Chapter 11

Supersymmetric Quantum Mechanics

In this section we examine simple $1+0$-dimensional supersymmetric field theories. In $1+0$ dimensions the Poisson-algebra reduces to time translations generated by the Hamiltonian H and the hermitian field and momentum operators $\phi(t)$ and $\pi(t)$ may be viewed as position and momentum operators of a point particle on the real line in the Heisenberg-picture. Hence susy field theories in $1+0$ dimensions are particular quantum mechanical systems [35]. Such systems are interesting in their own right since they describe the infrared-dynamics of supersymmetric field theories in finite volumes. In mathematical physics supersymmetric QM has been useful in proving index theorems for physically relevant differential operators [36]. There exist several extensive texts on susy quantum mechanics [37, 38, 39] in which the one-dimensional systems are discussed in detail. First we consider the simple Hamiltonian

$$H = H_B + H_F, \quad \text{where} \quad H_B = \omega a^\dagger a, \quad H_F = \omega b^\dagger b, \tag{11.1}$$

and a and b are bosonic and fermionic annihilation operators: $[a, a^\dagger] = 1$ and $\{b, b^\dagger\} = 1$. The *Fockspace* is generated by acting with the creation operators on the vacuum defined by

$$a|0\rangle = b|0\rangle = 0. \tag{11.2}$$

Using the commutation and anticommutation relations for the creation and annihilation operators one finds that besides the non-degenerate zero-energy ground state all excited states are *double degenerate* since $(a^\dagger)^n|0\rangle$ and $b^\dagger(a^\dagger)^{n-1}|0\rangle$ have both energy $E = n\omega$. Introducing the fermion number operator $N_F = b^\dagger b$ we see that there is always a bosonic state ($N_F = 0$) and a fermionic one ($N_F = 1$) with the same energy. This system is the simplest supersymmetric quantum mechanical system, namely the *supersymmetric harmonic oscillator* (we have set the mass to one and shall also set $\hbar = 1$ in what follows).

Let us now generalize the above Hamiltonian and consider

$$H = H_B + H_F, \quad \text{where} \quad H_B = \frac{1}{2}\left(p^2 + W^2\right) \quad \text{and} \quad H_F = W' b^\dagger b, \tag{11.3}$$

where $W(x)$ is an arbitrary function. Using the formula (10.25) and the corresponding bosonic result (2.32) yields the following path integral representation for the evolution kernel

$$K(t,q,q',\bar{\alpha},\alpha') = \int \mathcal{D}w\mathcal{D}\alpha\mathcal{D}\bar{\alpha}\, e^{iS[w,\alpha,\bar{\alpha}]}, \tag{11.4}$$

where one sums over all paths $w(t),\alpha(t),\bar{\alpha}(t)$ with

$$w(0)=q',\quad w(t)=q,\quad \alpha(0)=\alpha' \quad\text{and}\quad \bar{\alpha}(t)=\bar{\alpha}.$$

The action contains the familiar bosonic part S_B and an additional term depending on the Grassmann values path,

$$S = \int dt\,\mathcal{L} = S_B[w] + S_F[w,\alpha],$$

with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\dot{w}^2 - \frac{1}{2}W^2(w) + i\bar{\alpha}\dot{\alpha} - W'(w)\bar{\alpha}\alpha. \tag{11.5}$$

This models are supersymmetric. Under a supersymmetry transformation

$$\delta w = \bar{\epsilon}\alpha + \bar{\alpha}\epsilon,\quad \delta\alpha = -(i\dot{w}+W)\epsilon \quad \delta\bar{\alpha} = -\bar{\epsilon}(-i\dot{w}+W) \tag{11.6}$$

with constant anticommuting parameters $\epsilon,\bar{\epsilon}$, the variation of the Lagrange function is a total time-derivative,

$$\delta\mathcal{L} = \frac{d}{dt}\Big(\dot{w}\bar{\alpha}\epsilon - iW\bar{\epsilon}\alpha\Big) \tag{11.7}$$

and thus the action is invariant.

It has been observed by Nicolai [40] that the following transformation of the bosonic field

$$w(t) \longrightarrow y(t) = \dot{w}(t) + iW(w(t)) \tag{11.8}$$

for which

$$\frac{1}{2}y^2 = \frac{1}{2}\dot{w}^2 - \frac{1}{2}W^2 + iW\dot{w} \quad\text{and}\quad \frac{\delta y(t)}{\delta w(t')} = \left(\frac{d}{dt}+iW'\right)\delta(t-t') \tag{11.9}$$

simplifies the analysis considerable, due to supersymmetry. To see that we first note that

$$\mathcal{D}w\int \mathcal{D}\alpha\mathcal{D}\bar{\alpha}\, e^{-(\bar{\alpha}\dot{\alpha}+iW'\bar{\alpha}\alpha)} = \mathcal{D}w\det\left(\frac{d}{dt}+iW'\right) = \mathcal{D}y, \tag{11.10}$$

which means that the Jacobian of the bosonic transformation is canceled by the fermionic integral. We have been a bit sloppy with the boundary conditions, for a more detailed analysis of this point I refer you to the paper of Ezawa and Klauder [41]. Second we observe that

$$\frac{1}{2}\int y^2 dt = \frac{1}{2}\int\left(\dot{w}^2 - W^2\right)dt + i\int W\dot{w}dt = S_B + i\int_{q'}^{q} W(w)dw. \tag{11.11}$$

Inserting the last two identities into the evolution kernel (11.4) we see that this kernel is given by a Gaussian integral in terms of the new variables,

$$K(t,\ldots) = \exp\left(\int_{q'}^{q} W\right) \int \mathcal{D}y\; e^{iy^2/2}. \tag{11.12}$$

To obtain the partition function we continue to imaginary time $t = -i\tau$ such that the action changes into the Euclidean action

$$S_E = \int d\tau \mathcal{L} \quad \text{with} \quad \mathcal{L} = \frac{1}{2}\dot{w}^2 + \frac{1}{2}W^2 + \bar{\alpha}\dot{\alpha} + W'\bar{\alpha}\alpha, \tag{11.13}$$

and the supersymmetric transformations are modified to

$$\delta\alpha = (\dot{w} - W)\epsilon \quad \delta\bar{\alpha} = \bar{\epsilon}(-\dot{w} - W) \quad \text{with} \quad \delta\mathcal{L} = \frac{d}{d\tau}\left(\dot{w}\bar{\alpha}\epsilon - \bar{\epsilon}\alpha W\right). \tag{11.14}$$

Note that the transformation of w is unchanged. The *Nicolai map* of the Euclidean model reads

$$w(\tau) \longrightarrow y(\tau) = \dot{w}(\tau) + W(w(\tau)). \tag{11.15}$$

To obtain the 'partition function' one integrates over β-periodic paths $w(\tau)$ and β-*antiperiodic paths* $\bar{\alpha}(\tau), \alpha(\tau)$ (see below). Such finite temperature boundary conditions break supersymmetry which transforms periodic bosonic fields into periodic fermionic fields. Physically this is not surprising since a equilibrium state is not invariant under Lorentz transformation and hence cannot be supersymmetric. After all, supersymmetry is an extension of Lorentzsymmetry. If we instead integrate only over periodic paths then supersymmetry is not violated by the boundary condition. This corresponds to the Euclidean model and for $\beta \to \infty$ expectation values become vacuum expectation values. For periodic boundary conditions we can transform to Nicolai variables and obtain

$$Z_{\text{per}} = \oint \mathcal{D}y\; e^{-y^2/2}, \tag{11.16}$$

where one integrates over β-periodic paths $y(\tau)$. When treating the boundary conditions more carefully one can indeed show that for matrix elements the cancellation between the fermionic determinant and the bosonic Jacobian occurs for a certain definition of the fermionic path integral and thus the above formal manipulations are justified.

Note that the correlation functions can now be evaluated as

$$\langle w(\tau_1)\ldots w(\tau_n)\rangle = \int w(\tau_1)\ldots w(\tau_n) d\mu_0(y)$$

with the Gaussian measure as in (11.16) rather than with the complicated interaction measure. However, the moments are not that easy to calculate because $w(t)$ is generally a nonlinear and nonlocal function of the fluctuating y-path as determined by the inverse Nicolai map.