Chapter 15

External field problems

There are many interesting physical effects induced by external fields, e.g. the Coulomb scattering of a charged electron by a heavy nucleus, the electron-positron pair production in strong electric fields, the Hawking radiation emitted by a black hole and the Casimir effect induced by external gauge- and gravitational fields to mention only a few of them. One of the central objects to describe such phenomena is the S-matrix. So we shall first derive its path integral representation and apply the result to the calculation of the pair creation in strong electromagnetic fields.

15.1 The S-matrix

Assume that the Hamiltonian of a quantum mechanical system decomposes as $H = H_0 + V$, that is into a free part H_0 and an interaction term V which may depend on time. For example V could describe the coupling to a time-dependent external current. The transition from the Schrödinger to the interaction picture is achieved by the following unitary transformation

$$\psi_w = e^{itH_0/\hbar}\psi_s(t) = U_0(-t)\psi_s(t).$$

extf1 The time dependence of ψ_w follows from the evolution of ψ_s (2.14) as

$$i\hbar\dot{\psi}_w = U_0(-t)V\psi_s(t) = U_0(-t)VU_0(t)\psi_w = V_w(t)\psi_w(t). \tag{15.1}$$

Setting

$$\psi_w(t) = U_w(t,t')\psi_w(t') \tag{15.2}$$

the 2-parametric unitary operators U_w obey

$$i\hbar\dot{U}_w = V_wU_w \quad \text{and} \quad U_w(t',t') = Id. \tag{15.3}$$

The solution of this evolution equation is known to be

$$U_w(t,t') = T\exp\Big[-\frac{i}{\hbar}\int\limits_{t'}^{t} V_w(t')dt'\Big], \tag{15.4}$$

where we used a short hand notation for the Dyson serie

$$U_w(t,t') = \sum(-i)^n\frac{1}{n!}\int\limits_{[t,t']^n} dt_1\dots dt_n T\Big(V_w(t_1)\cdots V_w(t_n)\Big), \tag{15.5}$$

and the time ordering T is defined as

$$T\Big(A(t_1)\cdots A(t_n)\Big) = \sum_{\pi\in\sigma_n}\theta(t_{\pi(1)},\dots,t_{\pi(n)})\; A(t_{\pi(1)})\cdots A(t_{\pi(n)}). \tag{15.6}$$

The generalized step function θ is 1 if its arguments are in decreasing order and else it is 0. In other words, in the time ordered product of n operators the operator with the 'latest time' stands on the left, the one with the second-latest time follows and so on.

Between the asymptotic states there is the relation

$$\psi_w(\infty) = S\psi(-\infty) \quad \text{where} \quad S = U_w(\infty,-\infty), \tag{15.7}$$

and this defines the scattering matrix transforming asymptotic in-states in asymptotic out-states. The path integral representation is most easily obtained be rewriting (15.2) as

$$\psi_w(t) = U_0^{-1}(t)U(t,t')U_0(t')\psi_w(t'), \tag{15.8}$$

where U and U_0 are the full and free evolution operators in the Schrödinger picture.

15.2 Scattering in Quantum Mechanics

For quantum mechanical system we have already derived the path integral representation for the full and free evolution operators in (2.32) and (2.21). Inserting these results we obtain the S-matrix elements

$$\langle p|S|p'\rangle = \frac{1}{2\pi\hbar}\,e^{i(Et-E't')/\hbar}\int dxdy e^{i(p'y-px)/\hbar}K(t,x,t',y), \tag{15.9}$$

where of course $E = p^2/m$. Instead of developing the perturbations theory for the S-matrix by using the perturbative expansion for the evolution operator, we shall calculate it exactly for a time-dependent harmonic force. For a such a force the evolution kernel has been computed in (3.21). The Gaussian integrals over x and y yield

$$\langle\ldots\rangle = \sqrt{\frac{1}{2\pi im\hbar}}\sqrt{\frac{D}{1+\dot DD'}}\exp\Big[\frac{i}{\hbar}\Big(Et-E't'+\frac{D}{\dot DD'}(E'\dot D-ED'-pp'/m)\Big)\Big], \tag{15.10}$$

where $D = D(t,t')$ is the solution defined in (3.18) and $\dot{D}$ and D' denote the partial derivatives with respect to t and t' respectively. Let us take as an example a harmonic force which vanishes exponentially for large times, e.g.

$$\omega^2(t) = \frac{2a^2}{\cosh^2(at)}. \tag{15.11}$$

For this interaction the D function reads

$$D(t,t') = \tanh(at')\Big(t\tanh(at) - 1/a\Big) - \Big(t \leftrightarrow t'\Big) \tag{15.12}$$

Assuming $t' = -t$ and letting $t \to \infty$ one finds after expanding the D-function and its derivatives to leading order in t and e^{at} the result

$$\langle \dots \rangle = \sqrt{\frac{1}{2\pi i m\hbar}}\sqrt{\frac{e^{2at}}{8a}}\exp\Big(-\frac{ie^{2at}}{16am\hbar}(p+p')^2\Big)\exp\Big(\frac{i}{4am\hbar}((p+p')^2+2p^2+2p'^2)\Big).$$

Using the identity

$$\sqrt{\frac{\alpha}{i\pi}}e^{i\alpha\xi^2} \longrightarrow \delta(\xi) \qquad \text{for} \qquad \alpha\to\infty$$

we end up with

$$\langle p|S|p'\rangle = i\delta(p+p')e^{ip^2/m\hbar}. \tag{15.13}$$

for the exact S-matrix. One easily checks that $SS^\dagger = I$ as it must be. Note that a particle subject to a harmonic force with time-dependent coupling strength as defined in (15.11) reflected with probability one. This is a particular feature of the chosen coupling.

For systems which are not exactly soluble one has to retreat to some approximation, e.g. the ordinary perturbation theory in the coupling constant or the semiclassical approximation. To find the perturbative expansion of the S-matrix one inserts the perturbation serie (4.12) into (15.9) and this yields the well-known rules for the diagrammatic expansion of S-matrix elements. Similarly, the semiclassical expansion is obtained by inserting (6.40) into (15.9)

15.3 Scattering in Field Theory

Let us now turn to the corresponding problem in field theory. Let $\Phi(t,x)$ denote an interacting field. It could be a photon field in interaction with an external current, an electron-positron field interacting with a gauge field or any other field interacting with a source, another field or with itself. Further we denote the incoming free field by Φ_{in} which approximates Φ for $t\to-\infty$ in some weak limit. We now wish to construct the operator that realizes the time-dependent canonical transformation relating the interacting to the incoming field

$$\Phi(t,\boldsymbol{r}) = U^{-1}(t)\Phi_{\text{in}}(t,\boldsymbol{r})U(t), \tag{15.14}$$

and fulfills

$$\lim_{t\to-\infty} U(t) = \mathbb{1}. \tag{15.15}$$

The time evolutions of these fields are given by

$$\dot{\Phi} = i[H(t), \Phi] \quad \text{and} \quad \dot{\Phi}_{\text{in}} = i[H_0, \Phi_{\text{in}}] \tag{15.16}$$

and similarly for the corresponding momentum densities. Here $H(t) = H(\Phi(t), \pi(t), j(t))$ may depend on an external current and H_0 is the time-independent free Hamiltonian. It follows from these formulae that

$$U(t)H(\Phi(t), \pi(t), j(t))U^{-1}(t) = H(\Phi_{\text{in}}(t), \psi_{\text{in}}(t), j(t)). \tag{15.17}$$

It also follows that

$$\partial_t \Phi_{\text{in}} = \partial_t\Big(U\Phi U^{-1}\Big) = \dot{U}U^{-1}\Phi_{\text{in}} + iU[H,\Phi]U^{-1} - \Phi_{\text{in}}\dot{U}U^{-1}. \tag{15.18}$$

Now we may use (15.17) for the second term on the right hand side to find

$$\partial_t \Phi_{\text{in}} = [iH(\Phi_{\text{in}}, \pi_{\text{in}}, j) + \dot{U}U^{-1}, \Phi_{\text{in}}], \tag{15.19}$$

and similarly for the time derivative of ψ_{in}. Comparing this result with the time evolution determined by (15.16) we see that

$$\dot{U}U^{-1} + i\Big(H(\Phi_{\text{in}}, \psi_{\text{in}}, j) - H_0(\Phi_{\text{in}}, \pi_{\text{in}})\Big) \equiv \dot{U}U^{-1} + iH_I(t)$$

commutes with all in-fields and hence must we a multiple of the identity operator. This central operator will drop in normalized matrix elements and can be left out in the following. Thus the time dependence of U is determined by the interacting Hamiltonian H_I as follows

$$i\dot{U} = H_I(\Phi_{\text{in}}, \pi_{\text{in}}, j)U, \tag{15.20}$$

and its solution is given by

$$U(t) = T\exp\Big(-i\int_{-\infty}^{t} dt' H_I(t')\Big). \tag{15.21}$$

The S-matrix is obtained by letting $t \to \infty$:

$$S = \lim_{t\to\infty} T\exp\Big(-i\int_{-\infty}^{t} H_I(t')\Big). \tag{15.22}$$

In a theory without derivative-couplings one has

$$H_I(t) = \int d^3x \mathcal{H}_I(t, \boldsymbol{r}) = -\int d^3x \mathcal{L}_I(t, \boldsymbol{r}), \tag{15.23}$$

so that (15.22) can be recast in a (formally) manifest covariant form

$$S = T\, e^{i \int d^4x \mathcal{L}_I(x)}. \tag{15.24}$$

This is a rather formal representation of the scattering matrix. When one tries to calculate S (e.g. perturbatively) one encounters short-distance singularities which must be regularized. The treatment of these singularities is the subject of renormalization theory.

Let us now consider the electron-positron field in interaction with an external gauge fields. Its interaction Hamiltonian is given by (see (12.3) and below)

$$\mathcal{H}_I = -\mathcal{L}_I = -\bar{\psi}_{\text{in}}(x)\gamma^\mu \psi_{\text{in}}(x) A_\mu(x) \tag{15.25}$$

and this results in the expression

$$S = T \exp\Big[ie \int d^4x \bar{\psi}_{\text{in}}(x)\gamma^\mu \psi_{\text{in}}(x) A_\mu(x)\Big] \tag{15.26}$$

for the S-matrix.

Let us now calculate the matrix element $\langle 0_{\text{in}}|S|0_{\text{in}}\rangle$, which is to be interpreted as amplitude for emitting no pair. We expand in (15.26) in powers of the the electric charge,

$$\langle 0_{\text{in}}|S|0_{\text{in}}\rangle = \sum_{n=0}^\infty \frac{(ie)^n}{n!} \int dx_1 \dots dx_n \, \langle 0_{\text{in}}|T[(\bar{\psi}_{\text{in}} A\!\!\!/ \psi_{\text{in}})(x_1) \cdots (\bar{\psi}_{\text{in}} A\!\!\!/ \psi_{\text{in}})(x_n)]|0_{\text{in}}\rangle$$

This should be compared with the perturbation expansion of the path integral,

$$\begin{aligned}\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_0 + ie\int \bar{\psi} A\!\!\!/ \psi} \\ &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_0} \Big(1 + ie\int \bar{\psi} A\!\!\!/ \psi + \frac{(ie)^2}{2!}\int \bar{\psi} A\!\!\!/ \psi \int \bar{\psi} A\!\!\!/ \psi + \cdots\Big). \end{aligned} \tag{15.27}$$

According to (12.12) the moments are just the corresponding expectation values of the time-ordered fields. Hence we obtain the following simple looking path integral representation for the expectation value of the S-matrix in the in-vacuum (omitting the subscript 'in'):

$$\langle 0_{\text{in}}|S|0_{\text{in}}\rangle = \frac{1}{Z[0]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_0} \Big(1 + ie\int \bar{\psi} A\!\!\!/ \psi + \frac{(ie)^2}{2!}\int \bar{\psi} A\!\!\!/ \psi \int \bar{\psi} A\!\!\!/ \psi + \cdots\Big), \tag{15.28}$$

which according to (15.27) is, up to a A-independent normalization constant, just the full path integral. Hence we conclude, that

$$\langle 0_{\text{in}}|S|0_{\text{in}}\rangle = \frac{1}{Z[0]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} = \det \frac{iD\!\!\!\!/ - m + i\epsilon}{i\partial\!\!\!/ - m + i\epsilon} = \exp\left(iS_{\text{eff}}[A]\right). \tag{15.29}$$

This formula yields a direct physical interpretation of the fermionic determinant. Expanding

$$\log\Big(iS_{\rm eff}[A]\Big) = \log\det\Big(I + e\not{A}\frac{1}{i\not\partial - m + i\epsilon}\Big) \tag{15.30}$$

in powers of the electric charge reproduces the well-known perturbation expansion for the vacuum-vacuum amplitude (External A-lines attached to a fermionic loop).

In the last section we have computed this determinant for massless two-dimensional fermions exactly. Continuing the Euclidean result (12.50) back to Minkowski space-time (the inverse transformation of (12.17) on finds

$$\langle 0_{\rm in}|S|0_{\rm in}\rangle = \det\frac{i\not{D} + i\epsilon}{i\not\partial + i\epsilon} = \exp\Big[\frac{ie^2}{2\pi}\int F_{01}\frac{1}{\partial^2}F_{01}\Big] \tag{15.31}$$

for the vacuum to vacuum amplitude. Since this is a pure phase, no pairs are produced in the Schwinger model. This is not true anymore for massive fields. Also, this conclusion only holds for gauge-fields for which (12.50) is the correct formula for the fermionic path integral. We have already seen that this formula is only correct for gauge fields for which the Dirac operator has no zero modes, that is for gauge fields with flux less or equal to 1.

15.4 Schwinger-Effect

Let us now calculate the pair production rate of massive fermions in a constant electro-magnetic field. To compute the determinant of $i\not{D}-m$ we recall that the non-zero eigenvalues of $i\not{D}$ come always in pairs $\{\lambda,-\lambda\}$ so that in the determinant $\det(i\not{D}-m)$ they contribute $-\lambda^2+m^2$. Hence the determinant of $i\not{D}-m$ can be defined as the square root of the determinant of $-\not{D}^2-m^2$ (for the zero-modes this is true anyway). To compute the logarithm of the determinant we use the identity

$$\log(a/b) = \int\limits_0^\infty \frac{ds}{s}\Big(e^{is(b+i\epsilon)} - e^{is(a+i\epsilon)}\Big) \tag{15.32}$$

which yields

$$-\log\left(2iS_{\rm eff}[A]\right) = \int\frac{ds}{s}e^{-is(m^2-i\epsilon)}\int d^4x\Big(\langle x|e^{-is\not{D}^2}|x\rangle - \langle x|e^{-is\not\partial^2}|x\rangle\Big), \tag{15.33}$$

where we have used the (formal) identity $\log\det(A) = \text{tr}\,\log(A)$ and have represented the trace in the $|x\rangle$ basis. For a constant electric field in the 3-direction the only non-vanishing field strength components are

$$F_{03} = -F_{30} = E. \tag{15.34}$$

As potential we choose $A_\mu = (0,0,0,Ex^0)$ with constant E. In the present case the square of $\not{D}$ (see (8.69)) simplifies to

$$\not{D}^2 = D^2 + 2\Sigma_{03}F_{03} = \partial_0^2 - \partial_1^2 - \partial_2^2 - (\partial_3 - iEx^0)^2 - i\gamma^0\gamma^3 E. \tag{15.35}$$

Since the Pauli term in $\not{D}^2$ commutes with D^2, its exponential can be computed separately. Using $(\gamma^0\gamma^3)^2 = \mathbb{1}$, one finds

$$\exp\big(-\gamma^0\gamma^3 E\big) = \cosh(sE) - \sinh(sE)\gamma^0\gamma^3 \Longrightarrow \text{tr}\,(\ldots) = 4\cosh(sE),$$

so that the Dirac-trace of the heat kernel in (15.33) yields

$$\text{tr}_D\langle x|e^{-is\not{D}^2}|x\rangle = 4\cosh(sE)\langle x|e^{-isD^2}|x\rangle. \tag{15.36}$$

Now we are left with computing the heat kernel of D^2. For that purpose we observe that D^2 can be written as the sum of two 2-dimensional commuting operators

$$D^2 = -\Big(\partial_1^2 + \partial_2^2\Big) + \Big(\partial_0^2 - (\partial_3 - iEx^0)^2\Big) = -\Delta_{12} + D_{03}^2 \tag{15.37}$$

and thus its heat kernel is just the product of the two corresponding two-dimensional heat kernels

$$\langle x|e^{-isD^2}|x\rangle = \langle x^1,x^2|e^{is\Delta_{12}}|x^1,x^2\rangle K(s,x^0,x^3) = \frac{1}{4i\pi s}K(s,x^0,x^3), \tag{15.38}$$

where K the heat kernel belonging to D_{03}^2. To calculate this remaining heat kernel we first note that ∂_3 commutes with D_{03}. Thus they can be diagonalized simultaneously and the eigenfunctions have the form

$$D_{03}^2\psi_\lambda = \lambda\psi_\lambda \Longrightarrow \psi_\lambda = e^{ip_3x^3}\phi_\lambda, \quad \text{where} \quad \Big(\partial_0^2 + E^2(x^0 - \frac{p_3}{E})^2\Big)\phi_\lambda = \lambda\phi_\lambda. \tag{15.39}$$

It follows that the diagonal-elements of K are independent of the x^3. The remaining operator on the right hand side in (15.39) is just a shifted harmonic oscillator with imaginary frequency and thus has eigenvalues $-i(2n+1)E$ (we assume E to be positive, else we would have to write everywhere $|E|$. The minus sign is due to time ordering). Since the eigenvalues are independent of p_3 they are degenerate and apriori we can determine the trace of K only up to the multiplicity C of the eigenmodes as

$$\int dx^0dx^3\, K(s,x^0,x^3) = C\sum_{n=0}^\infty e^{-s(2n+1)E} = \frac{C}{2\sinh(sE)}. \tag{15.40}$$

However, recalling that for a vanishing electric field K is the free heat kernel,

$$< x^0,x^3|e^{-is\partial^2}|x^0,x^3\rangle = \sqrt{\frac{i}{4\pi s}\frac{-i}{4\pi s}} = \frac{1}{4\pi s},$$

(since phases relevant, we have emphasized that due to the $(+,-)$-signature in D_{03} the diagonal elements of $\langle\ldots\rangle$ are real), we can now easily determine C and find

$$\int dx^0 dx^3\, K(s,x^0,x^3) = \frac{EV_{03}}{4\pi\sinh(sE)}, \tag{15.41}$$

where V_{03} denotes the volume of the $(0,3)$ plane. Inserting now (15.36,15.38) and (15.41) into the general formula (15.33), we find for the real part $\int d^4x\, w(x) \equiv \Re\log(2iS_{\rm eff})$ the formula

$$\begin{aligned}\int d^4x\, w(x) &= \frac{V}{(2\pi)^2}\int_0^\infty \Re\Big(\frac{1}{i}e^{-is(m^2-i\epsilon)}\Big)\Big[E\coth(sE)-\frac{1}{s}\Big]\frac{ds}{s^2}\\ &= -\frac{V}{(2\pi)^2}\int e^{-\epsilon s}\sin(sm^2)\Big[E\coth(sE)-\frac{1}{s}\Big]\frac{ds}{s^2},\end{aligned} \tag{15.42}$$

where $V=V_{03}V_{12}$ is the volume of the four-dimensional Minkowski space-time. Since

$$|\langle 0_{\rm in}|S|0_{\rm in}\rangle|^2 = |e^{iS_{\rm eff}[A]}|^2 = e^{2\Re(iS_{\rm eff})} = e^{-\int d^4x\, w(x)}$$

measures the probability of emitting no pair, and

$$e^{-\int d^4x\, w(x)} \sim e^{-\sum \Delta V w(x_i)} \sim \prod\,(1-\Delta V w(x_i)),$$

we interpret $\Delta V w(x_i)$ as probability to create a pair in the volume element ΔV or $w(x)$ as a probability density for pair creation.

Note that the s-integral is convergent both in the ultraviolet (small s) and infrared (large s) regions, even after setting ϵ to zero. The last integrand is an even function in s for $\epsilon=0$ and the integral can be transformed into an integral over the real line $(-\infty,\infty)$. Thus we obtain

$$\begin{aligned}w(x) &= -\frac{1}{4(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{i}e^{ism^2}\Big[E\coth(Es)-\frac{1}{s}\Big]\frac{ds}{s^2} \;+\; cc\\ &= \frac{i}{16\pi^2}2\pi i \sum_{\substack{Residue\\ s_n=in\pi/E}} \Big[e^{ism^2}E\coth(sE)\frac{ds}{s^2}\Big] \;+\; cc\\ &= \frac{1}{8\pi}\sum_1^\infty \frac{E^2}{n^2\pi^2}e^{-n\pi m^2/E} \;+\; cc\,.\end{aligned} \tag{15.43}$$

Reinserting the electric charge we finally end up with

$$w(x) = \frac{\alpha E^2}{\pi^2}\sum_1^\infty \frac{1}{n^2}\exp\Big(-\frac{n\pi m^2}{|eE|}\Big), \tag{15.44}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. The analog calculation in two dimensions yields

$$w(x) = \frac{eE}{2\pi}\sum_1^\infty \frac{1}{n}\exp\Big(-\frac{n\pi m^2}{|eE|}\Big). \tag{15.45}$$

In these exact formulae for the pair creation density in a constant electric field the essential factor is non-perturbative $\sim \exp(-\pi m^2/eE)$ and can be interpreted as Gamov factor for the tunneling of an electron in an external electric field through a potential barrier. Such a factor cannot be gotten by ordinary perturbation theory, since $\exp(c/e)$ cannot be expanded in powers of the coupling constant. Unfortunately, pair creation in a constant electric field has not been observed since $|E| \ll m^2$ for realistic electric fields. Due to the exponential suppression factor the creation density is then too small.