Lattice Gauge Theories - An Introduction

Andreas Wipf

Theoretisch-Physikalisches Institut
Friedrich-Schiller-University Jena

seit 1558

Bayrischzell, März 2018

- weakly interacting systems :
  subsystems almost independent of each other
  - weakly correglated quantum systems
  - weakly interacting effective dof (quasi particles)
  - quantum electrodynamics
  - weak interaction
  - weak field gravity
  - strong interaction at high energies
- underlying Gaussian fixed point
- perturbations theory applicable

- strongly interacting systems:

  properties explained by strong correlations between subsystems
  - strongly correlated quantum systems
  - high temperature superconductivity
  - ultra cold atoms in optical lattices
  - spin systems near phase transitions
  - strong field gravity
  - strong interaction at low energies
- underlying interacting fixes point
- dependent on scale a theory can be weakly or strongly interacting
- needs non-perturbative methods

- soluble models
  - low dimensions:
    exactly soluble models: Ising-, Schwinger-, Thirring model, ...
  - high symmetry:
    conformal symmetry, supersymmetry, integrable systems, dualities,...
- approximations:

  mean field, strong coupling expansion, expansions for high/low temperataure, phenomenological models, ...
- functional methods:

  ∞-system of coupled Schwinger-Dyson equations

  functional renormalization group
- lattice formulation, ab-initio lattice simulation

  lattice-QFT $\Rightarrow$ particular statistical system

  powerful simulation methods of statistical physics

Gauge theories in continuum

- all fundamental theories = gauge theories
  - electrodynamics: abelian U(1) gauge theory
  - electroweak model: SU(2)×U(1) gauge theory
  - strong interaction: SU(3) gauge theory
  - gravity: gauge theory
- matter field $\phi(x) \in \mathcal{V}$, global gauge transform $\phi(x) \to \Omega\phi(x)$
- $\Omega \in \mathcal{G}$ gauge group
- invariant scalar product on $\mathcal{V}$: $(\Omega\phi, \Omega\phi) = (\phi, \phi)$
- invariant Lagrange density

$$\mathcal{L}(\phi, \partial_\mu\phi) = (\partial_\mu\phi, \partial_\mu\phi) - V(\phi)$$

- invariant potential $V(\Omega\phi) = V(\phi)$

- construction of locally gauge invariant theory

$$\phi(x) \longrightarrow \phi'(x) = \Omega(x)\phi(x), \quad \Omega(x) \in \mathcal{G}$$

- $\partial_\mu\phi$ wrong transformation property; need covariant derivative

$$D_\mu\phi = \partial_\mu\phi - \mathrm{i}gA_\mu\phi, \quad g \text{ coupling constant}$$

- needs new dynamical field $A_\mu \in \mathfrak{g}$ ($\mathfrak{g} =$ Lie algebra)
- requirement: $D_\mu\phi$ transforms as ϕ does $\Longrightarrow$

$$D'_\mu = \Omega D_\mu \Omega^{-1} \Longleftrightarrow A'_\mu = \Omega A_\mu \Omega^{-1} - \frac{\mathrm{i}}{g}\partial_\mu\Omega\Omega^{-1}$$

- field strength

$$F_{\mu\nu} = \frac{\mathrm{i}}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \in \mathfrak{g}$$

- transforms in adjoint representation

$$F_{\mu\nu}(x) \longrightarrow \Omega(x) F_{\mu\nu}(x) \Omega^{-1}(x)$$

$\mathcal{L}$ Lorentz invariant, parity invariant, gauge invariant $\Rightarrow$

$$\mathcal{L} = -\frac{1}{4} \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu} + \left(D_\mu \phi, D^\mu \phi\right) - V(\phi)$$

- principle of minimal coupling:
  begin with globally invariant theory, replace $\partial_\mu \to D_\mu$
  add Yang-Mills term $-\frac{1}{4}\mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$ (cp. electrodynamics)
- symmetries and particle content $\to$ Lagrange density (almost)

- $\mathcal{C}_{yx}$ path from x to y, parametrized $x(s)$
- parallel transport of ϕ along path:

$$0 = \dot{x}^\mu D_\mu \phi \Longleftrightarrow \frac{\mathrm{d}\phi(s)}{\mathrm{d}s} = \mathrm{i}gA_\mu(s)\dot{x}^\mu(s)\phi(s), \quad \phi(s) \equiv \phi\big(x(s)\big)$$

- cp. time-dependent Schrödinger equation
- let $x(0) = x$ and $x(1) = y \Rightarrow$

$$\phi(y) = \mathcal{P}\exp\left(\mathrm{i}g\int_0^1 \mathrm{d}s\, A_\mu(s)\dot{x}^\mu(s)\right)\phi(x)$$

parallel transport along path $\mathcal{C}_{yx}$

$$U(\mathcal{C}_{yx}, A) = \mathcal{P}\exp\left(\mathrm{i}g\int_{\mathcal{C}_{yx}} A\right) \in \mathcal{G}, \qquad A = A_\mu \mathrm{d}x^\mu$$

- paths $\mathcal{C}_{yx}$ and $\mathcal{C}_{zy}$ can be composed: $\mathcal{C}_{zy} \circ \mathcal{C}_{yx} = \mathcal{C}_{zx}$

$$U(\mathcal{C}_{zy} \circ \mathcal{C}_{yx}, A) = U(\mathcal{C}_{zy}, A) U(\mathcal{C}_{yx}, A)$$

- exists (useless?) nonabelian Stokes theorem
- gauge transformation

$$U(\mathcal{C}_{yx}, A') = \Omega(y)\, U(\mathcal{C}_{yx}, A)\, \Omega^{-1}(x)$$

- from x to y parallel transportet field

$$U(\mathcal{C}_{yx})\phi(x) \quad \text{transforms as} \quad \phi(y)$$

- gauge invariant objects (over-complete)

$$\begin{array}{rl} \mathrm{tr}\, U(\mathcal{C}_{xx}) & \text{holonomies} \\ \big(\phi(y), U(\mathcal{C}_{yx})\phi(x)\big) & \text{scalar products} \end{array}$$

- field theory in continuous spacetime $\mathbb{R}^d$ ill-defined (UV-divergences)
- spacetime continuum $\to$ discretize spacetime
  e.g. hypercubic lattice Λ with lattice constant a
- lattice sites, lattice links, lattice plaquettes, lattice cubes, ...
- minimal momentum $p = 2\pi/a$
  $\Rightarrow$ theory regularized in UV
- matter field $\phi(x) \to \phi_x,\ x \in \Lambda$ lattice field
- derivative $\to$ difference oparator or lattice derivative, e.g.

$$(\partial_\mu \phi)_x = \frac{1}{a}\{\phi(x + ae_\mu) - \phi(x)\}$$

- gauge theory: covariant lattice derivative

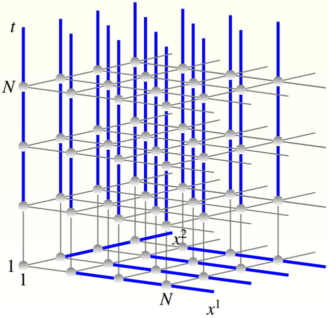
$$(D_\mu \phi)_x \equiv \frac{1}{a}\left\{\phi_{x+ae_\mu} - U_{x,\mu}\phi_x\right\}$$

- $U_{x,\mu}$ parallel transporter from x to $x + ae_\mu$
- lattice action for matter field ($a = 1$)

$$S_{\text{matter}} = \sum_{x,\mu} \left(D_\mu \phi_x, D_\mu \phi_x\right) + \sum_x V(\phi_x)$$

$$= -2\,\Re \sum_{x,\mu} \left(\phi_{x+e_\mu} U_{x,\mu} \phi_x\right) + \sum_x \Big(2d\left(\phi_x, \phi_x\right) + V(\phi_x)\Big)$$

- IR cutoff
  - finite lattice $\Lambda = \mathbb{Z}^4 \to N_t \times N^3$
  - needed in simulations
  - extrapolate to $N \to \infty$
- classical spin system
  - nearest neighbour interaction
  - S_{matter} real, positive
  - locally gauge invariant



- new dynamical compact field $U_{x,\mu} \in \mathcal{G}$
  parallel transporter along link from x to $x + ae_\mu$
- replaces dynamical noncompact field $A_\mu(x) \in \mathfrak{g}$
- relation via parallel transport

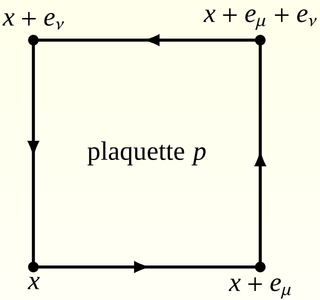
A_μ smooth on scale $a \Rightarrow$

$$U_{x,\mu} \approx \mathrm{e}^{\mathrm{i}g\,aA_\mu(x)} = \mathbb{1} + \mathrm{i}g\,aA_\mu(x) + \dots$$

- covariant derivative

$$\begin{aligned}(D_\mu\phi)_x &= \frac{1}{a}\{\phi_{x+ae_\mu} - (\mathbb{1} + \mathrm{i}g\,aA_\mu(x) + \dots)\phi_x\} \\ &= (\partial_\mu\phi)_x - \mathrm{i}g(A_\mu\phi)_x + O(a)\end{aligned}$$

- there are $O(a^2)$ improved lattice derivative



transporter $U_{\mu\nu}(x)$
around plaquette

- paralell transport around plaquette
  $p \sim (x,\mu,\nu)$

$$U_p = U_{x+e_\nu,-e_\nu} U_{x+e_\mu+e_\nu,-e_\mu} U_{x+e_\mu,e_\nu} U_{x,\mu}$$

- Baker-Hausdorff formula

$$U_{x,\mu} \approx \mathrm{e}^{\mathrm{i}agA_\mu(x)}, \quad a \ll 1$$
$$\Rightarrow U_p = \mathrm{e}^{\mathrm{i}a^2 gF_{\mu\nu}(x)+O(a^3)}$$

- transforms homogeneously

$$U_p(x) \to \Omega(x) U_p(x) \Omega^{-1}(x)$$

$$U_p + U_p^\dagger \approx 2\cdot\mathbb{1} - a^4 g^2 F_{\mu\nu}^2(x) + O(a^6)$$

- lattice action for gauge field configuration $U = \{U_{x,\mu}\}$

$$S_{\mathrm{W}}(U) = \frac{1}{g^2 N} \sum_p \mathrm{tr} \left\{ \mathbb{1} - \frac{1}{2} \left(U_p + U_p^\dagger \right) \right\} \qquad \text{(Wilson)} .$$

- in particular for $\mathcal{G} = SU(2)$

$$S_{\mathrm{W}} = \frac{1}{2g^2} \sum_p \mathrm{tr} \left(\mathbb{1} - U_p \right)$$

- improved lattice action (Symanzik)

$$S_{\mathrm{YM}} - S_{\mathrm{Sy}} = O(a^2)$$

$\Rightarrow$ faster convergence to continuum limit $a \to 0$

- functional integral over lattice gauge fields $\{U_{x,\mu}\} = \{U_\ell\}$

$$\int \mathcal{D}A_\mu(x) \xrightarrow{?} \int \prod_{(x,\mu)} \mathrm{d}U_{x,\mu} = \int \prod_\ell \mathrm{d}U_\ell, \quad \ell: \text{ link}$$

- action and measure must be gauge invariant
- recall $U_{x,\mu} \to \Omega_{x+e_\mu} U_{x,\mu} \Omega_x^{-1}$

gauge invariance $\Rightarrow \mathrm{d}U_{x,\mu}$ left- and right-invariant (normalized) Haar measure

- expectation values in pure lattice gauge theory

$$\langle \hat{O} \rangle = \frac{1}{Z} \int \prod_\ell \mathrm{d}U_\ell \, O(U) \, \mathrm{e}^{-S_\mathrm{W}(U)}$$

- partition function

$$Z = \int \prod_\ell \mathrm{d}U_\ell \, \mathrm{e}^{-S_\mathrm{W}(U)}$$

- consider irreducible representations $U \to \mathcal{R}(U)$ of compact $\mathcal{G}$, dim=$d_\mathcal{R}$
- **Peter-Weyl theorem:** *The functions $\{\mathcal{R}(U)^{ab}\}$ form an orthogonal basis on $L_2(dU)$, and*

$$(\mathcal{R}^{ab}, \mathcal{R}'^{cd}) \equiv \int \bar{\mathcal{R}}^{ab}(U)\mathcal{R}'^{cd}(U)\,\mathrm{d}U = \frac{\delta_{\mathcal{R}\mathcal{R}'}}{d_\mathcal{R}}\,\delta_{ac}\delta_{bd},$$

- **Lemma:** *The characters $\chi_\mathcal{R}(U) = \mathrm{tr}\,\mathcal{R}(U)$ form a ON-basis of invariant functions, $f(U) = f(\Omega U \Omega^{-1})$ in $L_2(dU)$, such that $(\chi_\mathcal{R}, \chi_{\mathcal{R}'}) = \delta_{\mathcal{R}\mathcal{R}'}$*
- **identities**

orthogonality: $$(\mathcal{R}^{ab}, \chi_{\mathcal{R}'}) = (\chi_{\mathcal{R}'}, \mathcal{R}^{ab}) = \frac{\delta_{\mathcal{R}\mathcal{R}'}}{d_\mathcal{R}}\,\delta_{ab}$$

gluing: $$\int d\Omega\, \chi_\mathcal{R}(U\Omega^{-1})\chi_{\mathcal{R}'}(\Omega V) = \frac{\delta_{\mathcal{R}\mathcal{R}'}}{d_\mathcal{R}}\chi_\mathcal{R}(UV)$$

cutting: $$\int d\Omega\, \chi_\mathcal{R}(\Omega U \Omega^{-1} V) = \frac{1}{d_\mathcal{R}}\chi_\mathcal{R}(U)\chi_\mathcal{R}(V)$$

decomposition of unity: $$\sum_\mathcal{R} d_\mathcal{R}\, \chi_\mathcal{R}(U) = \delta(\mathbb{1}, U)$$

- functional integral on finite d-dimensional lattice

$$dV \dim(\mathcal{G}) - \text{dimensional integral}$$

- SU(2) gauge theory, moderate hyper-cubic 16^4-lattice $\Rightarrow$

$$786\,432 - \text{dimensional integral}$$

- cannot be calculated numerically!
- stochastic methods
  - generate many configurations
  - distributed according to $\mathrm{e}^{-\text{action}}$
  - method of important sampling
  - Monte Carlo (MC) algorithms (Metropolis, Heat bath, ...)
  - with ferminions: expensive (hybrid MC + ...)

- only gauge invariant observables (Elitzur theorem)
- traces of parallel transporters along loops

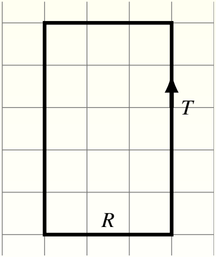
$$W[\mathcal{C}] = \mathrm{tr}\left(U_{\ell_n} \cdots U_{\ell_1}\right), \quad \mathcal{C} = \ell_n \circ \cdots \circ \ell_1 \quad \text{Wilson loops}$$

- $W[R, T]$ rectangular loop, edge lengths R, T
- static energy of a static $q\bar{q}$-pair separated by R

$$V_{q\bar{q}}(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W[R, T]\rangle$$

- string tension and Lüscher term

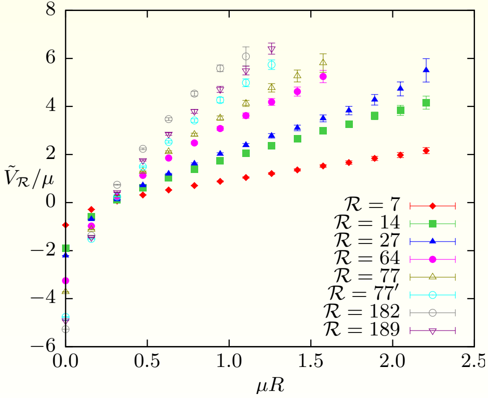
$$V_{q\bar{q}}(R) \sim \sigma R + \text{ const.} - \frac{c}{R} + o\left(R^{-1}\right)$$



- confinement $\Rightarrow \sigma > 0$

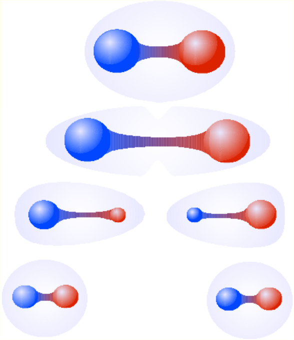
  $\Rightarrow$ $W \sim \exp(-\sigma RT)$ area law (strong coupling)
- only colorless (gauge invariant) states are seen

Click here

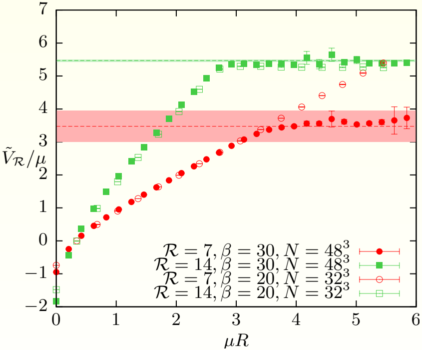


linear potentials for static quarks in different G_2 representations

- dynamical quarks
- meson, diquark $\bar{q}q \to$ 2 mesons, diquarks



- charges in adjoint or G_2
- energy scale = 2 m_{glueball}
- decay products: glue-lumps



- confinement: $\Rightarrow$ only colorless (gauge invariant) states are seen
- QCD: confinement at low temperature, no gluons
- glueballs = colourless bound states of gluons
- state by acting with interpolating operator on vacuum

$$|\psi(\tau)\rangle = \hat{\mathcal{O}}(\tau)|0\rangle, \quad \hat{\mathcal{O}}(\tau) = \mathrm{e}^{\tau\hat{H}}\mathcal{O}(0)\mathrm{e}^{-\tau\hat{H}}$$

- two-point function

$$G_E(\tau) = \langle 0|T\hat{\mathcal{O}}(\tau)\hat{\mathcal{O}}(0)|0\rangle = \sum_n |\langle 0|\hat{\mathcal{O}}|n\rangle|^2\mathrm{e}^{-E_n\tau}$$

- asymptotically large Euclidean time

$$G_E(\tau) \longrightarrow |\langle 0|\hat{\mathcal{O}}|0\rangle|^2 + |\langle 0|\hat{\mathcal{O}}|1\rangle|^2\mathrm{e}^{-E_1\tau}\Big(1 + O\big(\mathrm{e}^{-\tau(E_2-E_1)}\big)\Big)$$

- excited state with $\langle 0|\hat{\mathcal{O}}|1\rangle \neq 0 \rightarrow$ asymptotics
- $\hat{\mathcal{O}}|0\rangle$ and $|1\rangle$ should have same quantum numbers

  parity, angular momentum (cubic group), charge conjugation, ...
- glueballs: $\mathcal{O}$ combination of paralles transporters
- masses of glueballs in MeV

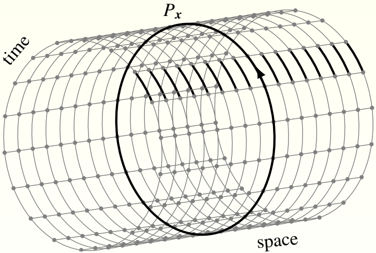
MC-simulation of Chen et al.

J^{PC}	0^{++}	2^{++}	0^{-+}	1^{+-}	2^{-+}	3^{+-}
m_G[MeV]	1710	2390	2560	2980	3940	3600

J^{PC}	3^{++}	1^{--}	2^{--}	3^{--}	2^{+-}	0^{+-}
m_G[MeV]	3670	3830	4010	4200	4230	4780

- partition function: β-periodic gauge fields

$$Z(\beta) = \oint \prod_{(x,\mu)} \mathrm{d}U_{x,\mu}\, \mathrm{e}^{-S_{\mathrm{W}}(U)}$$

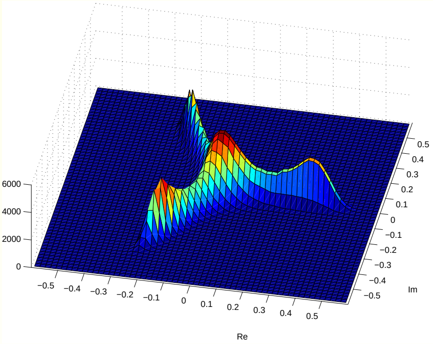


- $Z \Rightarrow$ thermodynamic potentials
- $T < T_c$: confinement $\rightarrow$ glueballs
- $T > T_c$: deconfinement $\rightarrow$ gluon plasma
- phase diagram, order of transition(s)
- order parameter: Polyakov loop $P_{\boldsymbol{x}}$

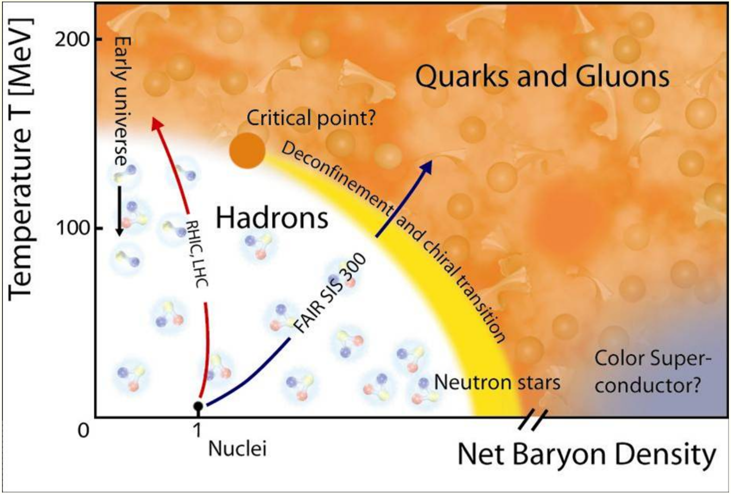
- center symmetry: non-periodic gauge trafo by center trafo
- order parameter: Polyakov loop $P_{\boldsymbol{x}}$

$$P_{\boldsymbol{x}} = \mathrm{tr}\left(\prod_{x_0=1}^{N_t} U_{(x_0,\boldsymbol{x}),0}\right)$$

- SU(3): center $= \mathbb{Z}_3$
- broken below T_c
- restored above T_c



histogram of Polyakov loop



expected phase diagram of QCD

fermions on the lattice

- functional approach: $\psi_\alpha(x)$ anticommuting

$$\{\psi_\alpha(x),\psi_\beta(y)\}=\{\bar\psi_\alpha(x),\bar\psi_\beta(y)\}=\{\psi_\alpha(x),\bar\psi_\beta(y)\}=0$$

- fermionic integration = multi-dimensional Grassmann integral

$$\int\mathcal{D}\psi\mathcal{D}\bar\psi\,\cdots\equiv\int\prod_x\prod_\alpha \mathrm{d}\psi_\alpha(x)\,\mathrm{d}\bar\psi_\alpha(x)\,\ldots$$

- expectation value of observable $\hat{A}$

$$\langle 0|\hat{A}|0\rangle=\frac{1}{Z_{\rm F}}\int\mathcal{D}\psi\mathcal{D}\bar\psi\,A(\bar\psi,\psi)\,\mathrm{e}^{-S_{\rm F}(\psi,\bar\psi)}$$

- partition function

$$Z_{\rm F}=\int\mathcal{D}\psi\mathcal{D}\bar\psi\,\mathrm{e}^{-S_{\rm F}}$$

- bilinear classical action S_{F} for the fermion field

$$S_{\mathrm{F}} = \int \mathrm{d}^d x\ \mathcal{L}(\psi, \bar{\psi}), \quad \mathcal{L} = \bar{\psi}(x) D \psi(x)$$

- Grassmann integration $\rightarrow$ determinant of fermion operator

$$Z_{\mathrm{F}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int \mathrm{d}^d x\, \bar{\psi}(x) D \psi(x)\right) = \det D$$

- corresponding formula for complex scalars

$$Z_{\mathrm{B}} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \exp\left(-\int \mathrm{d}^d x\, \bar{\phi}(x) A \phi(x)\right) = \frac{1}{\det A}$$

- Majorana fermions (susy)

$$Z_{\mathrm{F}} = \int \mathcal{D}\psi \exp\left(-\int \mathrm{d}^d x\, \psi(x) D \psi(x)\right) = \mathrm{Paff}(D)$$

- expectation values in full lattice gauge theory

$$\langle(U)\rangle = \frac{1}{Z}\int \mathcal{O}(U)\,\mathrm{d}\mu(U), \quad \mathrm{d}\mu(U) = \det(D)\,\mathrm{e}^{-S[U]}\mathcal{D}U, \quad Z = \int \mathrm{d}\mu(U)$$

- subtle: first order Dirac operator on lattice
- on finite lattice D (huge) matrix
- stochastic methods applicable if $\det(D)\,\mathrm{e}^{-S[U]} > 0$
- usually: D is γ_5-hermitean

$$\gamma_5 D \gamma_5 = D^\dagger$$

- eigenvalues come in complex conjugated pairs $\Rightarrow$ determinant real

$$P(\lambda) \equiv \det(\lambda - D) = \det \gamma_5(\lambda - D)\gamma_5 = \det\left(\lambda - D^\dagger\right) = P^*(\lambda^*)$$

- λ root $\Rightarrow \lambda^*$ root, real, not necessarily positive
- sign problem if det D changes sign
- example

$$D = \partial\!\!\!/ + m + \mathcal{O} \quad \gamma_5 \text{ hermitean} \Longleftrightarrow \partial_\mu = -\partial_\mu^\dagger,\ \mathcal{O} = \mathcal{O}^\dagger,\ [\gamma_5, \mathcal{O}] = 0$$

- natural choice

$$\left(\mathring{\partial}_\mu f\right)(x) = \frac{1}{2}\left(f(x+e_\mu) - f(x-e_\mu)\right)$$

- gauge theories: chiral symmetry for massless fermions

$$\mathrm{e}^{\mathrm{i}\alpha\gamma_5} D\, \mathrm{e}^{\mathrm{i}\alpha\gamma_5} = D \Leftrightarrow \{\gamma_5, D\} = 0$$

- naive Dirac operator

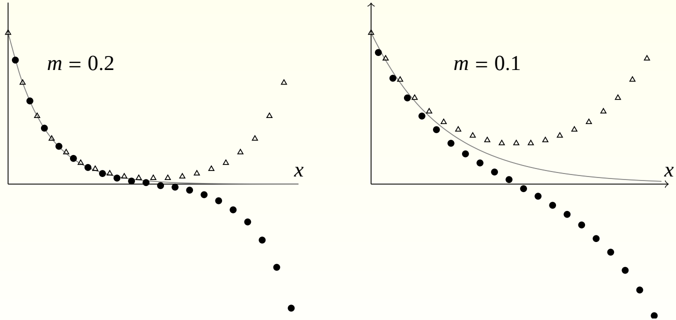
$$D = \gamma^\mu \mathring{\partial}_\mu + m$$

- γ_5-hermitean, chirally symmetric for $m = 0$

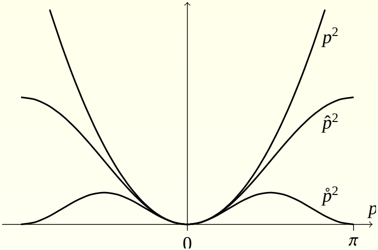
- doublers on lattice with N sites: fermion Green function

$$\langle x|\frac{1}{\mathring{\partial}+m}|0\rangle = \frac{1}{N}\sum_{n=1}^{N}\frac{e^{ip_nx}}{m+i\mathring{p}_n},\quad \mathring{p}_n=\sin p_n,\quad p_n=\frac{2\pi n}{N}$$

- Green function on the lattice with 40 sites.



fermion Green function on one-dimensional lattice with $N = 40$



- dispersion relations for $-\partial^2$
- p^2 : continuum relation
- $\mathring{p}^2$ from $\mathring{\partial}$
- $\hat{p}^2$ from nearest neighbor Laplacian ($\sim$ Wilson operator)

$$(\hat{\Delta} f)(x) = \sum_\mu \left(f(x+e_\mu) - 2f(x) + f(x-e_\mu)\right)$$

- $\mathring{\partial}_\mu \Rightarrow$ chiral and γ_5-hermitean $\mathring{\partial}\!\!\!/$
- doublers in spectrum

Theorem (Nielsen-Ninomiya)

exists no translational invariant D fulfilling

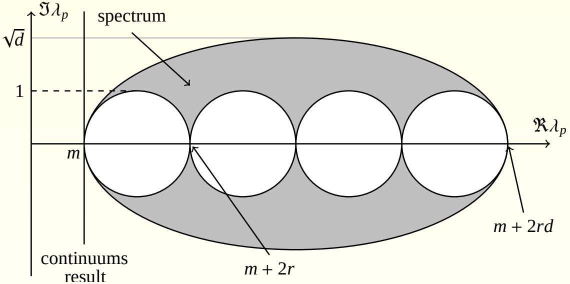
1. *locality:* $D(x-y) \lesssim \mathrm{e}^{-\gamma|x-y|}$,
2. *continuum limit:* $\lim_{a\to 0} \tilde{D}(p) = \sum_\mu \gamma^\mu p_\mu$,
3. *no doublers:* $\tilde{D}(p)$ *is invertible if* $p \neq 0$,
4. *chirality:* $\{\gamma_5, D\} = 0$.

- nice topological proof
- give up chiral invariance: Wilson fermions

$$S_\mathrm{w} = S_\mathrm{naive} - \frac{r}{2}\sum_x \bar{\psi}_x\, a\hat{\Delta}\psi_x = \sum_x \bar{\psi}_x D_\mathrm{w}\psi_x \ ,$$

- Wilson operator

$$D_\mathrm{w} = \gamma^\mu \mathring{\partial}_\mu - \frac{ar}{2}\hat{\Delta}$$



- γ_5 hermitian
- $\{\gamma_5, D\} \neq 0$
- complex eigenvalues in thermodynamic limit ($r = 1$)
- staggered fermions, Ginsparg-Wilson fermions

- lattice action

$$S_{\mathrm{F}} = \sum_x \bar{\psi}_x (D_{\mathrm{w}}\psi)_x$$

- gauge invariance: first parallel transport and then compare ($r = 1$)

$$\begin{aligned}(D_{\mathrm{w}})_{xy} &= (m+d)\delta_{xy} \\ &\quad - \frac{1}{2}\sum_\mu \Big((1+\gamma^\mu)U_{y,-\mu}\delta_{x,y-e_\mu} + (1-\gamma^\mu)U_{y,\mu}\delta_{x,y+e_\mu}\Big)\end{aligned}$$

- rescaling (Wilson)

$$\psi \to \frac{1}{\sqrt{m+d}}\,\psi$$

- gauge invariant action

$$S_{\mathrm{w}} = \sum_x \bar{\psi}_x\psi_x - \kappa\sum_{x,\mu}\Big(\bar{\psi}_{x-e_\mu}(1+\gamma^\mu)U_{x,-\mu}\psi_x + \bar{\psi}_{x+e_\mu}(1-\gamma^\mu)U_{x,\mu}\psi_x\Big)$$

- hopping parameter $\kappa = (2m+2d)^{-1}$

- lattice functional integrals

$$
\begin{aligned}
Z &= \int \prod_\ell \mathrm{d}U_\ell \prod_x \mathrm{d}\psi_x \mathrm{d}\bar{\psi}_x \, \mathrm{e}^{-S_\mathrm{g}(U) - S_\mathrm{F}(\psi,\bar{\psi})} \\
&= \int \prod_\ell \mathrm{d}U_\ell \, \det(D[U]) \, \mathrm{e}^{-S_\mathrm{g}(U)} \\
&= \int \prod_\ell \mathrm{d}U_\ell \, \mathrm{sign}(\det D) \, (\det M)^{1/2} \, \mathrm{e}^{-S_\mathrm{g}(U)}
\end{aligned}
$$

- $M = D^\dagger D \Rightarrow \det M \geq 0$.
- try stochastic method with

$$
\mathrm{d}\mu(U) = (\det M)^{1/2} \, \mathrm{e}^{-S_\mathrm{g}(U)} \mathcal{D}U
$$

- expectation values

$$\langle O[U]\rangle = \frac{\int \mathrm{d}\mu(U)\,\mathrm{sign}(\det D)\, O(U)}{\int \mathrm{d}\mu(U)\,\mathrm{sign}(\det D)}$$

- problem with re-weighing: sign(det D) may average to zero
- fermion determinant: method of pseudofermion fields

$$(\det M)^{1/2} = \int \prod_p \mathcal{D}\phi_p^\dagger \mathcal{D}\phi_p \,\mathrm{e}^{-S_{\rm PF}}, \quad S_{\rm PF} = \sum_{p=1}^{N_{\rm PF}} (\phi_p, M^{-q}\phi_p)$$

- $q \cdot N_{\rm PF} = 1/2$. If $\det D > 0 \Rightarrow$

$$Z = \int \prod_\ell \mathrm{d}U_\ell \mathcal{D}\phi\mathcal{D}\phi^* \;\mathrm{e}^{-S_{\rm g}(U) - S_{\rm PF}(U,\phi,\phi^\dagger)}$$

- HMC algorithm: force given by gradient of non-local $S_{\rm g} + S_{\rm PF}$

- rHMC dynamics $M^{-q} \to$ rational approximation

$$M^{-q} \approx \alpha_0 + \sum_{r=1}^{N_R} \frac{\alpha_r}{M + \beta_r}$$

- fermion correlators: S_F quadratic in $\psi \Rightarrow$ Wick contraction
- e.g. interpolating operator for pion

$$\mathcal{O}_\pi(t) = \sum_{\boldsymbol{x}} \bar{\psi}(t, \boldsymbol{x}) \tau \gamma_5 \psi(t, \boldsymbol{x})$$

- Wick-contraction

$$\langle 0 | \mathcal{O}_\pi^\dagger(t) \mathcal{O}_\pi(0) | 0 \rangle = \frac{1}{Z} \int \prod_\ell \mathrm{d}U_\ell \, G_F G_F \, \det(D[U]) \, \mathrm{e}^{-S_g(U)}$$
$$\sim \text{amplitude} \cdot \mathrm{e}^{-m_\pi t}$$

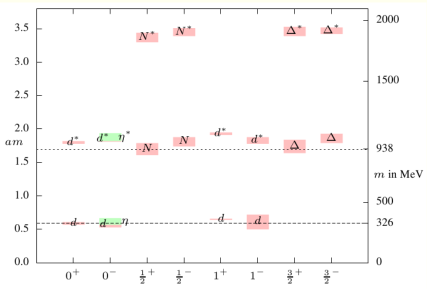
- $\Rightarrow$ masses of bound states: mesons, baryons, glueballs, . . .

mesons (baryon number 0)

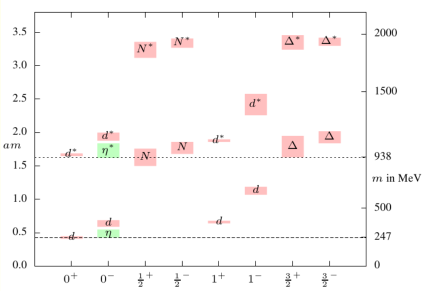
Name	$\mathcal{O}$	T	J	P	C
π	$\bar{u}\gamma_5 d$	SASS	0	-	+
η	$\bar{u}\gamma_5 u$	SASS	0	-	+
a	$\bar{u}d$	SASS	0	+	+
f	$\bar{u}u$	SASS	0	+	+
ρ	$\bar{u}\gamma_\mu d$	SSSA	1	-	+
ω	$\bar{u}\gamma_\mu u$	SSSA	1	-	+
b	$\bar{u}\gamma_5\gamma_\mu d$	SSSA	1	+	+
h	$\bar{u}\gamma_5\gamma_\mu u$	SSSA	1	+	+

- increase overlap with vacuum: smearing of sources and sinks
- diagonalization of correlation matrix

Ensemble	β	κ	$m_{d(0+)}a$	$m_N a$	$m_{d(0+)}$ [MeV]	a [fm]	a^{-1} [MeV]	MC
Heavy	1.05	0.147	0.59(2)	1.70(9)	326	0.357(33)	552(50)	7K
Light	0.96	0.159	0.43(2)	1.63(13)	247	0.343(45)	575(75)	5K



heavy ensemble



light ensemble

Wellegehausen, Maas, Smekal, AW (2013)

- simulations: stochastic, linear algebra, programming
- works for QCD at $T = 0$ and $T > 0$ (fermions β-anti-periodic)
- but: fermions difficult and expensive

  thermodynamic and continuum extrapolations: $N \to \infty$ and $a \to 0$

  realistic quark masses achieved
- problem: finite baryon density, $\det(D)$ complex

  $\Rightarrow$ conventional MC does not work
- simulations for supersymmetric YM theories
  - lattice breaks supersymmetry
  - some results of mass spectrum of $\mathcal{N} = 1$ SYM
  - new result on $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (8, 8)$
  - relevant for AdS/CFT (Gregory-Laflamme instability)
- books: Montvay-Münster, Rothe, Lang-Gattringer, AW, ...