Chapter 3

Lagrangian Symmetries of First-Class Hamiltonian Systems

In this chapter we show how the well-known local symmetries of Lagrangian systems, and in particular the diffeomorphism invariance, emerge in the Hamiltonian formulation. We show that only the constraints which are linear in the momenta generate Lagrangian symmetries. The nonlinear constraints (which we have, for instance, in (super)gravity and string theory) rather generate the dynamics. Only in a very special combination with 'trivial' compensating transformations proportional to the equations of motions do they lead to symmetry transformations. We reveal the importance of these special 'trivial' transformations for the interconnection theorems [7] which relate the symmetries of a system with its dynamics. In proving these theorems for general Hamiltonian systems, we shall see that there is a deep connection between the structure of the constraints and the dynamics. For example, in string theory some of the Hamiltonian equations and in gravity all of them are automatically follow if we demand that the constraints are satisfied everywhere and for any foliation of space time. We apply the developed formalism to concrete physically relevant systems, e.g. Yang-Mills theories, the relativistic particle, the bosonic string and gravity.

An interesting application of the considered formalism one could find in the quantized theories, in which we are ultimately interested. For example, in the functional integral approach it can be advantageous to consider the phase space integral as compared to the Lagrangian one. This is true in particular for diffeomorphism invariant theories where the question of the correct measure in the Lagrangian formulation becomes nontrivial. On the other hand, in phase space at least the q, p part of the measure is just the well-known Liouville measure. But then the question arises which symmetries (the ones generated by the constraints alone or the symmetries of the Lagrangian system) should we use to construct the 'correct' path integral or BRST charge. Only in the simple cases of the relativistic particle and supersymmetric particle it has been demonstrated that the results in both cases are the same [12]. For field theories it is still an open question whether the different quantization lead to equivalent results.

To see more clearly what are the problems with generally covariant systems we consider the simplest example, namely the relativistic particle. We describe the relativistic particle moving in 4-dimensional Minkowski spacetime by 4 scalar fields $\phi^\mu(t)$, $\mu = 0, 1, 2, 3$, in 1-dimensional 'spacetime'. The action for the relativistic particle takes the form

$$S = \frac{1}{2} \int \sqrt{g}[g^{00} \dot{\phi}^\mu \dot{\phi}_\mu - m^2] dt \tag{3.1}$$

where the overdot denotes differentiation with respect to time t and $\phi^\mu \phi_\mu = (\phi^0)^2 - \sum_1^3 (\phi^i)^2$. The m^2-term maybe viewed as 'cosmological constant' in 1-dimensional 'spacetime'.

S is invariant with respect to general coordinate transformations (reparametrization invariance). The infinitesimal form of these transformations reads

$$t \to t - \xi, \qquad g_{00} \to g_{00} + \mathcal{L}_\xi g_{00}, \qquad \phi^\mu \to \phi^\mu + \mathcal{L}_\xi \phi^\mu, \tag{3.2}$$

where $\mathcal{L}_\xi$ is the Lie-derivative. Introducing the lapse function $\mathcal{N}$ according to

$$g_{00} = \mathcal{N}^2 \tag{3.3}$$

we get the following transformation law from (3.2)

$$\delta \phi^\mu = \dot{\phi}^\mu \xi \quad \text{and} \quad \delta \mathcal{N} = \frac{d}{dt}(\mathcal{N}\xi) \ . \tag{3.4}$$

The action (3.1) leads to the primary constraint $\phi_1 = \pi_{g^{00}} = 0$ which in turn implies the secondary one

$$\phi_2 \equiv \gamma = \frac{1}{2}(\pi^\mu \pi_\mu - m^2), \tag{3.5}$$

where the π_μ are the momenta conjugated to the ϕ^μ,

$$\{\phi^\mu, \pi_\nu\} = \delta^\mu_\nu. \tag{3.6}$$

These are FCC. The partial gauge fixing $F_1 = g^{00} - 1 = 0$ and ϕ_1 form a conjugate second class pair and can be eliminated. Applying the standard procedure one finds then the following first order action

$$S = \int [\pi_\mu \dot{\phi}^\mu - \mathcal{N}\gamma] dt. \tag{3.7}$$

The Lagrangian multiplier $\mathcal{N}$ accompanying the constraint γ (the super-Hamiltonian) reintroduces the lapse function.

The action (3.7) is invariant with respect to the infinitesimal off mass-shell *gauge* transformations

$$\delta_\lambda \phi^\mu = \{\phi^\mu, \lambda\gamma\} = \pi^\mu \lambda, \quad \delta_\lambda \pi_\mu = \{\pi_\mu, \lambda\gamma\} = 0, \quad \delta_\mathcal{N} = \dot{\lambda}. \qquad (3.8)$$

With the identification $\lambda = \mathcal{N}\xi$ these transformations coincide with the diffeomorphism transformations (3.4), but *only on mass shell*:

$$\dot{\phi}^\mu = \mathcal{N}\pi^\mu, \qquad \dot{\pi}^\mu = 0. \qquad (3.9)$$

This is a general problem with diffeomorphism invariant theories. Infinitesimal diffeomorphisms involve time derivatives of the canonical variables which cannot be gotten by equal time commutators with FCC. Only on-shell can the transformations generated by the FCC be identified with the Lagrangian symmetries. On the technical side the difficulty of identifying gauge and diffeomorphism transformations can be traced back to the nonlinear dependence of the constraint on the momentum. This is the important difference between internal and spacetime symmetries. In the following section we shall see how the canonical transformations generated by the FCC must be modified to yield all Lagrangian symmetries.

3.1 Hamiltonian vs. Lagrangian symmetries

In this chapter I shall consider a general FC system, the first order action of which is given by (2.100). These actions describes both systems with a finite or infinite number of degrees of freedom if the following condensed notation [17] is assumed: For systems with a finite number of degrees of freedom a and i are discrete and for field theories they denote both internal indices and space-coordinates. To distinguish internal from composite indices we shall use tildes for the latter ones. For field theories $\tilde{i} = \{i, \vec{x}\}$ and $\tilde{a} = \{a, \vec{x}\}$, where i and a are some discrete (internal) indices. For a scalar field $q^{\tilde{i}}(t) = \varphi^x(t) = \varphi(x,t)$ and for a vector field $q^{\tilde{i}}(t) = A^{i,x}(t) = A^i(x,t)$. We adopt the Einstein convention and assume summation over discrete repeated indices and integration over continuous ones, for example

$$\xi^x p_{i,x}\dot{q}^{i,x} = \sum_i \int dx \xi(x) p_i(x) \dot{q}^i(x) \quad \text{but} \quad \xi^x p_i^x q^{ix} = \sum_i \xi(x) p_i(x) q^i(x).$$

Also, we shall not distinguish $q^{i,x}$ and q^i_x and use the position of the continuous index just to indicate when we should integrate. Sometimes it will be convenient to resolve the composite index $\tilde{i}$ (or $\tilde{a}$) as i, x (or a, x). If the system contains fermions then some of the variables $p, q, \mathcal{N}$ will be of Grassmannian type.

In particular the first order action reads

$$S_e = \int \Big(p_{\tilde{i}} \dot{q}^{\tilde{i}} - \mathcal{N}^{\tilde{a}} \gamma_{\tilde{a}} - H \Big) dt. \qquad (3.10)$$

For FC systems the constraints [1] and Hamiltonian form a closed algebra (possibly extended to fermionic variables, in which case the algebra is graded [14]):

$$\{\gamma_{\tilde{a}}, \gamma_{\tilde{b}}\} = t^{\tilde{c}}_{\tilde{a}\tilde{b}} \gamma_{\tilde{c}} \qquad \text{and} \qquad \{H, \gamma_{\tilde{a}}\} = t^{\tilde{b}}_{\tilde{a}} \gamma_{\tilde{b}}. \qquad (3.11)$$

The structure coefficients may depend on the canonical variables p, q.

The equation of motion resulting from the variation of the action (3.10) with respect to q, p and the Lagrangian multipliers $\mathcal{N}$

$$\delta S_e = \int \Big(\delta p_{\tilde{i}} EM(q^{\tilde{i}}) - \delta q^{\tilde{i}} EM(p_{\tilde{i}}) - \delta \mathcal{N}^{\tilde{a}} \gamma_{\tilde{a}} \Big) dt + \text{bound. terms} \qquad (3.12)$$

are

$$\begin{aligned} EM(q^{\tilde{i}}) &\equiv \dot{q}^{\tilde{i}} - \{q^{\tilde{i}}, \mathcal{N}^{\tilde{b}} \gamma_{\tilde{b}} + H\} = 0, \\ EM(p_{\tilde{i}}) &\equiv \dot{p}_{\tilde{i}} - \{p_{\tilde{i}}, \mathcal{N}^{\tilde{b}} \gamma_{\tilde{b}} + H\} = 0, \\ \gamma_{\tilde{a}} &= 0. \end{aligned} \qquad (3.13)$$

We use the abbreviations $EM(q)$ and $EM(p)$ for the equations of motion. Of course, on mass shell we have $EM = 0$, but off mass shell either $EM(q)$ or $EM(p)$ (or both) does not vanish.

To go from the Hamiltonian to the Lagrangian formalism we should express the momenta in terms of the velocities via the Hamiltonian equations $EM(q^{\tilde{i}}) = 0$. Thus not all off mass-shell trajectories of the Hamiltonian system can be considered in the Lagrangian formalism, but only those for which this equations hold. Hence one can say that the Lagrangian system lives only in the subspace $\mathcal{M}$ of the 'extended phase space' defined by the conditions

$$\mathcal{M} : EM(q) = \dot{q}^{\tilde{i}} - \{q^{\tilde{i}}, \mathcal{N}^{\tilde{b}} \gamma_{\tilde{b}} + H\} = 0. \qquad (3.14)$$

The action (3.10) invariant (up to boundary terms) with respect to the infinitesimal transformations generated by the constraints if the Lagrangian multipliers are transformed simultaneously [11, 2]

$$\begin{aligned} \delta_\lambda q^{ix} &= \{q^{ix}, \lambda^{\tilde{b}} \gamma_{\tilde{b}}\}, \\ \delta_\lambda p^x_i &= \{p^x_i, \lambda^{\tilde{b}} \gamma_{\tilde{b}}\}, \\ \delta_\lambda \mathcal{N}^{\tilde{a}} &= \dot{\lambda}^{\tilde{a}} - \lambda^{\tilde{b}} \mathcal{N}^{\tilde{c}} t^{\tilde{a}}_{\tilde{c}\tilde{b}} - \lambda^{\tilde{b}} t^{\tilde{a}}_{\tilde{b}}. \end{aligned} \qquad (3.15)$$

The parameters $\lambda^{\tilde{a}} = \lambda^a(\mathcal{N}, \vec{x}, t)$ are the parameters of the infinitesimal transformations. The order in which λ enters in (3.15) is important if some of the variables are of Grassmannian type. We shall only consider the case when the parameters λ depend explicitly on spacetime coordinates and Lagrangian multipliers, since this suffices to cover all known physically relevant theories[1]. Because of this $\mathcal{N}$-dependence we should keep λ inside the Poisson bracket even for purely bosonic theories since if we calculate the commutator of two subsequent infinitesimal transformations, then λ of the second transformation will depend on q, p if the structure constants depend on the canonical variables.

It is not difficult to see that the variation of the action (3.10) under these transformations leads only to the boundary terms

$$\delta_\lambda S = \left(p_{\tilde{i}}\delta_\lambda\, q^{\tilde{i}} - \lambda^{\tilde{a}}\gamma_{\tilde{a}}\right)\Big|_{t_i}^{t_f}. \tag{3.16}$$

This term can be removed even if the parameters λ do not vanish at the boundaries if we add to the action the total derivative of some function $Q(p, q)$ which satisfies the equation

$$\frac{\delta Q}{\delta q^{\tilde{i}}}\delta_\lambda q^{\tilde{i}} + \frac{\delta Q}{\delta p_{\tilde{i}}}\delta_\lambda p_{\tilde{i}} = \lambda^{\tilde{a}}\gamma_{\tilde{a}} - p_{\tilde{i}}\delta_\lambda q^{\tilde{i}},$$

The question which naturally arise here is the following: do the symmetry transformations (3.15) correspond to Lagrangian symmetries, that is are they, for instance, the diffeomorphism transformations in general relativity and string theory?

As we shall see below the answer is no if some constraints are nonlinear in the momenta. The reason is that the transformations (3.15) generated by a nonlinear constraint take a trajectory on $\mathcal{M}$ away from it and the transformed trajectory can not be viewed as a trajectory of the Lagrangian system.

Actually the set of infinitesimal off mass-shell transformations which leave the first order action invariant is much bigger than (3.15). Any infinitesimal transformation $(\delta q, \delta p, \delta\mathcal{N})$ orthogonal to the (functional) gradient $\nabla S = (-EM(p), EM(q), -C)$ leaves the action invariant [5, 31] , as can be easily seen from (3.12). Hence we could add to the transformations generated by the constraints for example any transformation of the form

$$\begin{aligned}
\delta q^{\tilde{i}} &= EM(q^{\tilde{j}})\xi_{\tilde{j}}^{\ \tilde{i}} + EM(p_{\tilde{j}})\eta^{\tilde{j}\tilde{i}}, \\
\delta p_{\tilde{i}} &= EM(p_{\tilde{j}})\xi_{\tilde{i}}^{\ \tilde{j}} + EM(q^{\tilde{j}})\zeta_{\tilde{j}\tilde{i}}, \\
\delta\mathcal{N}^{\tilde{a}} &= 0,
\end{aligned} \tag{3.17}$$

[1] In principle, we could consider more general transformations for which λ would also depend on the canonical variables. Then first order action is also invariant with respect to infinitesimal transformations generated by the constraints if the $\mathcal{N}^{\tilde{a}}$ are transformed as $\delta_\lambda\mathcal{N}^{\tilde{a}} = \partial_t\lambda^{\tilde{a}}(q, p, t) - \lambda^{\tilde{b}}\mathcal{N}^{\tilde{c}}t^{\tilde{a}}_{\tilde{c}\tilde{b}} - \lambda^{\tilde{b}}t^{\tilde{a}}_{\tilde{b}} - \mathcal{N}^{\tilde{b}}\{\gamma_{\tilde{b}}, \lambda^{\tilde{a}}\} - \{H, \lambda^{\tilde{a}}\}$.

where ξ are arbitrary 'matrices' (kernels) and the η, ζ are antisymmetric. Generically such transformations are nonlocal, and they exist for all systems even for those without any symmetries.

We will show that in all theories containing only one nonlinear constraint (e.g. gravity and string theory) we need only very special transformations from (3.17), namely

$$\delta_\xi q^{ix} = EM(q^{ix})\xi^x \quad \text{and} \quad \delta_\xi p_i^x = EM(p_i^x)\xi^x \tag{3.18}$$

to recover all Lagrangian symmetries. In the general case with several nonlinear constraints, e.g. in supergravity, one needs extra transformations from (3.17).

The infinitesimal transformations (3.18) are not important on their own, but in a very special combination with the transformations (3.15) generated by the FCC they lead to physically meaningful symmetries. To recover the Lagrangian symmetries we consider the combined transformations

$$\hat{I}_{\xi,\lambda}\, F(q,p,\mathcal{N}) = F(\hat{I}_{\xi,\lambda}\, q, \hat{I}_{\xi,\lambda}\, p, \hat{I}_{\xi,\lambda}\, \mathcal{N}), \qquad \hat{I}_{\xi,\lambda} = \hat{1} + \delta_{\xi,\lambda} + \cdots, \tag{3.19}$$

where

$$\begin{aligned} \delta_{\xi,\lambda}\, q^{ix} &= EM(q^{ix})\xi^x + \{q^{ix}, \lambda^{\tilde{b}}\gamma_{\tilde{b}}\}, \\ \delta_{\xi,\lambda}\, p_i^x &= EM(p_i^x)\xi^x + \{p_i^x, \lambda^{\tilde{b}}\gamma_{\tilde{b}}\}, \\ \delta_{\xi,\lambda}\, \mathcal{N}^{\tilde{a}} &= \dot{\lambda}^{\tilde{a}} - \lambda^{\tilde{b}}\mathcal{N}^{\tilde{c}} t^{\tilde{a}}_{\tilde{c}\tilde{b}} - \lambda^{\tilde{b}} t^{\tilde{a}}_{\tilde{b}}. \end{aligned} \tag{3.20}$$

The number of functions (ξ, λ^α) which appear here is equal to the number of constraints (per point of space) plus one. This seems strange since for all constrained theories the number of parameters for the symmetry transformations is equal to the number of constraints. To understand why we need the 'trivial' transformations (3.18) and to reveal the connections between the parameters ξ and λ^α we derive the conditions under which the transformations (3.20) are Lagrangian symmetries.

For that the transformations (3.20) should leave any trajectory in the subspace $\mathcal{M}$ in this subspace. The necessary conditions for this can be gotten by varying (3.14) as follows

$$\begin{aligned} \frac{d}{dt}(\delta_{\xi,\lambda} q^{\tilde{i}}) &= \frac{\delta^2(H + \mathcal{N}^{\tilde{e}}\gamma_{\tilde{e}})}{\delta p_{\tilde{i}}\delta q^{\tilde{j}}}\, \delta_{\xi,\lambda} q^{\tilde{j}} \\ &+ \frac{\delta^2(H + \mathcal{N}^{\tilde{e}}\gamma_{\tilde{e}})}{\delta p_{\tilde{i}}\delta p_{\tilde{j}}}\, \delta_{\xi,\lambda} p_{\tilde{j}} + \{q^{\tilde{i}}, \delta_{\xi,\lambda}\mathcal{N}^{\tilde{e}}\gamma_{\tilde{e}}\}. \end{aligned} \tag{3.21}$$

The transformations δq, δp and $\delta\mathcal{N}$ should satisfy the equation (3.21) on the hyper-surface $\mathcal{M}$.

Substituting (3.20) into (3.21) this condition simplifies to

$$\frac{\delta^2(H+\mathcal{N}^{\tilde{e}}\gamma_{\tilde{e}})}{\delta p_i^x \delta p_j^y} EM(p_j^y)\xi^y = \frac{\delta^2\gamma_{\tilde{e}}}{\delta p_i^x \delta p_j^y} EM(p_j^y)\lambda^{\tilde{e}} \tag{3.22}$$

and imposes a certain functional dependence between ξ and λ^a. If this condition is fulfilled the phase space transformations (3.20) can be interpreted as Lagrangian symmetries. At the same time the number of free functions becomes equal to the number of constraints as it should be.

Let us note that the 'trivial' transformations (3.17) and (3.18) do not satisfy (3.22) for off mass-shell trajectories if the Hamiltonian H and/or $\gamma_{\tilde{a}}$ are nonlinear in momenta. Hence these transformations alone cannot be identified with Lagrangian symmetries.

On the other hand, if some constraints $\gamma_{\tilde{a}}$ are nonlinear, then the transformations (3.15) generated by the nonlinear constraints also cannot satisfy (3.22). Hence they cannot be viewed as Lagrangian symmetries either. Only when they are taken in a very special combination with the 'trivial' transformations can one satisfy this condition. In other words, the 'trivial' transformation bring the trajectories back to $\mathcal{M}$. Also we shall see that the transformations (3.15) generate the dynamics for super-Hamiltonian systems. Now we would like to consider two important examples:

Gauge Invariance. If the constraints are linear and H at least quadratic in the momenta then only for $\xi^z = 0$ can equation (3.22) be satisfied [2]. So, in this case the transformations (3.15) generated by the constraints alone are also Lagrangian symmetries. We shall call them *gauge transformations.* For example, in Yang-Mills theories or the gauged Wess-Zumino-Novikov-Witten models investigated in the following chapter, all constraints are linear and the Lagrangian gauge transformations are indeed the transformations (3.15).

Let us add another remark concerning the gauge invariance. Assume that we start with the canonical Hamiltonian and that $\mathcal{N}^{\tilde{a}}$ is conjugate to the primary constraint, now denoted by $\Pi_{\tilde{a}}$. Then we know, that

$$\dot{\Pi}_{\tilde{a}} = \{\Pi_{\tilde{a}}, H_c\} = -\frac{\delta H_c}{\delta \mathcal{N}^{\tilde{a}}} \equiv -\gamma_{\tilde{a}}$$

are secondary constraints[3]. Let us further assume that the consistency condition does not lead to tertiary constraints. Thus we know that the canonical Hamiltonian must have the form

$$H_c = H + \mathcal{N}^{\tilde{a}}\gamma_{\tilde{a}}$$

[2]If H and all constraints are at most linear in momenta, as it is the case for the Chern-Simons theories, then the Hamiltonian system is strongly degenerate.

[3]we have chosen the last sign such as to agree with the other conventions in this chapter

On the other hand we may eliminate the second class pairs $\mathcal{N}^{\tilde{a}}, \Pi_{\tilde{a}}$ by setting them to zero. Then we remain with the *extended* Hamiltonian on the partially reduced phase space

$$H_e = H_c(\mathcal{N} = \Pi = 0) + \mathcal{N}^{\tilde{a}}\gamma_{\tilde{a}} = H_c$$

where we denoted the Lagrangian multipliers of the secondary FCC again by $\mathcal{N}$. Let us know define the first class constraint

$$G = \mu^{\tilde{a}}\Pi_{\tilde{a}} + \lambda^{\tilde{a}}\gamma_{\tilde{a}} \tag{3.23}$$

Now we prove the following

Lemma 2 $\dot{G} = f^{\tilde{a}}\Pi_{\tilde{a}} \iff$ *G generates gauge transformations*

Indeed, requiring that the coefficients of the secondary constraints in

$$\begin{aligned} \dot{G} &= \dot{\mu}^{\tilde{a}}\Pi_{\tilde{a}} + \dot{\lambda}^{\tilde{a}}\gamma_{\tilde{a}} + \mu^{\tilde{a}}\{\Pi_{\tilde{a}}, H_p\} + \lambda^{\tilde{a}}\{\gamma_{\tilde{a}}, H_p\} \\ &= \dot{\mu}^{\tilde{a}}\Pi_{\tilde{a}} + (\dot{\lambda}^{\tilde{a}} - \mu^{\tilde{a}} - t^{\tilde{a}}_{\tilde{b}}\lambda^{\beta} + t^{\alpha}_{\tilde{b}\tilde{c}}\lambda^{\tilde{b}}\mathcal{N}^{\tilde{\gamma}})\gamma_a \end{aligned} \tag{3.24}$$

vanish, fixes the $\mu^{\tilde{a}}$ as function of the $\lambda^{\tilde{a}}$. This the yields the FCC

$$G = (\dot{\lambda}^{\tilde{a}} - t^{\tilde{a}}_{\tilde{b}}\lambda^{\tilde{b}} - t^{\tilde{a}}_{\tilde{b}\tilde{c}}\mathcal{N}^{\tilde{b}}\lambda^{\tilde{c}})\Pi_a + \lambda^{\tilde{a}}\gamma_{\tilde{a}} \tag{3.25}$$

which indeed generates the correct gauge transformations (3.15) for systems with linear constraints.

Reparametrization invariance. Usually the reparametrization invariance of a Lagrangian system, if it exists, is identified with the gauge invariance (3.15) in the Hamiltonian formalism. As we shall see they are actually very different and this identification can only be made on mass shell.

If some constraints are nonlinear then it is obvious that the transformations generated by the constraints only ($\xi = 0$) do not satisfy the condition (3.22) and hence are not Lagrangian symmetries. However, in all known theories with nonlinear constraints $H = 0$ and the condition (3.22) can be satisfied if we impose some functional dependence between λ and ξ in (3.20). Thus the nonlinear constraints generate the Lagrangian symmetry only in very special combination with 'trivial' ξ-transformations. The reason for that is the following: a transformation generated by a nonlinear constraints takes off mass-shell trajectories away from the subspace $\mathcal{M}$ and the extra compensating transformation returns the trajectories back to it. More explicitly taking $\lambda^{\tilde{e}}$ to be $\lambda^{ez} = \mathcal{N}^{ez}\xi^z$ in (3.22) we reduce this equation to

$$\mathcal{N}^{ez}(\xi^y - \xi^z)\frac{\delta^2\gamma_{ez}}{\delta p_i^x \delta p_j^y}EM(p_j^y) = 0. \tag{3.26}$$

One sees at once that if

$$\frac{\delta^2 \gamma_{ez}}{\delta p_i^x \delta p_j^y} \sim \delta(z-y) \tag{3.27}$$

then even for nonlinear constraints the equation (3.26) is satisfied off mass shell ($EM(p) \neq 0$). From that it follows immediately that the transformations (3.20) with $\lambda^{ez} = \mathcal{N}^{ez}\xi^z$ are Lagrangian symmetries if $H = 0$. We shall call the corresponding invariance reparametrization invariance: $\hat{R}_\xi = \hat{I}_{\xi,\lambda^{ez}=\mathcal{N}^{ez}\xi^z}$. The explicit form of the reparametrization transformations read

$$\begin{aligned} \delta_\xi q^{ix} &= \dot{q}^{ix}\xi^x + (\xi^y - \xi^x)\{q^{ix}, \mathcal{N}^{by}\gamma_{by}\} \\ \delta_\xi p_i^x &= \dot{p}_i^x \xi^x + (\xi^y - \xi^x)\{p_i^x, \mathcal{N}^{by}\gamma_{by}\} \\ \delta_\xi \mathcal{N}^{ax} &= (\mathcal{N}^{ax}\xi^x)^{\cdot} - \xi^y \mathcal{N}^{by}\mathcal{N}^{cz} t^{ax}_{cz,by}. \end{aligned} \tag{3.28}$$

These transformations are the correct ones for theories with non-linear constraints. For example, for the relativistic particle the transformations (3.28) (and not the gauge transformations (3.8) generated by the FCC alone) coincide with (3.4) on $\mathcal{M}$.

Algebra of transformations Clearly, the infinitesimal transformations can only be exponentiated to finite ones if they for a close algebra, that is the commutator of two subsequent transformations should be a transformation of the same type. So let us calculate the result for the commutator of two subsequent infinitesimal transformations (3.20) with parameters ξ_1, λ_1 and ξ_2, λ_2, respectively. For an arbitrary algebraic function $F(q,p)$ of the canonical variables a rather lengthy but straightforward calculation yields the commutator

$$\begin{aligned} &[\hat{I}_{\xi_2\lambda_2}, \hat{I}_{\xi_1\lambda_1}]\, F^x(q,p) = \Big(\frac{\delta F^x}{\delta q^{iz}} EM(q^{iz}) + (q \to p)\Big)(\dot{\xi}_1^z \xi_2^z - \xi_1^z \dot{\xi}_2^z) \\ &+\Big((\xi_2^x - \xi_2^y)\lambda_1^{\tilde{c}} - \mathcal{N}^{\tilde{c}}\xi_2^x \xi_1^y - (1 \leftrightarrow 2)\Big)\Big(\{F^x, \frac{\delta \gamma_{\tilde{c}}}{\delta q^{jy}}\} EM(q^{jy}) + (q \to p)\Big) \\ &-(\xi_2^x \xi_1^y - \xi_1^x \xi_2^y)\Big(\{F^x, \frac{\delta H}{\delta q^{jy}}\} EM(q^{jy}) + (q \to p)\Big) + \{F^x, \bar{\lambda}^{\tilde{c}}\gamma_{\tilde{c}}\} \end{aligned} \tag{3.29}$$

and correspondingly for the Lagrangian multipliers one has

$$\begin{aligned} [\hat{I}_{\xi_2\lambda_2}, \hat{I}_{\xi_1\lambda_1}]\mathcal{N}^{\tilde{a}} \;&=\; (\hat{I}_{\bar{\lambda}} - 1)\mathcal{N}^{\tilde{a}} + \lambda_2^{\tilde{d}}\lambda_1^{\tilde{c}}\Big(\dot{t}^{\tilde{a}}_{\tilde{c}\tilde{d}} - \{t^{\tilde{a}}_{\tilde{c}\tilde{d}}, \mathcal{N}^{\tilde{e}}\gamma_{\tilde{e}} + H\}\Big) \\ -(\lambda_2^{\tilde{c}}\xi_1^x - \lambda_1^{\tilde{c}}\xi_2^x) \;&\Big(\frac{\delta}{\delta q^{ix}}(\mathcal{N}^{\tilde{b}} t^{\tilde{a}}_{\tilde{b}\tilde{c}} + t^{\tilde{a}}_{\tilde{c}})EM(q^{ix}) + 2(q \to p)\Big), \end{aligned} \tag{3.30}$$

where we have introduced

$$\bar{\lambda}^{\tilde{a}} = \lambda_1^{\tilde{e}}\lambda_2^{\tilde{b}} t^{\tilde{a}}_{\tilde{b}\tilde{e}} + \frac{\delta \lambda_2^{\tilde{a}}}{\delta \mathcal{N}^{\tilde{b}}}\delta_{\lambda_1}\mathcal{N}^{\tilde{b}} - \frac{\delta \lambda_1^{\tilde{a}}}{\delta \mathcal{N}^{\tilde{b}}}\delta_{\lambda_2}\mathcal{N}^{\tilde{b}}. \tag{3.31}$$

In deriving (3.29,3.30) we used the identities

$$(\lambda_1^{\tilde{c}}\lambda_2^{\tilde{d}} - \lambda_2^{\tilde{c}}\lambda_1^{\tilde{d}})(\{t^{\tilde{a}}_{\tilde{e}\tilde{c}}, \gamma_{\tilde{d}}\} + t^{\tilde{b}}_{\tilde{e}\tilde{c}} t^{\tilde{a}}_{\tilde{b}\tilde{d}}) = \lambda_1^{\tilde{c}}\lambda_2^{\tilde{d}}(t^{\tilde{b}}_{\tilde{c}\tilde{d}} t^{\tilde{a}}_{\tilde{e}\tilde{b}} - \{t^{\tilde{a}}_{\tilde{c}\tilde{d}}, \gamma_{\tilde{e}}\}),$$

and

$$(\lambda_1^{\tilde{c}}\lambda_2^{\tilde{d}} - \lambda_2^{\tilde{c}}\lambda_1^{\tilde{d}})(\{t^{\tilde{a}}_{\tilde{c}}, \gamma_{\tilde{d}}\} + t^{\tilde{b}}_{\tilde{c}} t^{\tilde{a}}_{\tilde{b}\tilde{d}}) = \lambda_1^{\tilde{c}}\lambda_2^{\tilde{d}}(t^{\tilde{b}}_{\tilde{c}\tilde{d}} t^{\tilde{a}}_{\tilde{b}} - \{t^{\tilde{a}}_{\tilde{c}\tilde{d}}, \mathcal{H}\}),$$

which are consequences of the Jacobi identities for $\{\{\gamma_{\tilde{b}}, \lambda_1^{\tilde{c}}\gamma_{\tilde{c}}\}, \lambda_2^{\tilde{e}}\gamma_{\tilde{e}}\}$ and for $\{\{H, \lambda_1^{\tilde{c}}\gamma_{\tilde{c}}\}, \lambda_2^{\tilde{e}}\gamma_{\tilde{e}}\}$.[4] Also we took into account that if the canonical variables are transformed, $\tilde{q} = q + \triangle q$ and $\tilde{p} = p + \triangle p$, then the Poisson bracket of some quantities $A(\tilde{q}, \tilde{p})$ and $B(\tilde{q}, \tilde{p})$ with respect to $\tilde{q}, \tilde{p}$ are connected with the Poisson bracket of $A(q,p)$ and $B(q,p)$ with respect to the old variables in first order in $\triangle q$, $\triangle p$ in the following manner

$$\begin{aligned} \{A(\tilde{q},\tilde{p}), B(\tilde{q},\tilde{p})\}_{\tilde{q},\tilde{p}} &= \{A(q,p), B(q,p)\}_{q,p} \\ + \frac{\delta}{\delta q^{\tilde{i}}}(\{A,B\})\triangle q^{\tilde{i}} &+ (q \to p) + O(\triangle q^2, \triangle p^2). \end{aligned} \tag{3.32}$$

We stress that when we are performing the second transformation in (3.29,3.30) which follows the first one, then we must use the transformed variables. In particular, instead of $\lambda_2(\mathcal{N}, x, t)$ we must take $\lambda_2(\hat{I}_{\lambda_1}\mathcal{N}, x, t)$. This explains the appearance of the last terms in (3.31)

When the structure coefficients $t^{\tilde{a}}_{\tilde{b}\tilde{c}}$ do not depend on the canonical variables then $\bar{\lambda}$ also does not depend on them and $\dot{t}^{\tilde{a}}_{\tilde{b}\tilde{c}} = 0$. Thus, in this case the commutator of two transformations generated by the FCC only ($\xi = 0$) yields again a transformation generated by the constraints. Hence, *if the structure coefficients do not depend on the canonical variables then the transformations generated by the constraints form a closed algebra off mass-shell.* On the other hand, if the structure coefficients depend on the canonical variables that does not automatically imply that the algebra of transformations will not close. Actually, the q, p-dependence in the formula (3.31) for $\bar{\lambda}$ can, in principle, be canceled against an appropriate choice of the $\mathcal{N}$-dependence of λ. Actually this takes place for gravity, where some of the structure coefficients depend on q. Also the last terms in (3.30) vanish in this case on $\mathcal{M}$ and the algebra of transformations generated only by the FCC is closed, but only on $\mathcal{M}$ where the Lagrangian system lives.

The algebra of transformations (3.29,3.30) can also be closed in all relevant cases even when $\xi \neq 0$ if the $\lambda^{\tilde{b}}$ and ξ are related in a certain way. The resulting transformations are actually the transformations corresponding to Lagrangian symmetries when some FCC are nonlinear.

[4] For simplicity we consider only the bosonic case from now on.

An interesting question to which I have no general answer is the following: what are the conditions to exponentiate the infinitesimal transformations to finite ones. For the relativistic particle, string and for gravity the finite transformations for the corresponding Lagrangian systems are just the familiar symmetries. These finite symmetries can then be formulated in the Hamiltonian formulation and this way one can find the finite transformation in the first order formalism. But in general it is not clear whether the closing of the algebra of infinitesimal transformations is sufficient to make them finite. I suppose that this cannot be the case since for a free *non-relativistic particle*, which very probably does not admit any known finite local symmetry, the transformations (3.20) with $\lambda = 0$ form a closed algebra. This difficult and very important question (i.e. for the functional integral) what are the conditions such that the transformations can be made finite needs further investigation.

Constraints and the equations of motion. There is a very interesting and non-trivial connection between the equations of motion $EM(q) = EM(p) = 0$ and the constraints $\gamma_{\tilde{a}} = 0$. Clearly, since $\dot{\gamma}_{\tilde{a}} = 0$ the classical trajectories will stay on Γ_c. Inversely, in some theories (e.g. gravity) we can get the equations of motions if we only demand that the constraints are fulfilled for all t (i.e. everywhere) and that the symmetry transformations do not destroy this property. For example, in diffeomorphism invariant theories this means that we demand that the constraints are valid everywhere and for any choice of space-like hyper-surfaces, because the symmetry transformations can be interpreted as a change of foliation of space-time.

It is very easy to arrive at this conclusion using the developed formalism. Let us consider how the constraints change under the symmetry transformations (3.20):

$$\begin{aligned} \delta_{\xi,\lambda}\gamma_{\tilde{a}} &= \frac{\delta\gamma_{\tilde{a}}}{\delta q^{ix}}\delta_{\xi,\lambda}q^{ix} + \frac{\delta\gamma_{\tilde{a}}}{\delta p_i^x}\delta_{\xi,\lambda}p_i^x \\ &= \frac{\delta\gamma_{\tilde{a}}}{\delta q^{ix}}EM(q^{ix})\xi^x + \frac{\delta\gamma_{\tilde{a}}}{\delta p_i^x}EM(p_i^x)\xi^x + \lambda^{\tilde{c}}t^{\tilde{b}}_{\tilde{a}\tilde{c}}\gamma_{\tilde{b}}. \end{aligned} \tag{3.33}$$

For the known theories the constraints are local in (q,p) and involve only space derivatives of q up to second and p up to first order. It follows then that the structure of the functional derivative of the constraints have the form

$$\begin{aligned}\frac{\delta\gamma_{ay}}{\delta q^{ix}} &= A_{ia}\delta(x,y) + B^{\alpha}_{ia}\frac{\partial}{\partial y^{\alpha}}\delta(x,y) + D^{\alpha\beta}_{ia}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\delta(x,y) \\ \frac{\delta\gamma_{ay}}{\delta p^{x}_{i}} &= E^{i}_{a}\delta(x,y) + F^{i\alpha}_{a}\frac{\partial}{\partial y^{\alpha}}\delta(x,y),\end{aligned} \tag{3.34}$$

where $A, B, \ldots$ are functions of q^y and p^y. Substituting (3.34) into (3.33) a straightforward calculation yields

$$\begin{aligned}\delta_{\xi,\lambda}\gamma_{ay} &= (\dot{\gamma}_{ay} + \mathcal{N}^{\tilde{b}}t^{\tilde{c}}_{\tilde{b},ay}\gamma_{\tilde{c}} + t^{\tilde{c}}_{ay}\gamma_{\tilde{c}})\xi_y \\ &+ \lambda^{\tilde{c}}t^{\tilde{b}}_{ay,\tilde{c}}\gamma_{\tilde{b}} + \Big(B^{\alpha}_{ia}EM(q^{iy}) + F^{i\alpha}_{a}EM(p^{y}_{i})\Big)\frac{\partial\xi^y}{\partial y^{\alpha}} \\ &+ D^{\alpha\beta}_{ia}\Big(2\frac{\partial EM(q^{iy})}{\partial y^{\alpha}}\frac{\partial\xi^y}{\partial y^{\beta}} + EM(q^{iy})\frac{\partial^2\xi^y}{\partial y^{\alpha}\partial y^{\beta}}\Big).\end{aligned} \tag{3.35}$$

Now we can reformulate our question in the following manner: when can the equations of motion (or some of them) be the consequence of the equations

$$\gamma_{\tilde{a}} = 0 \quad \text{and} \quad \delta_{\xi,\lambda}\gamma_{\tilde{a}} = 0. \tag{3.36}$$

The first condition just means that the constraints are fulfilled everywhere and the second one that this statement does not depend on the chosen foliation.

From (3.33) we can immediately conclude that the equations of motion can be derived from (3.36) only if the following necessary conditions are satisfied:

- Some of the constraints should be nonlinear in the momenta, since, as we showed earlier, only in this case should we use the extra 'trivial' transformations (and consequently $\xi \neq 0$).
- The system should have an infinite number of degrees of freedom. Otherwise there are no spatial derivatives of ξ and the pieces which are proportional to the equations of motion are absent.
- The constraints should involve spatial derivatives of the p and/or the q. Else all coefficients B, F, D in (3.34) vanish and the pieces proportional to the equations of motion are again absent.

If we demand that (3.36) holds for arbitrary ξ, then from (3.33,3.34) we immediately get the following set of equations

$$\begin{aligned}D^{\alpha\beta}_{ia}EM(q^{iy}) &= 0 \\ B^{\alpha}_{ia}EM(q^{iy}) + 2D^{\beta\alpha}_{ia}\frac{\partial EM(q^{iy})}{\partial y^{\beta}} &= 0 \\ F^{i\alpha}_{a}EM(p^{y}_{i}) &= 0\end{aligned} \tag{3.37}$$

which can be solved to obtain the equations of motion. The equations of motion which we can get from (3.37) depends on the properties of the matrices D, B, F. Now we will briefly review how the general results apply to particular systems:

Systems with a finite number of degrees of freedom: In this case no equations of motion follow from (3.36) even if $\xi \neq 0$ since there are no spatial derivatives of ξ.

Gauge theories: All of the constraints are linear in the momenta and therefore the "trivial" transformations (3.17) are absent. Consequently, none of the equations of motion can be obtained from (3.36).

Bosonic string: One constraint is nonlinear in the momenta and hence $\xi \neq 0$. The matrices F, D are identically zero in this case and $B \neq 0$. Then only some relations between the $EM(q)$ follow from (3.36).

Gravity: This is the most interesting case. One constraint is nonlinear and leads to $\xi \neq 0$ for the diffeomorphism transformations. The matrices F and D are non-singular. As is clear from (3.37) all Hamiltonian equations follow then from (3.36), that is the whole dynamics of general relativity in the Hamiltonian formulation is hidden in the requirement that the constraints are satisfied everywhere and for any foliation. Let us stress that in distinction to [7] we did not assume $EM(q) = 0$. These equations are also consequences of eqs. (3.36) and thus the interconnection theorem has been proved entirely within the Hamiltonian formalism.

3.2 Yang-Mills Theories

We have seen that there is a big difference between systems with internal symmetries and those which are generally covariant. All constraints in theories in the first class are linear and generate the symmetries. The most important theories with linear constraints are the Yang-Mills theories. In this section I consider YM theories [1] without coupling to matter and emphasize the role of the constraints [30, 46]. Pure *non-Abelian* YM theories are interesting in their own right and they are non-trivial.

The YM action for the gauge fields is

$$S = -\frac{1}{4} \int \mathrm{tr}\,[F_{\mu\nu} F^{\mu\nu}] d^3x dt, \tag{3.38}$$

where [5] the field strength is Lie-algebra valued,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad , \quad A_\mu = A_\mu^a T_a \quad , \quad [T_a, T_c] = i f_{ab}^c T_c, \tag{3.39}$$

and the action is invariant under local gauge transformations

$$A_\mu \longrightarrow e^{-i\theta} A_\mu e^{i\theta} + i e^{-i\theta} \partial_\mu e^{i\theta} \tag{3.40}$$

with $\theta = \theta^a(\vec{x}, t) T_a$. The infinitesimal form of these gauge transformations is

$$\delta_\theta A_\mu^a = -(\partial_\mu \theta^a + f_{bc}^a A_\mu^b \theta^c) = -(D_\mu \theta)^a. \tag{3.41}$$

The local gauge invariance implies generalized Bianchi identities $D_\mu D_\nu F^{\mu\nu} = 0$ and renders the system singular. Among the field equations $D_\mu F^{\mu\nu} = 0$ there are some containing second time-derivatives of A,

$$D_\mu F^{\mu i} = 0 \quad , \quad i = 1, 2, 3 \tag{3.42}$$

and which therefore are dynamical equations of motion. The others

$$D_\mu F^{\mu 0} = D_i F^{i0} \quad \text{or} \quad \phi_m(A, \dot{A}) = \partial_i F_m^{i0} + f_{ab}{}^m A_i^a F_b^{i0} = 0, \tag{3.43}$$

where $m = 1, \ldots, N$=dim(Gauge Group), are *Lagrangian constraints.* No further constraints appear since the time derivatives of the ϕ_m vanish on account of the field equations and the constraints themselves.

The canonical momenta conjugate to the A's are

$$\pi_a^\mu = -F_a^{0\mu} \quad , \quad \{A_\mu^a(\vec{x}), \pi_b^\nu(\vec{y})\} = \delta_b^a \delta_\mu^\nu \delta(\vec{x} - \vec{y}). \tag{3.44}$$

Since the field strength tensor is antisymmetric we obtain N *primary constraints*

$$\phi_m(A, \pi) = \pi_m^0 \approx 0. \tag{3.45}$$

After a partial integration the canonical Hamiltonian is found to be

$$H = \int dx \left(\frac{1}{2} \pi_i^a \pi_i^a + \frac{1}{4} F_{ij}^a F_{ij}^a - A_0^a D_i \pi_a^i \right), \tag{3.46}$$

and determines the time evolution

$$\dot{F} = \{F, H_p\} \quad , \quad H_p = H + \int d\vec{x}\, u^m \phi_m. \tag{3.47}$$

We need to check the consistency of the primary constraints:

$$\dot{\phi}_m = \{\phi_m, H_p\} = 0 \Longrightarrow \tilde{\phi}_m = (D_i \pi^i)_m \approx 0. \tag{3.48}$$

[5] $a, b, \ldots$ denote internal indices, $\mu, \nu \ldots$ space-time indices. The T_a are hermitian generators and the structure constants f_{ab}^c are totally antisymmetric.

These N secondary constraints are the generalizations of the *Gauss constraint* in electrodynamics.

The only non-trivial Poisson brackets of the algebra of constraints are

$$\{\tilde{\phi}_m(\vec{x}), \tilde{\phi}_n(\vec{y})\} = f_{mn}^{\ \ p}\tilde{\phi}_p(\vec{x})\delta(\vec{x}-\vec{y}). \tag{3.49}$$

The algebra is closed and therefore the $2N$ constraints $(\phi_m, \tilde{\phi}_n)$ form a FC system. Their Poisson brackets with H are computed to be

$$\{\phi_m, H\} = \tilde{\phi}_m \approx 0 \quad , \quad \{\tilde{\phi}_m, H\} = -f_{mn}^{\ \ p}A_0^n\tilde{\phi}_p \approx 0. \tag{3.50}$$

Let us now investigate the relation between the Hamiltonian gauge symmetries generated by the FCC and the Lagrangian gauge transformations (3.41). A general combination of the FCC $\phi = \int(\epsilon_1^m\phi_m + \epsilon_2^m\tilde{\phi}_m)$ generates the canonical symmetries

$$\begin{aligned} \delta A_\mu^a &= \{\phi, A_\mu^a\} = \delta_\mu^0\epsilon_1^a - \delta_\mu^i D_i\epsilon_2^a \\ \delta\pi_a^\mu &= \{\phi, \pi_a^\mu\} = \delta_i^\mu f_{ab}^{\ \ c}\epsilon_2^b\pi_c^i + \int \phi_m\{\epsilon_1^m, \pi_a^m\}, \end{aligned} \tag{3.51}$$

where we have already anticipated that ϵ_1 depends on A_0. From (3.41) we read off how the ϵ's must be chosen to correspond to Lagrangian gauge transformations. We find that the particular combination

$$G = D_0\theta^m\phi_m - \theta^m\tilde{\phi}_m \tag{3.52}$$

generates those transformations. Both primary and secondary FCC enter the Lagrangian gauge transformations similarly as for the CS theory.

Alternatively we can introduce gauge invariant variables, e.g. the Wilson loops [53], or fix the gauge. To fix the gauge freedom we need $2N$ gauge fixing conditions on the phase space variables (A, π). Contrary to the situation in electrodynamics the gauge fixing in YM theories is rather subtle due to the Gribov problem. Let $F_a(A_\mu)$ be local gauge fixings (which we assume not to depend on the momenta). Then the following problem may arise:
There are several field $A_\mu^{(j)}$ which are related by finite gauge transformations and all of them obey the gauge fixing.

This happens for the Coulomb (background) gauge conditions [28]. It already happens for QED_2 on the Euclidean torus where an arbitrary gauge field can be decomposed as in (2.118). The local condition $\partial^\mu A_\mu$ eliminates the gauge function λ but does not constrain the q_i. But 2π and $q_i + 2\pi$ are gauge equivalent configuration and this freedom cannot be fixed by a local gauge conditions. This is an example to a more general situation which has been proven by Singer [44]: For compactified YM-theories no global continuous gauge choice of the (local) form $F_a(A) = 0$ exists which completely specifies the gauge. This is due to the nontrivial topological nature of the fibration $\mathcal{A} \to \mathcal{C} = \mathcal{A}/G$, where $\mathcal{A}$ is the affine space of gauge potentials and G the group of local gauge transformations. For mathematical investigations concerning these structures I refer to [5, 38]. There were attempt to circumvent the Gribov problem by restricting the gauge potential to lie within the Gribov horizon [54]. Unfortunately, until now all attempts to make this idea in a functional integration rigorous failed.

Rather then dwelling on the various gauge fixings, their merits and drawbacks, and to which we come back in the functional quantization of gauge theories, let me make here some remarks about the variational problem.

The primary FCC ϕ_m are sort of uninteresting, since the SC pair (π^0, A_0) can easily be eliminated. The Dirac bracket for the remaining variables are just the Poisson bracket.

After this elimination we find the first order action

$$S = \int \left[\pi^i_a \dot{A}^a_i - \mathcal{N}^a \gamma_a - \frac{1}{2}(\pi^i_a \pi^a_i + B^i_a B^a_i)\right] \quad , \quad \gamma_a = \tilde{\phi}_a = (D_i \pi^i)^a, \quad (3.53)$$

with multiplier fields $\mathcal{N}_a$. This form of the action is the one which is usually met in the literature (for example, in gravity one does not keep the momenta conjugated to the lapse and shift functions in the first order action). After having eliminated one pair of canonical variables one may wonder how one can recover the full set of Lagrangian gauge transformation (3.41). Of course, that is exactly what we have achieved earlier. Indeed, applying (3.15) we obtain the following symmetry transformations for the system (3.53)

$$\begin{aligned}
\delta \vec{A}^{\tilde{a}} &= \{\vec{A}^{\tilde{a}}, \lambda^{\tilde{b}} \gamma_{\tilde{b}}\} = -(\vec{D}\lambda)^{\tilde{a}} \\
\delta A^0_{\tilde{a}} &= \delta \mathcal{N}^{\tilde{a}} = \dot{\lambda}^{\tilde{a}} - t^{\tilde{a}}_{\tilde{b}\tilde{c}} A^{0\tilde{b}} \lambda^{\tilde{c}} \\
\delta \vec{\pi}_{\tilde{a}} &= \{\vec{\pi}_{\tilde{a}}, \lambda^{\tilde{b}} \gamma_{\tilde{b}}\} = -f^a_{bc} \vec{\pi}^{bx} \lambda^{cx},
\end{aligned} \qquad (3.54)$$

where we have set $A^a_\mu = (A^a_0, \vec{A}^a)$ and $\pi^i_a = \vec{\pi}_a$. These transformations correspond to symmetries of the corresponding Lagrangian system since the constraints are linear in the momenta. The transformations (3.54) coincide with (3.41) if we identify $\lambda = \theta$ and hence the whole group of gauge transformations (including time dependent ones) is generated by the secondary FCC. It is easy to verify that the transformations for the momenta follow from the first equation in (3.54) if we use the relation between velocities $\vec{A}_{\tilde{a},t}$ and momenta $\vec{\pi}_{\tilde{a}}$ (the first Hamiltonian equation) which defines the subspace $\mathcal{M}$ where the Lagrangian system lives. To compare the symmetries in the Lagrangian and Hamiltonian formulations we need to use these equations. However, the Lagrangian system lives in the subspace $\mathcal{M}$ while the transformations (3.54) can be viewed as symmetries in the whole phase space.

The transformations (3.54) can be made finite in phase space off the hyper-surface $\mathcal{M}$. Actually the action (3.53) is invariant under the global transformation (2.123) if simultaneously the momenta are transformed as

$$\pi \longrightarrow e^{-i\theta}\pi e^{i\theta}.$$

To prove this we do not need to use any of the Hamiltonian equations. So this symmetry holds for all trajectories in phase space.

3.3 Generally covariant theories

Here we apply the general results about the relation between Hamiltonian and Lagrangian symmetries to the bosonic string and gravity [6]

3.3.1 The bosonic string

The bosonic string propagating in a D-dimensional flat target space can be viewed as the theory for D mass-less scalar fields ϕ^μ, $\mu = 0, \ldots, D-1$ on a 2-dimensional world-sheet spacetime with metric $g_{\alpha\beta}$. The action for this theory can be written in an invariant form with respect to diffeomorphism transformations as [10]

$$S = -\frac{1}{2}\int \sqrt{-g}g^{\alpha\beta}\frac{\partial\phi^\mu}{\partial x^\alpha}\frac{\partial\phi_\mu}{\partial x^\beta}\, d^2x, \tag{3.55}$$

where $x^\alpha \equiv (t, x)$ are the coordinates in the 2-dimensional spacetime. To simplify the formulas we shall skip the target-space index μ since it always appears in a trivial way and can easily be reinserted.

The diffeomorphism transformations which are manifest off mass-shell symmetries of the action (3.55) are

$$x^\alpha \to x^\alpha - \xi^\alpha, \qquad g_{\alpha\beta} \to g_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta}, \qquad \phi \to \phi + \mathcal{L}_\xi \phi, \tag{3.56}$$

where ξ^α is the infinitesimal parameter of the transformations. In addition the action is invariant with respect to Weyl transformations

$$g_{\alpha\beta} \to \Omega^2(x)g_{\alpha\beta} \quad \text{and} \quad \phi \to \phi. \tag{3.57}$$

To arrive at the first order formulation it is convenient to use the $1+1$-decomposition for the world-sheet metric as [3]

$$g_{\alpha\beta} = -(\mathcal{N}^2 - \mathcal{N}^1\mathcal{N}_1)dt^2 + 2\mathcal{N}_1 dx dt + \gamma_{11}dx^2, \tag{3.58}$$

[6] In this section we shall use the sign convention $(-,+,+,\ldots)$ for the signature of space-time as favored by most relativists

where $\mathcal{N}$ and $\mathcal{N}_1$ are the lapse and shift functions, respectively. We rise and lower the spatial index '1' using the metric $\gamma_{11} \equiv \gamma$ of the 1-dimensional hyper-surface t=constant in 2-dimensional spacetime. Correspondingly we have

$$\gamma^{11} = \frac{1}{\gamma}, \quad \mathcal{N}^1 = \frac{1}{\gamma}\mathcal{N}_1, \quad \sqrt{-g} = \mathcal{N}\sqrt{\gamma}. \tag{3.59}$$

Using (3.56) an easy calculation yields the following explicit transformations laws for

$$\mathcal{N}^0 = \frac{\mathcal{N}}{\sqrt{\gamma}}, \tag{3.60}$$

$\mathcal{N}^1$ and ϕ under diffeomorphism transformations $x^\alpha \to x^\alpha - \xi^\alpha,\ \xi^\alpha = (\xi^0, \xi^1)$:

$$\begin{aligned}
\delta\mathcal{N}^0 &= \delta(\frac{\mathcal{N}}{\sqrt{\gamma}}) = (\mathcal{N}^0\xi^0)^{\cdot} + \mathcal{N}^{1\prime}(\mathcal{N}^0\xi^0) - \mathcal{N}^1(\mathcal{N}^0\xi^0)' \\
&\qquad +\mathcal{N}^{0\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^0(\xi^1 + \mathcal{N}^1\xi^0)', \\
\delta\mathcal{N}^1 &= (\xi^1 + \mathcal{N}^1\xi^0)^{\cdot} + \mathcal{N}^{1\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^1(\xi^1 + \mathcal{N}^1\xi^0)' \\
&\qquad +\mathcal{N}^{0\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^0(\xi^1 + \mathcal{N}^1\xi^0)', \\
\delta\phi &= \dot{\phi}\xi^0 + \phi'\xi^1.
\end{aligned} \tag{3.61}$$

Here dot and prime mean the differentiations with respect to the time and space coordinates $x^0 = t$ and $x^1 = x$, respectively. The transformation law for the momentum π conjugate to ϕ,

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\sqrt{\gamma}}{\mathcal{N}}(\dot{\phi} - \mathcal{N}^1\phi') \tag{3.62}$$

follows immediately from (3.61):

$$\delta\pi = \dot{\pi}\xi^0 + (\pi\xi^1)' + (\mathcal{N}^1\pi + \mathcal{N}^0\phi')\xi^{0\prime}. \tag{3.63}$$

In the first order Hamiltonian formulation the action (3.55) takes the form

$$S = \int (\pi\dot{\phi} - \mathcal{N}^a\gamma_a)dxdt, \tag{3.64}$$

where the Lagrangian multipliers $\mathcal{N}^a$ are just the functions defined in (3.59,3.60) (that is they are the lapse and shift functions up to $\sqrt{\gamma}$). The constraints

$$\gamma_0 = \frac{1}{2}(\pi^2 + \phi'^2), \qquad \text{and} \qquad \gamma_1 = \pi\phi' \tag{3.65}$$

form a closed algebra, i.e. are FCC, with respect to the standard Poisson brackets $\{\phi(x), \pi(y)\} = \delta(x, y)$:

$$\begin{aligned}
\{\gamma_i(x), \gamma_i(y)\} &= \gamma_1(x)\frac{\partial}{\partial x}\delta(x,y) - \gamma_1(y)\frac{\partial}{\partial y}\delta(x,y) \quad \text{i=0,1} \\
\{\gamma_0(x), \gamma_1(y)\} &= \gamma_0(x)\frac{\partial}{\partial x}\delta(x,y) - \gamma_0(y)\frac{\partial}{\partial y}\delta(x,y).
\end{aligned} \tag{3.66}$$

Rewriting these relations in terms of the light-cone constraints $\gamma_0 \pm \gamma_1$ we immediately recognize them as Virasoro algebra [4].

Concerning the symmetries we first note that the Weyl symmetry (3.57) takes the trivial form in the Hamiltonian formalism

$$\mathcal{N}^0 = \frac{\mathcal{N}}{\sqrt{\gamma}} \to \frac{\Omega\mathcal{N}}{\Omega\sqrt{\gamma}} = \mathcal{N}^0, \qquad \mathcal{N}^1 = \frac{\mathcal{N}_1}{\sqrt{\gamma}} \to \mathcal{N}^1, \tag{3.67}$$

so that all variables in the first order action are Weyl invariant.

Because one of the constraints, namely γ_0, is quadratic in the momentum, we need to combine gauge and reparametrization transformations as in (3.20) to recover the diffeomorphism invariance (3.61,3.62) in the Hamiltonian formalism. For the bosonic string the explicit transformation (3.20) reads

$$\begin{aligned}
\delta\mathcal{N}^0 &= \dot{\lambda}^0 + \mathcal{N}^{1\prime}\lambda^0 - \mathcal{N}^1\lambda^{0\prime} + \mathcal{N}^{0\prime}\lambda^1 - \mathcal{N}^0\lambda^{1\prime} \\
\delta\mathcal{N}^1 &= \dot{\lambda}^1 + \mathcal{N}^{1\prime}\lambda^1 - \mathcal{N}^1\lambda^{1\prime} + \mathcal{N}^{0\prime}\lambda^0 - \mathcal{N}^0\lambda^{0\prime} \\
\delta\phi &= (\dot{\phi} - \mathcal{N}^0\pi - \mathcal{N}^1\phi')\xi + \pi\lambda^0 + \phi'\lambda^1, \\
\delta\pi &= (\dot{\pi} - (\mathcal{N}^0\phi' + \mathcal{N}^1\pi)')\xi + (\phi'\lambda^0)' + (\pi\lambda^1)',
\end{aligned} \tag{3.68}$$

where we need to assume that the parameters are related by the condition (3.22). This condition is solved if we express the parameters $\xi, \lambda^0, \lambda^1$ in terms of two independent parameters as

$$\xi = \xi^0, \qquad \lambda^0 = \mathcal{N}^0\xi^0 = \frac{\mathcal{N}}{\sqrt{\gamma}}\xi^0, \qquad \lambda^1 = \xi^1 + \mathcal{N}^1\xi^0, \tag{3.69}$$

and then we immediately recognize the transformations (3.68) as diffeomorphism transformations (3.61,3.62) without using the Hamiltonian equations. Once again we emphasize that the transformations (3.68) are infinitesimal symmetry transformations on the whole phase space whereas the transformations (3.61,3.62) are applicable only to trajectories on $\mathcal{M}$.

As a first step toward exponentiating the infinitesimal transformations (3.68), i.e. make them finite, we should check their algebra. Using the formulas for the particular choice (3.69) of parameters it easy to find that the commutator of two subsequent transformations $\hat{I}_{\xi,\lambda} \equiv \hat{I}(\vec{\xi})$, where $\vec{\xi} = (\xi^0, \xi^1)$ becomes

$$[\hat{I}(\vec{\eta}), \hat{I}(\vec{\xi})] = \hat{I}(\mathcal{L}_{\vec{\eta}}\vec{\xi}) - \hat{1},$$

completely off mass shell. Hence the algebra of transformations (3.68) forms a (infinite dimensional) Lie-algebra even off the subspace $\mathcal{M}$.

The last remark concerns the connection between the constraints and the equations of motion for the string theory. Calculating the first functional derivative of the constraints with respect to the canonical variables we see that the B and E coefficients in (3.34) are

$$B_0 = E_1 = \phi'_y \quad , \quad B_1 = E_0 = \pi_y, \tag{3.70}$$

while the D and F coefficients vanish. Then the eqs.(3.37) reduce to

$$\phi'^{\mu} EM(\phi_\mu) = 0 \quad \text{and} \quad \pi^{\mu} EM(\phi_\mu) = 0 \tag{3.71}$$

where μ is the target-space index. From these equations we cannot conclude that all eqs. of motion should be satisfied. However, they put certain restrictions on the allowed $EM(\phi)$. Since the coefficients F are equal zero (the constraints do not involve any spatial derivatives of the momenta) the requirement that the constraints are satisfied everywhere and for any foliation does not tell us anything about the eqs. of motion $EM(\pi) = 0$. We will see in the next section that the interconnection theorem, which we just discussed, is much more interesting in gravity.

3.3.2 Gravity

General relativity without matter has the action [7]

$$S = \int R\sqrt{-g}d^4x \tag{3.72}$$

and is invariant with respect to coordinate (or diffeomorphism) transformations, the infinitesimal form of which read

$$x^\alpha \to x^\alpha - \xi^\alpha, \qquad g_{\alpha\beta} \to g_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta}. \tag{3.73}$$

Rewriting the metric $g_{\alpha\beta}$ in the $3+1$-split form [3]

$$ds^2 = -(\mathcal{N}^2 - \mathcal{N}_i\mathcal{N}^i)dt^2 + 2\mathcal{N}_i dx^i dt + \gamma_{ij}dx^i dx^j, \tag{3.74}$$

where $\mathcal{N}$ is the lapse function, $\mathcal{N}_i$ are the shift functions, $\mathcal{N}_i = \gamma_{ij}\mathcal{N}^j$, and γ_{ij} is the metric of the 3-dimensional hyper-surface Σ_t of constant time t, we derive from (3.73) the following explicit transformations for $\mathcal{N}$, $\mathcal{N}^i$, and γ_{ij}:

$$\begin{aligned}
\delta\mathcal{N} &= (\mathcal{N}\xi^0)^{\cdot} - \mathcal{N}^i(\mathcal{N}\xi^0)_{,i} + \mathcal{N}_{,m}(\xi^m + \mathcal{N}^m\xi^0)\,, \\
\delta\mathcal{N}^i &= (\xi^i + \mathcal{N}^i\xi^0)^{\cdot} - (\xi^i + \mathcal{N}^i\xi^0)_{,m}\mathcal{N}^m + \mathcal{N}^i{}_{,k}(\xi^k + \mathcal{N}^k\xi^0) \\
&\quad - \mathcal{N}\gamma^{ij}(\mathcal{N}\xi^0)_{,j} + \gamma^{ij}\mathcal{N}_{,j}(\mathcal{N}\xi^0)\,, \\
\delta\gamma_{ij} &= (\dot\gamma_{ij} - \mathcal{N}_{i|j} - \mathcal{N}_{j|i})\xi^0 + {}^{(3)}\mathcal{L}_{\xi+\mathcal{N}\xi^0}\gamma_{ij}.
\end{aligned} \tag{3.75}$$

[7] we adapt the sign and units conventions in [13]

Here the comma denotes ordinary differentiation with respect to the corresponding space coordinate, the bar denotes covariant derivative in the 3 dimensional space Σ_t with metric γ_{ij}, γ^{ij} is the inverse 3-dimensional metric on Σ_t and ${}^{(3)}\mathcal{L}$ is the Lie derivative in Σ_t. This Lie derivative is to be taken in the direction $\xi + \mathcal{N}\xi^0 \equiv \{\xi^i + \mathcal{N}^i\xi^0\}$.

In the first order Hamiltonian formalism the ADM action for pure gravity takes the form [8]

$$S = \int (\pi^{ij}\dot{\gamma}_{ij} - \mathcal{N}^s\mathcal{H}_a)d^3x dt, \tag{3.76}$$

where π^{ij} are the momenta conjugated to γ_{ij} and the 4 Lagrangian multipliers are

$$\mathcal{N}^0 = \mathcal{N}, \qquad \text{and} \qquad \mathcal{N}^i = \gamma^{ij}\mathcal{N}_j \tag{3.77}$$

that is the lapse and shift function. Correspondingly the constraints $\mathcal{H}_a$ are [3, 13]

$$\mathcal{H}_0 = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{\gamma}\,{}^{(3)}R, \qquad \mathcal{H}_i = -2\gamma_{ij}\pi^{jl}_{|l}, \tag{3.78}$$

where

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}), \qquad \gamma = \det(\gamma_{ij}) \tag{3.79}$$

is the metric in super-space [13] and ${}^{(3)}R$ the intrinsic curvature of the hyper-surface Σ_t of constant time t. With the help of the fundamental Poisson brackets

$$\{\gamma_{ij}(x), \pi^{kl}(y)\} = \delta_i^{(k}\delta_j^{l)}\,\delta(x,y) = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(x,y) \tag{3.80}$$

one checks that the constraints (3.78) are first class [13]

$$\begin{aligned}
\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} &= \gamma^{ij}(x)\mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \gamma^{ij}(y)\mathcal{H}_j(y)\frac{\partial}{\partial y^i}\delta(x,y) \\
\{\mathcal{H}_i(x), \mathcal{H}_0(y)\} &= \mathcal{H}_0(x)\frac{\partial}{\partial x^i}\delta(x,y) \qquad (3.81) \\
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \mathcal{H}_i(y)\frac{\partial}{\partial y^j}\delta(x,y).
\end{aligned} \tag{3.82}$$

Let us note that if we add matter (covariantly coupled to gravity) to (3.72) then the constraints contain extra pieces, but their algebra remains unchanged. Another interesting observation is the following: If we use $\sqrt{\gamma}\mathcal{H}_0$ instead of $\mathcal{H}_0$ as a constraint then the algebra of constraints looks very much

[8] in this section we denote the constraints by $\mathcal{H}_a$, a notation which is widely used in gravity [13, 8, 9]

like a natural generalization of the Virasoro algebra (3.66) to four dimensions. It is a nontrivial problem where the diffeomorphism invariance of the original action (3.72) is hidden in the first order Hamiltonian reformulation of gravity. There have been various attempts to find this symmetry (see, for instance [8, 9])

Three of the constraints, namely the $\mathcal{H}_i$, are linear in momenta, so they should generate transformations which coincide with diffeomorphism transformations. This has been realized for time independent transformations some time ago [6]. However, the fourth constraint, namely $\mathcal{H}_0$, is quadratic in the momenta and hence cannot generate a symmetry of the corresponding Lagrangian system according to our general results. Only combined with a compensating transformation does it generate the symmetry we are looking for. Since the Hamiltonian is zero, this symmetry is exactly the reparametrization invariance (3.28). Assuming that the parameters in (3.20) are connected such that the condition (3.22) is satisfied, we can write this off shell symmetry transformation for gravity in the following explicit manner

$$\begin{aligned}
\delta\mathcal{N} &= \dot{\lambda}^0 - \mathcal{N}^j\lambda^0{}_{,j} + \mathcal{N}_{,j}\lambda^j, \\
\delta\mathcal{N}^i &= \dot{\lambda}^i - \mathcal{N}^j\lambda^i{}_{,j} + \mathcal{N}^i{}_{,j}\lambda^j - \mathcal{N}\gamma^{ij}\lambda^0{}_{,j} + \gamma^{ij}\mathcal{N}_{,j}\lambda^0, \\
\delta\gamma_{ij} &= EM(\gamma_{ij})\xi + \{\gamma_{ij}, \lambda^{\tilde{a}}\mathcal{H}_{\tilde{a}}\} \\
&= EM(\gamma_{ij})\xi + \frac{1}{\sqrt{\gamma}}(2\pi_{ij} - \gamma_{ij}\pi)\lambda^0 + {}^{(3)}\mathcal{L}_\lambda\gamma_{ij}
\end{aligned} \tag{3.83}$$

and

$$\delta\pi^{ij} = EM(\pi^{ij})\xi + \{\pi^{ij}, \lambda^{\tilde{a}}\mathcal{H}_{\tilde{a}}\}. \tag{3.84}$$

Here the 5 parameters ξ, λ^α are to be expressed in terms of the four independent parameters ξ^α as

$$\xi = \xi^0, \qquad \lambda^0 = \mathcal{N}\xi^0, \qquad \lambda^i = \xi^i + \mathcal{N}^i\xi^0 \tag{3.85}$$

and then it becomes evident that (3.83) is identical to (3.75). Again we need not use any of the Hamiltonian equations. A rather lengthy calculation shows that the transformation law one finds for the momenta by using their definition in terms of γ_{ij}, $\mathcal{N}_k$ and (3.75) coincides with (3.84) also off mass shell.

Thus we found that in gravity the three constraints which are linear in the momenta generate the diffeomorphism transformations while the forth constraint $\mathcal{H}_0$ does it only in a particular combination with the 'trivial' transformation (3.18). This nonlinear in momenta constraint itself is responsible for the origin of the dynamics in $\mathcal{M}$ in the super-Hamiltonian reformulation of gravity.

In gravity the structure coefficients depend on the canonical variables and one might expect that the algebra of infinitesimal transformations (3.83-

3.85) cannot close in this case. Fortunately, this expectation is not confirmed. In particular, in the formula (3.31) for the $\bar{\lambda}$-parameter this γ-dependence of the various terms on the right hand side cancels for the concrete choice (3.85) for the $\mathcal{N}$-dependence of the parameters λ. The price we pay for that is the explicit dependence of the parameters of transformations on the Lagrangian multipliers, but not on the canonical variables γ, π. Starting from the general formulas (3.29-3.31) a straightforward but rather lengthy calculation shows that the transformations (3.83-3.85) form a Lie algebra completely off mass shell:

$$[\hat{I}(\eta), \hat{I}(\xi)] = \hat{I}(\mathcal{L}_\xi \eta) - \hat{1}, \qquad \xi = (\xi^0, \dots, \xi^3), \quad \eta = (\eta^0, \dots, \eta^3), \quad (3.86)$$

where ξ^0, ξ^i and η^0, η^i are defined in (3.85), as it should be for diffeomorphisms. The formula (3.86) holds even for paths which are not in $\mathcal{M}$.

There is a deep connection between the constraints and equations of motion in gravity. Calculating the derivative of the constraints in this case we shall find that all of the coefficients $A, \cdots, F$ in (3.34) do not vanish.

In particular, taking into account that the index k in the formulas (3.34,3.15) is a composite one, $i \equiv (j,k)$; a, b run over the same spatial index l and calculating the derivatives of $\mathcal{H}_i$ with respect to π^{jk} and $\mathcal{H}_0$ with respect to γ_{np} we find

$$F^l_{ijk} = -2\gamma_{i(j}\delta^l_{k)} \quad \text{and} \quad D^{nplk}_0 = -G^{nplk}, \quad (3.87)$$

where G^{nplk} is the inverse of the superspace-metric, $G^{nplk}G_{lkij} = \delta^{(n}_i \delta^{p)}_j$. Then the first and last equations in (3.37) take the form

$$G^{nplk} EM(\gamma_{np}) = 0 \quad \text{and} \quad \gamma_{ij} EM(\pi^{jl}) = 0. \quad (3.88)$$

Since the determinants det G and det γ are not equal zero the eqs. (3.88) have the unique solution

$$EM(\gamma_{np}) = 0 \quad \text{and} \quad EM(\pi^{jl}) = 0. \quad (3.89)$$

The remaining equations in (3.37) are then automatically fulfilled. Thus, we see that in general relativity the whole dynamics follows from the requirement that the constraints are satisfied everywhere and they are preserved under diffeomorphisms.

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