Symmetries, Reductions and Quantization of Gauge Theories

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Abstract

This Schrift provides a rather self-contained review and some recent results about gauge theories. The first part contains an introduction into the classical theory of constrained systems stressing the relation between symmetries in the Hamiltonian and Lagrangean formulations. Then I present some new developments about the classical and quantum Hamiltonian reduction of Wess-Zumino-Novikov-Witten theories to generalized Toda theories or of Kac-Moody algebras to $\mathcal{W}$ algebras. In the second part some new results about 2-dimensional gauge theories and in particular the chiral symmetry breaking and thermodynamic of such models is investigated. In the last chapter the functional Schrödinger equation for fermions in external gauge fields is studied and solutions to the time-dependent functional Schrödinger equation and explicit expressions for the ground state functional are given.

Contents

Chapter 1

Introduction

The mother of field theories, namely quantum electrodynamics, is a gauge theory. All other fundamental field theories in physics are also invariant with respect to some group of *local symmetry transformations.* For Yang-Mills theories these are the gauge transformations, for string theory and gravity space-time diffeomorphisms and for supersymmetric theories local supersymmetry transformations. Unfortunately we still have a poor understanding of these classes of field theories. What we know best are the perturbative expansions in powers of Planck's constant or the gauge coupling constant. This presumably divergent formal power series expansions compare well with experimental data. However, we must admit that often we have to rely on not well understood tricks and methods, e.g. on partial resummations in high temperature QCD, to arrive at meaningful results.

In this Schrift I shall not be concerned about perturbation expansions and Feynman diagrams for gauge theories. Rather I shall be more concerned about algebraic structures, symmetries and non-perturbative aspects. The most general mathematical setting for gauge theories is Dirac's constraint formalism within the Hamiltonian formulation, since any gauge theory is a theory with first class constraints. Indeed, for *gauge theories* or more generally *singular systems* the local symmetry relates different solutions stemming from the same initial conditions and the general solution of the equations of motion contains arbitrary time-dependent functions. Hence there is a continuous set of accelerations which belong to the same initial position and velocity and we expect that all accelerations correspond only to a subset of initial conditions. This subset is defined by the Lagrangian constraints so that all gauge theories are systems with constraints. In the Hamiltonian formalism this means that there are conditions on the allowed initial momenta and positions. These conditions must then be conserved by the time evolution, and this consistence requirement may lead to further constraints [2].

For gauge theories with *internal symmetries* all constraints are linear in the momenta and the Hamiltonian does not vanish. The local symmetry transformations are generated by the first class constraints.

For (non-topological) *generally covariant theories* at least one constraint is quadratic in the momenta and there are canonical variables for which the Hamiltonian H itself is a constraint, usually called *super-Hamiltonian.* This leads to the question whether H generates the dynamical time-evolution or kinematic local symmetries as the other first class constraints. This question is very much related to the interpretation of time in generally covariant theories. We shall see that in super-Hamiltonian systems the constraints which are nonlinear in the momenta do not generate kinematic symmetries but rather generate the dynamics of such systems.

In a gauge theory the gauge-related configurations must be identified and this identification may be achieved either by introducing gauge-invariant variables or by fixing the gauge freedom. In most cases the theory gets more complicated on the reduced space of gauge fixed configurations. Indeed, recent progress has been made in the quantization of nonlinear Toda-type theories by interpreting them as gauged fixed versions of much simpler gauged Wess-Zumino-Novikov-Witten theories [1]. The WZNW $\rightarrow$ Toda Hamiltonian reduction is achieved by imposing certain first class, conformally invariant constraints on the Kac-Moody (KM) currents of the WZNW theories. The structure of the symmetry reductions to the reduced phase space contains all information of the reduced theories. For example, one easily understands the appearance of *non-linear $\mathcal{W}$ symmetry algebras* [23] as they are just the projection of linear Kac-Moody symmetry algebras to the reduced phase space.

There are several alternative ways of *quantizing* systems with constraints. In the conventional *Schrödinger representation* quantization (see e.g. [11]) the dynamical quantities are expressed in terms of fixed-time canonical variables. The quantum field theory involves states $|\Psi\rangle$ that are realized in the Schrödinger representation as functionals $\Psi(\varphi)$ of a c-number field $\varphi(\vec{x})$. The field operator acts on these states by multiplication, while the canonical momentum operator acts by functional differentiation

$$\Phi(\vec{x})|\Psi\rangle \longrightarrow \varphi(\vec{x})\Psi(\varphi) \quad \text{and} \quad \pi(\vec{x})|\Psi\rangle \longrightarrow \frac{1}{i}\frac{\delta}{\delta\varphi(\vec{x})}\Psi(\varphi).$$

The first class constraints, e.g. the Gauss constraint in quantum electrodynamics, should annihilate the physical states and this way define the physical Hilbert space.

Since the Lorentz invariance is blurred in the non-covariant Schrödinger representation the regularization program is not very transparent. This may explain why this physically intuitive way of quantizing gauge theories has not been popular in the past. However, at least for pure Yang-Mills theories it has been shown that the Schrödinger functional can be renormalized by adding the usual counter-terms to the action plus a set of further terms that are integrals of local polynomials in the field and its derivatives over the boundary of space-time [20, 14, 15, 21].

More popular has been the *path integral quantization,* in particular after the important contributions of Slavnov, Faddeev and Popov and later of Becchi, Rouet, Stora and Tyutin. For constraint systems Faddeev's expression

$$\int \mathcal{D}\phi\mathcal{D}\pi\delta(\gamma)\delta(F)|\det\{\gamma,F\}|\exp\Big(\frac{i}{\hbar}\int_t^{t\prime} d\tau d^3x(\pi\dot{\phi}-\mathcal{H})\Big),$$

where the F's are gauge fixing conditions for the gauge transformations generated by the first class constraints γ, is the starting point for perturbative expansions after introducing ghosts and auxiliary fields to rewrite the determinant and gauge-fixing delta function. Any serious attempt to *calculate* in the standard model involves Faddeev-Popov ghosts, objects which appear in covariant gauges in order to guarantee gauge invariance. However, when addressing *non-perturbative questions*, e.g. the chiral symmetry breaking in gauge theories with fermions or the confinement problem in QCD, one must be cautious in summing over all gauge field configurations, including those with windings. Recently arguments have been put forward which show that, depending on the current quark masses, configurations with windings may be essential in finite-volume QCD [12].

In the *second chapter* of this Habilitationsschrift I review the classical theory of constraints systems. First we discuss *singular Lagrangian theories* and in particular the off-shell Bianchi identities and show that all gauge theories are constrained systems. Then some important facts about *constrained Hamiltonian systems* are reviewed and discussed. In particular primary/secondary and first/second class constraints, the generalized Legendre transformation and the Dirac-Bergman algorithm are introduced. Then we discuss the reduced phase space for first and second class systems. Here the important Dirac bracket for second class (SC) systems, the concept of observables and gauge transformations for first class (FC) systems and the first order formalism for mixed SC and FC systems are discussed. The developed formalism is then applied to the Abelian Chern-Simons model with sources [8, 22]. It has been argued that these type of models catch the long wavelength features of the Quantum-Hall effect [7, 6] and high temperature superconductors [16]. Here I am not elaborating on these very important aspects but rather use these models just to illustrate the general constraint formalism. This way the reader may become acquainted with the constrained dynamics by way of example. We will go through Dirac's program, step by step, and will arrive at a finite dimensional reduced phase space. Also, we argue that in the 'compact' version the only observables are

the Wilson-loops.
This review about Hamilton's formalism for constraint systems in the second chapter is an extended version of a series of lectures given by the author at the 1993-Bad Honnef meeting on quantum gravity.

In chapter 3 I investigate the relation between Lagrangian symmetries and the Hamiltonian gauge transformations generated by the first class constraint [9]. We shall see that for generally covariant theories the latter must be supplemented by transformations which vanish on-shell in order to recover the Lagrangian symmetries. The precise relation between the gauge transformations in the Lagrangian and Hamiltonian form is derived for general gauge gauge theories. We will see that the Hamiltonian gauge transformations which can be identified with Lagrangian symmetries form a closed algebra off mass-shell. Also, we shall see that in general relativity the whole dynamics follows from the requirement that the constraints are satisfied everywhere and they are preserved under diffeomorphisms [10]. More generally, I will discuss for which theories the equations of motion follow from the local symmetries. The general results are then applied to relevant theories, i.e. Yang-Mills theories, string theory and Einstein's theory of gravity. Some of the results are new and have not previously been published. I feel that the results offered are somewhat novel.

Recently it has been discovered that conformal Toda field theories can be naturally viewed as Hamiltonian reductions of the Wess-Zumino-Novikov-Witten (WZNW) theory [3, 1, 17] The main feature of the WZNW theory is its affine Kac-Moody (KM) symmetry, which underlies its integrability [13]. The reduction WZNW $\rightarrow$ generalized Toda theories, which we consider in chapter 4, is achieved by imposing certain first class and conformally invariant constraints on the KM currents. The constrained theory is a gauge theory and the gauge transformations are generated by the imposed first class constraints. The reduced phase space carries then a chiral $\mathcal{W}$-algebra as its Poisson bracket structure. This algebra is related to the phase space of the generalized Toda theory in the same way as the KM algebra is related to the phase space of the WZNW theory. This way of looking at Toda theories has numerous advantages, e.g. it allows for an easy construction of the general solution to the nonlinear Toda-field equation, the $\mathcal{W}$-algebra of Toda theory arises immediately as the algebra formed by the gauge invariant polynomials of the constrained KM currents and their derivatives and finally there are natural gauges which facilitate the analysis of the the theory.
In section 4.1 we gauge the WZNW theory, study the Hamiltonian structure of the resulting gauge theory and give Lie-algebraic condition for the constraints, which generate the gauge transformations, to be first class. Next we derive the effective theories for the gauge invariant fields. These turn out to be generalizations of the well-known Toda theories. In section 4.2 we give Lie-algebraic conditions for the resulting theories to be conformally

invariant and for the gauge invariant function to be generated by polynomials. The Poisson bracket algebras of these gauge invariant polynomials, the so-called $\mathcal{W}$-algebras have been introduced by Zamolodchikov [23] and are non-linear extensions of the Virasoro algebra. In the rest of the chapter I present a systematic study of the conformally invariant Hamiltonian reductions of the WZNW-theory. In particular we shall construct the nonlinear effective field theories which possess the $\mathcal{W}$ algebras as symmetry algebras, investigate the quantum reduction of WZNW theories and finally derive the general formula for the central charge of the reduced conformal field theories. The results presented in this chapter have been obtained in a series of papers [3, 17, 1, 4, 18, 5] with various collaborators.

It is supposed that 2-dimensional $U(1)$-gauge theories mimic certain aspects of one-flavor QCD [12]. In particular, gauge fields with windings, the so-called instantons, should be responsible for the non-vanishing condensate in both theories. In chapter 5 an idealized interacting 2-dimensional $U(1)$ gauge theory is investigated in detail. For certain values of the coupling constants the theory reduces to the gauged Thirring model, the Schwinger model or conformal fields coupled to a background curvature. Similarly as QCD the model possess so-called θ-vacua, field configurations with windings and shows a chiral symmetry breaking at finite temperature. Due to the non-trivial topology of the configuration space a careful quantization of these generalized gauged Thirring models at finite temperature turns out to be rather subtle. For example, when introducing a chemical potential for the conserved $U(1)$-charge, there arise ambiguities in the definition of fermionic determinants [19]. Also, the same problem arises if one introduces twisted boundary conditions for the Dirac-fermions.
By using functional techniques I shall solve the finite temperature and density model and in particular derive the exact equation of state and explicit temperature and curvature dependence of the chiral condensate. It turns out the condensate vanishes exponentially for high temperature and/or big curvature of space time. Indeed, we can associate an effective temperature to the curvature and this way arrive at a non-perturbative identification of the Hawking temperature in deSitter space time. If the electric charge is set to zero then the model reduces to a generalization of the ordinary conformally invariant Thirring model. Besides the Virasoro algebra the model contains an $U(1)$ Kac-Moody symmetry algebra. At the end of chapter 5 I investigate the conformal structure of these un-gauged models and determine the conformal weights and $U(1)$-charges of the fundamental fields.

In the last chapter of this Schrift I discuss the Schrödinger picture for fermionic fields [11] in external gauge fields for both stationary and time-dependent problems. I give formal results for the ground state and the solution of the time-dependent Schrödinger equation for QED in arbitrary dimensions, while more explicit results are obtained in two dimensions. For

both the mass-less and massive Schwinger model I give an explicit expression for the ground state functional as well as for the expectation values of energy, electric and axial charge. I also give the corresponding results for non-Abelian fields. Then I solve the functional Schrödinger equation for a constant external field in four dimensions and obtain the amount of particle creation. Next, the Schrödinger equation for arbitrary external fields for mass-less QED_2 is solved and a careful discussion of the anomalous particle creation rate follow. Finally, I discuss some subtleties connected with the interpretation of the quantized Gauss constraint.
At the end of each chapter I added the references relevant for that chapter.

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