partial_+ b\, b^{-1}. \tag{4.125}$$

This equality [42, 18] holds because the constrained current J and $(M+\beta)$ are related by a gauge transformation, as we have seen. By choosing some test function $f(x^+)$, we now associate to $W_{\text{eff}}(\beta)$ the following transformation of the field $b(x)$:

$$\delta_W b(y) = \Big[\int d^2x\, f(x^+) \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}\Big] \cdot b(y)\ , \tag{4.126}$$

and we wish to show that $\delta_W b$ is a symmetry of the action $I^H_{\text{eff}}(b)$. Before proving this, we notice, by combining the definition in (4.126) with (4.125), that $(\delta_W b)b^{-1}$ is a polynomial expression in f, β and their ∂_+-derivatives up to some finite order.

We start the proof by noting that the change of the action under an arbitrary variation δb is given by the formula

$$\begin{aligned} \delta I^H_{\text{eff}}(b) &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\,,\, b(y)\frac{\delta I^H_{\text{eff}}}{\delta b(y)}\rangle \\ &= -\int d^2y\, \langle \delta b\, b^{-1}(y)\,, \partial_-\beta(y) + [b(y)\tilde{M}b^{-1}(y), M]\rangle\,. \end{aligned} \tag{4.127}$$

In the next step, we use the field equation to replace $\partial_-\beta$ by $-[b\tilde{M}b^{-1}, M]$ in the obvious equality

$$\partial_- W_{\text{eff}}(x) = \int d^2y\, \langle \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}, \partial_-\beta(y)\rangle, \tag{4.128}$$

and then, from the fact that $\partial_- W_{\text{eff}} = 0$ on-shell, we obtain the following identity:

$$\int d^2y\, \langle \frac{\delta W_{\text{eff}}(x)}{\delta\beta(y)}, [b(y)\tilde{M}b^{-1}(y), M]\rangle = 0\,, \tag{4.129}$$

Of course, the previous argument only implies that (4.129) holds on-shell. However,we now make the crucial observation that (4.129) is an *off-shell identity*, i.e., it is valid for any field $b(x)$ not only for the solutions of the field equation. This follows by noticing that the object in (4.129) is a local expression in $b(x)$ containing only x^+-derivatives. In fact,any such object which vanishes on-shell has to vanish also off-shell, because one can find solutions of the field equation for which the x^+-dependence of the field b is prescribed in an arbitrary way at an arbitrarily chosen fixed value of x^-.

By using the above observation, it is easy to show that $\delta_W b$ in (4.126) is indeed a symmetry of the action. First, simply inserting (4.126) into (4.127), we have

$$\delta_W I^H_{\rm eff}(b) = -\int d^2x\, f(x^+) \int d^2y\, \langle \frac{\delta W_{\rm eff}(x)}{\delta\beta(y)}, \partial_-\beta(y) + [b(y)\tilde{M}b^{-1}(y), M]\rangle.$$

We then rewrite this equation as

$$\delta_W I^H_{\rm eff}(b) = -\int d^2x\, f(x^+)\partial_- W_{\rm eff}(x),$$

with the aid of the identities (4.129) and (4.128). Hence the integrand is a total derivative and this then proves that

$$\delta_W I^H_{\rm eff}(b) = 0\,.$$

One can also see, from equation (4.126), that $W_{\rm eff}$ is the Noether charge density corresponding to the symmetry transformation $\delta_W b$ of $I^H_{\rm eff}(b)$.

4.5.4 BRST formalism for WZNW reductions

Since the constrained WZNW theory can be regarded as the gauged WZNW theory (4.8), one is naturally led to construct the BRST formalism for the theory as a basis for quantization.Below we discuss the BRST formalism based on the gauge symmetry (4.6) and thus return to the general situation where no relationship between the two subalgebras, Γ and $\tilde{\Gamma}$, is supposed.

Prior to the construction we here note how the conformal symmetry is realized in the gauged WZNW theory when there is an operator H satisfying the condition (4.62). (For simplicity, in what follows we discuss the symmetry associated to the left-moving sector.) In fact, with such H and a chiral test function $f^+(x^+)$ one can define the following transformation,

$$\begin{aligned} \delta g &= f^+\partial_+ g + \partial_+ f^+ H g, \\ \delta A &= f^+\partial_+ A + \partial_+ f^+[H, A], \\ \delta\tilde{A} &= f^+\partial_+\tilde{A} + \partial_+ f^+\tilde{A}, \end{aligned} \tag{4.130}$$

which leaves the gauged WZNW action $I(g, A, \tilde{A})$ invariant. This corresponds exactly to the conformal transformation in the constrained WZNW theory generated by the Virasoro density L_H in (4.59), as can be confirmed by observing that (4.130) implies the conformal action (4.60) for the current with $f(x^+) = f^+(x^+)$. We shall derive later the Virasoro density as the Noether charge density in the BRST system.

Turning to the construction of the BRST formalism, we first choose the space $\Gamma^* \subset \mathcal{G}$ which is dual to Γ with respect to the Cartan-Killing form (and similarly $\tilde{\Gamma}^*$ dual to $\tilde{\Gamma}$). Following the standard procedure [31] we introduce two sets of ghost, anti-ghost and Nakanishi-Lautrup fields, $\{c \in \Gamma,\, \bar{c}_+, B_+ \in \Gamma^*\}$ and $\{b \in \tilde{\Gamma},\, \bar{b}_-, B_- \in \tilde{\Gamma}^*\}$. The BRST transformation corresponding to the (left-sector of the) local gauge transformation (4.6) is given by

$$\begin{array}{rclcc} \delta_{\rm B} g &=& -cg &,& \delta_{\rm B}\bar{c}_+ = iB_+, \\ \delta_{\rm B} A &=& D_- c &,& \delta_{\rm B} B_+ = 0, \\ \delta_{\rm B} c &=& -c^2 &,& \delta_{\rm B}(\text{others}) = 0, \end{array} \tag{4.131}$$

with $D_- = \partial_- + [A, \quad]$. and $D + - = \partial_+ - [\tilde{A}, \quad]$. After defining the BRST transformation $\bar{\delta}_{\rm B}$ for the right-sector in an analogous way, we write the BRST action by adding a gauge fixing term and a ghost term to the gauged action,

$$I_{\rm BRST} = I(g, A, \tilde{A}) + I_{\rm gf} + I_{\rm ghost}.$$

The additional terms can be constructed by the manifestly BRST invariant expression,

$$\begin{aligned} I_{\rm gf} + I_{\rm ghost} &= -i\kappa(\delta_{\rm B} + \bar{\delta}_{\rm B}) \int d^2x\, (\langle \bar{c}_+, A\rangle + \langle \bar{b}_-, \tilde{A}\rangle) \\ &= \kappa \textstyle\int d^2x\, (\langle B_+, A\rangle + \langle B_-, \tilde{A}\rangle + i\langle \bar{c}_+, D_- c\rangle + i\langle \bar{b}_-, D_+ b\rangle), \end{aligned} \tag{4.132}$$

where we have chosen the gauge fixing conditions as $A_\pm = 0$. Then the path-integral for the BRST system is given by

$$Z = \int d\mu(g)\, d\tilde{A} dA dc\, d\bar{c}_+ db\, d\bar{b}_- dB_+ dB_-\; e^{iI_{\rm BRST}}, \tag{4.133}$$

which, upon integration of the ghosts and the Nakanishi-Lautrup fields, reduces to (4.119). (Strictly speaking, for this we have to generalize the gauge fixing conditions in (4.119) to be dependent on the gauge fields.) By this construction the nilpotency, $\delta_{\rm B}^2 = 0$, and the BRST invariance of the action, $\delta_{\rm B} I_{\rm BRST} = 0$, are easily checked.

It is, however, convenient to deal with the simplified BRST theory obtained by performing the trivial integrations of $A_\pm$ and $B_\pm$ in (4.133),

$$I_{\rm BRST}(g, c, \bar{c}_+, b, \bar{b}_-) = S_{\rm WZ}(g) + i\kappa \int d^2x\, (\langle \bar{c}_+, \partial_- c\rangle + \langle \bar{b}_-, \partial_+ b\rangle). \tag{4.134}$$

We note that this effective BRST theory is not merely a sum of a free WZNW sector and free ghost sector as it appears, but rather it consists of the two interrelated sectors in the physical space specified by the BRST charge defined below. At this stage the BRST transformation which leaves the simplified BRST action (4.134) invariant reads

$$
\begin{aligned}
&\delta_{\rm B} g = -cg \quad , \quad \delta_{\rm B}\bar{c}_+ = -\pi_{\Gamma^*}\Big[i(\partial_+ g\, g^{-1} - M_-) + (c\bar{c}_+ + \bar{c}_+ c)\Big], \\
&\delta_{\rm B} c = -c^2 \quad , \quad \delta_{\rm B}(\text{others}) = 0,
\end{aligned}
\tag{4.135}
$$

where $\pi_{\Gamma^*} = \sum_i |\gamma_i^*\rangle\langle\gamma_i|$ is the projection operator onto the dual space Γ^* with the normalized bases, $\langle\gamma_i, \gamma_j^*\rangle = \delta_{ij}$. From the associated conserved Noether current, $\partial_- j_+^{\rm B} = 0$, the BRST charge $Q_{\rm B}$ is defined to be

$$
Q_{\rm B} = \int dx^+ j_+^{\rm B}(x) = \int dx^+ \langle c, \partial_+ g\, g^{-1} - M - c\bar{c}_+\rangle. \tag{4.136}
$$

The physical space is then specified by the condition,

$$
Q_{\rm B}|\text{phys}\rangle = 0.
$$

In the simple case of the WZNW reduction which leads to the standard Toda theory, the BRST charge (4.136) agrees with the one discussed earlier [7].

In the case where there is an H operator which guarantees the conformal invariance, the BRST system also has the corresponding conformal symmetry,

$$
\begin{aligned}
\delta g &= f^+\partial_+ g + \partial_+ f^+ H g \quad , \quad \delta b = f^+\partial_+ b, \\
\delta c &= f^+\partial_+ c + \partial_+ f^+[H, c] \quad , \quad \delta\bar{b}_- = f^+\partial_+\bar{b}_-, \\
\delta\bar{c}_+ &= f^+\partial_+\bar{c}_+ + \partial_+ f^+(\bar{c}_+ + [H, \bar{c}_+])
\end{aligned}
\tag{4.137}
$$

inherited from the one (4.130) in the gauged WZNW theory. If the H operator further provides a grading, one finds from (4.137) that the currents of grade $-h$ have the (left-) conformal weight $1-h$, except the H-component, which is not a primary field. Similarly, the ghosts c, $\bar{c}_+$ of grade h, $-h$ have the conformal weight h, $1-h$, respectively, whereas the ghosts b, $\bar{b}$ are conformal scalars. Now we define the total Virasoro density operator $L_{\rm tot}$ from the associated Noether current, $\partial_- j_+^{\rm C} = 0$, by

$$
\int dx^+ j_+^{\rm C}(x) = \frac{1}{\kappa}\int dx^+ f^+(x^+) L_{\rm tot}(x).
$$

The (on-shell) expression is found to be the sum of the two parts, $L_{\rm tot} = L_H + L_{\rm ghost}$, where L_H is indeed the Virasoro operator (4.59) for the WZNW part, and

$$
L_{\rm ghost} = i\kappa(\langle\bar{c}_+, \partial_+ c\rangle + \partial_+\langle H, c\bar{c}_+ + \bar{c}_+ c\rangle), \tag{4.138}
$$

is the part for the ghosts. The conformal invariance of the BRST charge,

$\delta Q_{\mathrm{B}} = 0$, or equivalently, the BRST invariance of the total conformal charge, $\delta_{\mathrm{B}} L_{\mathrm{tot}} = 0$, are readily confirmed.

Let us find the Virasoro center of our BRST system. The total Virasoro center c_{tot} is given by the sum of the two contributions, c from the WZNW part and c_{ghost} from the ghost one. The Virasoro center from L_H is given by

$$c = \frac{k \dim \mathcal{G}}{k+g} - 12k\langle H, H\rangle, \tag{4.139}$$

where k is the level of the KM algebra and g is the dual Coxeter number. On the other hand, the ghosts contribute to the Virasoro center by the usual formula,

$$c_{\mathrm{ghost}} = -2\sum_{\Gamma}[1 + 6h(h-1)], \tag{4.140}$$

where the summation is performed over the eigenvectors of ad_H in the subalgebra Γ. (One can confirm (4.140) by performing the operator product expansion with L_{ghost} in (4.138).)

4.5.5 The Virasoro center in 2 examples

By elaborating on the general result of the previous section, we here derive explicit formulas for the total Virasoro center in two important special cases of the WZNW reduction.

The generalized Toda theory $I^H_{\mathrm{eff}}(b)$ In this case the summation in (4.140) is over the eigenstates of ad_H with eigenvalues $h > 0$, since $\Gamma = \mathcal{G}^H_{>0}$. We can establish a concise formula for c_{tot}, (4.143) below, by using the following group theoretic facts.

First, we can assume that the grading operator $H \in \mathcal{G}$ is from the Cartan subalgebra of the complex simple Lie algebra $\mathcal{G}_c$ containing $\mathcal{G}$. Second, the scalar product $\langle\ ,\ \rangle$ defines a natural isomorphism between the Cartan subalgebra and the space of roots, and we introduce the notation $\vec{\delta}$ for the vector in root space corresponding to H under this isomorphism. More concretely, this means that we set $H = \sum_i \delta_i H_i$ by using an orthonormal Cartan basis, $\langle H_i, H_j\rangle = \delta_{ij}$. Third, we recall the *strange formula* of Freudenthal-de Vries [21], which (by taking into account the normalization of $\langle\ ,\ \rangle$ and the duality between the root space and the Cartan subalgebra) reads

$$\dim\,\mathcal{G} = \frac{12}{g}|\vec{\rho}|^2\ , \tag{4.141}$$

where $\vec{\rho}$ is the Weyl vector, given by half the sum of the positive roots.

Fourth, we choose the simple positive roots in such a way that the corresponding step operators, which are in general in $\mathcal{G}_c$ and not in $\mathcal{G}$, have non-negative grades with respect to H.

By using the above conventions, it is straightforward to obtain the following expressions

$$\begin{aligned} \sum_{h>0} 1 = \dim \Gamma = \frac{1}{2}(\dim \mathcal{G} - \dim\, \mathcal{G}_0^H), \qquad & \sum_{h>0} h = 2(\vec{\rho} \cdot \vec{\delta}), \\ \sum_{h>0} h^2 = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_H)^2 = g\langle H, H\rangle = g|\vec{\delta}|^2, & \end{aligned} \tag{4.142}$$

for the corresponding terms in (4.140). Substituting these into (4.140) and also (4.141) into (4.139), one can finally establish the following nice formula of the total Virasoro center [39]:

$$\boxed{c_{\mathrm{tot}} = c + c_{\mathrm{ghost}} = \dim\, \mathcal{G}_0^H - 12\Big|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\Big|^2.} \tag{4.143}$$

In particular, in the case of the reduction leading to the standard Toda theory (4.102) the result (4.143) is consistent with the one directly obtained in the reduced theory [35, 9] [10].

The $\mathcal{W}_S^{\mathcal{G}}$-algebra for half-integral $sl(2)$ embeddings For $sl(2)$ embeddings the role of the H is played by M_0 and in the half-integral case we have $\Gamma = \mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}} = \mathcal{G}_{>0} - \mathcal{Q}_{\frac{1}{2}}$. It follows that the value of the total Virasoro center can now be obtained by subtracting the contribution of the 'missing ghosts corresponding to $\mathcal{Q}_{\frac{1}{2}}$, which is $\frac{1}{2}\dim\, \mathcal{G}_{\frac{1}{2}}$, from the expression in (4.143). We thus obtain that in this case

$$\boxed{c_{\mathrm{tot}} = N_t - \frac{1}{2}N_s - 12\Big|\sqrt{k+g}\,\vec{\delta} - \frac{1}{\sqrt{k+g}}\,\vec{\rho}\Big|^2,} \tag{4.144}$$

where

$$N_t = \dim\, \mathcal{G}_0\ , \qquad \text{and} \qquad N_s = \dim\, \mathcal{G}_{\frac{1}{2}}\ ,$$

are the number of tensor and spinor multiplets in the decomposition of the adjoint of $\mathcal{G}$ under the $sl(2)$ subalgebra $\mathcal{S}$, respectively. We note that, as proven by Dynkin [34], it is possible to choose a system of positive simple roots so that the grade of the corresponding step operators is from the

[10] more precisely, the center (4.143) agrees with that of refs. [35, 9] if the "coupling constant" of the Toda theory k is replaced by $k+g$. The cause of the shift in the WZNW reduction is discussed, e.g. in [27]

set $\{0, \frac{1}{2}, 1\}$, and that $\vec{\delta}$ is $(\frac{1}{2}\times)$ the so called *defining vector* of the $sl(2)$ embedding in Dynkin's terminology.

As has been mentioned in section 4.3.1, Bais *et al* [5] (see also [45]) studied a similar reduction of the KM algebra for half-integral $sl(2)$ embeddings where all the current components corresponding to $\mathcal{G}_{>0}$ are constrained from the very beginning. In their system, the constraints (4.88) of $\mathcal{G}_{\frac{1}{2}}$, being inevitably second-class, are modified into first-class by introducing an auxiliary field to each constraint of $\mathcal{G}_{\frac{1}{2}}$. Accordingly, the auxiliary fields give rise to the extra contribution $-\frac{1}{2}\mathrm{dim}\ \mathcal{G}_{\frac{1}{2}}$ in the total Virasoro center. It is clear that adding this to the sum of the WZNW and ghost parts (which is of the form (4.143) with M_0 substituted for H), renders the total Virasoro center of their system identical to that of our system, given by (4.144). This result is natural if we recall the fact that their reduced phase space (after complete gauge fixing) is actually identical to ours. It is obvious that our method, which is based on purely first-class KM constraints and does not require auxiliary fields, provides a simpler way to reach the identical reduced theory.

The W_n^l-algebras By using the results of section 4.3.2 we can easily compute the Virasoro center of the W_n^l algebras. We consider the conformal structure given by L_{M_0}, where M_0 is the $sl(2)$ generator (4.94), and introduce ghosts for FCC defined by Γ, eq. (4.98). The contribution to the Virasoro center from L_{M_0} is given by

$$c = \frac{(n^2-1)k}{k+n} - km(m+1)[3n-(2m+1)\,l].$$

Taking into account the multiplicities of the grades in Γ, we find from (4.140)

$$\begin{aligned} c_{\mathrm{ghost}} &= -2\mathrm{dim}\ \mathcal{D}_0 + \mathrm{dim}\ \mathcal{P}_{1/2} - 2\sum_{i=1}^{m}[l+6i(i-1)]\,\mathrm{dim}\ \mathcal{G}_i \\ &= -(m^3+4m^2+3m+1)\,l^2 - n^2(3m^2+2) \\ &\quad +[n(2m^3+3m^2+6m+2)+1]\,l. \end{aligned} \tag{4.145}$$

This result disagrees with the one obtained for W_n^2 in ref. [8], where instead of our L_{M_0} a different L_H was adopted for defining the conformal structure and a set of auxiliary fields has been introduced to render the constraints first class. This disagreement is not surprising because of the ambiguity in defining the conformal structure of W_n^l, i.e. in choosing H in (4.59), which eventually reflects in the value of c. In addition, there is also an arbitrariness in the number of auxiliary fields introduced, and the Virasoro center agrees only when one uses the minimal number of fields (with the same H).

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