Symmetries and the methods of quantum field theory: Supersymmetry on a space-time lattice

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1 Introduction

Symmetries are one of the essential concepts of modern theoretical physics. The basic laws and principles of nature are represented by symmetries [1]. A theory describing nature must hence be compatible with these symmetries. Quantum field theory unifies the principles of quantum mechanics with the laws of special relativity. These laws are represented by the relativistic space-time symmetries (Poincaré symmetry). Thus the Poincaré symmetry is the basic symmetry of quantum field theory. In order to respect the laws of relativity it cannot be reduced, but only extended by other symmetries.

The symmetries can be found in the experimental results. For example, certain scattering processes do not occur since they violate a symmetry. Based on the observations of the experiments one can hence find out the symmetries and construct a theoretical description. In this way the basis of the current quantum field theoretical description of nature, the standard model, was found. Despite its great success, it is, however, in some respects an unsatisfactory and incomplete description. One reason can be found in the Higgs sector of the model. The mass of the Higgs particle must be much lighter than the next physical scale (Planck mass or GUT scale). This introduces large quantum corrections that shift this mass towards the larger scale unless they are cancelled by the parameters of the model. This cancellation can only be achieved by an "unnatural" fine-tuning of these parameters [2]. The explanation of the astronomical data of the cosmic microwave background is another problem of the standard model [3, 4]. In fact, these data are a strong indication for the existence of dark matter that cannot be described by the standard model. A different motivation for the extension of the standard model is that it does not involve a description of the gravitational force.

There were many attempts to extend the standard model and its symmetries. However, it turned out that not all extensions of the space-time symmetries are allowed. Too many symmetry requirements will restrict the scattering amplitudes of an experiment too much to allow a realistic result. The very profound analysis of this fact is given by the Coleman-Manula theorem [5]. According to this theorem only an extension of the space-time symmetries by an internal symmetry, which does not change the spin or the mass of the particle, is possible. The only exception not covered by this theorem is a symmetry that connects fermionic and bosonic fields, a supersymmetry [6]. A first field theoretical realisation of such a symmetry was found in terms of the Wess-Zumino model [7].

Supersymmetric extensions can help to resolve the problems of the standard model. The cancellation of fermionic and bosonic divergences resolves the problem of the light Higgs mass. The new particles implied by the extensions are candidates for the dark matter. In addition, supergravity, as constructed from a local version of supersymmetry, is capable for a description of the gravitational force [8, 9]. Furthermore, supersymmetry plays an important role in string theory, and its investigation is also attractive from

the mathematical point of view. However, since supersymmetry is not visible in the experimental results, it must be broken at the scale of our present investigations.

Although quantum field theory provides a very attractive description of the nature, many relevant quantities cannot be calculated exactly. In some cases supersymmetry allows the exact calculation of quantities that are not accessible in other theories. The most prominent example is the low energy effective action in $\mathcal{N} = 2$ super Yang-Mills theory [10, 11]. In the general situation one has to rely on approximation methods. The well-known method of perturbation theory does not allow for a complete analysis of the theory. It is, e. g., not possible to investigate spontaneous supersymmetry breaking with this kind of approximation. A very successful method that provides an insight into the non-perturbative sector are the numerical lattice simulations.

Therefore, it is not only important to know that certain symmetries are compatible with the general physical principles in quantum field theory. The symmetry must also be respected by the approximation methods. The investigation of supersymmetry in the context of certain approximation methods is the topic of the present thesis. The main emphasis lies on the compatibility of lattice calculations with supersymmetry. A review of other investigations in the context of supersymmetric lattice calculations can be found in [12, 13, 14, 15]. It is very important to have this non-perturbative tool at hand for many investigations of supersymmetric theories. On the other hand, the symmetry is generically broken in the discretisation of the continuum theory. Therefore, a detailed analysis is necessary to find out, how supersymmetry can still be respected in these calculations. Furthermore, also alternative methods are considered here that can be used to study non-perturbative effects like supersymmetry breaking.

Although the main concern of the thesis is supersymmetry, some of the results apply to an arbitrary linear global symmetry. It turns out that supersymmetry is in some respects more difficult to handle in the approximations than other symmetries. One reason for this is that supersymmetry acts nontrivially on the space-time symmetries. It hence comprises properties of the space-time and the internal symmetries.

This thesis is organised as follows. In chapter 2 the symmetries are first analysed from the classical point of view. Some classical theories are introduced to illustrate the discussion. The general properties, exact relations, and symmetries of the quantum effective action are considered in section 2.2. This effective action contains all properties of the quantum theory. Only in very simple cases it can be calculated exactly. Some approximation methods commonly used to calculate quantum observables are introduced in section 2.3. Chapter 3 is devoted to general aspects of the construction of a discretised supersymmetric action. The problem is analysed in a classical way and in the context of lattice perturbation theory. In chapter 4 a number of different lattice realisations are constructed from the supersymmetric examples introduced in section 2.1.4 according to the

results of this analysis. The results of numerical lattice simulations with these realisations can also be found in this chapter. Only these can confirm the results of chapter 3. To get a theoretical prediction that does not rely on a perturbative analysis, the approach of Ginsparg-Wilson is generalised and applied to lattice supersymmetry in chapter 5. Alternative methods are discussed in chapter 6 and 7. The first one contains an application of an alternative approach to the loop expansion in supersymmetric theories, whereas in the latter the method of the functional renormalisation group flow is used.

2 Classical and quantum symmetries

In this chapter the concept of symmetries is introduced first in a classical field theory. The group of space-time symmetries is based on the principles of special relativity. To deduce the representation of the symmetries in the theory from these principles one has to take into account that eventually a quantisation of the classical theory is considered. This allows the introduction of fermionic fields. From this point of view supersymmetry, which connects fermions and bosons, is a further generalisation and extension of the classical concepts of relativity. The basic (global) classical symmetries of a four-dimensional quantum field theory, with the main emphasis on supersymmetry, are briefly reviewed in this chapter. Then examples of some classical theories and their corresponding symmetries are given. The main focus are supersymmetric models in one and two dimensions as toy models for the fourdimensional theory. The general aspects that can be found in this work are illustrated by these models.

After these considerations basic concepts of the quantum field theory are introduced and the implications of the classical symmetries are given. This presentation contains already some derivations and discussions that plays a role in the follwing chapters. At the end of this discussion some methods for approximative calculations of quantum field theory are introduced.

2.1 Symmetries of the classical action

Classical symmetries are transformations that leave the action – that means the physical laws – of a considered model invariant. According to the theorem of Coleman and Mandula the continuous symmetries of quantum field theory are restricted. They consist of space-time symmetries, internal symmetries and supersymmetry.[1] The action in quantum field theory is a functional of spacetime dependent fields, so the symmetry is represented on a multiplet of these fields.

2.1.1 Space-time and internal symmetries

A foundation of all physical descriptions is the equivalence principle. It states that in all inertial frames the same physical laws apply. The corresponding symmetry group, which transforms the inertial frames into each other, is the Poincaré group, the group of spacetime symmetries. An invariance of the action under this symmetry group is crucial for every physical theory. It is defined as the composition of the translation in spacetime and Lorentz transformations, that leaves the metric of the spacetime invariant. This symmetry must be respected so it can only be extended if more general symmetries of a physical theory should be considered. The simplest extension is the inclusion of an internal

[1] For massless models the space-time symmetries are enlarged by the so called conformal group. This is not considered here.

symmetry, that does not interfere with the spacetime symmetries. Hence this symmetry transforms fields that are in the same representation of the Lorentz group into each other. These are continuous symmetries; additional possible discrete symmetry transformations are C (charge conjugation), P (parity inversion) and T (time reversal).

The generators of the symmetries correspond to observables, measurable quantities, that characterise the state of the system like the momentum or the angular momentum. In a quantum theory these observables are represented by hermitian operators on a Hilbert space. An exponentiation of these hermitian operators leads to unitary operators that define the symmetry transformations. This representation of the classical symmetries by unitary operators in a quantum theory does not cover all the differences that appear in the translation from a classical to a quantised system. In the quantum theory the equivalence principle means that the observers of two systems that are connected by a symmetry transformation should measure the same probability for a certain state. The identification of all the states that lead to the same probability yield a projective space. A representation of a group on this space is called a projective representation. Fortunately, according to the theorems of Wigner and Bargmann all projective representations of a Lie group can be identified with unitary representations of its universal covering group with a central extension [16, 17]. Hence instead of considerations of the projective representations one can retain the above discussion of unitary representations allowing an extended symmetry group. The universal covering group of the Lorentz group has the same generators but allows also representations of half integer spin. This is the reason why these representations are possible in quantum physics and not in the macroscopic world. The corresponding fields are called spinor or fermionic fields. According to the spin statistics theorem the spinor quantum field operators must obey anticommutation instead of commutation rules. In the path integral description of the quantum theory a classical action is used to calculate quantum observables. This action must, however, take account of the fermionic operators. For this reason the introduction of anticommuting fields (of Grassmann numbers) into the classical action is necessary.

2.1.2 Supersymmetry

The new concept of fermionic, i. e. anticommuting, operators and fields is in that sense introduced by the quantum nature of the theory. This extension of concepts demands also a revision of the above considerations about possible symmetries of a physical theory. If we introduce fermionic objects as field operators, these objects should in principle also be allowed as symmetry generators. This symmetry is called supersymmetry. It translates fermionic into bosonic fields and acts hence nontrivially on the Lorentz group. According to the theorem of Coleman and Mandula this is not allowed for any other symmetry.

Some basic facts of supersymmetry can be illustrated with some relations from the

fourdimensional supersymmetry algebra:

$$\left\{Q_i\,,\,\bar{Q}_j\right\} = 2(\delta_{ij}\gamma^\mu P_\mu + i\mathrm{Im}Z_{ij} + i\gamma_5\mathrm{Re}Z_{ij}) \tag{2.1}$$

$$\left[Q_i\,,\,P_\mu\right] = 0\;;\;\left[Q_i\,,\,M_{\mu\nu}\right] = \frac{1}{2}\Sigma_{\mu\nu}Q_i\;;\;\left[Q_i\,,\,R\right] \propto (r)_{ij}Q_j. \tag{2.2}$$

The fermionic operators Q are fourcomponent spinors in a Majorana representation and have hence a nontrivial commutation relation with the generators of the Lorentz transformations M. The commutator of two supercharges is, apart from possible central charges Z, an infinitesimal translation P. In addition internal symmetry generators R (R-symmetry) can act on the supercharges.

The supercharges are represented on a multiplet of bosonic and fermionic fields. In some cases a superspace representation can be used. The space is then extended by fermionic coordinates and instead of the multiplet one has a superfield that depends on the coordinates of this extended space. The supercharges are linear combinations of space-time derivatives multiplied by Grassmann coordinates and derivatives with respect to the fermionic coordinates.

2.1.3 Supersymmetric actions

If all supersymmetry variations of a Lagrangian, $\delta\mathcal{L}$, are zero it must be constant. This follows from the fact that the commutator of two supercharges represents a derivative operator. A nontrivial invariant action is hence only possible for $\delta L = \partial^\mu K_\mu$. It can be obtained from a polynomial of superfields on which some operators commuting with the supercharges are applied. An integration over the fermionic coordinates generates a Lagrangian that transforms, as required, into a total derivative. This is a consequence of the basic rules of the Grassmann integration that acts in the same way as a derivative and squares to zero. So only the space-time derivative part of the supercharge remains. Such a Lagrangian corresponds to the highest component of a multiplet, which is also transformed into a total derivative.

The supersymmetry transformations are so far represented by linear operators. On the other hand most of the nontrivial interaction terms for fermionic and bosonic fields have a different form. They can only be related by nonlinear transformations. The reason for the linear representation is that the considered supersymmetric actions contain auxiliary fields. If these fields are replaced according to their classical equation of motion the transformations become nonlinear and the action obtains a more common form. So the auxiliary field can be seen as a linearisation method for the supersymmetry transformations.

2.1.4 Examples

The main subject of this work is the investigation of supersymmetric quantum field theories. As an illustration and for further investigations low dimensional models are used. In addition several aspects of general global symmetries in quantum field theories are covered. Especially with respect to the lattice calculations chiral symmetry provides a good nontrivial example and is introduced here in its classical form.

Chiral symmetry

In the classical, or tree, approximation the chiral symmetry is a simple example of an internal symmetry. A massless fermionic action term in four dimensions can be constructed according to

$$S[\bar{\psi},\psi] = \int d^4x\ \bar{\psi}(x)\not{\mathfrak{D}}\psi(x)\,. \tag{2.3}$$

The $\not{\mathfrak{D}}$ may contain also the dependence on a backround gauge field, A_μ. In this case $\mathfrak{D}_\mu = \partial_\mu + A_\mu$ and the usual gauge field term with the field strength $F_{\mu\nu}$ is added

$$S = \int d^4x \left(\bar{\psi}(x)\not{\mathfrak{D}}\psi(x) + \frac{1}{2g^2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}\right)\,. \tag{2.4}$$

It is easy to prove that the action is invariant under the transformations generated by γ_5 (with $\psi \to \psi' = e^{-i\epsilon\gamma_5}\psi$ and $\bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{-i\epsilon\gamma_5}$) since this matrix anticommutes with the gamma matrices. The invariance follows from[2]

$$S[\bar{\psi}',\psi'] = \int d^4x \bar{\psi}(x)e^{-i\epsilon\gamma_5}\not{\mathfrak{D}}e^{-i\epsilon\gamma_5}\psi(x) = S[\bar{\psi},\psi]\,. \tag{2.5}$$

The symmetry is broken when a mass term $\bar{\psi}m\psi$ is added to the Lagrangian.

A one-dimensional supersymmetric action

The one-dimensional equivalent of the Poincaré algebra contains only the translation into the time direction. A supersymmetry algebra in one dimension is

$$\left\{Q\,,\,\bar{Q}\right\} = 2P \quad \left[P\,,\,Q\right] = \left[P\,,\,\bar{Q}\right] = 0 \tag{2.6}$$

The one-dimensional (bosonic) superfield has the following expansion in the Grassmann coordinates θ $\bar{\theta}$

$$\Phi(t,\theta,\bar{\theta}) = \varphi + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta F\,. \tag{2.7}$$

[2] In Minkowski space this is in accordance with $\bar{\psi} = \psi^\dagger\mathcal{A}$ with $\mathcal{A} = \gamma_0$. In Euclidian space one treats the ψ and $\bar{\psi}$ as independent instead of using $\mathcal{A} \propto \mathbb{1}$ or $\mathcal{A} \propto \gamma_5$.

It contains the real bosonic field φ, the fermions ψ and $\bar{\psi}$ as well as the auxiliary field F. The algebra is represented on the superfields by

$$Q = i\partial_{\bar{\theta}} + \theta\partial_t\,, \quad \bar{Q} = i\partial_\theta + \bar{\theta}\partial_t\,, \quad P = i\partial_t\,, \tag{2.8}$$

which means that the components of the multiplet are transformed under the supersymmetry transformations according to

$$\delta\varphi = i\bar{\varepsilon}\psi - i\bar{\psi}\varepsilon\,, \quad \delta\psi = (\partial_t\varphi - iF)\varepsilon\,, \quad \delta\bar{\psi} = \bar{\varepsilon}(\partial_t\varphi + iF)\,, \quad \delta F = -\bar{\varepsilon}\partial_t\psi - \partial_t\bar{\psi}\varepsilon\,. \tag{2.9}$$

As explained a Lagrangian that transforms into a total derivative corresponds to the highest component of a superfield. A multiplication of superfields or an application of operators commuting with Q and $\bar{Q}$ on them yields again a superfield. In particular Q and $\bar{Q}$ anticommute with the covariant derivatives

$$D = i\partial_{\bar{\theta}} - \theta\partial_t\,, \quad \bar{D} = i\partial_\theta - \bar{\theta}\partial_t\,. \tag{2.10}$$

Obviously, these operators fulfil

$$\left\{D\,,\,\bar{D}\right\} = -2i\partial_t\,. \tag{2.11}$$

The projection onto the highest component corresponds to an integration over all Graßmann coordinates. The action is hence obtained from an integration over the whole superspace. In the present case we consider

$$\begin{aligned} S &= \int d\theta d\bar{\theta} dt \left[\frac{1}{2}\Phi(t,\theta,\bar{\theta})K\Phi(t,\theta,\bar{\theta}) + iW(\Phi(t,\theta,\bar{\theta}))\right] && (2.12)\\ &= \int dt \left(\frac{1}{2}(\partial_t\varphi)^2 - i\bar{\psi}\partial_t\psi + \frac{1}{2}F^2 + iFW'(\varphi) - i\bar{\psi}W''(\varphi)\psi\right)\,, && (2.13) \end{aligned}$$

where $W(\Phi)$ is a polynomial in Φ and $K = \frac{1}{2}(D\bar{D} - \bar{D}D)$. Since the auxiliary field F has only an algebraic equation of motion it can be replaced with the solution $F = -iW'$ to get the so-called on-shell action

$$S_{\text{on}} = \int dt \left(\frac{1}{2}(\partial_t\varphi)^2 - i\bar{\psi}\partial_t\psi + \frac{1}{2}W'(\varphi)^2 - i\bar{\psi}W''(\varphi)\psi\right)\,. \tag{2.14}$$

In this form one recognises easily the kinetic terms and the bosonic potential $\frac{1}{2}W'(\varphi)^2$ together with an additional Yukawa interaction between fermions and bosons. This re-

placement changes the transformation into a nonlinear form:

$$\delta\varphi = i\bar{\varepsilon}\psi - i\bar{\psi}\varepsilon\,, \quad \delta\psi = (\partial_t\varphi - W'(\varphi))\varepsilon\,, \quad \delta\bar{\psi} = \bar{\varepsilon}(\partial_t\varphi + W'(\varphi))\,. \tag{2.15}$$

Note that without the i in iFW' one would need a term $-\frac{1}{2}F^2$ to arrive at the same on-shell action. With a real F such a term in an Euclidean action is unbounded from below and the solution of the equations of motion corresponds to a maximum of the potential. This situation is quite generic for an Euclidean off-shell action in a supersymmetric theory. In the present work the introduction of the auxiliary field is, however, treated as a linearisation method for the transformations and the physical situation is described by the on-shell action. Consequently the (unphysical) auxiliary field can be complex. Whenever a term like $-F^2$ or $-|F|^2 = -F\bar{F}$ appears in the action F and $\bar{F}$ are supposed to be replaced with iF and $i\bar{F}$ and the field equations then correspond to a minimum of the potential. In a path integral the transition from the off-shell to the on-shell action is done by an integration of the auxiliary field. This integration is exact since the field appears only quadratically in the action.

For an explanation of further conventions used in later chapters note the factor $-i$ in front of the fermionic contribution can be neglected. It introduces just an irrelevant constant factor in the path intergal. Thus the action

$$S_{\text{on}} = \int dt \mathcal{L}_{on} = \int dt \left(\frac{1}{2}(\partial_t\varphi)^2 + \bar{\psi}\partial_t\psi + \frac{1}{2}W'(\varphi)^2 + \bar{\psi}W''(\varphi)\psi\right)\,, \tag{2.16}$$

with the supersymmetry transformations

$$\delta\varphi = \bar{\varepsilon}\psi + \bar{\psi}\varepsilon\,, \quad \delta\psi = (\partial_t\varphi - W'(\varphi))\varepsilon\,, \quad \delta\bar{\psi} = -\bar{\varepsilon}(\partial_t\varphi + W'(\varphi))\,, \tag{2.17}$$

describes the same physical situation. The Hamiltonian of this model can be easily derived by a Legendre transformation of $\mathcal{L}_{on}$, (2.16). It is

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}W'(\varphi)^2 + \bar{\psi}\psi W''(\varphi)\,. \tag{2.18}$$

π is the conjugated momentum to φ.

$\mathcal{N} = 2$ Wess-Zumino model in two Euclidean dimensions

For the two-dimensional $\mathcal{N} = 2$ Wess-Zumino model two different formulations are used in the present work. The first one is a complex formulation with the spinors in a Weyl-representation ($\gamma_0 = \sigma_1$, $\gamma_1 = -\sigma_2$, $\gamma_* = i\gamma_0\gamma_1 = \sigma_3$). With these matrices one can

introduce derivatives in complex space

$$\frac{1}{2}\not{\partial} = \begin{pmatrix} 0 & \frac{1}{2}(\partial_0 + i\partial_1) \\ \frac{1}{2}(\partial_0 - i\partial_1) & 0 \end{pmatrix} := \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} \tag{2.19}$$

The complex two-component spinor fields of this model can be decomposed according to $\psi = (\psi_1, \psi_2)^T$ and $\bar{\psi} = (\bar{\psi}^1, \bar{\psi}^2)$. The bosonic field ϕ and the auxiliary field F are both complex and the off-shell supersymmetry transformations are

$$\delta\phi = \bar{\psi}^1\varepsilon_1 + \bar{\varepsilon}_1\psi^1, \qquad \delta\bar{\psi}^1 = -\frac{1}{2}F\bar{\varepsilon}_1 - \partial\phi\bar{\varepsilon}_2, \qquad \delta\psi^1 = -\frac{1}{2}F\varepsilon_1 + \bar{\partial}\phi\varepsilon_2, \tag{2.20}$$

$$\delta\bar{\phi} = \bar{\psi}^2\varepsilon_2 + \bar{\varepsilon}_2\psi^2, \qquad \delta\bar{\psi}^2 = -\bar{\partial}\bar{\phi}\bar{\varepsilon}_1 - \frac{1}{2}\bar{F}\bar{\varepsilon}_2, \qquad \delta\psi^2 = \partial\bar{\phi}\varepsilon_1 - \frac{1}{2}\bar{F}\varepsilon_2. \tag{2.21}$$

with the invariant action

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\bar{\phi} - \frac{1}{2}|F|^2 + \frac{1}{2}F\,W' + \frac{1}{2}\bar{F}\,\bar{W}' \right. \\ \left. + \bar{\psi}(\not{\partial} + W''P_+ + \bar{W}''P_-)\psi\right) , \tag{2.22}$$

with the chiral projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_*)$ and a holomorphic superpotential $W(\phi)$. The corresponding on-shell theory has the action

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\bar{\phi} + \frac{1}{2}|W'|^2 + \bar{\psi}(\not{\partial} + W''P_+ + \bar{W}''P_-)\psi\right) . \tag{2.23}$$

and the transformations are

$$\delta\phi = \bar{\psi}^1\varepsilon_1 + \bar{\varepsilon}_1\psi^1, \qquad \delta\bar{\psi}^1 = -\frac{1}{2}\bar{W}'\bar{\varepsilon}_1 - \partial\phi\bar{\varepsilon}_2, \qquad \delta\psi^1 = -\frac{1}{2}\bar{W}'\varepsilon_1 + \bar{\partial}\phi\varepsilon_2, \tag{2.24}$$

$$\delta\bar{\phi} = \bar{\psi}^2\varepsilon_2 + \bar{\varepsilon}_2\psi^2, \qquad \delta\bar{\psi}^2 = -\bar{\partial}\bar{\phi}\bar{\varepsilon}_1 - \frac{1}{2}W'\bar{\varepsilon}_2, \qquad \delta\psi^2 = \partial\bar{\phi}\varepsilon_1 - \frac{1}{2}W'\varepsilon_2. \tag{2.25}$$

For this work the W' is chosen to be $m\phi + g\phi^2$. A shift of the bosonic field makes the additional symmetries of the model more explicit: $W'(\phi) = W'(\tilde{\phi} - \frac{m}{2g}) = g\tilde{\phi}^2 - \frac{m^2}{4g}$. Thus at $m = 0$ the bosonic potential $\frac{1}{2}|W'(\tilde{\phi} - \frac{m}{2g})|^2$ has a $U(1)$ ($\tilde{\phi} \to e^{i\alpha}\tilde{\phi}$) and a Z_2^P ($\tilde{\phi} \to \bar{\tilde{\phi}}$) symmetry that is broken down to $Z_2^R \times Z_2^P$ ($Z_2^R : \tilde{\phi} \to -\tilde{\phi}$) if m is nonzero. In the fermionic part the chiral rotation ($\psi \to e^{i\gamma_*\alpha/2}\psi$; $\bar{\psi} \to \bar{\psi}e^{i\gamma_*\alpha/2}$) or, respectively, the discrete transformation ($\psi \to i\gamma_*\psi$; $\bar{\psi} \to \bar{\psi}i\gamma_*$) compensates the $U(1)$ or Z_2^R.[3] To compensate the Z_2^P transformation in the fermionic part the fermions are changed according to $\psi \to \gamma_0\psi$ and $\bar{\psi} \to \bar{\psi}\gamma_0$. In addition a parity inversion ($x_1 \to -x_1$) has to be applied.[4] The whole action is hence invariant under the combination of these

[3]Note that in these transformations ψ and $\bar{\psi}$ were treated as independent fields.

[4]The Z_2^P transformation is a parity inversion where the imaginary part of ϕ is an axial field.

bosonic and fermionic transformations.

The complex formulation is in particular useful for analytic calculations. For the lattice simulations a real formulation is much easier to handle. The physical properties are the same as in the above complex case. For the matrices the Majorana representation ($\gamma_0 = \sigma_3$, $\gamma_1 = \sigma_1$, $\gamma_* = -\sigma_2$) is chosen and the complex field is decomposed into two real fields ($\phi = \varphi_1 + i\varphi_2$). The action becomes in this formulation

$$S = \int d^2x \left(\frac{1}{2}(\partial_\mu \varphi_1 \, \partial^\mu \varphi_1 + \partial_\mu \varphi_2 \, \partial^\mu \varphi_2) + \frac{1}{2} V(\varphi_1, \varphi_2) \right. \\ \left. + \bar{\psi}(\not{\partial} + 2g\varphi_1 \mathbb{1} + 2gi\gamma_* \varphi_2)\psi \right) \quad (2.26)$$

with the bosonic potential,

$$V(\varphi_1, \varphi_2) = \left(m^2 + 2mg\varphi_1 + g^2(\varphi_1^2 + \varphi_2^2)\right)(\varphi_1^2 + \varphi_2^2)\,, \quad (2.27)$$

that has two minima, $(\varphi_1, \varphi_2) = (0, 0)$ and $(\varphi_1, \varphi_2) = (-\frac{m}{g}, 0)$, and is invariant under $\varphi_1 \to -\varphi_1 - \frac{m}{g}$ (Z_2^R) and $\varphi_2 \to -\varphi_2$ (Z_2^P). The corresponding fermionic transformations are the same as mentioned above, apart from the different representation of the gamma matrices.

2.2 Symmetries of the effective action

So far the symmetries of the classical actions in quantum field theory were discussed. The physical situation is characterised by expectation values of certain operators (observables). The classical action determines the dynamic of the classical fields. The dynamic of the quantum observables is determined by the Hamiltonian. In a quantum field theory the considered observables are combinations of field operators and are related to the n-point functions. The generating functional of all connected one particle irreducible n-point functions is the effective action. The effective action contains the information of the quantum system. In this chapter I derive some basic quantities and relations for the description of the quantised system.

For convenience I have to explain some conventions: The fields φ, ϕ, ϕ_s used in this chapter represent multiplets containing bosonic or fermionic fields. The sources j are also assumed to be composed of sources for each of the multiplet components. In a product the summation over all indices (if doubly encountered or unspecified) and an integration over unspecified spatial components is assumed, e. g. $j\varphi = j^i\varphi^i = \sum_i \int d^{\mathrm{D}}y \, j^i(y)\varphi^i(y)$. A prime at a functional denotes the functional differentiation with respect to the argument ($S'[\varphi]^i(x) = S'[\varphi^i(x)] = \frac{\delta S[\varphi]}{\delta \varphi^i(x)}$). For fermionic fields the functional derivative are first

applied from the left then from the right and so on[5].

2.2.1 Observables, the effective action, and the path integral

Schwinger functional and the path integral

In quantum field theory the important observables are correlation functions. They are connected, for example, with the correlations of in and out states in a scattering process. Under certain assumptions a Wick rotation from the Minkowski space to a Euclidian theory is possible, where the correlation functions can be derived from the path integral using the (Euclidian) classical action S ($\mathcal{T}$ is the usual time ordering operator):

$$\begin{aligned}\langle\varphi^{i_1}(t_1,y_1)\varphi^{i_2}(t_2,y_2)\ldots\rangle &:= \langle 0|\mathcal{T}\hat{\varphi}^{i_1}(t_1,y_1)\hat{\varphi}^{i_2}(t_2,y_2)\ldots|0\rangle_E \\ &= \frac{1}{\mathcal{Z}[0]}\int \mathcal{D}\varphi\,\varphi^{i_1}(t_1,y_1)\varphi^{i_2}(t_2,y_2)\ldots e^{-S[\varphi]},\ \text{where } \mathcal{Z}[j]=\int\mathcal{D}\varphi\, e^{-S[\varphi]+j\varphi}\,. \quad (2.28)\end{aligned}$$

$Z[j]$ is the generating functional of these correlation functions. In the same way $W[j]$, the so-called Schwinger functional, is the generating functional of the connected part of these functions. It is defined as the logarithm of $\mathcal{Z}[j]$:

$$W[j]=\log\int\mathcal{D}\varphi\, e^{-S[\varphi]+j\varphi}=\log\mathcal{Z}[j]\,. \quad (2.29)$$

The effective action

The Legendre transform of the Schwinger functional is the effective action, Γ. For a functional this transformation is done according to

$$\Gamma[\phi]=\sup_j\left(j\phi-W[j]\right)=j[\phi]\phi-W[j[\phi]]\,, \quad (2.30)$$

where the supremum of the expression is formally obtained with $j[\phi]$ from the inversion of

$$\phi[j]=\frac{\delta W[j]}{\delta j}=\langle\varphi\rangle_j\,. \quad (2.31)$$

Since $W[j]$ is convex the Legendre transformation can be inverted.

A functional derivative of the generating functional with respect to the field ϕ yields the quantum equations of motion

$$\frac{\delta\Gamma}{\delta\phi(x)}=-\int d^{\mathrm{D}}y\frac{\delta W[j]}{\delta j[\phi(y)]}\frac{\delta j[\phi(y)]}{\delta\phi(x)}+\int d^{\mathrm{D}}y\;\phi(y)\frac{\delta j[\phi(y)]}{\delta\phi(x)}+j[\phi(x)]=j[\phi(x)] \quad (2.32)$$

The extremum of the effective action is hence a solution of $j[\phi(x)]=0$ and because of

[5] e. g. $S''[\varphi]^{ij}(x,y)=\frac{\delta^2 S[\varphi]}{\delta\varphi^i(x)\delta\varphi^i(y)}:=\overrightarrow{\frac{\delta}{\delta\varphi^i(x)}}S[\varphi]\overleftarrow{\frac{\delta}{\delta\varphi^i(y)}}$

equation (2.31) ϕ is at this point equal to $\langle\varphi\rangle$.

The second derivative of the effective action can be calculated from equation (2.31) and (2.32) to be

$$\frac{\delta^2 W}{\delta j^i(x_1)\delta j^j(x_2)}\Big|_{j=j[\phi]} = \frac{\delta\phi^i[j](x_1)}{\delta j^j(x_2)} = \left(\frac{\delta^2\Gamma}{\delta\phi^i\delta\phi^j}\right)^{-1}(x_1,x_2)\,. \tag{2.33}$$

This expression shows that, if $\phi = \langle\varphi\rangle$ ($j = 0$), the second derivative of Γ is the connected two-point function. More generally the effective action serves also as a generating functional. Higher derivatives at the point $\phi = \langle\varphi\rangle$ (the minimum of Γ) generate the (connected) one particle irreducible (1PI) correlation functions.

From the path integral definition of the Schwinger functional one can derive a path integral form of the effective action[6]

$$e^{-\Gamma[\phi]} = e^{W[j[\phi]]-j[\phi]\phi} = \int \mathcal{D}\varphi\, e^{-S[\varphi]+j[\phi](\varphi-\phi)} = \int \mathcal{D}\varphi\, e^{-S[\varphi+\phi]+\Gamma'[\phi]\varphi}\,. \tag{2.34}$$

This is, however, a complicated equation that contains a functional differentiation and integration.

Now I derive an expression for the differences between the quantum effective action and the classical action. Similar expressions were found in [18, 19]. To that end we reorder the terms in equation (2.34) and obtain

$$e^{-(\Gamma[\phi]-S[\varphi_s]-S'[\varphi_s](\phi-\varphi_s))} = \int \mathcal{D}\varphi\, e^{-(S[\varphi+\varphi_s]-S[\varphi_s]-S'[\varphi_s]\varphi)+(\Gamma'[\phi]-S'[\varphi_s])(\varphi+\varphi_s-\phi)} \tag{2.35}$$

This equation can be reformulated in terms of a new effective action Γ_{φ_s} defined by $\Gamma_{\varphi_s}[\phi-\varphi_s] := \Gamma[\phi] - S[\varphi_s] - S'[\varphi_s](\phi-\varphi_s)$:

$$e^{-\Gamma_{\varphi_s}[\phi-\varphi_s]} = \int \mathcal{D}\varphi\, e^{-S_{\varphi_s}[\varphi]+\Gamma'_{\varphi_s}[\phi-\varphi_s](\varphi+\varphi_s-\phi)}\,. \tag{2.36}$$

Hence Γ_{φ_s} is the effective action for the following classical action S_{φ_s} (derivative action[7]),

$$S_{\varphi_s}[\varphi] := S[\varphi+\varphi_s] - S[\varphi_s] - S'[\varphi_s]\varphi\,. \tag{2.37}$$

This action represents a Taylor expansion of $S[\varphi]$ around $\varphi = \varphi_s$ without the first and second contribution. If φ_s is constant it roughly resembles the original action with modified coupling constants depending on φ_s. , For $\varphi_s = \phi$ the definition of Γ_{φ_s} simplifies

[6] In the path integral the invariance under a shift $\varphi \to \varphi + \varphi_s$ is assumed.
[7] This notion corresponds to the derivative Lagrangian defined in [20].

to

$$\Gamma[\phi] = S[\phi] + \Gamma_\phi[0]\,. \tag{2.38}$$

This formula states that the difference between the classical and the effective action can be derived from a modified theory with field dependent coupling constants. This theory is evaluated at a stationary point ($S'_{\varphi_s}[0] = 0$). This point is a classical minimum if $S''_{\varphi_s}[0] = S''[\varphi_s] \geq 0$. When also $\Gamma_{\varphi_s}[\phi]$ has its minimum at $\phi = 0$ equation (2.38) means that the difference between classical and effective action are the (1PI) vacuum correlation functions of S_ϕ.

The renormalisation group flow for the effective action

In the path integral all possible fluctuations contribute. The idea of the renormalisation group flow is a successive integration of the fluctuations. In that way an interpolation between the classical and the effective action can be obtained. To that end we include an additional contribution, the regulator part S_k, in the path integral. It depends on a parameter k. Subtracting the classical effects of this contribution we define the following effective action

$$\Gamma_k[\phi] = \sup_j(j\phi - W[k,j]) - S_k[\phi] = \Gamma[k,\phi] - S_k[\phi]\,,$$
$$\text{with } W[k,j] = \log \int \mathcal{D}\varphi\, e^{-S[\varphi]-S_k[\varphi]+j\varphi}\,. \tag{2.39}$$

$\Gamma[k,\phi]$ is the effective action of the regularised theory (regularised effective action). It is calculated with a regularised version of the classical action $S_R := S + S_k$. In contrast to the regularised effective action, $\Gamma_k[\phi]$ is no longer convex. With these definitions at hand, we derive from equation (2.34)

$$\begin{aligned} e^{-\Gamma_k[\phi]} &= \int \mathcal{D}\varphi\, e^{-S[\varphi+\phi]+\frac{\delta\Gamma[k,\phi]}{\delta\phi}\varphi - S_k[\varphi+\phi]+S_k[\phi]} \\ &= \int \mathcal{D}\varphi\, e^{-S[\varphi+\phi]+\frac{\delta\Gamma_k}{\delta\phi}\varphi} e^{-(S_k[\varphi+\phi]-S'_k[\phi]\varphi-S_k[\phi])}\,. \end{aligned} \tag{2.40}$$

S_k should fulfil the following conditions:

(1) If k approaches a cutoff Λ the factor $e^{-(S_k[\varphi+\phi]-S'_k[\phi]\varphi-S_k[\phi])}$ should resemble $\delta[\varphi]$ in function space. This means that $\lim_{k\to\Lambda}\Gamma_k[\phi] = S[\phi]$. Hence the classical action is obtained at $k = \Lambda$. Λ is assumed to be large, $\Lambda \to \infty$.

(2) When k approaches zero S_k should vanish. Consequently we get the effective action in this limit, $\lim_{k\to 0}\Gamma_k[\phi] = \Gamma[\phi]$.

In this way Γ_k interpolates between the classical and the quantum effective action.

Differentiating W_k with respect to the parameter k yields

$$\partial_k W[k,j] = -\frac{\int \mathcal{D}\varphi\,(\partial_k S_k[\varphi]) e^{-S[\varphi]-S_k[\varphi]+j\varphi}}{\mathcal{Z}_k[j]} = -\langle(\partial_k S_k[\varphi])\rangle_j\,. \tag{2.41}$$

The right hand side can be expressed in terms of connected correlation functions. These are functional derivatives of W_k. The result is hence an equation containing only functional derivatives of W_k and derivatives with respect to k but no path integral. We want a similar equation for the flow of the effective action. From (2.30) we immediately conclude that if W and Γ depend on an additional parameter k the relation $\partial_k\Gamma[\phi] = -\partial_k W[j[\phi]]$ holds. In the definition of Γ_k the part S_k of the classical contribution was subtracted. Therefore we get

$$\partial_k\Gamma_k[\phi] = \partial_k\langle S_k[\varphi]\rangle_{j[\phi]} - \partial_k S_k[\phi]\,. \tag{2.42}$$

The correlation functions on the right hand side can in this case be expressed in terms of 1PI correlation functions. These are obtained in a functional differentiation of Γ_k+S_k. A particularly simple case is if S_k is just a quadratic functional, $S_k[\varphi] = \int d^{\mathrm{D}}p\varphi(-p)R_k(p)\varphi(p)$. One obtains

$$\begin{aligned}\partial_k\Gamma_k[\phi] &= \int d^{\mathrm{D}}p\left(\partial_k R_k(p)\langle\varphi(-p)\varphi(p)\rangle_{j[\phi]} - \phi(-p)\partial_k R_k(p)\phi(p)\right)\\ &= \int d^{\mathrm{D}}p\partial_k R_k(p)\langle\varphi(-p)\varphi(p)\rangle_{j[\phi],1\mathrm{PI}} = \frac{1}{2}\mathrm{Str}\left(\partial_k R_k(\Gamma_k''[\phi]+R_k)^{-1}\right)\end{aligned} \tag{2.43}$$

(the Str includes also the integration of the momentum p). This equation for the flow of the effective action is sometimes called Wetterich equation [21]. It defines a renormalisation group flow for the effective action. A solution yields for a given classical action at $k=\Lambda$ the corresponding effective action at $k=0$. As in equation (2.38) $\Gamma[k,\phi]$ can be derived from the vacuum correlation functions of the derivative action derived from $S[\varphi]+S_k[\varphi]$. The expressions of (2.43) are nothing but a differentiation of these vacuum correlation with respect to k:

$$\partial_k\Gamma_k[\phi] = \partial_k(\Gamma[k,\phi]-S_k[\phi]) = \partial_k(S[\phi]+S_k[\phi]+\Gamma_\phi[k,0]-S_k[\phi]) = \partial_k\Gamma_\phi[k,0]\,. \tag{2.44}$$

Expansions of the effective action for supersymmetric theories

The discussion of certain expansion schemes of the effective action of a supersymmetric theory is necessary to find, later on, a suitable expansion scheme for the flow equations of the effective action. The applicability of these different schemes is also discussed in the context of the loop expansion.

The effective action is a functional of all fields and their derivatives. The concrete form of this action is restricted by Lorentz invariance. Therefore not all combinations of

the derivatives and the fields are allowed. An appropriate expansion of the effective action is according to the order of the derivative operators (momentum). For a scalar bosonic theory this leads to

$$\Gamma[\phi, \partial\phi, \partial^2\phi, \ldots] = \int d^{\mathrm{D}}x U(\phi(x)) + \int d^{\mathrm{D}}x (\partial_\mu \phi^i)(\partial^\mu \phi^j) Z_{ij}(\phi) + \ldots , \tag{2.45}$$

and the fermionic case is similar with a $\not{\partial}$-term as the first order. The zeroth order term U corresponds to the effective potential. It is defined in the limit of constant fields ($\phi(x) = \phi = \text{const.}$) with an additional division by the volume Ω. This limit (and the division) is denoted here for all quantities comparable to the effective potential by small letters, like $u = \lim_{\phi(x)\to\phi} U[\phi(x)]/\Omega$. u plays an important role in the considerations of symmetries.

In a superspace representation of a supersymmetric action superfields and their covariant derivatives appear. For the off-shell effective action the expansion in terms of covariant derivatives is hence an alternative to the above expansion scheme. It leads to[8]

$$\Gamma[\Phi, D\Phi, \bar{D}\Phi, D\bar{D}\Phi \ldots] = i \int d^D z W_{\text{eff}}(\Phi(z)) + \int d^D z DZ(\Phi)\bar{D}Z(\Phi) + \ldots . \tag{2.46}$$

The zeroth order of this expansion is the effective superpotential W_{eff}. For the classical action the separation into a part $S_{S,k}$, quadratic in the covariant derivatives, and the superpotential W, containing no covariant derivatives, is done according to such an expansion. For a comparison with U, U_S is defined as W_{eff} after the integration of the Grassmann coordinates.[9]

The zeroth order of the expansion (2.45) contains arbitrary powers of the auxiliary field. Hence it is invariant only under the zero-mode supersymmetry transformations, obtained when the derivative terms in the transformations are neglected. The auxiliary field is, itself, invariant under these transformations. In the expansion in terms of covariant derivatives (2.46), on the other hand, each contribution is supersymmetric. In other words: In an expansion of the effective action that is term by term supersymmetric the higher orders in the auxiliary field should be treated on the same level as the derivatives of the other fields.

Similarly, in the derivative expansion of the on-shell theory, each order is, itself, not supersymmetric. The first term of the expansion (2.45) for the on-shell theory U_{on} is invariant under the on-shell version of the zero-mode supersymmetry; but not under the complete symmetry. To arrive at U_{on} for a given U, the equation $\frac{\delta\Gamma}{\delta F} = 0$ must be solved to eliminate the auxiliary field. (This is the same as setting $j_F = 0$.) All orders in the

[8] z denotes the coordinates of the superspace. The i is inserted for a comparison with our conventions in supersymmetric quantum mechanics.

[9] For supersymmetric quantum mechanics $u_S = iFW'_{\text{eff}}(\varphi)$.

auxiliary field are relevant in this step. On the other hand, to arrive at the on-shell counterpart of U_S, here called $U_{S,\mathrm{on}}$, only a solution of linear equations of motion for the auxiliary field is necessary. These are obtained when the classical part $S_{S,k}$ is added to U_S. $U_{S,\mathrm{on}}$ is an approximation of the on-shell effective potential U_{on}, where higher orders of the auxiliary field are neglected.

2.2.2 Symmetries and Ward identities

With the help of the Schwinger functional the results of a symmetry transformation on the correlation functions can be investigated in a quite general context. The path integral with a transformed field φ' must be the same as the one with φ since it is a mere redefinition of the field. This holds true even if the two fields are connected by a continuous transformation, $\varphi' = e^{\varepsilon\mathcal{M}}\varphi$. For an infinitesimal symmetry transformation this means [10]

$$Z[j] = \int \mathcal{D}\varphi'\, e^{-S[\varphi']+j\varphi'} = \int \mathcal{D}\varphi\ \mathrm{Sdet}\left(e^{\varepsilon\mathcal{M}}\right) e^{-S[\varphi]-\varepsilon\delta S[\varphi]+j(\varphi+\varepsilon\delta\varphi)} \approx$$
$$\approx Z[j] + \varepsilon \int \mathcal{D}\varphi\,(\mathrm{Str}\mathcal{M} - \delta S[\varphi] + j\delta\varphi) e^{-S[\varphi]+j\varphi}\,. \quad (2.47)$$

Because of this elementary identity the following expression must vanish

$$\langle(\mathrm{Str}\mathcal{M} - \delta S[\varphi] + j\delta\varphi)e^{j\varphi}\rangle = 0\,. \quad (2.48)$$

Suppose the first term does not contribute and the action is invariant under the transformation. A functional differentiation with respect to j at $j = 0$ leads to a set of relations between correlation functions called Ward-Takahashi identities, e. g.

$$\langle\delta\varphi\rangle = 0\,; \quad \langle\varphi(\delta\varphi) + (\delta\varphi)\varphi\rangle = 0 \ \ldots \quad (2.49)$$

These relations are the consequences of the symmetry on the level of the correlations functions, i. e. the observables. Form the above expression, (2.48), one can also derive the consequences for the effective action setting $j = j[\phi]$ and using the quantum equations of motion:[11]

$$\frac{\delta\Gamma}{\delta\phi}\langle\delta\varphi\rangle_{j[\phi]} = \frac{\delta\Gamma}{\delta\phi}\mathcal{M}\phi = \langle(\mathrm{Str}\mathcal{M} - \delta S[\varphi]\rangle_{j[\phi]}\,. \quad (2.50)$$

This equation shows that, provided the right hand side of the equation vanishes, the effective action and not only the classical action is invariant under the symmetry. It is then called a Slavnov-Taylor identity [22].

[10] A definition of the superdeterminant, Sdet, and its relation to the supertrance, Str, can be found in the appendix A.4.

[11] Note that the first equality applies for linear symmetries only.

2.2.3 Symmetries in the renormalisation group flow

In the Slavnov-Taylor-identities, (2.50), one gets an additional contribution if the term S_k in the flow equations (2.43) is not invariant under the symmetry,

$$\frac{\delta\Gamma}{\delta\phi}\langle\delta\phi\rangle_{j[\phi]} = \frac{\delta\Gamma}{\delta\phi}\mathcal{M}\phi = \frac{\delta\Gamma_k}{\delta\phi}\mathcal{M}\phi + \frac{\delta S_k[\phi]}{\delta\phi}\mathcal{M}\phi = \langle -\delta S_k[\varphi] - \delta S[\varphi] + \mathrm{Str}\mathcal{M}\rangle_{j[\phi]}\,. \quad (2.51)$$

The symmetry of initial and endpoint of flow, i. e. the classical and effective action, is not affected. Instead of the explicit breaking by a non-invariant action or path integral measure the term $\langle\delta S_k[\varphi]\rangle_{j[\phi]}$ represents a mild breaking of the symmetries. It is called a modified Slavnov-Taylor identity and has been studied, e. g., in [23, 24, 25]. This applies for the exact solution of the renormalisation group flow.

2.2.4 Symmetries, divergences and regulators: anomalies

The path integral as defined above is not in all cases a well-defined expression. The integration involves field configurations with large fluctuations and high momentum modes. These configurations can lead to divergent terms. This is not a mere problem of the path integral formulation. It also appears in the operator formalism. There divergences are introduced in terms of singular expressions (δ distributions) in the (anti)commutation relations of the field operators after the canonical quantisation. A term $\langle\mathrm{Str}\mathcal{M}\rangle = \langle\int d^n x\,\mathrm{str}\mathcal{M}\delta(x-x)\rangle$, (2.47), is in the same sense not well defined. In most cases the trace over the field indices of the matrix $\mathcal{M}$ vanishes but an additional divergent factor appears. To evaluate this expression it has to be regulated. This time the regulator is not merely a tool for the successive integration of the configurations in the path integral. Eventually the result should not depend on the choice of regulator. The dependence is absorbed into some parameters of the theory in a renormalisation procedure. These parameters must be fixed by some physical input. Hence the predictive power of the theory is lost in favour of the cancellation of the divergences.

Although the dependence on the regulator is eventually removed it has still consequences for the symmetries. For the removal of the divergences the regulator has to be chosen according to the symmetry and invariance of the theory. The regulator can, however, not always be chosen in accordance with all the desired properties of the theory. One example is a chiral symmetric theory that should at the same time be gauge invariant. For example with the gauge invariant regulator $\not{\mathfrak{D}}^2/m^2$ ($m\to\infty$) one has, cf. [26],

$$\langle\mathrm{Str}\mathcal{M}\rangle \to \lim_{m\to\infty}\langle\int\frac{d^4p}{(2\pi)^4}\gamma_5 e^{-\not{\mathfrak{D}}^2(p)/m^2}\rangle = \langle\frac{-1}{16\pi^2}\int d^4x\mathrm{tr}\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x)\rangle$$
$$= \langle n_+ - n_-\rangle\,. \quad (2.52)$$

The last expression contains the index, the number of fermionic zero modes with positive (n_+) or negative (n_-) chirality for a given gauge backround. This anomaly appears also for other regulators that respect the gauge invariance [27].

2.3 Methods of quantum field theory

The above derivations are all exact relations for the effective action and the quantum observables. Unfortunately a direct evaluation of these quantities is in most cases not possible. One has to rely on certain approximative calculations. I sketch in the following basic aspects of different approaches for such calculations.

2.3.1 Feynman diagrams

A well-known method for the calculation of correlation functions are Feynman diagrams. The action is separated into a dominant part S_0 and a perturbation, λS_λ with a small parameter λ, usually the coupling constant. The approach is basically a formal expansion in the coupling constant. The generating functional $\mathcal{Z}[j]$ is represented in the following way

$$\begin{aligned}\mathcal{Z}[j] &= \int \mathcal{D}\varphi\, e^{-S_0[\varphi]-\lambda S_\lambda[\varphi]+j\varphi} = e^{-\lambda S_\lambda[\frac{\delta}{\delta j}]} \int \mathcal{D}\varphi\, e^{-S_0[\varphi]+j\varphi} := e^{-\lambda S_\lambda[\frac{\delta}{\delta j}]}\mathcal{Z}_0[j] \\ &\approx \left(1-\lambda S_\lambda\left[\frac{\delta}{\delta j}\right]+\frac{\lambda^2}{2}(S_\lambda\left[\frac{\delta}{\delta j}\right])^2+\ldots\right)\mathcal{Z}_0[j] \qquad (2.53)\end{aligned}$$

In the weak coupling expansion the dominant part is the quadratic free theory. $\mathcal{Z}_0$ is therefore given as

$$\begin{aligned}\mathcal{Z}_0[j] = \int \mathcal{D}\varphi\, e^{-\frac{1}{2}\int d^Dx\, \varphi(x)K\varphi(x)+j\varphi} = \operatorname{Sdet}\mathcal{K}\, e^{\frac{1}{2}\int d^Dx_1 d^Dx_2\, j(x_1)G_0(x_1-x_2)j(x_2)} \\ \text{with } (\mathcal{K}G_0)(x_1-x_2)=\delta(x_1-x_2)\,. \qquad (2.54)\end{aligned}$$

The Feyman rules are a well-known graphical representation of this weak coupling expansion. The application of $S\left[\frac{\delta}{\delta j}\right]$ is represented by vertices. The lines represent propagators of free particles generated by the differentiation of $\mathcal{Z}_0[j]$. In this representation a 1PI correlation function is a diagram that cannot be decomposed into two separate parts by cutting a single line. Hence an approximation of the 1PI correlation functions is derived in the weak coupling expansion. An approximation for the effective mass, the pole of the propagator in momentum space, is deduced from the (connected) 1PI two-point function

$\Sigma(p)$ and the free propagator G_0 since

$$G(p) = \underset{G_0(p)}{\text{—}} + \underset{G_0(p)\Sigma(p)G_0(p)}{\text{—●—}} + \underset{G_0(p)\Sigma(p)G_0(p)\Sigma(p)G_0(p)}{\text{—●—●—}} + \cdots$$

$$= G_0(p) \sum_{n=0}^{\infty} (\Sigma(p)G_0(p))^n = (\mathcal{K}(p) - \Sigma(p))^{-1} . \quad (2.55)$$

The effective action is the generating functional of the 1PI correlation functions, and an approximation of it can be reconstructed from this result. In fact, a common expansion of the effective action is (cf. [28])

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \ldots d^D x_n \Gamma^{(n)}(x_1, \ldots, x_n) \big(\phi(x_1) - \langle\varphi(x_1)\rangle\big) \cdots$$

$$\cdots \big(\phi(x_n) - \langle\varphi(x_n)\rangle\big) . \quad (2.56)$$

Each of these contributions can be calculated from the 1PI correlation functions: $\Gamma^{(1)} = 0$, $\Gamma^{(2)} = K + \Sigma$, etc. The whole derivation of the Feynman rules can also be done in superspace (for details cf. [29]).

2.3.2 The loop expansion

The loop expansion is a saddle point expansion of the path integral around the classical solutions of the field equations [28]. Such a solution minimises the classical action so it has a larger contribution to the path integral than other field configurations. The exponent of the path integral has a factor of $1/\hbar$ in front of the action and the source term. In natural units this factor is one, but a domination of the classical contribution can still be understood as the formal limit $\hbar \to 0$. $\hbar$ is the expansion parameter of the loop expansion. Effectively, the loop expansion corresponds to a re-summation of Feynman graphs up to a certain loop level with an arbitrary number of vertices.

With the help of (2.38) one derives another representation of the loop expansion. It is the diagrammatic expansion of the vacuum diagrams defined by S_ϕ. The first ($\hbar$) contribution of this expansion is the logarithm of $\operatorname{Sdet} K_\phi$ with the quadratic part K_ϕ of S_ϕ, see (2.54). The $\hbar^n$ contribution is a summation of 1PI vacuum diagrams with n loops. Note that for the representation in terms of vacuum graphs, it is necessary to have the minimum of $\Gamma_\phi[\tilde{\phi}]$ at $\tilde{\phi} = 0$. This is the reason why the loop expansion fails in a nonconvex region of the classical potential [18]. A physical interpretaion of this fact can be found in [30].

In the loop expansion the field ϕ is usually constant, so one derives an approximation of the effective potential. The approach can be applied in a more general situation, but the calculation of the vacuum diagrams is then more difficult. For constant ϕ the

ϕ dependent contributions of $S_\phi[\varphi]$ amounts to a (field dependent) modification of the coupling constants and masses.

Using the expansion in λ of the last section one can determine the effective action up to a certain order of the expansion 2.56 from the (1PI) diagrams. In this case the diagrams are calculated only up to a certain order of the coupling constant. In contrast, the loop calculation yields an approximation of the effective action that contains arbitrary orders of the fields. However, the (1PI) vertices of the diagrammatic calculations contain the full momentum dependence; not only the effective potential as obtained in the loop expansion.

2.3.3 Lattice calculations

The lattice calculation is an approximative evaluation of the path integral that is not based on a formal expansion. Instead of the continuous D-dimensional space-time a hypercubic lattice is considered. The lattice sites ($x_n = \sum_\mu n_\mu a_\mu e_\mu$; $n \in \Lambda^{\mathrm{D}} \subset \mathbb{Z}^{\mathrm{D}}$; n_μ the component of n in μ direction) are separated in μ direction by the distance a_μ. In this direction the number of these sites is N_μ. In the path integral the integration variables φ_n are fields on the sites.[12] They can be seen as continuum fields evaluated at these points. The action is discretised and becomes a function of φ_n configurations. In this way one arrives at the lattice action S_L. Derivative operators are replaced by discretised approximations of the continuum derivatives. The examples considered here are[13]

$$\begin{array}{ll} \text{symmetric derivative} & (\nabla_\mu^{(s)}\varphi)_n = \frac{1}{2a_\mu}(\varphi_{n+e_\mu} - \varphi_{n-e_\mu}) \\ \text{forward derivative} & (\nabla_\mu^{(+)}\varphi)_n = \frac{1}{a_\mu}(\varphi_{n+e_\mu} - \varphi_n) \\ \text{backward derivative} & (\nabla_\mu^{(-)}\varphi)_n = \frac{1}{a_\mu}(\varphi_n - \varphi_{n-e_\mu}) \\ \text{SLAC derivative} & (\nabla_\mu^{\mathrm{SLAC}}\varphi)_n = \sum_{m_\mu=0}^{N_\mu-1}(-1)^{n_\mu-m_\mu}\frac{\pi/(a_\mu N_\mu)}{\sin(\pi(n_\mu-m_\mu)/N_\mu)}\varphi_{n+e_\mu m_\mu} \end{array} \tag{2.57}$$

The SLAC derivative[14], [32, 33, 34], is derived from a discretisation of the Fourier space representation of the continuum derivative operator, $\nabla_\mu(p) = p_\mu$. After the discretisation in momentum space a lattice Fourier transformation yields $\nabla_\mu^{\mathrm{SLAC}}$. (For the conventions of the Fourier transformation on the lattice see A.3. The lattice momentum of one field is restriced to the first Brillouin zone (BZ) because of the discretisation.)

The classical continuum limit is determined by $a \to 0$ in the classical action. In this

[12] The (common) conventions (cf. [31]) for the lattice operators are explained in A.2. Throughout this work the number of lattice points N_μ is, for convenience, assumed to be odd in all directions μ. e_μ is the unit vector in μ direction. If not further specified, the sum of $\sum_n$ runs always over the whole lattice and contains a factor of the lattice spacing $\prod_\mu a_\mu$.

[13] $(\nabla_\mu\varphi)_n$ means $\sum_m(\nabla_\mu)_{nm}\varphi_m = \prod_{\mu=0}^{D-1} a_\mu \sum_{n_\mu=0}^{N_\mu-1}(\nabla_\mu)_{nm}\varphi_m$.

[14] This derivative is also-called DWY derivative (Drell, Weinstein, and Yankielowicz).

limit the discretised derivative operators reproduce the continuum derivative $(\nabla_\mu \varphi) \stackrel{a\to 0}{\longrightarrow} (\partial_\mu \varphi)(x_n)$. By construction, the lattice action turns into the continuum action in the classical continuum limit. Some basic properties of the action, like hermiticity, should be already realised at a finite lattice spacing. Therefore, the left and right derivative are only used in the hermitian combination $\nabla^{(+)}\nabla^{(-)}$.

Flinally, the path integral is approximated[15] by a large number of ordinary integrals in the lattice theory,

$$\langle \mathcal{O}(\varphi)\rangle = \frac{1}{\mathcal{Z}[0]}\int \mathcal{D}\varphi \mathcal{O}(\varphi) e^{-S[\varphi]} = \lim_{\substack{N\to\infty \\ a\to 0}} \langle \mathcal{O}(\varphi)\rangle_L \,,$$
$$\text{with } \langle \mathcal{O}(\varphi)\rangle_L = \frac{1}{\mathcal{Z}_L[0]}\int \prod_n^N d\varphi_n \mathcal{O}(\varphi_n) e^{-S_L[\varphi_n]}\,, \quad (2.58)$$

and $\mathcal{Z}_L[j] = \int \prod_n^N d\varphi_n e^{-S_L[\varphi_n]+\sum_n j_n\varphi_n}$. In general the N-dimensional integral cannot be performed analytically or by the numerical integration techniques like Simpson's method. Only Monte-Carlo simulations allow a numerical computation of it. These simulations generate successively, in a Markov chain, a large number N_{fc} of field configurations. With a certain update algorithm a new configuration is calculated from the previous one. If the conditions of detailed balance and ergodicity are fullfilled by the update algorithm, the configurations are eventually distributed according to the measure $\frac{e^{-S_L[\varphi_n]}}{\mathcal{Z}_L[0]}$ [35, 36]. Expectation values for observables can hence be calculated in terms of

$$\langle O(\varphi)\rangle_L = \lim_{N_{\text{fc}}\to\infty} \frac{1}{N_{\text{fc}}} \sum_n^{N_{\text{fc}}} O(\varphi_n)\,. \quad (2.59)$$

It must be stressed that in the definition of the lattice path integral, (2.58), the lattice action was constructed with a correct classical continuum limit. Due to the divergences in the quantum theory the classical continuum limit does, however, not guarantee the correct continuum limit for the observables. More precisely the transition from the continuum to the lattice theory consists of two steps. In the first one a finite volume, a hypercube with length $L_\mu = aN_\mu$ in μ direction, is considered instead of the infinite continuum.[16] The next step introduces the finite lattice spacing. This situation corresponds to a renormalisation group flow: the finite volume introduces infrared regulator for the modes; the finite lattice spacing an ultraviolet cutoff.[17] In the lattice theory the observables are measured. Eventually the regulator is removed: an extrapolation towards the limit of infinite volume

[15] As detailed later on the

[16] In this thesis periodic boundary conditions are applied for this volume, in accordance with supersymmetry.

[17] A precise definition for the renormalisation group step from the continuum to the lattice is given in chapter 5.

(thermodynamic limit) and vanishing lattice spacing (continuum limit) is performed. For this extrapolation a smooth behaviour with respect to this limit is needed.

Note that the complete configuration space is sampled only if N_{fc} tends to infinity. The finiteness of N_{fc}, like for a regulator or an additional term in the action, restricts the considered configurations in the path integral. Ideally the statistical error induced by this effect is of the order $\sqrt{N_{\mathrm{fc}}}^{-1}$ and hence very small. It can, however, be that certain domains of the configuration space with a low action and consequently a large contribution to the path integral are widely separated from each other. Then all of these contributions are sampled only for a very large N_{fc} (when a tunnelling between them occurs).

Symmetries are an essential property of quantum field theories. If possible, the symmetries should also be present in the lattice theory. In some cases the symmetries can not be realised in the same way as in the continuum. From the classical point of view, there are a number of possible representations of the symmetry on the lattice. All of these representations differ from each other by operators that vanish for $a \to 0$. However, not all of these lattice representations of the symmetry ensure also a symmetric quantum theory in the continuum limit. The classical continuum limit is altered by the divergent contributions of the quantum theory.

Another important property of the continuum field theory is its locality. Clearly the lattice action involves the interaction of fields separated by the lattice spacing and is not local in the continuum sense. The continuum locality must be recovered in the continuum limit. In the context of a lattice theory "local" refers to properties that ensure such a local continuum limit.

The strongest possible requirement to ensure the continuum locality is "ultralocality". That means that the interaction of fields maximally separated by a fixed number of lattice spacing occurs in the lattice action. This is in some cases a too strict requirement. For the usual condition of locality on the lattice the interaction strength has to decay at least exponentially with the separation of the lattice fields. The width of this exponential decay has to scale with the inverse lattice spacing. For a lattice operator this condition means analyticity and periodicity in momentum space.

From the classically point of view the locality is always ensured as the lattice action approaches the local continuum action in the classical continuum limit. This local classical continuum limit is only spoilt by the divergent quantum contributions. What kind of divergent contributions appear depends on the considered theory. For some of these theories even less restrictive requirements than the above mentioned lattice locality and ultralocality can ensure a local continuum limit. The mentioned lattice derivative operators (2.57) are all ultralocal except for the non-local SLAC derivative.

2.3.4 Flow equations for a truncated effective action

A different approach is based on the renormalisation group flow of the effective action, (2.43). Although this equation contains no path integral it can, in most cases, not be solved exactly. In addition, all possible operators can be included in the effective action. Thus it is difficult to find a proper representation or expansion of it. Sometimes the physical properties of the theory suggest a certain expansion of the effective action. As explained above, a derivative (or covariant derivative) expansion is a common representation, another one is shown in (2.56). To get an approximation of the renormalisation group flow, a certain expansion of the effective action is considered. The flow for the set of operators that appear in the truncation is calculated. This means that the contribution of every operator generated on the right hand side of (2.43) and not contained in the truncation is neglected. One generically arrives in this way at a set of coupled nonlinear (partial) differential equations. These can be solved with numerical methods.

The best way to respect the symmetries of the quantum field theory in these calculations is the application of a symmetric regulator. If in addition the truncation respects the symmetry, all necessary conditions for a symmetric result are fulfilled. When a symmetric regulator cannot be chosen, the modified Slavnov-Taylor identities (2.51) should be respected. These ensure a symmetric endpoint of the flow.

3 The lattice formulation of supersymmetry

This chapter is devoted to the formulation of a supersymmetric theory on the lattice. It is the first step towards the numerical calculation of quantum field theoretical observables. As we have seen the symmetries should, if possible, be realised on the lattice as well as in the continuum theory. Thus I first try to find a realisation of supersymmetric lattice theory. The first step for the formulation of the lattice theory, the restriction to a finite volume, can be easily done (except for spontaneously broken supersymmetry with massless Goldstone modes). Supersymmetry is respected as long as the periodic boundary conditions ensure the irrelevance of a total derivative term. The discretisation, the introduction of a UV-cutoff, is a more severe problem in a supersymmetric theory. The most obvious reason is the violation of the Leibniz rule. There a second, less severe, problem is associated with the so-called fermion doubling. To solve these problems, the nonlocal operator, like the SLAC derivative, are found to be useful. I investigate such nonlocal realisations afterwards in lattice perturbation theory. A correct local continuum limit is found for the models considered later on in the simulations.

3.1 The failure of the Leibniz rule and supersymmetry breaking on the lattice

The main source of supersymmetry breaking on the lattice is failure of the Leibniz rule for any discretised derivative operator. I discuss possible solutions of this problem in this section. The results of [37, 38, 39, 40] are generalised and compared with the requirements of the lattice simulations that are the subject of the next chapter. The discretisation of the supersymmetry transformation is done straight forwardly in this section: the same lattice derivative operator replaces the continuum derivative in the fermionic and bosonic kinetic part of the action as well as in the supersymmetry transformations. This derivative operator is assumed to be antisymmetric and translational invariant. The doubling problem of such an operator is discussed separately in section 3.2.

3.1.1 Locality and the Leibniz rule

A generic supersymmetric action is invariant up to a total derivative term. Such a term vanishes if the correct boundary conditions are fulfilled. This is true in the continuum, where one can use partial integration and the Leibniz rule. For periodic boundary conditions on the lattice the former can be applied[1] , whereas the latter is violated by any lattice derivative operator. Thus the basic source of the supersymmetry breaking on the lattice is the violation of the Leibniz rule. This violation appears when the product of continuum fields $\varphi^{(1)}(x)\varphi^{(2)}(x)$ is replaced by a product of fields at the same lattice point

[1] The lattice counterpart of $\int d^{\mathrm{D}}x(\varphi^{(1)}\partial^{\mu}\varphi^{(2)} - (\partial^{\mu}\varphi^{(1)})\varphi^{(2)}) = 0$ still holds true for an antisymmetric operator as explained below.

$\varphi_n^{(1)}\varphi_n^{(2)}$.

In considering the supersymmetric examples mentioned in section 2.1.4 the transformation of the action leads to terms of the form

$$\delta S \propto \int d^{\mathrm{D}}x\varepsilon^{\alpha}\left[(\partial_{\mu}\psi_{\alpha})W'(\varphi)+\psi_{\alpha}W''(\varphi)\partial_{\mu}\varphi\right] . \tag{3.1}$$

More generally, one arrives in a supersymmetric theory at

$$\delta S \propto \int d^{\mathrm{D}}x\varepsilon^{\alpha}\left[(\partial_{\mu}\psi_{\alpha})\varphi^{(1)}\cdots\varphi^{(n_{\mathrm{f}})}+\psi_{\alpha}(\partial_{\mu}\varphi^{(1)})\cdots\varphi^{(n_{\mathrm{f}})}+\ldots+\psi_{\alpha}\varphi^{(1)}\cdots(\partial_{\mu}\varphi^{(n_{\mathrm{f}})})\right] . \tag{3.2}$$

This can be understood from the superspace representation of the supercharges, which contain Graßmann and space-time derivatives ($Q \propto \partial_{\theta}+\Theta^{\mu}\partial_{\mu}$ with Θ^{μ} depending on the Graßmann coordinates θ). The action can be represented without loss of generality by a product of n_{f} fields integrated over superspace. The supercharges generate a change of this action. It is

$$\delta S \propto \int \prod_{\alpha} d\theta_{\alpha} d^{\mathrm{D}}x \left[(\varepsilon Q\Phi^{(1)})\Phi^{(2)}\cdots\Phi^{(n_{\mathrm{f}})}+\Phi^{(1)}(\varepsilon Q\Phi^{(2)})\cdots\Phi^{(n_{\mathrm{f}})}+\ldots\right] , \tag{3.3}$$

and for periodic boundary conditions it is identically zero if the Leibniz rule is fulfilled. The above expression (3.2) is a component respresentation of this term.

The Leibniz rule is, however, violated by a generic lattice derivative operator. When the continuum derivative is replaced by a discrete derivative operator the Leibniz rule becomes

$$(\nabla^{\mu}(\varphi^{(1)}\varphi^{(2)}))_m = \varphi_m^{(1)}(\nabla^{\mu}\varphi^{(2)})_m + (\nabla^{\mu}\varphi^{(1)})_m\varphi_m^{(2)} . \tag{3.4}$$

This relation does not hold for any lattice derivative operator, but in the zero momentum sector, i. e. after the summation of m, it is still valid.[2] The right hand side of (3.4) is zero after the summation. The left hand side then corresponds to the lattice analog of a partial integration, which ensures the supersymmetric invariance of a quadratic lattice action.

In the higher than quadratic case the expression (3.2) is not zero on the lattice since for a product of three ore more fields the analog of (3.4) is violated even after the summation. To show this in detail, let us consider such a product of fields in momentum space. The

[2]If one splits the antisymmetric matrix ∇ according to (A.11) one can find the modified Leibniz rule for each of the component: $\sum_n \nabla^{(r)}_{mn}(\varphi^{(1)}_{n+r}\varphi^{(2)}_n) = (\sum_n \nabla^{(r)}_{mn}\varphi^{(1)}_{n+r})\varphi^{(2)}_{m+r} + \varphi^{(1)}_m(\sum_n \nabla^{(r)}_{mn}\varphi^{(2)}_n)$ (in one dimension). Thus the Leibniz rule is fulfilled up to translations, which are not relevant under the summation.

repeated application of (3.4) results in[3]

$$(\nabla^\mu(\varphi^{(1)}\cdots\varphi^{(n_\mathrm{f})}))(p_s) = \sum_{k_1\ldots k_{n_\mathrm{f}}} \delta(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}}-p_s)(\nabla^\mu(p_{k_1})+\ldots+\nabla^\mu(p_{k_{n_\mathrm{f}}}))\varphi^{(1)}\cdots\varphi^{(n_\mathrm{f})}\,. \quad (3.5)$$

Supersymmetric invariance implies that this equation must hold for $p_s = 0$ and all fields $\varphi^{(i)}$. This is equivalent to [4]

$$\delta(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}})(\nabla^\mu(p_{k_1})+\ldots+\nabla^\mu(p_{k_{n_\mathrm{f}}})) = \sum_{i=1}^{n_\mathrm{f}-1}\nabla^\mu(p_{k_i}) - \nabla^\mu(\sum_{i=1}^{n_\mathrm{f}-1} p_{k_i}) = 0\,. \quad (3.6)$$

The locality of the action is analysed in the thermodynamic limit, where the discrete momentum becomes continuous, $p_{k_i} \to p_i$. Then locality is corresponds to the analyticity and periodicity in momentum space (cf. appendix A.3). For convenience the modulus of $\sum_{i=1}^{n_\mathrm{f}-1} p_i^\mu$ is first assumed to be smaller than the lattice cutoff $\Lambda_L^\mu = \frac{\pi}{a_\mu}$ for all directions μ. Later on I relax this condition. For now these assumptions fix the solution of equation (3.6) to[5]

$$\nabla^\mu(p) = c_1 p^\mu\,. \quad (3.7)$$

The only possible solution is hence nonlocal. It is (apart from irrelevant constants) the SLAC-derivative.

One can try to avoid this nonlocality of the lattice action with the introduction of a modified interaction term. For example, the product of three fields is represented on the lattice according to

$$\int d^\mathrm{D}x\,\varphi^{(1)}(x)\varphi^{(2)}(x)\varphi^{(3)}(x) \overset{\text{on the lattice}}{\longrightarrow} \sum_{m_1,m_2,m_3}\tilde{C}_{m_1,m_2,m_3}\varphi^{(1)}_{m_1}\varphi^{(2)}_{m_2}\varphi^{(3)}_{m_3}\,. \quad (3.8)$$

In Fourier space this ansatz brings equation (3.6) into the form (cf. (A.13))

$$\tilde{C}(p_1,p_2,p_3)(\nabla^\mu(p_1)+\nabla^\mu(p_2)+\nabla^\mu(p_3)) = 0\,, \quad (3.9)$$

which is solved by

$$\tilde{C}(p_1,p_2,p_3) = \delta(\nabla^\mu(p_1)+\nabla^\mu(p_2)+\nabla^\mu(p_3))\,. \quad (3.10)$$

For the symmetric derivative such a solution was proposed in [37]. However, with a

[3] Cf. appendix A.3 for the transformation to momentum space.

[4] The dispersion relation of ∇^μ_{mn} is periodically continued for momenta larger than the lattice cutoff (c.f. section A.3).

[5] $c_1 = 1$ in the continuum limit; additional constant contributions are zero because ∇^μ is antisymmetric.

derivative different from (3.7) the solution breaks the translational invariance on the lattice. A possible solution is to accept a modification of this invariance [39]. Here this property is, however, considered to be even more important than locality. A translational invariant choice is $\tilde{C}_{n_1,n_2,n_3} = C_{(n_1-n_2),(n_1-n_3)}$. The product of three fields in (3.8) is then represented by (cf. (A.16))

$$\sum_{m_1,m_2,m_3} C_{m_3-m_1,m_3-m_2}\left[(\nabla^\mu\varphi^{(3)})_{m_3}\varphi^{(1)}_{m_1}\varphi^{(2)}_{m_2} + \varphi^{(3)}_{m_3}(\nabla^\mu\varphi^{(1)})_{m_1}\varphi^{(2)}_{m_2} + \varphi^{(3)}_{m_3}\varphi^{(1)}_{m_1}(\nabla^\mu\varphi^{(2)})_{m_2}\right]$$
$$= \int_{p_1,p_2,p_3} \varphi^{(3)}(-p_1-p_2)C(p_1,p_2)(\nabla(p_1)+\nabla(p_2)-\nabla(p_1+p_2))\varphi^{(1)}(p_1)\varphi^{(2)}(p_2)\,.$$

Obviously, this approach can be generalised to the situation of n_f fields, and the condition for supersymmetric invariance, cf. (3.6), becomes[6]

$$C(p_1,\ldots,p_{n_\mathrm{f}-1})\left[\sum_{i=1}^{n_\mathrm{f}-1}\nabla(p_i) - \nabla(\sum_{i=1}^{n_\mathrm{f}-1}p_i)\right] = 0\,. \tag{3.11}$$

This corresponds to the Leibniz rule for $n_\mathrm{f}-1$ fields with the modified lattice product

$$(\varphi^{(1)} * \varphi^{(2)} * \cdots * \varphi^{(n_\mathrm{f}-1)})_l := \sum_{m_1,\ldots,m_{n_\mathrm{f}-1}} C_{l-m_1,\ldots,l-m_{n_\mathrm{f}-1}}\varphi^{(1)}_{n_1}\cdots\varphi^{(n_\mathrm{f}-1)}_{m_{n_\mathrm{f}-1}}\,. \tag{3.12}$$

The advantage of the modified lattice product is that $C(p_1,\ldots,p_{n_\mathrm{f}})$ can vanish instead of the terms in the square bracket can vanish in (3.11). However, if C is local it must be analytic for all of the momenta and can hence be zero only on isolated points. The exclusion of these isolated points does not allow for local solutions of (3.6), and the generalisation does not allow for a local invariant action. Either we get a nonlocal product or a nonlocal derivative operator. In case of three fields this is the key observation of [38].

To investigate further requirements for a lattice fromulation with such a generalised action, I go back to a finite lattice and discrete momentum modes. This reveals another problem of the suggestion in [37], i. e. (3.10) with the symmetric derivative. It then corresponds to a projection onto the trivial solutions, $p_{k_1} = 0$, $p_{k_2} = 0$, or $p_{k_3} = 0$. Consequently, the nonlocal interaction term allows for every finite lattice only a trivial interaction with at least one field at zero momentum, i. e.

$$\int d^\mathrm{D}x\,\varphi^{(1)}(x)\varphi^{(2)}(x)\varphi^{(3)}(x) \overset{\text{on the lattice}}{\longrightarrow} \sum_k \big[\varphi^{(1)}(0)\varphi^{(2)}(-p_k)\varphi^{(3)}(p_k)$$
$$+\varphi^{(1)}(-p_k)\varphi^{(2)}(0)\varphi^{(3)}(p_k) + \varphi^{(1)}(-p_k)\varphi^{(2)}(p_k)\varphi^{(3)}(0)\big]\,. \tag{3.13}$$

[6]The lattice operator C is symmetric under an exchange of its arguments.

Although for a continuous momentum the correct classical continuum limit is shown in [37], such a solution is not useful for the lattice simulations. There should be an increasing number of combinations of p_{k_1} and p_{k_2} that are included in the interaction, i. e. $\tilde{C}(p_{k_1}, p_{k_2}, -p_{k_2} - p_{k_3}) \neq 0$. A correct continuum limit demands that these p_{k_1} and p_{k_2} are in a region near zero in momentum space.

This leads us to an additional condition for the lattice formulations with a generalised product in the interaction term. For the correct continuum limit there must be a region around zero in momentum space with $C(p_{k_1}, \ldots, p_{k_{n_\mathrm{f}-1}}) \neq 0$, when the $p_{k_1} \ldots p_{k_{n_\mathrm{f}-1}}$ are chosen from this region. This region has to increase as a becomes smaller. At least in this region the derivative operator must hence have the dispersion relation of a SLAC-type derivative, (3.7). In the thermodynamic limit $\nabla^\mu(p)$ is hence either nonanalytic[7] inside the first BZ or it is similar to the SLAC derivative. If ∇ is not the SLAC derivative then C has to project onto the region where ∇ has the dispersion relation (3.7). In both cases ∇ is nonlocal.

As a last step I remove the restriction $\left|\sum_{i=1}^{n_\mathrm{f}-1} p_{k_i}^\mu\right| < \Lambda_L^\mu$. In this case even a SLAC-type derivative violates the Leibniz rule, cf. (3.6), because $\nabla^\mu(p)$ is periodically continued (in case of the SLAC derivative with a discontinuity at $p^\mu = \Lambda^\mu$ cf. (B.2)). The violation is for $(2l^\mu - 1)\Lambda_L^\mu < \sum_{i=1}^{n_\mathrm{f}-1} p_{k_i}^\mu < (2l^\mu + 1)\Lambda_L^\mu$ $(\forall \mu;\, l^\mu \in \mathbb{Z})$ given by

$$\sum_{i=1}^{n_\mathrm{f}-1} p_{k_i} - \sum_{i=1}^{n_\mathrm{f}-1} \nabla(p_{k_i}) = 2\Lambda_L \sum_\mu l^\mu \,. \tag{3.14}$$

Contrary to [38] and in accordance with the suggestion for a supersymmetric lattice action in [41] even for a SLAC-type derivative one needs a nonlocal C to ensure the Leibniz rule. This C has to vanish when the sum of the momenta is larger than the lattice cutoff.

I conclude with a stronger No-Go statement than found in [38]: *In order to get an interacting supersymmetric lattice theory one needs a nonlocal derivative operator and a nonlocal interaction term. The dispersion relation of the derivative operator has to agree with a SLAC-type derivative for an increasing number of momentum modes. When the SLAC derivative is used the nonlocal interaction term restricts the sum of the momenta to the first BZ.*

As a sidemark I note that the introduction of a cutoff in the theory is in conflict with a nonlinear symmetry, such as the on-shell representation of supersymmetry. When we start with some field with momenta restriced below a certain cutoff a generic nonlinear transformation rule like

$$\delta\varphi^{(i)}(x) = \sum_{jk} M^{ijk} \varphi^{(j)}(x) \varphi^{(k)}(x) \tag{3.15}$$

[7] Note that either $\nabla(p) = c_1 p$ only on isolated points or everywhere for an analytic function.

is represented in Fourier space as

$$\delta\varphi^{(i)}(p) = \sum_{jk} M^{ijk} \int \frac{d^{\mathrm{D}}q_1 d^{\mathrm{D}}q_2}{(2\pi)^{2\mathrm{D}}} \delta(p - q_1 - q_2)\varphi^{(j)}(q_1)\varphi^{(k)}(q_2)\,. \tag{3.16}$$

Thus the right hand side of this equation generates higher momentum modes than the cutoff in the variations. The lattice Fourier transformation leads to a periodic delta function and consequently maps the higher momentum modes onto the first Brillouin zone. So the cutoff is not violated. As a result the modes $\delta\varphi(p)$ cover twice the first BZ, when the momentum of the considered $\varphi(p)$ cover it once. After the introduction of the auxiliary field the same problem appears in the equations of motion for this field. The nonlocal interaction term introduced above to get a supersymmetric lattice action with the SLAC-derivative changes this situation: In the on-shell transformation it restricts the right hand side of (3.16) to a momentum below the cutoff.

An explicit realisation

With theses results I proceed a further step towards real lattice simulations. In this way I put some "flesh on the bones" of the so far rather abstract statements. The supersymmetric action is composed of a quadratic part and the interacting terms containing higher powers of the fields. To make contact with the above findings the lattice theory is first considered in an off-shell representation. Since anyway a nonlocal lattice derivative is needed one can, for convenience, choose the SLAC derivative in the lattice action and the transformations. The translation of a product of continuum fields into a nonlocal lattice term, cf. (3.8), can be done for each term of the action separately. A quadratic term needs hence no further modification. According to the power n_{f} of the fields (no matter if they are bosonic, fermionic, or auxiliary) that appear in the interaction the appropriate C must be chosen. The Cs are decomposed into a product of their one-dimensional counterparts (here called C_1),

$$C(p_{k_1}, p_{k_2}, \ldots, p_{k_{n_{\mathrm{f}}-1}}) = \prod_{\mu} C_1(p^{\mu}_{k_1}, p^{\mu}_{2}, \ldots, p^{\mu}_{k_{n_{\mathrm{f}}-1}})\,. \tag{3.17}$$

With the SLAC derivative this ansatz solves (3.11) for

$$C_1(p^{\mu}_{k_1}, p^{\mu}_{k_2}, \ldots, p^{\mu}_{k_{n_{\mathrm{f}}-1}}) := \begin{cases} 0 & \text{if } |\sum_{i=1}^{n_{\mathrm{f}}-1} p_{k_i}| > \Lambda^{\mu}_{L} \\ 1 & \text{otherwise} \end{cases}\,. \tag{3.18}$$

One can define a nonperiodic delta as done in [41],

$$\delta^{\Lambda}(p_{k_1}, p_{k_2}, \ldots, p_{k_{n_{\mathrm{f}}}}) := \delta(p_{k_1} + \ldots + p_{k_{n_{\mathrm{f}}}}) C(p_{k_1}, p_{k_2}, \ldots, p_{k_{n_{\mathrm{f}}-1}})\,. \tag{3.19}$$

For the purpose of the lattice simulation, an appropriate representation of this nonperiodic delta function is needed. Since C can in higher dimension be represented as a product of its one-dimensional counterpart, it is enough to look for a one-dimensional nonperiodic delta function.

Consider a lattice with the $(n_\mathrm{f}-1)$-fold number of lattice points compared to the original lattice. The lattice spacing of this finer lattice should be $a/(n_\mathrm{f}-1)$. Hence the periodicity of a Fourier space delta of this lattice ($\delta^{n_\mathrm{f}-1}$) is $(n_\mathrm{f}-1)\Lambda_L$. The boundary of the first BZ of this finer lattice cannot be reached with the sum of the momenta $p_{k_1}\ldots p_{k_{n_\mathrm{f}-1}}$. They are hence never folded back and the $\delta^{n_\mathrm{f}-1}(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}})$ is equal to one only when $\sum_{i=1}^{n_\mathrm{f}-1} p_{k_i}$ exactly matches p_{n_f}, and otherwise zero. Thus it represents a nonperiodic delta.

This shows that a one-dimensional interaction term with a product of n_f fields is represented on the lattice in the following way

$$\begin{aligned}\int dx\varphi^{(1)}(x)\cdots\varphi^{(n_\mathrm{f})}(x) &\overset{\text{on the lattice}}{\to} \sum_{k_1,\ldots,k_{n_\mathrm{f}}} \delta^{n_\mathrm{f}-1}(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}})\varphi^{(1)}(p_{k_1})\cdots\varphi^{(n_\mathrm{f})}(p_{k_{n_\mathrm{f}}}) \\ &= \sum_{k_1,\ldots,k_{n_\mathrm{f}}} \frac{a}{n_\mathrm{f}-1} \sum_{m=0}^{(n_\mathrm{f}-1)N-1} e^{-i\frac{a}{n_\mathrm{f}-1}m(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}})}\varphi^{(1)}(p_{k_1})\cdots\varphi^{(n_\mathrm{f})}(p_{k_{n_\mathrm{f}}})\,. \quad (3.20)\end{aligned}$$

Going back from Fourier space to a real space representation one arrives at[8]

$$\begin{aligned}\int dx\varphi^{(1)}(x)\cdots\varphi^{(n_\mathrm{f})}(x) &\overset{\text{on the lattice}}{\to} \\ &\frac{a}{n_\mathrm{f}-1} \sum_{n=0}^{(n_\mathrm{f}-1)N-1} \tilde\varphi_n^{(1)}\cdots\tilde\varphi_n^{(n_\mathrm{f})} \;\text{ with }\; \tilde\varphi_n^{(i)} = \sum_n \mathcal{F}_{nm}\varphi_m^{(i)}\,. \quad (3.21)\end{aligned}$$

The $\big((n_\mathrm{f}-1)N\big)\times N$ matrix $\mathcal{F}_{nm}$ translates all of the fields into fields on the finer lattice. It comprises a Fourier transformation on the lattice with N lattice points and an inverse Fourier transformation on the larger lattice with $(n_\mathrm{f}-1)N$ lattice points. The matrix elements read explicitly

$$\mathcal{F}_{nm} = \frac{\sin(\pi(m-n/(n_\mathrm{f}-1))}{aN\sin(\frac{\pi}{N}(m-n/(n_\mathrm{f}-1)))} \text{ for } m\neq n/(n_\mathrm{f}-1) \quad \text{and 1 otherwise.} \qquad (3.22)$$

This is easily generalised to the higher-dimensional case. In this way one obtains for each term in the action a lattice counterpart. Combining the terms one obtains a (fully) supersymmetric off-shell lattice action. As in the continuum theory the auxiliary field can be integrated out. The on-shell counterpart of this action defines also a supersymmetric lattice theory. As presented here, I have successfully reformulated the suggestion of [41]

[8] The ordinary definition of the Fourier representation of the fields $\varphi_n^{(i)}$ is employed.

in a way that can be used in the simulations. The results of simulations with such a supersymmetric lattice action can be found in the next chapter.

3.1.2 Partial realisation of the supersymmetry

As we have seen it is not possible to find a local lattice theory invariant under the full supersymmetry. In theories with extended supersymmetry one has more than one independent supercharge. Since it is not possible to get the full supersymmetry, one may then try to implement a part of the symmetry. The basic assumption of this approach is that the realised part of the supersymmetry ensures the restoration of the broken part in the continuum limit. In other words, the partial realisation reduces the fine tuning problem. One possible way to implement the partial realisation of supersymmetry is due to the Nicolai map [42, 43]. This maps the path integral of a supersymmetric theory onto the one with a Gaussian measure. The basic (formal) idea of this approach is a cancellation of the bosonic Jacobi determinant obtained in the transformation of the bosonic part by the fermion determinant (cf. appendix C). Only in some cases a local representation of this map has been found. Concrete examples are discussed in the next chapter.[9]

There are also other ways to implement a partial realisation of the supersymmetry that cannot be discussed here. These approaches use orbifolding techniques, [47, 48], or Dirac-Kähler fermions , [49, 50]. For a review cf. [51].

3.1.3 Modifications of the lattice symmetry transformation

Instead of an adjustment of the action also the symmetry transformations can be modified such that a given lattice action is invariant. These modified transformations approach the continuum supersymmetry transformations in the continuum limit. From the classical point of view it is easy to find such transformations for a given lattice action since this lattice action is equal to the supersymmetric continuum action in the classical continuum limit. Hence these modified supersymmetry transformations can be found at least in terms of a series. One can start, e. g., with an action separated into a quadratic part $S_0[\varphi]$ and interaction terms $\lambda S_1[\varphi]$. The one defines the modified symmetry transformations as $\delta_{\text{modified}}\varphi := \delta_0\varphi + \lambda\delta_1\varphi + \lambda^2\delta_1\varphi + \ldots$. δ_0 is the supersymmetry transformation we have used above, with the same antisymmetric lattice derivative as it appears in the action. Consequently, the quadratic term is invariant under these transformations. Due to the violation of the Leibniz rule the interaction term is not invariant under δ_0. The next order $\delta_1\varphi$ is chosen in such a way as to compensate this violation, namely $S_0'[\varphi]\delta_1\varphi = S_1'[\varphi]\delta_0\varphi$. Iterating this procedure, one obtains order by order the series representation of

[9] Note that despite their local form these implementations typically violate reflection positivity of the lattice action. This property of the continuum theory is usually esteemed to be crucial for a lattice theory [44, 45, 46]. Form this point of view it is not enough to recover it in the continuum limit; it should be respected by the lattice action. For lattice supersymmetry it may, nevertheless, be necessary to relax these restrictions.

the modified transformations. Each order compensates the violation of the previous one. In the classical continuum limit this series tends (order by order) towards the continuum supersymmetry transformations. It can, however, not be assumed that the symmetry is also recovered in the continuum limit of the quantum theory. This is achieved when all the Ward identities (2.49) of the modified symmetry approach the Ward identities of the continuum supersymmetry. In a two-dimensional example this has been checked with perturbative methods (to all orders of perturbation theory) in [52]. A non-perturbative approach to find a modified lattice supersymmetry can be found in chapter 5.

3.2 The fermion doubling and supersymmetry breaking on the lattice

So far we have assumed an antisymmetric lattice derivative operator such as the symmetric derivative. For the realisation of dynamic fermions on the lattice an additional discussion of the doubling problem is necessary. It is well-known that a naive discretisation, i. e. with $\nabla^{(s)}$, introduces a doubling of the fermion species in the continuum limit. All local representations with an antisymmetric derivative operator share this problem.[10] The only way out can be a symmetric term ($m^{(W)}(-p) = m^{(W)}(p)$). Such a term can be interpreted as a momentum dependent mass. To remove the doubling problem, this mass diverges at the additional zero modes (doublers) in the continuum limit. The doublers have then no dynamic contribution to the action and are "freezed out". The most prominent example of such a term is the Wilson mass. It is

$$m^{(W)} = \frac{ar}{2} \sum_{\mu} \nabla_{\mu}^{(-)} \nabla_{\mu}^{(+)} \tag{3.23}$$

The lattice analogue of (2.3) ($\hat{\not{\mathcal{D}}} = \not{\nabla} + m^{(W)}$)

$$S[\psi, \bar{\psi}] = \sum_{n,m} \bar{\psi}_n \hat{\not{\mathcal{D}}}_{nm} \psi_m \,, \tag{3.24}$$

is with the additional mass term no longer invariant under the chiral symmetry. More generally, one cannot find a local[11] chirally symmetric lattice Dirac operator $\hat{\not{\mathcal{D}}}$ that has no doubling modes in the continuum limit. This fact is known as the Nielson-Ninomiya theorem [53, 54]. An approach to circumvent this problem is discussed in chapter 5.

In the case of supersymmetry the doubling modes also lead to so far unconsidered

[10] In fact if $\not{\nabla}(p) = -\not{\nabla}(-p)$ is periodic and analytic (or continuous) it must have at least one additional zero apart from $p = 0$ in the BZ. This zero leads to a doubling mode in the continuum limit. This is also true if instead of the strict antisymmetry only a sign change appears at $p = 0$ as necessary for the continuum limit $\not{\nabla}(p) = \gamma_\mu p^\mu$.

[11] Here locality can be understood in an even more general sense: continuous and periodic is enough.

problems. According to the standard lattice formulation they appear only in the fermionic sector. Consequently, there are no longer the same number of degrees of freedom in the bosonic and fermionic sector, and supersymmetry is broken. Form the classical point of view this problem is resolved when a Wilson-type mass removes the doublers. Nevertheless, a lattice supersymmetry similar to the one of section 3.1.1 is violated even in the free theory.

Furthermore, in perturbative calculations of the supersymmetric continuum theory some bosonic and fermionic divergences cancel each other. The reason for these cancellations is the relation between the bosonic and fermionic vertices and propagators. If the same kind of cancellations should appear in the lattice perturbation theory, the same relation has to be valid on the lattice. In the standard formulation the Wilson mass has no bosonic counterpart. Even more, the cancellation of the fermionic and bosonic contributions in perturbation theory cannot always be recovered in the continuum limit. The Wilson mass becomes irrelevant for $a \to 0$ only at the centre of the BZ and diverges at the doublers. To find out the relevance of certain contributions in lattice perturbation theory, one can use the lattice degree of divergence defined by Reisz [55].[12] A typical one loop diagram in lattice perturbation theory is represented as an integral of the loop momentum p constrained to the first BZ. When the lattice degree of divergence is negative, the continuum limit can be obtained in a naive way: The integrand in the limit of $a \to 0$ and an unconstrained momentum integration yields the corresponding continuum expression. In such a naive continuum limit the contribution of the Wilson mass disappears (the essential part for this limit comes from the vicinity of $p = 0$). A typical one loop contribution of the Wilson mass in the fermionic sector is

$$\int \frac{d^{\mathrm{D}}p}{(2\pi)^{\mathrm{D}}} \frac{m^{(W)}(p)}{\sum_\mu \nabla_\mu^{(s)}(p)\nabla_\mu^{(s)}(p) + (m + m^{(W)}(p))^2} \,. \tag{3.25}$$

Its lattice degree of divergence is $\mathrm{D} - 1$. This leads already in one dimension to a finite nonvanishing contribution in the continuum limit, although in the continuum fermion loops are cancelled by boson loops. In one dimension this effect is investigated in detail in [56] and is also discussed in chapter 4.

Different from this standard formulation in section 3.1 the same derivative operator is used for fermions and bosons. This introduces doublers, which can only be removed with Wilson-type mass terms, in both sectors. Similar lattice formulations were investigated, e. g., in [57, 52]. The Wilson mass is then just a modification of the usual mass term in the superpotential.

The SLAC derivative opens another possibility to circumvent this problem. This

[12] As usual for lattice perturbation theory the calculation is carried out in the thermodynamic limit.

nonlocal derivative has no doublers. Consequently, a Wilson mass term is not needed.

Note that the described adjustment of the fermionic and bosonic mass terms alone does not in general imply a supersymmetric continuum limit. This has been shown in the perturbation theory of the fourdimensional Wess-Zumino model [41]. Although the masses and derivatives are the same for fermions and bosons non-supersymmetric counterterms are needed in this case to get a supersymmetric continuum limit.

3.3 Perturbative investigations of Wess-Zumino models with the SLAC derivative

The above investigations have shown that the SLAC derivative has features that are appropriate for the realisation of a supersymmetric theory on the lattice. On the other hand, this derivative is nonlocal. The interaction strength between two lattice points decays only polynomial and not exponentially with their distance. In [58] it has been found that the renormalisation of lattice QED with SLAC fermions needs (in the continuum sense) nonlocal and not Lorentz covariant counterterms in the continuum limit. A (in the lattice sense) nonlocal lattice representation is can hence be problematic. In [59] a different conclusion was drawn that was criticised in [60]. I cannot go into the details of this general discussion. The problem arises especially in gauge theories. It seems that behind it stands the general contradiction between chiral symmetry and a local gauge invariant regulator.

In the present work supersymmetric Wess-Zumino models are considered. Since gauge symmetries are absent, it is much easier to handle. I consider here the perturbation theory of the models containing the SLAC derivative with a local and nonlocal interaction terms. According to [59] in the first case no nonlocal or Lorentz violating counterterms appear in any dimension. There is, however, still an ongoing debate about this result. With reference to the one loop perturbation theory the applicability of the SLAC derivative in a lattice theory is often questioned. Therefore, I show the renormalisability and correct continuum limit of the one loop lattice perturbation theory of a twodimensional Wess-Zumino model.

For such a superrenormalisable theories no additional problems are expected at any higher loop level. In addition, the continuum degree of divergence of all the diagrams in this less than one. In the BPHZ renormalisation scheme [61] the counterterms are identified with the divergent parts of a power series of the diagram in terms of the external momentum. If the degree of divergence is less than one, the counterterms follow just from the first term of this series, the limit of zero external momentum.

On the lattice a similar renormalisation procedure was introduced in [62]. This can, however, not be applied for the SLAC derivative. Not even the well-known lattice power counting theorem of Reisz is valid here since it applies only for integrands that are smooth

periodic functions of the loop momentum.[13] The argumentation is, nevertheless, closely related to the renormalisation procedure of Reisz. The subtraction of the divergences is also done in a similar way as in the BPHZ scheme.

Note that in both of the considered lattice representations with the SLAC derivative the usual cancellation of fermionic and bosonic divergences appear also in lattice perturbation theory. Thus the $N = 2$ Wess-Zumino model contains no divergent terms. The discussion applies, however, also for the $\mathcal{N} = 1$ Wess-Zumino model and similar two-dimensional theories with bosonic and fermionic fields.

3.3.1 Renormalisability of a two-dimensional Wess-Zumino model with a local interaction term

An explicit consideration of all the one-loop contributions that appear in a model with scalar bosons and fermions realised with the SLAC derivative can be found in appendix B. Here I only review some basic arguments concerning the renormalisability. There are two ways to realise the perturbation theory with the SLAC derivative. In the first case the vertices contain, as in the continuum, a momentum conservation. Then the propagators in lattic perturbation theory with the SLAC derivative are the same as in the continuum for all momenta below the lattice cutoff. Due to the momentum conservation at the vertices, however, a momentum greater than allowed by the cutoff can appear. The momentum space representation of the SLAC derivative is then periodically continued in terms of a saw-tooth function.

In the other realisation a periodic momentum conservation appears at each vertex. Then the propagators are the same as in the continuum and no periodic continuation is needed. The integration of loop momenta is in both cases resticed to the BZ. This is in conflict with the Euclidian rotation symmetry that demands, e. g., a restriction of only the modulus of the momentum.

In either of the representations there appear additional contributions when the sum of the momenta that flows into a vertex is larger than the cutoff. Apart from the noncovariant cutoff these contributions are the difference between the lattice perturbation theory and its continuum counterpart. In the twodimensional Wess-Zumino model it turns out that these contributions are not relevant in the continuum limit. In addition the difference between the covariant and the lattice regularisation does not introduce counterterms that do not appear in the continuum perturbation theory. Thus the considered Wess-Zumino models (and more general models of this type without supersymmetry) are renormalisable when the SLAC derivative is applied.

[13] As usual the calculations of lattice perturbation theory are carried out in the thermodynamic limit where the number of lattice points tends to infinity and the lattice momentum becomes continuous.

3.3.2 The case with a nonlocal interaction term

The nonlocal interaction term introduces an effective cutoff at each vertex. Instead of the momentum conservation with a periodic delta the "nonperiodic delta" (3.19) appears. Consequently, along each line the momentum is constraint to the first BZ ($-\frac{\pi}{a} \leq p_\mu \leq \frac{\pi}{a}$ $\forall\mu$). Apart from this constraint the perturbative expansion on the lattice is the same as in the continuum since the SLAC derivative has the ordinary continuum dispersion relation for momenta below the cutoff. In this way one arrives at a cutoff regularised perturbation theory. When the cutoff is removed in the continuum limit, the continuum perturbation theory is approached by all nondivergent integrals. This cutoff is, however, not introduced in a covariant way since it restricts the momentum to the BZ and instead of restricting only the modulus of the momentum in accordance with the Euclidian rotation symmetry. For the integrals that diverge at $\Lambda_L \to \infty$ one still has to prove that none of them needs a non covariant or nonlocal counterterm. A lattice version of the BPHZ scheme can be applied since the integrals are smooth functions for all momenta below the cutoff. It is hence enough to subtract the parts of a power series in the momentum that diverges in the continuum limit. In the one loop lattice perturbation theory of two dimensions only the zero momentum part is, as in the continuum, divergent. Thus the subtracted counterterm contains no momentum dependence and is of the same form as in the continuum. At least in the lowdimensional case no difficulties can be expected from this lattice realisation.

In higher dimensions the counterterms involve also higher terms of the expansion and hence a nontrivial momentum dependence. Nevertheless, it was found in [63] that also the perturbation theory of the fourdimensional Wess-Zumino model with the nonlocal lattice realisation can be consistently renormalised.

4 Lattice simulations in low dimensional supersymmetric theories

So far I have discussed general aspects of supersymmetry on the lattice. The investigations were based on the discretisation of the classical action and lattice perturbation theory. Although it was possible to make statements about the applicability of certain lattice formulations, for complete analysis also the nonperturbative sector of the theory must be considered. At the end only a real lattice simulation can show wether or not these formulations of supersymmetric theories can be used and give reliable results. According to the last chapter nonlocal formulations of the lattice theory are in some respects favourable for the simulation of supersymmetry. Thus these formulations should be included in the investigations. In addition, supersymmetric lattice simulations must contain dynamical fermions. These two conditions demand a large numerical effort in the simulations. Nevertheless, the investigations should not be restricted to certain lattice actions because of the accessible computer power. This can, at least for some of the lattice formulations, only be achieved in lowdimensional theories. Therefore, the main subject of this chapter are the lattice simulations of one- and two-dimensional supersymmetric theories.

4.1 Supersymmetric quantum mechanics

The quantisation of the one-dimensional supersymmetric model introduced in 2.1.4 has been the subject of many intresting investigations [64]. Its main feature is a bosonic and a fermionic subspace with degenerate spectra. It is important in the investigation of spontaneous supersymmetry breaking [65, 66]. In this thesis only the case without a spontaneously broken supersymmetry is considered, since the main task is the analysis of the breaking due to the lattice regularisation. The considered classical action is found in (2.16) with the superpotential $W(\varphi) = \frac{m}{2}\varphi^2 + \frac{g}{3}\varphi^3$. A discussion of supersymmetric quantum mechanics in the operator formalism is shown in appendix D. As explained there, the standard numerical methods of quantum mechanics can be applied to determine the mass gap and the effective potential with a high precision. These are, therefore, further on called the exact results. A part of the presented results was published in [67].

4.1.1 Different lattice formulations

The model of supersymmetric quantum mechanics is used to compare a number of different lattice formulations. As we have already seen in chapter 3 the discretised version of the continuum action (2.16),

$$S_L = \frac{1}{2}\sum_n \left((\nabla\varphi)_n^2 + W_L'(\varphi)_n^2\right) + \sum_{n,m} \bar{\psi}_n \left(\nabla_{nm} + W_L''(\varphi)_{nm}\right)\psi_m \,, \tag{4.1}$$

breaks (for a local $W'_L(\varphi)$) the supersymmetry due to the violation of the Leibniz rule.[1] According to section 3.1.1 the discrete supersymmetry transformations have the following form (for later convenience they are separated into two parts):

$$\begin{aligned}
&\delta^{(1)}\varphi_n = \bar{\varepsilon}\psi_n\,; \quad \delta^{(1)}\psi_n = 0\,; \quad \delta^{(1)}\bar{\psi}_n = -\bar{\varepsilon}(\nabla\varphi + W'_L(\varphi))_n \\
&\delta^{(2)}\varphi_n = \bar{\psi}_n\varepsilon\,; \quad \delta^{(2)}\psi_n = (\nabla\varphi - W'_L(\varphi))_n\varepsilon\,; \quad \delta^{(1)}\bar{\psi}_n = 0\,.
\end{aligned} \tag{4.2}$$

The variation of this action under the symmetry transformation is (cf. equation (3.2))

$$\begin{aligned}
\delta^{(1)}S_L &= -\bar{\varepsilon}\sum_{n,m}\left(\psi_n W''_L(\varphi)_{nm}(\nabla\varphi)_m + \nabla_{mn}\psi_n W'_L(\varphi)_m\right) \\
&\text{and} \quad \delta^{(2)}S_L = -\varepsilon\sum_n\left(\bar{\psi}_n W''_L(\varphi)_n(\nabla\varphi)_n - \bar{\psi}_n\nabla_{nm}W'_L(\varphi)_m\right)\,.
\end{aligned} \tag{4.3}$$

This is nonzero because of the violation of the Leibniz rule, except for the free theory. In addition the doubling problem for the antisymmetric derivative ∇ must be solved. In the simulations the superpotential is $W(\varphi) = \frac{m}{2}\varphi^2 + \frac{g}{4}\varphi^4$ with a positive mass parameter m and coupling strength g.

A naive discretisation (naive Wilson model)

The usual way to discretise a lattice theory uses a left derivative for the bosonic fields and a Wilson-derivative for the fermionic ones. This resolves the doubling problem of S_L. The corresponding lattice action is

$$S_{\text{naiv}} = \frac{1}{2}\sum_n\left((\nabla^{(-)}\varphi)_n^2 + W'(\varphi_n)^2\right) + \sum_{n,m}\bar{\psi}_n\left(\nabla^{(s)}_{nm} + m^{(W)}_{nm} + W''(\varphi)_n\delta_{nm}\right)\psi_m\,. \tag{4.4}$$

Obviously, a discretised version of the continuum supersymmetry (with $\nabla^{(+)}$ or $\nabla^{(s)}$) is broken. In the free theory a degeneracy between fermion and boson mass can only be found in the continuum limit but not at any finite lattice spacing. Moreover the continuum cancellations between fermionic and bosonic loops do not appear in the lattice theory, as found in section 3.2. Investigations of such a model can also be found in [56].

The Wilson mass inside the superpotential (unimproved Wilson model)

According to section 3.2 the Wilson mass should better be included into the superpotential and the same derivative operators should be used for fermions and bosons. This leads to a mass degeneracy between fermions and bosons in the free theory at a finite lattice

[1] ∇ is an arbitrary antisymmetric lattice derivative operator. As we have done in section 3.1.1 the same derivative operator is applied for fermions and bosons and in the supersymmetry transformations. If not further specified by the considered model W'_L is the continuum superpotential ($W_L(\varphi)_n = W'(\varphi_n)$) and, consequently, $W''_L(\varphi)_{nm} = W''(\varphi_n)\delta_{nm}$. The fermionic part can always be written as $\sum_{n,m}\bar{\psi}_n(K_f)_{nm}\psi_m$ with the fermion matrix $K_f(\varphi)$.

spacing. For the present model the resulting lattice formulation is

$$S_{\mathrm{WS}} = \frac{1}{2}\sum_n \left((\nabla^{(s)}\varphi)_n^2 + ((m^{(W)}\varphi)_n + W'(\varphi_n))^2\right) + \sum_{n,m}\bar{\psi}_n\left(\nabla^{(s)}_{nm} + m^{(W)}_{nm} + W''(\varphi_n)\delta_{nm}\right)\psi_m\,. \quad (4.5)$$

From this formulation one can infer that in the discretised version of the supersymmetry transformations $\nabla^{(s)}$ should be used as derivative operator and $(m^{(W)}\varphi)_n + W'(\varphi_n)$ replaces the continuum superpotential $W'(\varphi(x_n))$:

$$W'_L(\varphi) = (m^{(W)}\varphi)_n + W'(\varphi_n); \quad W''_L(\varphi)_{nm} = m^{(W)}_{nm} + W''(\varphi_n)\delta_{nm}\,. \quad (4.6)$$

Now the free theory is invariant under the transformations; but the interacting theory is not even invariant under a part of the supersymmetry. The model has also been considered in [57].

A lattice action with the SLAC derivative (unimproved SLAC model)

To resolve the fermion doubling problem without a Wilson-type mass term or the introduction of additional fermionic degrees of freedom the application of a nonlocal derivative operator is needed. One example of such a nonlocal operator is the SLAC derivative. A lattice formulation with the SLAC derivative is

$$S_{SLAC} = \frac{1}{2}\sum_n \left((\nabla^{\mathrm{SLAC}}\varphi)_n^2 + W'(\varphi_n)^2\right) + \sum_{n,m}\bar{\psi}_n\left(\nabla^{\mathrm{SLAC}}_{nm} + W''(\varphi_n)\delta_{nm}\right)\psi_m\,. \quad (4.7)$$

One gets a mass degeneracy in the free theory without a modification of the superpotential. An additional advantage of this formulation is that $O(a)$ contributions due to the Wilson mass are not present. Thus lattice artifacts are, consequently, expected to be much smaller than for S_{WS}.

Nicolai-improved lattice actions (improved Wilson and improved SLAC model)

In the models considered so far not even a part of the supersymmetry is present on the lattice. One can, however, find a formulation in which half of the supersymmetry is present. This is, as explained in section 3.1.2, a feature of models with extended supersymmetry. The Nicolai map can be used to construct such an action, cf. appendix C. It is

$$S_{NI1} = \frac{1}{2}\sum_n (\nabla\varphi + W'_L(\varphi))_n^2 + \sum_{n,m}\bar{\psi}_m(\nabla + W''_L(\varphi))_{nm}\psi_m\,, \quad (4.8)$$

and invariant under the symmetry transformations $\delta^{(1)}$, (4.2). The second supersymmetry transformation (4.2) leads to

$$\delta^{(2)} S_{NI1} = 2\delta^{(2)} S_L \,. \tag{4.9}$$

Thus the action is invariant under one supersymmetry. The violation of the second supersymmetry is, however, twice as big compared to the one of S_L. The operators that appear in the second variation are of the same form in both formulations. One may, however, hope that due to the one preserved supersymmetry the violation of the Ward identities of $\delta^{(2)}$ is now smaller. That means the corresponding expectation value $\langle \mathcal{O}(\varphi, \bar{\psi}, \psi)\delta S\rangle$ for some operator $\mathcal{O}$ is supposed to be smaller for S_{NI1} than for S_L.

The difference between S_L and S_{NI1} are terms that are surface terms in the continuum limit,

$$S_{NI1} - S_L = \sum_n W_L'(\varphi)_n (\nabla\varphi)_n \quad \to \int dt W'(\varphi)\partial_t\varphi = \int dt \partial_t W(\varphi)\,, \tag{4.10}$$

and vanish due to the periodic boundary conditions. From the variations of this term under the supersymmetry transformations, $\delta(S_{NI1} - S_L) = \delta^{(2)} S_L - \delta^{(1)} S_L$, one easily observes that the action $S_{NI2} = S_L - S_{NI1} + S_L$ is invariant under the second symmetry, $\delta^{(2)}$.[2]

With the SLAC derivative and $W_L'(\varphi) = W'(\varphi)$ ($W_L''(\varphi) = W'(\varphi)\delta_{nm}$) S_{NI1} defines the action of the improved SLAC model:

$$S_{NISLAC} = \frac{1}{2}\sum_n (\nabla^{\mathrm{SLAC}}\varphi + W'(\varphi))_n^2 + \sum_{n,m} \bar{\psi}_m (\nabla^{\mathrm{SLAC}}_{nm} + W''(\varphi)\delta_{nm})\psi_m \,. \tag{4.11}$$

The symmetric derivative as ∇ and a superpotential that removes the doublers $W_L'(\varphi_n) = (m^{(W)}\varphi)_n + W'(\varphi_n)$ ($W_L''(\varphi_n)_{nm} = m^{(W)}_{nm} + W''(\varphi_n)\delta_{nm}$) leads to the action of the improved Wilson model,

$$S_{NIW} = \frac{1}{2}\sum_n (\nabla^{(s)}\varphi + m^{(W)}\varphi + W_L'(\varphi))_n^2 + \sum_{n,m} \bar{\psi}_m (\nabla^{(s)}_{nm} + m^{(W)}_{nm} + W''(\varphi)\delta_{nm})\psi_m \,. \tag{4.12}$$

Fermion determinants and the Stratonovich discretisation (Stratonovich model)

An other important problem of the Wilson derivative is the inaccurate reproduction of the fermionic determinant in the continuum limit. With the help of standard methods the fermionic determinant can be calculated in the continuum, [68, 69]. A (divergent)

[2] For convenience only S_{NI1} is considered here.

prefactor can be absorbed in the free determinant and one obtains

$$\det\left(\frac{\partial_\tau + W''(\phi(\tau))}{\partial_\tau + m}\right) = \frac{\sinh\left(\frac{1}{2}\int_0^\beta d\tau\, W''(\phi(\tau))\right)}{\sinh(\frac{\beta}{2}m)}\,. \tag{4.13}$$

For the superpotential considered here this expression is positive for all field configurations. On the other hand, if W (and hence also W'') is odd, the determinant takes both signs.[3] With the Wilson derivative one gets, instead,

$$\begin{aligned}\det\left(\frac{\nabla^{(s)}_{nm} + m^{(W)}_{nm} + W(\varphi_n)\mathbb{1}}{\nabla^{(s)}_{nm} + m^{(W)}_{nm} + m\mathbb{1}}\right) &= \frac{\prod\left(1 + aW''(\phi_n)\right) - 1}{(1+am)^N - 1} \\ &\xrightarrow{N=\beta/a\to\infty} \frac{e^{\int_0^\beta d\tau\, W''(\phi(\tau))\, d\tau/2}}{e^{\beta m/2}} \det\left(\frac{\partial_\tau + W''(\phi(\tau))}{\partial_\tau + m}\right)\,. \end{aligned}\tag{4.14}$$

Hence the continuum determinant is not correctly reproduced on the lattice.

Let us now turn to a lattice action that correctly reproduces the continuum determinant. In the fermionic part of the action the superpotential is also included in the off diagonal elements according to

$$S_{FS} = \sum_{n,m} \bar\psi_x \left(\nabla^{(s)}_{nm} + m^{(W)}_{nm} + \frac{1}{2}W''(\sigma_n)\left(\delta_{nm} + \delta_{n-1,m}\right)\right)\psi_m\,, \tag{4.15}$$

where $\sigma_n = \frac{1}{2}(\varphi_n + \varphi_{n+1})$. Instead of equation (4.14) one now finds the correct continuum result,

$$\begin{aligned}\det\left(\frac{\nabla^{(s)}_{nm} + m^{(W)}_{nm} + \frac{1}{2}W''(\sigma_n)\left(\delta_{nm} + \delta_{n-1,m}\right)}{\nabla^{(s)}_{nm} + m^{(W)}_{nm} + m\left(\delta_{nm} + \delta_{n-1,m}\right)}\right) &= \frac{\prod(1 + \frac{a}{2}W''(\sigma_n)) - \prod(1 - \frac{a}{2}W''(\sigma_n))}{\prod(1+\frac{a}{2}m) - \prod(1-\frac{a}{2}m)} \\ &\xrightarrow{N=\beta/a\to\infty} \det\left(\frac{\partial_\tau + W''(\phi(\tau))}{\partial_\tau + m}\right)\,. \end{aligned}\tag{4.16}$$

This suggests $W''_L(\varphi)_{nm} = m^{(W)}_{nm} + W''(\sigma_n)(\delta_{nm} + \delta_{n-1,m})$ in this model. This is obtained from the derivative $W'_L(\varphi)_n = (m^{(W)}\varphi)_n + W'(\sigma_n)$ with respect to φ_m. The Nicolai improved bosonic part that follows from the fermionic matrix is

$$S_{BS} = \frac{1}{2}\sum_n \left((\nabla^{(s)}\varphi)_n + (m^{(W)}\varphi)_n + W'(\sigma_n)\right)^2\,. \tag{4.17}$$

The corresponding action,

$$S_{Strat} = S_{BS} + S_{FS}\,, \tag{4.18}$$

[3] This is important since the Witten-index of the theory is known to be zero which can only be achieved with a sign change of the determinant.

then defines the so-called Stratonovich model.

Note that in one dimension one has $(\nabla^{(+)}\varphi)_n = (\nabla^{(s)}\varphi)_n + (m^{(W)}\varphi)_n$. Hence the difference between the bosonic part of S_{NIW} and the bosonic part of S_{naiv} is the surface term

$$\frac{1}{2}\sum_{n,m} W'\big(\varphi_n\big)\big(\varphi_n - \varphi_{n-1}\big)\,. \tag{4.19}$$

It is formulated in the Ito prescription. In the current formulation we obtain, instead, the term

$$\frac{1}{2}\sum_{n} W'(\sigma_n)((\nabla^{(s)} + m^{(W)})\varphi)_n = \frac{1}{2}\sum_{n} W'\big(\frac{1}{2}(\varphi_n + \varphi_{n-1})\big)\big(\varphi_n - \varphi_{n-1}\big)\,. \tag{4.20}$$

This corresponds to the formulation of the surface term according to the Stratonovich prescription [70].

With the above W'_L the supersymmetry $\delta^{(1)}$ is realised on the lattice because the action is constructed in the improved formulation. The second supersymmetry ($\delta^{(2)}$) is, however, violated even in the free theory. To define a supersymmetry that is broken only in the interacting theory I use $W'_L(\varphi)_n = (m^{(W)}\varphi)_n + W'(\varphi_n + \varphi_{n+1})$ in the second symmetry transformation for this special model.

A model preserving all the continuum supersymmetries on the lattice (full supersymmetric model)

According to section 3.1.1 there is also a way to realise not only a part but the full supersymmetry of the continuum theory on the lattice. Besides the nonlocal SLAC derivative also a nonlocal interaction term is needed. The easiest way to obtain this model is to start with the superpotential $W(\Phi)$, in our case $\frac{m}{2}\Phi^2 + \frac{g}{4}\Phi^4$. To construct the off-shell action, the superpotential is integrated over the whole superspace. The basic idea of this approach is to replace the space-time integration of the term $\lambda\Phi^{n_f}$ in the superpotential with the nonlocal product

$$\int dt\lambda(\Phi(t,\theta,\bar\theta))^{n_f} \to \lambda \sum_{m_1,\ldots,m_{n_f}} C_{m_1\ldots m_{n_f}} \Phi_{m_1}(\theta,\bar\theta)\cdots\Phi_{m_{n_f}}(\theta,\bar\theta)$$

$$= \frac{\lambda a}{n_f - 1} \sum_{n=0}^{(n_f-1)N-1} (\tilde\Phi_n(\theta,\bar\theta))^{n_f}\,, \tag{4.21}$$

where $\tilde{\Phi}_n = \sum_n \mathcal{F}_{nm}\Phi_m$ with $\mathcal{F}_{nm}$ given in (3.22). After the integration of Grassmann coordinates one obtains $\sum_n F_n \tilde{W}'(\varphi)_n$[4] with

$$\tilde{W}'(\varphi)_n = \frac{a\lambda n_f}{n_f - 1} \sum_{m=0}^{(n_f-1)N-1} \mathcal{F}_{mn}(\tilde{\varphi}_m)^{n_f-1} \tag{4.22}$$

The superpotential is a sum of terms of the form $\lambda(\Phi)^{n_f}$. The above translation is done for every term, except the quadratic part.[5] Thus in case of the full supersymmetric model we arrive at the following lattice version of the superpotential

$$W'_L(\varphi)_n = m\varphi_n + \frac{ag}{3} \sum_{m=0}^{3N-1} \mathcal{F}_{mn}(\tilde{\varphi}_m)^3 \,. \tag{4.23}$$

From this kind of potential we deduce for the fermion matrix

$$W''_L(\varphi)_{nm} = m\delta_{nm} + \frac{ag}{3} \sum_{m_1=0}^{3N-1} \mathcal{F}_{m_1 n}\mathcal{F}_{m_1 m}(\tilde{\varphi}_{m_1})^2 \,. \tag{4.24}$$

When the SLAC derivative is used the improvement term, just as expected, vanishes. Hence the Nicolai improved version of this model and the unimproved version are identical. The action is

$$S_{\mathrm{SUSY}} = \frac{1}{2} \sum_n \left((\nabla^{\mathrm{SLAC}}\varphi)_n^2 + W'_L(\varphi)_n^2 \right) + \sum_{n,m} \bar{\psi}_n \left(\nabla^{\mathrm{SLAC}}_{nm} + W''_L(\varphi)_{nm} \right) \psi_m \,. \tag{4.25}$$

4.1.2 Simulation details

Supersymmetry demands for the simulation of dynamical fermions. The quenched approximation severely breaks the symmetry, and fermionic contributions is fluctuate too strongly to allow for an efficient reweighting of the fermion determinant. Therefore, the full contribution of the fermions to the path integral must be included. Since the fermion determinant is in the present case always positive the fermionic fields can be integrated out without a sign problem. The resulting determinant can be included into the action

$$S_{\mathrm{eff}}(\varphi) = S_B(\varphi) - \log\det K_f(\varphi) \tag{4.26}$$

(S_B is the bosonic action of a given model). Even if we start with ultralocal lattice action, the resulting S_{eff} involves the interaction of all lattice points. Therefore, local update algorithms are inefficient in this case. For that reason the HMC algorithm [71] is applied

[4] F_n is the auxiliary field on the lattice.
[5] There the matrix $\mathcal{F}$ would just disappear according to eqn. (A.19).

to the dynamical fermion simulations. The update algorithm follows a molecular dynamics trajectory determined by

$$\dot{\varphi}_n = \frac{\partial \mathcal{H}}{\partial \pi_n}\,, \quad \dot{\pi}_n = -\frac{\partial \mathcal{H}}{\partial \varphi_n}\,, \tag{4.27}$$

and the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{n=0}^{N-1} \pi_n + S(\varphi)\,. \tag{4.28}$$

The phase space is obtained by introducing a conjugate momentum π_n for every field φ_n. The numerical solutions of the differential equations (4.27) are computed with a standard leap frog algorithm. The fermionic contribution $\partial S_{\text{eff}}(\varphi)/\partial\varphi_n$ is calculated from

$$\frac{\partial}{\partial\varphi_n}(\log\det K_f) = \frac{\partial}{\partial\varphi_n}(\operatorname{tr}\log K_f) = \operatorname{tr}\left(\left(\frac{\partial K_f}{\partial\varphi_n}\right) K_f^{-1}\right)\,, \tag{4.29}$$

where

$$\frac{\partial K_f}{\partial\varphi_n} = \frac{\partial W_L''(\varphi)_{nm}}{\partial\varphi_n}\,. \tag{4.30}$$

For all models, except the Stratonovich and the full supersymmetric model, the trace collapses since W_L'' is diagonal.

Note that because of the nonlocal interaction term the computational cost of the full supersymmetric model is much higher than for all the other models. In the present formulation it is, however, possible to use this formulation at least in one dimension. If we would try to perform the sum over all the entries of C it would be too expensive. Instead, I first calculate the fields on the finer lattice, $\tilde{\varphi}$, and obtain W_L' and W_L'' from these fields. This reduces the numerical cost of the nonlocal interaction term.

4.1.3 The effective mass on the lattice

An important indication of an intact supersymmetry is the equivalence of bosonic and fermionic masses, as it is predicted by the supersymmetry algebra. Therefore, the comparison of these masses is a first verification of the supersymmetry in the continuum limit. They are depicted as a function of the lattice spacing in figure 4.1 with a linear extrapolation to the continuum limit.

The effective masses of the theory are determined, in the usual manner, from the decay of the correlation functions,

$$G_n^{(b)} = \langle\varphi_n\varphi_0\rangle \quad \text{and} \quad G_n^{(f)} = \langle\psi_0\bar{\psi}_n\rangle = \langle(K_f(\varphi))_{0n}^{-1}\rangle\,, \tag{4.31}$$

at large distances. The asymptotic behaviour of the two-point functions is determined by $\sum_i c_i e^{-(E_i-E_0)x_n}$ (E_i the energy eigenvalues of the system, c_i constants). At large x_n the main contribution comes from the mass gap $m_{\text{eff}} = E_1 - E_0$. The fermionic two-point

function in a logarithmic representation is, as expected, approximately linear for large x_n. Thus a linear fit in the logarithmic representation can be used to determine the fermionic mass.

For the bosonic two-point functions the periodic boundary conditions induce a slight modification of the general form. At a large enough distance of x_n from zero and L it gets an additional contribution to become the symmetric function $\cosh(m_{\text{eff}}(x_n - aN/2))$. At distances that are well enough separated from zero and $aN/2$ (or $aN/2$ and L) this additional contribution is small and the cosh function can be approximated by $e^{-m_{\text{eff}}x_n}$ as in the fermionic case. Thus in this region the bosonic mass can be determined, as the fermionic one, by a linear fit in the logarithmic representation.

The lattice sizes were in a range between $N = 15$ and $N = 243$. The size of the lattice was kept fix $L = 10m^{-1} = Na$. The number of independent configurations used for the measurements was between 2.5×10^5 and 4×10^5. For the full supersymmetric model, due to the more expensive calculations, only a number of 5×10^4 independent configurations was available.

The formulations with the SLAC derivative demand some further remarks. Already in the free theory the fermionic and bosonic propagator of the SLAC derivative shows a superposition of the exponential decay by small oscillations. For example, the free fermionic propagator of the SLAC derivative in momentum space is $\frac{1}{ip+m}$. Because of the lattice cutoff the real space propagator is changed from the expected exponential decay, but the original form is reobtained in the continuum limit,

$$\begin{aligned} G_n^{(f)} &= \sum_k \frac{e^{ip_k x_n}}{ip_k + m} \\ &\overset{N\to\infty}{\longrightarrow} \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} \frac{e^{ipx_n}}{ip+m} = \frac{i}{2\pi}(\text{Ei}((m - i\frac{\pi}{a})x_n) - \text{Ei}((m + i\frac{\pi}{a})x_n))e^{-mx_n} \\ &\overset{a\to 0}{\longrightarrow} \int \frac{dp}{2\pi} \frac{e^{ipx}}{ip+m} = e^{-mx}\,, \end{aligned} \tag{4.32}$$

(for positive values of x). The additional contribution ($\text{Ei}(x) = -\int_{-x}^{\infty} dt\ \frac{e^{-t}}{t}$) leads to the observed oscillatory behaviour. Thus the basic reason for the deviation from the continuum propagator is a truncation in momentum space. This effect is known as the Gibbs phenomenon: the truncation in momentum space representation of a periodic and continuous function leads to an error in real space that is exponentially suppressed with the number of contributing Fourier modes n. If the function is non-periodic or has a discontinuity, the error is proportional to $n^{-\delta}$, for some positive δ at the regions away from the discontinuities. There are many well-known filtering techniques that can increase δ. The optimal filtering technique that was proposed in [72] even leads to an exponential

suppression of the error and was applied to the fermionic two-point function. An example for a filtered fermionic two-point function is shown in figure 4.1 (a). The filtered two-point function is much closer to the exponential decay. Thus the mass can be obtained in the usual way after the application of the filter. For the bosonic two-point function similar oscillations can appear but they are less pronounced and can be neglected.

In figure 4.1(c) shows the effect that was already found in section 3.2: The Wilson mass generates additional contributions in the correlation functions of S_{naiv} that are unbalanced by bosonic counterparts. Therefore, already in this simple model the bosonic and fermionic masses are different even in the continuum limit. Supersymmetry is broken on the lattice and not restored in the continuum limit. This effect has been identified with the one loop contributions of the Wilson mass in [56]. The unimproved SLAC model (S_{SLAC}) circumvents this problem via the introduction of a nonlocal operator that needs no Wilson mass to remove the doublers.

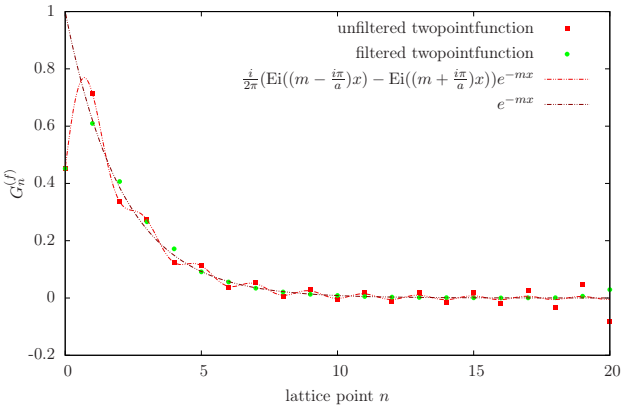
An other way of a sufficient realisation to achieve a mass degeneracy on the lattice is shown in figure 4.1 (d) in terms of the unimproved Wilson model (S_{WS}). Here the Wilson mass is included in the superpotential and a balance between fermionic and bosonic contributions is achieved. However, due to the $O(a)$ contributions of the Wilson mass, the lattice results are rather far away from the continuum values compared to the SLAC models.

With respect to the bosonic and fermionic masses there is no large difference between the improved realisations with one supersymmetry and the unimproved ones. This is shown for the Wilson models in figure 4.1 (e). The model that realises both supersymmetries (S_{SUSY}) is, according to the masses, similar to the unimproved and improved SLAC model, cf. figure 4.1 (b). Note that due to the low statistic the error is much larger, but especially for the fermionic masses a good agreement with the improved SLAC model can be observed.

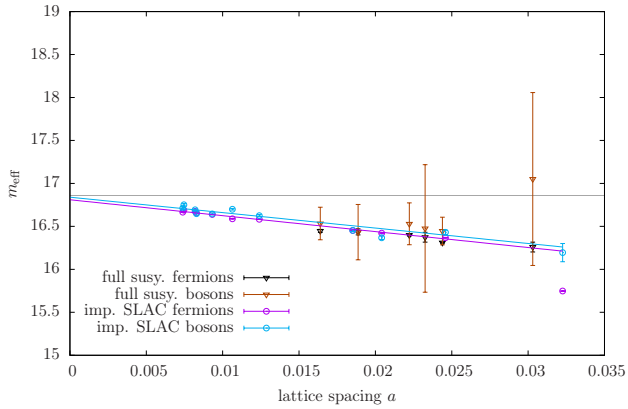
For the Stratonovich model the $O(a)$ contribution is, with respect to the Wilson model, reduced. The masses are much closer to the continuum (figure 4.1(f)).

Except for the naive Wilson model all mass extrapolations agree very well with the mass gap $(E_0 - E_1)$ determined with the numerical methods of quantum mechanics.

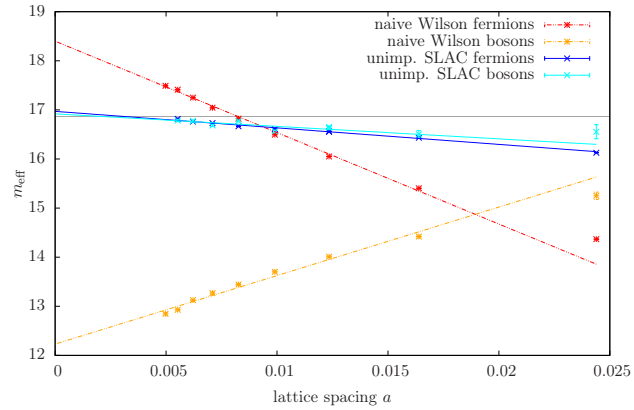
(a) The free fermionic lattice twopointfunction of the SLAC model compared with their thermodynamic and continuum limit ($m = 10$, $N = 21$)



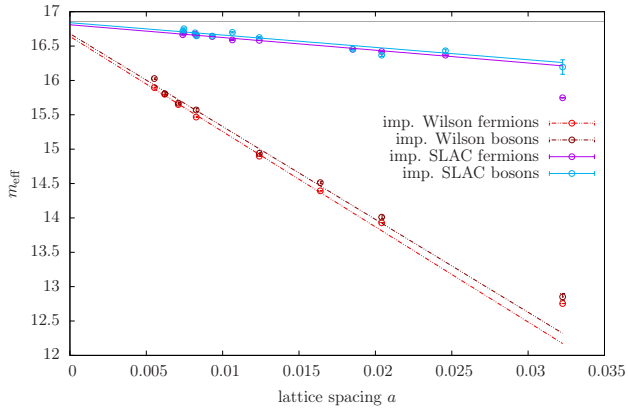
(b) A comparison of the masses of the improved SLAC model and the full supersymmetric model ($m = 10$, $g = 100$)



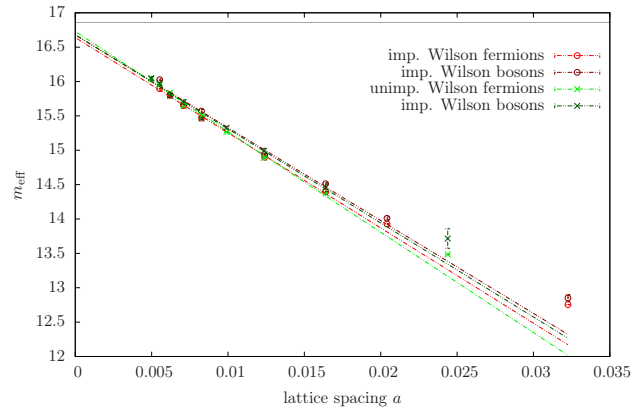
(c) A comparison of the masses of the unimproved SLAC model and the naive Wilson model ($m = 10$, $g = 100$)



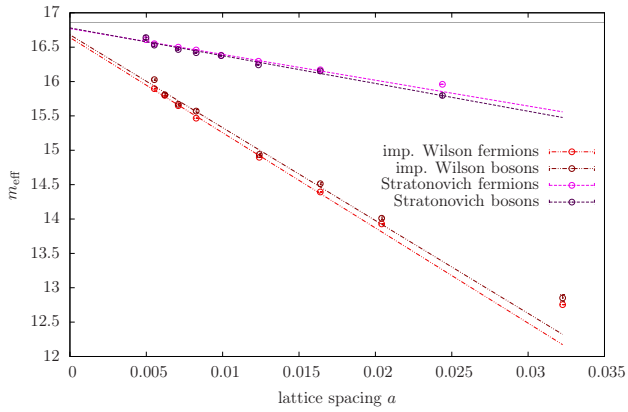
(d) A comparison of the masses of the improved SLAC model and the improved Wilson model ($m = 10$, $g = 100$)



(e) A comparison of the masses of the improved Wilson model and the unimproved Wilson model ($m = 10$, $g = 100$)



(f) A comparison of the masses of the improved Wilson model and the Stratonovich model ($m = 10$, $g = 100$)



model	m_{bos}	m_{ferm}	model	m_{bos}	m_{ferm}
Slac impr.	16.84 ± 0.03	16.81 ± 0.01	Slac unimpr.	16.92 ± 0.07	16.97 ± 0.03
Wilson impr.	16.86 ± 0.07	16.64 ± 0.03	Wilson unimpr.	16.68 ± 0.05	16.73 ± 0.04
Wilson naive	12.23 ± 0.08	18.04 ± 0.05	Stratonovich	16.78 ± 0.04	16.77 ± 0.02
full susy.	16.58 ± 0.16	16.69 ± 0.07	exact	16.865	16.865

Figure 4.1: (a) shows an example of the Gibbs-phenomenon for the free fermionic two-point functions of the SLAC models ($N = 21$ and in the thermodynamic limit (cf. eqn. (4.32))). The filtered two-point functions are (for large enough N) a good approximation of the continuum function e^{-mx}. (b)-(f) compare the masses obtained from a linear fit in a logarithmic representation. For the SLAC models the filtered fermionic functions were used. The table below lists the result of the linear extrapolation to the continuum limit (omitting the first two points) together with the error of the linear fit.

4.1.4 Lattice measurements of the Ward identities

After the masses are considered, now a more precise sign of the supersymmetry is investigated. As shown in section 2.2.2 the symmetries are reflected in the Ward identities of the observables. For the lattice simulations I consider Ward identities that are accessible with a good precision. These are the Ward identities that involve the two-point function and are obtained from the observables

$$\mathcal{R}^{(1)}_{n-m} = \langle \varphi_n \, \delta^{(1)} \bar{\psi}_m \rangle + \langle \bar{\psi}_n \delta^{(1)} \varphi_m \rangle = \langle \psi_n \bar{\psi}_m \rangle - \langle \varphi_n (\partial\varphi)_m \rangle - \langle \varphi_n W'_L(\varphi)_m \rangle \tag{4.33}$$

$$\mathcal{R}^{(2)}_{n-m} = \langle \varphi_n \, \delta^{(2)} \psi_m \rangle + \langle \psi_n \delta^{(2)} \varphi_m \rangle = \langle \varphi_n \, (\partial\varphi)_m \rangle - \langle \varphi_n W'_L(\varphi)_m \rangle - \langle \bar{\psi}_x \psi_y \rangle \, . \tag{4.34}$$

According to the supersymmetry these observables must be identically zero. Equation (2.48) shows that for a broken symmetry one instead obtains $\mathcal{R}^{(1)}_{n-m} = \langle \varphi_n \bar{\psi}_m \, \delta^{(1)} S \rangle$ and $\mathcal{R}^{(2)}_{n-m} = \langle \varphi_n \psi_m \, \delta^{(2)} S \rangle$. Since the naive model shows a supersymmetry breaking already in terms of the masses it is not considered in these more precise measurements. For all other models the Ward identities become smaller for a decreasing a or a decreasing g. Within the errors they hence vanish in the continuum limit. This is a strong indication of a restoration of supersymmetry. Already at a finite lattice spacing they are rather small. This is why in the similar measurements in [57] the accuracy was not high enough to discriminate the violation at a finite lattice spacing. Thus one could not tell whether or not there is at all a breaking at a finite lattice spacing.

A careful measurement is necessary to obtain reliable information of these small quantities. To pronounce the effect a rather small lattice was chosen ($N = 21$ and $N = 15$) and a high coupling strength ($g/m^2 = 8$). A high statistic was needed (4 independent runs with 10^6 independent configurations except for the improved and unimproved SLAC model (10 indep. runs of 10^6 config.) and the $N = 15$ lattice with improved and unimproved SLAC as well as the full supersymmetric model (8 indep. runs of 10^5 config.)). To estimate the zero singal the free Ward identities were measured with the same statistic. They are identically zero for the considered models since the breaking occurs only in the interacting case. Only if in the interacting case a larger deviation than indicated by the free Ward identities occurs, one can recognise it as an indication of a broken supersymmetry on the lattice.

In figure 4.2 the result of these measurements is depicted. For the unimproved Wilson and SLAC model both Ward identities are violated at a finite lattice spacing. They are only restored in the continuum limit. The supersymmetry breaking on the lattice is also visible in terms of the observables. In the improved Wilson and SLAC models one supersymmetry is, as expected, visible also at a finite lattice spacing since $\mathcal{R}^{(1)}_n$ is identically zero. However, the one realised supersymmetry does not lead to an improvement of the second one. On the contrary the violation of the second supersymmetry, as indicated by

the deviation of $\mathcal{R}_n^{(2)}$ from zero, is larger for the improved models. This can be seen as an indication that partial realisation of supersymmetry on the lattice improves the realised supersymmetries in favour of the remaining part of the transformations.

Most importantly, for the full supersymmetric model the realisation of the symmetry is visible also in terms of the Ward identities. Within the small statistical errors both Ward identities are fullfilled, whereas there is a clear signal for the supersymmetry breaking for the improved and unimproved SLAC models with the same parameters. *This is the first measurement of a full supersymmetric theory on the lattice.*

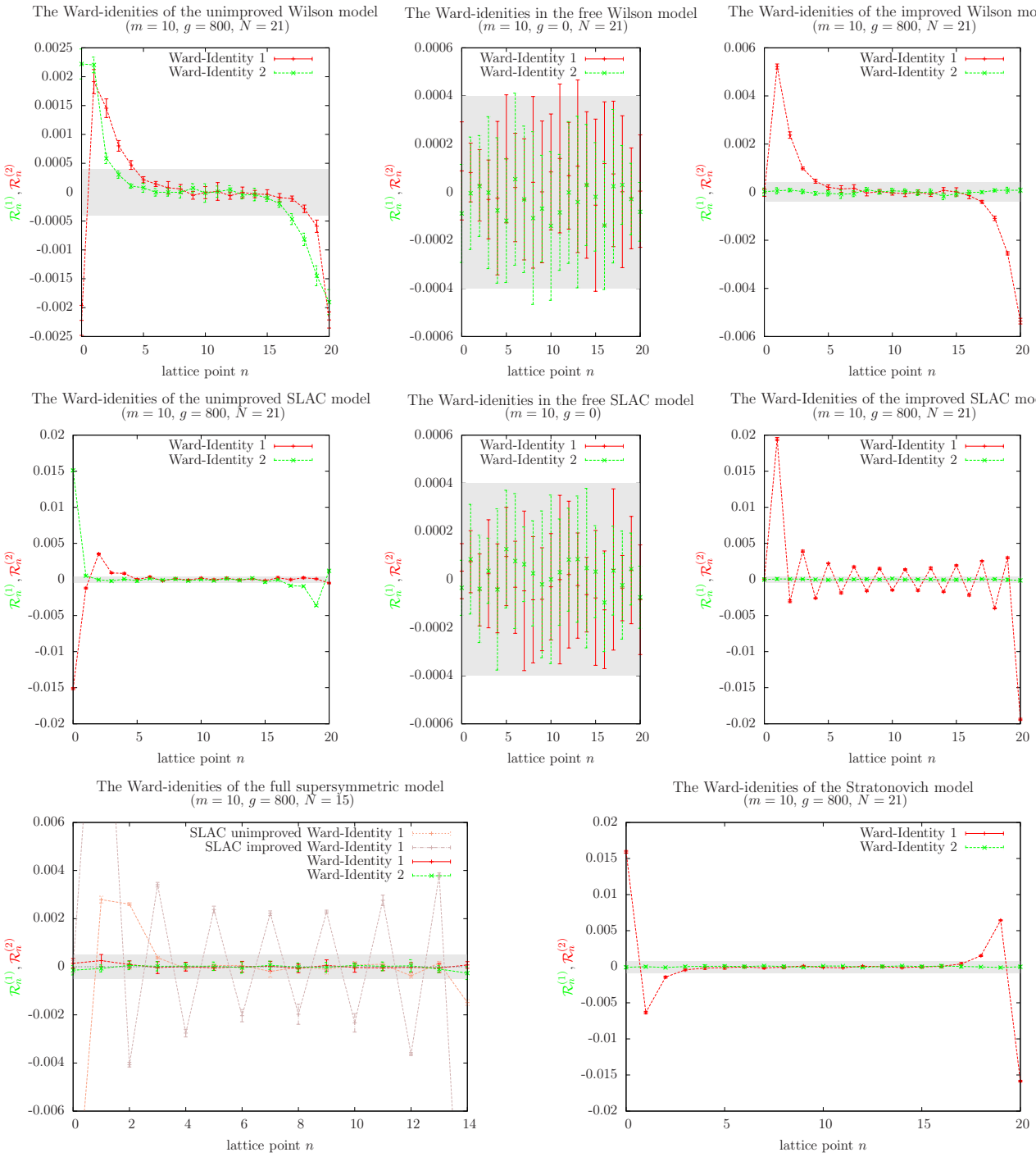


Figure 4.2: The Ward-identities of the different lattice models. To estimate the fluctuations of a fullfilled Ward-identity they were first measured for each model in the free ($g = 0$) case. The free Ward-identities should vanish in all of the considered models and their deviation from zero is indicated as a gray line in the graphs. For the Wilson and the Stratonovich discretisation on the lattice the measurements were performed with 4 independent runs of 10^6 independent configurations. For the SLAC lattice models with 10 independent runs and of 10^6 indep. configurations. All of these were done on a lattice of $N = 21$ and in the interacting case with $g/m^2 = 8$. For the full supersymmetric model the lattice size was $N = 15$ and 8 independent runs with 10^5 indep. configurations were used and for comparison also the first Ward-identity of the SLAC-models with the same parameters is shown.

4.2 Two-dimensional $\mathcal{N} = 2$ Wess-Zumino model

The classical two-dimensional $N = 2$ Wess-Zumino model was presented in section 2.1.4. Now the quantum features of this model are investigated with certain approximation methods. This low dimensional model shares already an important feature with other, more complicated, supersymmetric models: Due to the cancellation of the bosonic and fermionic loop corrections, the effective superpotential remains unchanged by the quantum effects. This is called the nonrenormalisation theorem [73, 74, 75]. In the two-dimensional case these cancellations even render the model finite. Whereas, when only the bosonic part of the theory was considered, the logarithmic divergences would need to be renormalised by the introduction of appropriate counterterms in the Lagrangian. For this model only the determination of the masses on the lattice and, for comparison, in the weak coupling expansion can be discussed here. An earlier numerical analysis of a similar model in two dimensions with local derivative operators can be found in [76, 77]. The investigation presented here are partly published in [78].

4.2.1 Different lattice formulations

A naive discretisation with an unbalanced Wilson mass term is not considered in this model. This would induce nonvanishing tadpole contributions. In addition, the formulation of a model that preserves all the supersymmetries according to section 3.1.1 is far beyond reach because of the high numerical costs. I consider here four different lattice discretisation of the action (2.26). Two of them allow for a partial realised supersymmetry on the lattice. The symmetric derivative with a Wilson term and the nonlocal SLAC derivative is used. Although in the simulations the formulation of the model in terms of real quantities (2.26) is used, I discuss the discretisation, for convenience, in terms of the complex fields. Indeed one can easily recover the real version of the model by $\phi_n = (\varphi_1)_n + i(\varphi_2)_n$ and a transition from the Weyl to the Majorana representation of the γ matrices. The superpotential has the form $W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$ with positive m and g. According to the discretisation used in chapter 3 the lattice version of the continuum action (2.23) is

$$S_L = \sum_n \left(\frac{1}{2}(\nabla_\mu \phi)_n (\nabla^\mu \phi)_n + \frac{1}{2} W'_L(\phi)_n \bar{W}'_L(\bar{\phi})_n \right) + \sum_{nm} \bar{\psi}_n (\nabla + W''_L(\phi) P_+ + \bar{W}''_L(\bar{\phi}) P_-)_{nm} \psi_m \,. \quad (4.35)$$

In this two-dimensional model the operators ∇ and $\bar{\nabla}$ should represent $\frac{1}{2}(\nabla_0 - i\nabla_1)$ and $\frac{1}{2}(\nabla_0 + i\nabla_1)$, respectively, and no one-dimensional derivative. The supersymmetry

transformations on the lattice are

$$\delta\phi_n = \bar{\psi}^1_n\varepsilon_1 + \bar{\varepsilon}_1\psi^1_n, \quad \delta\bar{\psi}^1_n = -\frac{1}{2}\bar{W}'_L(\phi)_n\bar{\varepsilon}_1 - (\nabla\phi)_n\bar{\varepsilon}_2, \quad \delta\psi^1_n = -\frac{1}{2}\bar{W}'_L(\phi)_n\varepsilon_1 + (\bar{\nabla}\phi)_n\varepsilon_2,$$
$$\delta\bar{\phi}_n = \bar{\psi}^2_n\varepsilon_2 + \bar{\varepsilon}_2\psi^2_n, \quad \delta\bar{\psi}^2_n = -(\bar{\nabla}\bar{\phi})_n\bar{\varepsilon}_1 - \frac{1}{2}W'_L(\phi)_n\bar{\varepsilon}_2, \quad \delta\psi^2_n = (\nabla\bar{\phi})_n\varepsilon_1 - \frac{1}{2}W'_L(\phi)_n\varepsilon_2.$$

A breaking of supersymmetry appears here again when the Leibniz rule is violated and no non-local interaction is used. A different formulation that preserves the supersymmetries with $\varepsilon_1 = \varepsilon_2 = 0$ and $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}$ is[6]

$$S_{NI} = \sum_n \left(\frac{1}{2}(\nabla_\mu\phi)_n(\nabla^\mu\phi)_n + \frac{1}{2}W'_L(\phi)_n\bar{W}'_L(\bar{\phi})_n + W'_L(\phi)_n(\nabla\phi)_n + \bar{W}'_L(\phi)_n(\bar{\nabla}\bar{\phi})_n \right) + \sum_{nm}\bar{\psi}_n(\nabla + W''_L(\phi)P_+ + W''_L(\bar{\phi})P_-)_{nm}\psi_m\,. \quad (4.36)$$

The difference between the improved and unimproved model is the improvement term

$$\Delta S = S_{NI} - S_L = \sum_n \left(W'_L(\phi)_n(\nabla\phi)_n + \bar{W}'_L(\phi)_n(\bar{\nabla}\bar{\phi})_n \right)\,. \quad (4.37)$$

The continuum counterpart of this term vanishes in the continuum for periodic boundary conditions. In the cases considered here the improvement term on the lattice is in the real formulation

$$\Delta S = g\sum_n \Big(\big((\nabla_0\varphi_1)_n + (\nabla_1\varphi_2)_n\big)\big((\varphi_1)^2_n + (\varphi_2)^2_n\big) - \big((\nabla_0\varphi_2)_n - (\nabla_1\varphi_1)_n\big)(\varphi_1)_n(\varphi_2)_n \Big)\,. \quad (4.38)$$

Its contribution gets hence smaller the smaller g is.

The improved and unimproved Wilson model

In case of the unimproved Wilson model the action (4.35) is used with a symmetric derivative $\nabla_\mu = \nabla^{(s)}_\mu$ and a superpotential that includes a Wilson mass term and consistently removes the doublers. It is

$$W'_L(\phi)_n = (m^{(W)}\phi)_n + W'(\phi)\delta_{nm}, \qquad W''_L(\phi)_n = m^{(W)}_{nm} + W'(\phi)\delta_{nm}, \quad (4.39)$$
$$\bar{W}'_L(\bar{\phi})_n = (m^{(W)}\bar{\phi})_n + \bar{W}'(\bar{\phi})\delta_{nm}, \qquad W''_L(\bar{\phi})_n = m^{(W)}_{nm} + \bar{W}'(\bar{\phi})\delta_{nm}\,. \quad (4.40)$$

It is clear that the Z^R_2 symmetry of the model is broken because of the nondiagonal additional mass term. It can only be recovered in the continuum limit. The improved Wilson model uses the general improved action (4.36) with the same specifications for the derivative operator and the superpotential. This improved model was also considered in

[6] In the fermionic part a summation over the spinor indices is always understood: $\sum_{\alpha,\beta}\bar{\psi}_\alpha(K_f)_{\alpha\beta}\psi_\beta$.

[76].

The improved and unimproved SLAC model

In the SLAC models the nonlocal SLAC derivative is used ($\nabla_\mu = \nabla_\mu^{\mathrm{SLAC}}$). Since the nonlocal derivative already removes the doublers, the continuum superpotential can be used:

$$W'_L(\phi)_n = W'(\phi)\delta_{nm}, \qquad W''_L(\phi)_n = W'(\phi)\delta_{nm}, \tag{4.41}$$

$$\bar{W}'_L(\bar{\phi})_n = \bar{W}'(\bar{\phi})\delta_{nm}, \qquad W''_L(\bar{\phi})_n = \bar{W}'(\bar{\phi})\delta_{nm}\,. \tag{4.42}$$

Again the improved model is the general improved lattice action (4.36) with the same specifications.

4.2.2 Some details of the simulation

As in the one-dimensional case the simulations do not yield proper results without full dynamical fermions so the standard HMC algorithm was applied. In this case, however, the inversion of the fermion determinant for the fermionic contribution of the force (4.29) on large lattices is too slow to achieve a high enough statistic within the accessible computer time. To circumvent this restriction, a pseudo fermion algorithm was applied. In this algorithm the fermionic determinant is rewritten in terms of bosonic fields. In the present case real bosonic fields χ were introduced to arrive at $\sum_{nm} \chi_n (K_f K_f^T)^{-1}_{nm} \chi_m = \eta^T \eta$ instead of $|\det(K_f)|$. For a further improvement of the simulation also higher order integrators in the molecular dynamic and Fourier acceleration were applied. All these algorithmic details were already discussed at length in the PhD thesis of T. Kästner [79]. The lowdimensional supersymmetric models turned out to be a good playground for the simulations of dynamical fermions with different algorithms.

The fermion determinant can in this model in principle have also a negative sign. Such a sign change is neglected in the pseudo fermion algorithm. However, for a sign change of the determinant at least one eigenvalue of the fermion matrix must change its sign. If this happens in a continuous way, in the intermediate configurations very small (even zero) eigenvalues are expected. Consequently, the contribution $\log\det K_f$ is very large so the intermediate configurations are suppressed by a large (even divergent) S_{eff}. In the real simulations a relevant number of the sign changes occurs only for a rather large coupling, $g/m > 1$.

4.2.3 Problems with the improved actions

The improvement term should become irrelevant in the continuum limit since its continuum counterpart is zero for periodic boundary conditions. This is observed also in the simulations at weak coupling (g/m small), where for the most of obtained configurations

the ΔS is much smaller than S_L. Consequently, also $\langle \Delta S \rangle$ is small. The smaller the coupling and the lattice spacing the less important becomes ΔS for the overall value of the action.

This good behaviour is, however, not observed for all values of the simulation parameters. At a larger value of g/m the improvement term can become the dominant part of the improved action. In fact, a certain transition is observed during the simulations, as shown in figure 4.3 (a). At first only configurations with a small ΔS are obtained. After some time the picture changes: The improvement term dominates. In addition, a larger value of the fermion determinant is observed. It is clear that the improvement term can be that dominant only when the high momentum modes are enhanced. The domination of high momentum modes and the improvement term should clearly be excluded since it is unphysical.

To exclude unphysical contributions, one has to check that $\langle \Delta S \rangle$ remains small in the simulation. More precisely, one has to look at the Monte-Carlo history for a transition to configurations with the dominant improvement term. It is observed that this transition appears the earlier the larger the dimensionless parameters g/m and am are. It can, nevertheless, not be in general excluded for any parameter range and any of the improved models. The discretised improvement term seems to allow for this additional phase and a contribution of it is only avoided because of the finite length of the Monte-Carlo history.

4.2.4 Masses

The masses were determined from the two-point functions. Instead of the full bosonic two-point functions (all correlators between φ_1, φ_2) the following observable is considered

$$G_t^{(b)} = \frac{1}{N_0 N_1^2} \sum_{(n_1)^0 - (n_2)^0 = t} \sum_{(n_1)^1=0}^{N_1-1} \sum_{(n_2)^1=0}^{N_1-1} \langle (\varphi_2)_{n_1} (\varphi_2)_{n_2} \rangle \,. \tag{4.43}$$

The averaging summation over the lattice points in space direction $(n)^1$ is the same as a projection onto zero spacial momentum. Because of the translational invariance, all measured correlations with the same distance $(n_1)^0 - (n_2)^0$ should be the same. The averaging over these correlations is hence done only to decrease the statistical error. In a similar way one can define fermionic observables

$$G_t^{(f)} = \frac{1}{N_0 N_1^2} \sum_{(n_1)^0 - (n_2)^0 = t} \sum_{(n_1)^1=0}^{N_1-1} \sum_{(n_2)^1=0}^{N_1-1} \sum_{\alpha=1}^{2} \langle (\psi_\alpha)_{n_1} (\bar{\psi}_\alpha)_{n_2} \rangle \,. \tag{4.44}$$

Again the fermionic correlators are calculated from the inverse of K_f. Both correlation functions are proportional to $\cosh(m_{\text{eff}} a_0 (t - N_0/2))$. This can already be seen from the calculation of these functions in perturbation theory. Quite analogous to the one

dimensional case the masses are obtained from a fit of the two-point functions.

The model has a Z_2^R symmetry as shown in section 2.1.4. It is also known that this symmetry is spontaneously broken in the quantum theory. On the lattice we can, however, consider only a finite volume. For a finite volume spontaneous symmetry breaking does not occur [36, 80]. Only at an infinite volume the tunnelling amplitude between the two degenerate minima of the classical theory is suppressed enough. Then they can be considered as two degenerate ground states. The transition to the spontaneously broken phase is not visible at any finite lattice size but occurs only for infinite volume. Thus the thermodynamic limit cannot be extrapolated from the lattice data in a straight forward way. To circumvent this problem, the following prescription is applied: The classical minima are at $(\varphi_1 = 0; \varphi_2 = 0)$ and $(\varphi_1 = -m/g; \varphi_2 = 0)$. We can separate, therefore, the configurations in two classes assigned to either of the two minima. For the first class the lattice average of φ_1 is larger than $-m/(2g)$. The second class has a average of φ_1 that is smaller than this value and belongs to the minima at $(\varphi_1 = -m/g; \varphi_2 = 0)$. For this second class the Z_2^R symmetry transformation is applied. In this way we get effectively only the contribution of one of the vacua already at a finite volume.[7] Similar strategies were applied in [81]. However, one has to check the scaling of the observables in this limit since they achieve an additional contribution. In case of the two-point function these finite volume effects can be neglected for the weak couplings. This was checked explicitly in a finite volume analysis of the observables. At a larger coupling the finite volume effects must be treated more carefully.

For the determination of the masses from the two-point functions a fit with the cosh function was applied in a region well enough separated from $t = 0$ and $t = N - 1$. This procedure was applied in the Wilson case. For the SLAC models again oscillations were visible. In the propagators of the free theory a certain averaging procedure over a number of lattice points was observed to yield the best results for the masses; details can be found in [78]. Since in the one-dimensional case and in the free theory the masses obtained from the SLAC derivative proved to be very close to the continuum results, an extrapolation to the continuum was not applied here. Instead, only one lattice size was investigated and a "perfect" scaling of the SLAC derivative was assumed. The extrapolation of the Wilson derivative is depicted in figure 4.3(b). As assumed the SLAC derivative agrees well with the extrapolated Wilson result.

The masses obtained in the mentioned way are all in good agreement with the perturbative result (cf. appendix G), as shown in figure 4.3(c) and 4.3(d). This is quite in contrast to the one-dimensional model where at the same value of the dimensionless parameter in the weak coupling expansion (one dimension g/m^2; two dimensions g/m) a

[7] Even for the Wilson models the strategy is applied although the symmetry is broken at a finite lattice spacing.

larger deviation is visible.

Further investigations show a splitting of the fermionic and bosonic masses at a high coupling. Its origin is a finite size effect since it decreases for larger volumes. This indicates that the above strategy, necessary to measure quantities in a spontaneously broken phase, is in conflict with supersymmetry. Thus, appart from the usual breaking mechanisms of supersymmetry on the lattice, one has to be careful with the additional techniques applied in the simulations. In the present case the thermodynamic limit of the lattice results indicates a restoration of the supersymmetry in terms of mass-degeneracy. In the future this will be investigated more carefully, also in terms of the Ward identities.

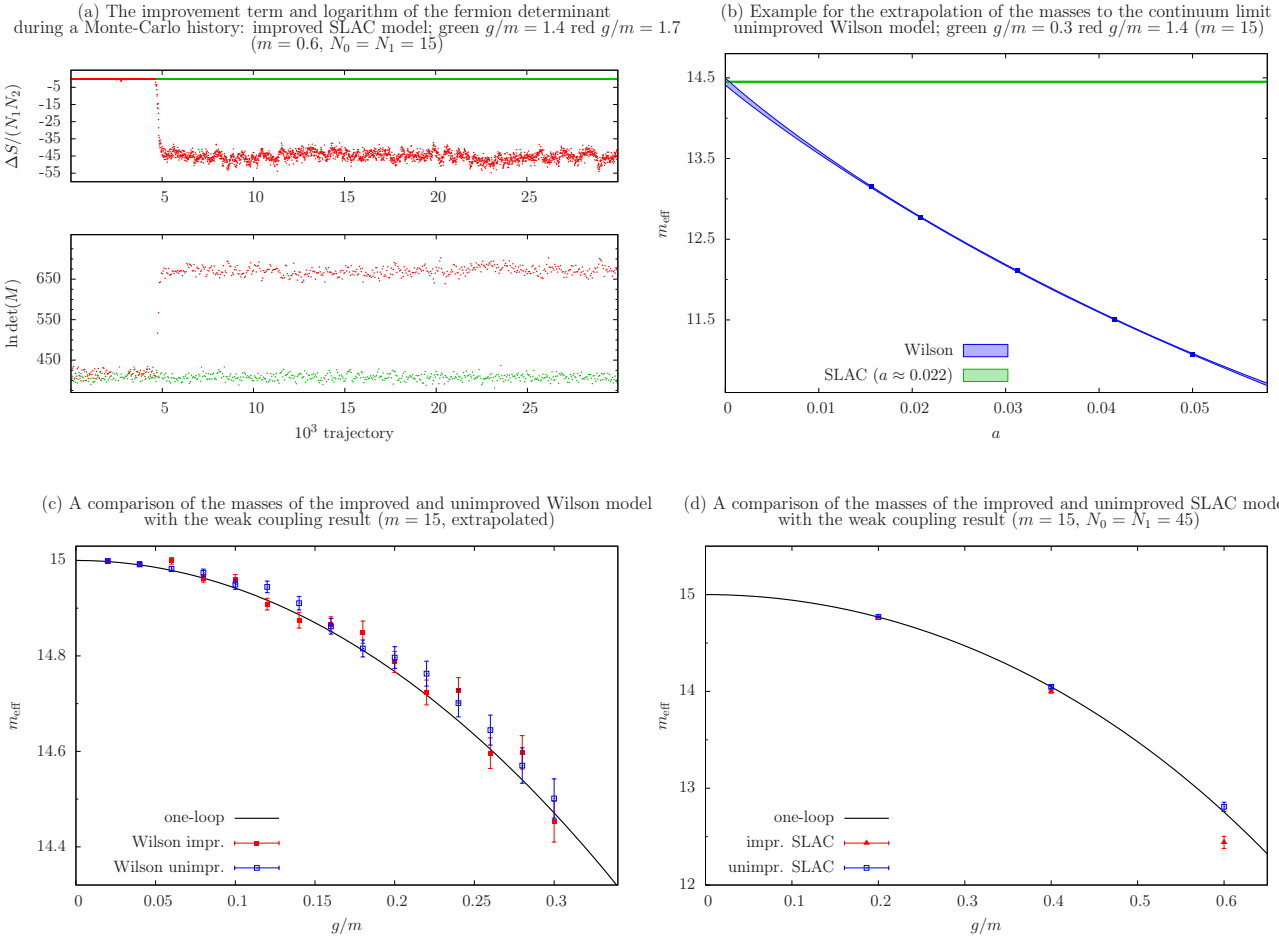


Figure 4.3: In the first picture, (a), the breakdown of the simulation of an improved model is shown. After a certain number of configurations are generated with a rather small improvement term the situation changes and the dominant contribution is generated by this term. The second picture, (b), shows an extrapolation to the continuum limit. For the Wilson models not only a linear extrapolation was used. Instead $m_{\text{eff}}(a) = m_{\text{eff}} + Aa + Ba^{3/2}$ was applied which was observed to be the best approximation for the extrapolation to the continuum of the masses obtained from the free propagator. Figure (c) shows the results of the extrapolation in comparison to the weak coupling approximation. In figure (d) the SLAC results are shown in a larger range of the coupling strength. At large coupling a small splitting of bosonic and fermionic masses is observed. This is, however, due to finite volume effects.

5 The generalisation of the Ginsparg-Wilson relation and supersymmetry

Let us take a closer look at the problem of finding a supersymmetric realisation on the lattice. In the previous chapters the discretisation of the theory is derived in a rather classical way, without any reference to the quantum nature, expressed, e. g., in terms of the path integral. The fields and operators of the action are just replaced by some discretised counterparts. In the same classical way the symmetries are discretised. As shown in chapter 3.1.1, a discretisation of the symmetry transformations with the same operators as in the local action fails. A symmetry of the lattice action can, however, still be found in terms of an expansion, cf. section 3.1.3. These discrete symmetries allow for a classical supersymmetric continuum limit. In contrast, they do not guarantee the symmetry in the continuum limit of the quantum theory. To find out the symmetry in this limit, the discretised theory is analysed, afterwards, in lattice perturbation theory and numerical simulations.

Here a different strategy is applied. The main goal is the derivation of certain relations for the discretised action, that ensure the correct quantum properties of the lattice theory. For this reason not only the classical action must be dicretised. Instead, the whole continuum path integral must be mapped onto a lattice path integral.

This approach was first applied by Ginsparg and Wilson for the chiral symmetry [82]. As discussed in section 3.2, there is no local chiral fermion action on the lattice without a doubling of the species. The situation seems in that way much similar to supersymmetry, where no local symmetric realisation can be found. The Ginsparg-Wilson relation provides a controlled breaking of the symmetry on the lattice. It ensures the correct reproduction of the chiral symmetry and its anomaly in the continuum limit. In that way it represents the symmetry on the lattice. I derive here a generalisation of this approach that holds for any linear global symmetry. Later on I specialise it to chiral symmetry to reproduce their findings.

Note that the map from the continuum onto the lattice is a renormalisation group step. The theory is mapped onto another one that contains a smaller number of degrees of freedom. The Ginsparg-Wilson relation is in that sense a modified Slavnov-Tailor identity as mentioned in section 2.2.3. Studies of a generalised Ginsparg-Wilson approach can also be found in [83, 84, 85, 86]. These results include the first derivation of a generalised Ginsparg-Wilson relation is that applies to arbitrary lattice actions and arbitrary global symmetries (including supersymmetry). They were published in [87].

Throughout this chapter a summation (without an additional factor of the lattice spacing) is understood, whenever lattice indices or multiplet indices are doubly encountered. ϕ_n is the field on the lattice and $\varphi(x)$ its continuum counterpart.

5.1 The Ginsparg-Wilson relation for a general global symmetry

In order to integrate out certain degrees of freedom a regulator term is introduced in the path integral. In the Ginsparg-Wilson approach it connects the lattice degrees of freedom with the continuum fields. This regulator can break the symmetries of the continuum theory. The symmetry is, however, broken only in a mild way that means it is restored in the continuum limit (cf. section 2.2.3).

The regulator consists of two parts: in the first step an averaging of the continuum field is introduced. This is necessary to reduce the number of degrees of freedom of the theory to a finite number of lattice fields. The second step introduces a quadratic regulator term, in which the lattice fields are connected with the averaged continuum fields via a blocking kernel α. This quadratic term is used instead of a direct assignment (with a "delta") of the lattice fields to the averaged fields. As is shown in the quadratic solutions the introduction of the blocking kernel ensures the locality of the lattice action.

5.1.1 Blocking procedure and the Wilsonian effective action

The averaging of the continuum fields is done with a function f that is peaked around zero, where the main contribution to the averaging should come from.[1] The averaged fields ϕ_f resulting from this procedure are defined as

$$\phi_f^i(x_n) := \int d^{\mathrm{D}}x\, f(x_n - x)\, \varphi^i(x)\,. \tag{5.1}$$

The index i labels the multiplet components of φ on which the symmetry operator acts in a linear way.

With these averaged continuum fields, the blocked lattice action $S[\phi]$ is derived from a given continuum action by[2]

$$e^{-S_{\mathrm{w}}[\phi]} = \mathfrak{N}(\alpha) \int \mathcal{D}\varphi e^{-S[\varphi] - \frac{1}{2}(\phi_n^i - \phi_f^i(x_n))\alpha_{nm}^{ij}(\phi_m^j - \phi_f^j(x_m))}\,. \tag{5.2}$$

The quadratic regulator term connects the lattice and the continuum fields.

In order to find out the meaning of these two steps, I derive the relation between the lattice and continuum generating functional. As in the continuum, all lattice observables can be derived from a differentiation of the lattice generating functional,

$$Z_L[j] = \int \prod_n d\phi_n e^{-S_{\mathrm{w}}[\phi] + j\phi}\,, \tag{5.3}$$

[1] In addition f should have the dimension inverse to the D-dimensional integral, such that the original and blocked fields have the same dimension.

[2] The normalisation is such that $\mathfrak{N}(\alpha)^{-1} = \mathrm{Sdet}(\alpha) = \int \prod_n d\phi_n e^{-\frac{1}{2}\phi_n^i \alpha_{nm}^{ij} \phi_m^j}$, see section A.4. Later on this factor is absorbed into $\mathcal{D}'\varphi$. In addition α should fulfil $\tilde{\phi}^i \alpha_{ij} \phi^j = \phi^i \alpha_{ij} \tilde{\phi}^j$ for two multiplets ϕ and $\tilde{\phi}$.

with respect to j_n. Inserting (5.2) and preforming a Gaussian integration, one immediately arrives at

$$Z_L[j] = e^{\frac{1}{2}j\alpha^{-1}j} \int \mathcal{D}\varphi e^{-S[\varphi]+j\phi_f} \,. \tag{5.4}$$

This shows that with an action defined by (5.2) one measures on the lattice averaged continuum observables, apart from an additional Gaussian contribution. There are many properties of the theory that can be determined already from these averaged quantities. Moreover, the continuum limit of such a lattice theory is a safe procedure: The averaging function must get narrower and narrower. This just indicates that the resolution of our measurements is increased with a smaller lattice spacing.[3] In addition α^{-1} must vanish in this limit, for quantities containing a contribution of this matrix. The blocked action S_{w} (or Wilsonian effective action) obtained in this way is hence also-called a perfect lattice action. Note that this action is similar to $\Gamma[k, \phi]$ in section 2.2.1.

5.1.2 A generalisation of the approach of Ginsparg and Wilson

In most cases the blocked action $S[\phi]$ cannot be directly calculated from the continuum action according to equation (5.2) and the form of this action may also not be convenient for the numerical computations. Hence one has to allow for more general actions. The main properties of the continuum theory should, however, be correctly resembled by the lattice counterpart. That means with respect to these properties there should be no difference between the considered lattice action and the perfect action.

In our case the symmetry of the continuum theory is such a property. Therefore I show how the continuum symmetry is represented at the level of the perfect action. The result is a relation that contains the information of the continuum symmetry. This is similar to the Ward identities that represent the symmetries at the level of the effective action. The two steps (eq. (5.1) and (5.2)) in the blocking have a different implications in the derivation of the relation. The starting point is a symmetry transformation in the continuum

$$\delta\varphi^i = \mathcal{M}^{ij}\varphi^j \,. \tag{5.5}$$

By definition, the continuum action is invariant under such a transformation and the averaged fields are transformed as

$$\delta\phi^i_f(x_n) = \int d^{\mathrm{D}}x f(x - x_n)\mathcal{M}^{ij}\varphi^j \,. \tag{5.6}$$

[3] Indeed with a detector that has a finite resolution in space-time one would measure such averaged observables.

Suppose now the following additional condition is fullfilled for the symmetry relations

$$\delta\phi_f^i(x_n) = \int d^{\mathrm{D}}x f(x-x_n)\mathcal{M}^{ij}\varphi^j \stackrel{!}{=} M_{nm}^{ij}\int d^{\mathrm{D}}x f(x-x_m)\varphi_m^j = M_{nm}^{ij}\phi_f^j(x_m)\,. \quad (5.7)$$

In the derivation of Ginsparg and Wilson only chiral symmetry was considered. In that case it is rather trivial to fulfil this condition, as will be shown in more detail later on. It defines a naive symmetry transformation,

$$\delta\phi_n^i = M_{nm}^{ij}\phi_m\,, \quad (5.8)$$

on the lattice.

Such a naive transformation of the lattice fields ($\phi_n^i \to \phi_n^i + \delta\phi_n^i$) is now performed. In addition, the continuum fields are transformed with a continuum symmetry transformation ($\varphi_n^i \to \varphi_n^i + \delta\varphi_n^i$). The result is (up to linear order in the transformations)

$$M_{nm}^{ij}\phi_m^j\frac{\delta}{\delta\phi_n^i}e^{-S_{\mathrm{w}}[\phi]} = \int \mathcal{D}'_\delta\varphi\; e^{-S[\varphi]}\; M_{nm}^{ij}(\phi-\phi_f)_m^j\frac{\delta}{\delta\phi_n^i}e^{-\frac{1}{2}(\phi-\phi_f)\alpha(\phi-\phi_f)}\,. \quad (5.9)$$

It is assumed that the continuum action is invariant and the $\mathcal{D}'_\delta$ indicates a possible additional change of the measure, e. g., due to an anomaly. Such a change can be formally represented in terms of the Jacobi determinant of the continuum symmetry transformation. Up to linear order it leads to the additional contribution $\langle \mathrm{STr}\mathcal{M}\rangle$ on the right hand side.[4] The averaged fields ϕ_f^i, depending explicitly on the continuum fields, can be replaced by a derivative with respect to the lattice fields according to

$$\begin{aligned}(\phi-\phi_f)_m^j\frac{\delta}{\delta\phi_n^i}e^{-\frac{1}{2}(\phi-\phi_f)\alpha(\phi-\phi_f)} = \\ -\left((-1)^{|\phi^i|}\delta_{mn}\delta^{ij} + \alpha^{-1}{}^{jk}_{mr}\frac{\delta}{\delta\phi_r^k}\frac{\delta}{\delta\phi_n^i}\right)e^{-\frac{1}{2}(\phi-\phi_f)\alpha(\phi-\phi_f)}\,. \quad (5.10)\end{aligned}$$

The $(-1)^{|\phi^i|}$ is (-1) for every fermionic field ϕ^i and 1 otherwise. The derivatives with respect to the lattice fields can be pulled in front of the path integral. In that way one finally arrives at

$$M_{nm}^{ij}\phi_m^j\frac{\delta S_{\mathrm{w}}}{\delta\phi_n^i} = (M\alpha^{-1})_{nm}^{ij}\left(\frac{\delta S_{\mathrm{w}}}{\delta\phi_m^j}\frac{\delta S_{\mathrm{w}}}{\delta\phi_n^i} - \frac{\delta^2 S_{\mathrm{w}}}{\delta\phi_m^j\delta\phi_n^i}\right) + (\mathrm{Str}M - \langle\mathrm{Str}\mathcal{M}\rangle)\,. \quad (5.11)$$

This relation depends only on the lattice fields and the blocked lattice action. It has no direct reference to the continuum any more. It is a remnant of the continuum symmetry on the lattice.[5] It is a property of any blocked (perfect) lattice action that is derived

[4] The expression $\langle\mathrm{Str}\mathcal{M}\rangle$ stands for $\int\mathcal{D}\varphi\;\mathrm{Str}(\mathcal{M})\;e^{-S[\varphi]-\frac{1}{2}(\phi-\phi_f)\alpha(\phi-\phi_f)+S_{\mathrm{w}}[\phi]}$ (cf. section 2.2.4).

[5] Or in more physical words: The way the symmetry is represented in the measurements of the men-

from a symmetric continuum action. Thus, if a lattice theory fulfils this condition, it is compatible with the continuum symmetry. To get a symmetric lattice theory, relation (5.11) is solved for the lattice action $S_{\rm w}$.

While the left hand side of the relation (5.11) is just the naive symmetry variation of the action $S_{\rm w}$, the right hand side constitutes some nontrivial modification of it. The last term is the difference between the continuum anomaly and a lattice contribution to it.[6] Apart from this terms the relation is reduced to the naive invariance

$$M^{ij}_{nm}\phi^j_m \frac{\delta S_{\rm w}}{\delta \phi^i_n} = 0\,, \tag{5.12}$$

if a symmetric blocking matrix, $\alpha = \alpha_S$, is chosen. Such a blocking matrix fulfils the condition

$$M\alpha_S^{-1} \pm (M\alpha_S^{-1})^T = 0\,. \tag{5.13}$$

The minus sign appears whenever fermionic fields are transformed into fermionic fields by the naive symmetry transformation (5.8). Furthermore, if an additional matrix α_S^{-1} is added to α^{-1}, the symmetry relation (5.11) remains unchanged. [7]

5.2 General solutions for a quadratic action

The situation is much simplified for a quadratic action like

$$S_{\rm w} = \frac{1}{2}\phi^i_n K^{ij}_{nm}\phi^j_m\,, \tag{5.14}$$

with the kernel K comprising kinetic and mass terms. Although this is seemingly a simple situation, it contains already the nontrivial result Ginsparg and Wilson [82] have found for the chiral symmetry. There the gauge fields are treated as a background and a relation for the remaining quadratic fermion part of the action is derived. K can also in our case be dependent on some background fields.

In this case the relation (5.11) simplifies to

$$\phi M^T K \phi = \phi\, K^T (M\alpha^{-1})^T K\, \phi - {\rm Tr}\,(M\alpha^{-1})\, K^T + ({\rm Str}M - \langle {\rm Str}\mathcal{M}\rangle)\,. \tag{5.15}$$

For the present discussion the last two terms are neglected. The remaining part is equiv-

tioned detector with a finite resultion.

[6] On the lattice there still remains an integration of a, with respect to the continuum, reduced number of degrees of freedom. The related measure of this integration can carry a part of the continuum anomaly.

[7] Note that in the approach of [85] to a supersymmetric Ginsparg-Wilson relation a symmetric blocking kernel is chosen and (5.7) is not considered. Thus the resulting relation did, in fact, not differ from the classical considerations of the last chapters.

alent to the matrix identity[8]

$$M^T K \pm (M^T K)^T = K^T (M\alpha^{-1})^T K \pm (K^T (M\alpha^{-1})^T K)^T \,. \tag{5.16}$$

If this condition is solved for K, a lattice action compatible with the continuum symmetry is derived. A further reformulation of the relation,

$$(K^{-1})^T M^T \pm M K^{-1} = (\alpha^{-1})^T M^T \pm M\alpha^{-1} \,, \tag{5.17}$$

shows that it resembles a compensation between the symmetry breaking of the action and the symmetry breaking of the blocking kernel. This reformulation also indicates that the general solution (5.11) for a quadratic action is

$$K^{-1} = \alpha^{-1} - \alpha_S^{-1} \,. \tag{5.18}$$

The K^{-1} is hence uniquely determined by α^{-1} up to symmetry-preserving terms α_S^{-1}. To identify the relation as a symmetry of the lattice action, I introduce the matrix

$$M_{\text{def}} := M \left(\mathbb{1} - \alpha^{-1} K \right) \,. \tag{5.19}$$

Equation (5.16) is then nothing but the invariance of the lattice action under the transformation $\delta\phi = M_{\text{def}}\phi$:

$$M_{\text{def}}^T K \pm (M_{\text{def}}^T K)^T = 0 \,. \tag{5.20}$$

Hence the relation represents a lattice symmetry. The condition for the lattice action is the invariance under this particular symmetry. The matrix α is, apart from some hermiticity requirements, so far unspecified. One may chose an appropriate α for a given K.

However, the continuum action must emerge from the lattice action in the continuum limit. Thus not all K and α may be chosen. Furthermore, a well behaved choice for K should be local. With the solution (5.18) of the symmetry relation M_{def} is

$$M_{\text{def}} = -M\alpha_S^{-1} K \,. \tag{5.21}$$

The matrix α_S^{-1} is constrained only by the requirement (5.13). Hence, even the solution $M_{\text{def}} = 0$ ($K^{-1} = \alpha^{-1}$) seems possible. A suitable lattice symmetry operator should carry some information about the (local) continuum symmetry. One reason for this is that a symmetry operator also defines an observable, that can be measured on the lattice to get information about its continuum value. Hence there are in fact two conditions that

[8]The minus sign again applies whenever the naive symmetry transforms fermions into fermions.

constain our choices of K and α: *The action and the symmetry operator M_{def} are required to be local and approach their continuum counterparts in the continuum limit.*

5.2.1 Chiral Symmetry

To illustrate the above general findings, the explicit example of chiral symmetry is considered. The result is the well known Ginsparg-Wilson relation. The corresponding symmetry defined by M_{def} was already found by Lüscher [88] as a reformulation of this relation.

Consider an action of a field multiplet with two fermionic fields: $\phi = (\psi, \bar{\psi}^T)$. K is given in terms of the Dirac operator $\hat{\not{D}}$,

$$\frac{K}{a^d} = \begin{pmatrix} 0 & -\hat{\not{D}}^T \\ \hat{\not{D}} & 0 \end{pmatrix} , \tag{5.22}$$

The lattice action (5.14) is thus a discretisation of (2.3). As we have seen in section 2.1.4, the continuum action is invariant under the chiral symmetry transformations generated by

$$\mathcal{M}\varphi = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5^T \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} , \tag{5.23}$$

with $\gamma_5^\dagger = \gamma_5$. In the present case the additional requirement (5.7) is trivially fulfilled since the generator acts only on the multiplet index. The naive transformation is hence the same as the continuum transformation. The matrix α is chosen to be

$$\frac{\alpha}{a^d} = \begin{pmatrix} 0 & -\alpha_1^T \\ \alpha_1 & 0 \end{pmatrix} , \tag{5.24}$$

with a general α_1. In order to get a real lattice action both, $\mathcal{D}$ and α_1, must be hermitian. With this setup the general matrix relation (5.16) becomes

$$\{\hat{\not{D}}\,,\,\gamma_5\} = \hat{\not{D}}\{\gamma_5\,,\,\alpha_1^{-1}\}\hat{\not{D}}\,. \tag{5.25}$$

As indicated in the general case (5.20) it can be rewritten in terms of a deformed symmetry with M_{def} given as

$$M_{\mathrm{def}} = \begin{pmatrix} \gamma_{5,\mathrm{def}} & 0 \\ 0 & (\bar{\gamma}_{5,\mathrm{def}})^T \end{pmatrix} . \tag{5.26}$$

It is intresting to note that instead of a modification of γ_5 in $\mathcal{M}$ the two different matices

$$\gamma_{5,\mathrm{def}} = \gamma_5(1 - \alpha_1^{-1}\hat{\not{\mathcal{D}}})\,, \quad \text{and} \quad \bar{\gamma}_{5,\mathrm{def}} = (1 - \hat{\not{\mathcal{D}}}\alpha_1^{-1})\gamma_5 \tag{5.27}$$

appear in the deformed symmetry transformations. Note that for hermitian α_1^{-1} and $\hat{\not{\mathcal{D}}}$ $\bar{\gamma}_{5,\mathrm{def}} = \gamma_{5,\mathrm{def}}^\dagger$ holds. With these matices the symmetry can be rewritten as

$$\bar{\gamma}_{5,\mathrm{def}}\hat{\not{\mathcal{D}}} + \hat{\not{\mathcal{D}}}\gamma_{5,\mathrm{def}} = 0\,. \tag{5.28}$$

This is the mentioned Ginsparg-Wilson-Lüscher symmetry. The simplest choice for α_1 is $1/a$, $\alpha_1 = 1/a\mathbb{1}$, where $\mathbb{1}$ is diagonal with respect to the lattice sites and the identity in Dirac space. The well know result of this specification,

$$\{\hat{\not{\mathcal{D}}}\,,\,\gamma_5\} = 2a\hat{\not{\mathcal{D}}}\gamma_5\hat{\not{\mathcal{D}}}\,, \tag{5.29}$$

is the Ginsparg-Wilson relation. For this choice of α the requirements of the locality and continuum limit of the action and the deformed symmetry generator are fulfilled when $\hat{\not{\mathcal{D}}}$ is local.

A more general solution for a general $\hat{\not{\mathcal{D}}}$ (cf. (5.18)) is

$$\alpha_1^{-1} = \hat{\not{\mathcal{D}}}^{-1} + \alpha_{1,S}^{-1}\,, \tag{5.30}$$

where the symmetric part satisfies $\alpha_{1,S}^{-1}\gamma_5 + \gamma_5\alpha_{1,S}^{-1} = 0$. $\hat{\not{\mathcal{D}}}$ is assumed to be local. Then the locality condition of M_{def} demands a local α_1. This is in accordance with the assumptions made in [89, 90]. Only the symmetric part of $\hat{\not{\mathcal{D}}}^{-1}$ can be absorbed in α_S^{-1}. Hence a necessary condition for a Dirac operator on the lattice is that the non-symmetric part $\mathbb{1}\frac{1}{\mathrm{D}}\mathrm{tr}(\hat{\not{\mathcal{D}}}^{-1})$ is local. This is not the case for the massless Wilson-Dirac operator.

Finally I will add a short note about the meaning of the ϕ independent part of equation (5.15) for the chiral symmetry. It is in this case

$$\mathrm{Tr}\,\gamma_5\alpha_1^{-1}\hat{\not{\mathcal{D}}} + (\mathrm{Tr}_{\mathrm{lattice}}\,\gamma_5 - \langle\mathrm{Tr}_{\mathrm{cont}}\,\gamma_5\rangle) = 0\,. \tag{5.31}$$

From the lattice trace we get no contribution $\mathrm{Tr}_{\mathrm{lattice}}\,\gamma_5 = 0$. When the simple blocking matrix $\alpha \propto \mathbb{1}$ is chosen, this expression becomes

$$2\,\mathrm{Tr}\,\gamma_5\hat{\not{\mathcal{D}}} = (n_+ - n_-)_{\mathrm{lattice}} = \langle\mathrm{Tr}_{\mathrm{cont}}\,\gamma_5\rangle = (n_+ - n_-)_{\mathrm{continuum}}\,. \tag{5.32}$$

When the blocking corresponds to a gauge-invariant regulator of the continuum trace as in eqn. (2.52). The relation defines in this way a lattice index of a Dirac operator.

5.2.2 Explicit solution for a quadratic action

For the quadratic continuum action, $S[\varphi] = \frac{1}{2}\varphi^i_x \tilde{K}^{ij}_{xy}\varphi^j_y$, the path integral in equation (5.2) can be calculated to obtain the blocked lattice action $S_{\mathrm{w}}[\phi] = \frac{1}{2}\phi K \phi$. It is a simple Gaussian integration. The result of this integration is called the fixed point operator [91] and reads in momentum space

$$K(p_k) = \left(\sum_{l\in\mathbb{Z}} \frac{f^*(p_k + l\frac{2\pi}{a}) f(p_k + l\frac{2\pi}{a})}{\tilde{K}(p_k + l\frac{2\pi}{a})} + \alpha^{-1}(p_k) \right)^{-1} . \tag{5.33}$$

Note that such a solution of the Ginsparg-Wilson relation was already mentioned in [82].

In many cases $f(x)$ is considered to be the averaging over one lattice spacing, e.g. in one dimension

$$f(x) = \begin{cases} 1/a & \text{if } |x| < a/2 \\ 0 & \text{otherwise} \end{cases} , \tag{5.34}$$

which means $f(p_k) = \frac{2}{La}\frac{\sin(p_k a/2)}{p_k}$. Such an averaging was applied in [92] to construct a free supersymmetric (perfect) lattice theory. It must be stressed that, since the constraint (5.7) was not considered there, the symmetry properties of the resulting S_{w} cannot be expressed in terms of a lattice symmetry involving only lattice fields: equation (5.7) demands for the derivative operator appearing in the supersymmetry transformations

$$\sum_m \nabla_{nm}\phi(am) = \frac{1}{a}(\varphi(an + a/2) - \varphi(an - a/2)) \tag{5.35}$$

and this cannot be fulfilled for any ∇_{nm} since the transformation involves the continuum fields.

To interpret the right hand side of equation (5.35) a new field was introduced in [92], which is defined to be $\frac{1}{a}\varphi(an + a/2)$ at the lattice point an. Then the lattice fields are transformed into such fields under the supersymmetry transformations. They are rather a continuum than a blocked lattice quantity. The correct supersymmetric continuum limit is ensured in this approach because the lattice action is a direct solution of the blocking. But this property is in this approach not expressed in terms of a lattice symmetry. A well defined lattice symmetry is desirable as a guiding principle for the construction of a more general lattice action.

Note, as a sidemark, that for vanishing α^{-1} the solution (5.33) becomes nonlocal as found in [93]. A vanishing α^{-1} is in principle allowed since it emerges in the limit of diverging eigenvalues of α. This corresponds to a $\prod_{ni}\delta(\phi^i_n - \phi^i_f(x_n)$ in the path integral. In such a δ-like blocking the averaged fields are directly associated with a corresponding lattice field. The SLAC derivative is one of these nonlocal solutions. One gets this

derivative operator from (5.33) when $f(p)$ is one for all momenta below the lattice cutoff and otherwise zero. In real space is corresponds to $f(x) = 2\sum_n \cos(p_n x)$. Thus the SLAC derivative is in that sense a perfect derivative operator. Only the nonlocality of the emerging solutions rules out a zero α^{-1}.

5.3 The additional constraint for a supersymmetric theory

As seen in section 5.2.1 it is rather trivial to fulfil the additional constraint (5.7) when the symmetry acts, like chiral symmetry, only on the multiplet index. Facing the application of the Ginsparg-Wilson approach in the case of supersymmetry, one has to consider derivative operators in the continuum symmetry transformations. In this case it is a nontrivial task to satisfy the additional constraint. For this problem it is enough to consider the one dimensional case, since it can be easily generalised. The requirement is then that a lattice derivative operator ∇ and an averaging function f exists such that

$$\nabla_{nm} \int dx\; f(am - x)\, \varphi(x) = \int dx\; f(an - x)\, \partial_x \varphi(x) \tag{5.36}$$

holds.[9] A partial integration and a Fourier transformation leads to[10]

$$f(p_q)\big[\nabla(p_q) - i\, p_q\big] = 0 \quad \forall\, q \in \mathbb{Z}\,. \tag{5.37}$$

That simply means the Fourier components of the averaging function must vanish at all momenta, where the dispersion relation of $\nabla(p)$ deviates from the continuum dispersion relation. Since $\nabla(p)$ is a periodic function they can have nonzero values only in the first BZ. However, the momentum of the averaged fields is, according to equation (5.1), restricted below the largest momentum of the averaging function. It is hence not desirable to have an averaging function with vanishing Fourier components for all momenta above a cutoff that is smaller than the lattice cutoff $\frac{\pi}{a}$. For all momenta above such an artificial cutoff the lattice fields will not be coupled to an averaged field and only a quadratic action $\frac{1}{2}\phi\alpha\phi$ remains for these modes.

After all the only remaining way to realise the additional requirement for supersymmetry is a derivative operator that agrees for all momenta below the lattice cutoff with the continuum dispersion relation. Only in this case a sensible lattice theory results from the blocking. The appearance of a nonlocal derivative operator might not be that surprising. In the additional requirement the blocking matrix α is not taken into account and we already know that for a vanishing α also the solutions of the perfect action, equation (5.33), become nonlocal. In the definition of the lattice symmetry operator (5.19) the naive symmetry is supplemented by a dependence on the matrix α. To investigate the

[9] Since unimportant for the present considerations, the multiplet indices were neglected.
[10] Details can be found in appendix A.3.

lattice symmetry and specify the matrix α a special supersymmetric example must be chosen. For simplicity supersymmetric quantum mechanics is considered here.

5.4 Supersymmetric quantum mechanics

The example of supersymmetric quantum mechanics with a quadratic action might seem rather trivial. However, the considerations of the chiral case show that already a quadratic theory leads to nontrivial solutions. Thus the following investigations might be seen as the supersymmetric version of the overlap solution for chiral symmetry.

5.4.1 The setup for a onedimensional supersymmetric theory

The basic setup for a onedimensional supersymmetric theory has already been discussed in section 2.1.4. For convenience I use sightly different conventions for the formulation of the Lagrangian and the symmetry in the following discussion. The field multiplet is

$$\varphi_x = (\chi(t), F(t), \psi(t), \bar{\psi}(t))^T \,, \tag{5.38}$$

where χ is a bosonic field, F the auxiliary field, and the complex ψ and $\bar{\psi}$ define the fermionic fields. The continuum action has the following form

$$S[\varphi] = \int dt \Big[\frac{1}{2}(\partial_t \chi)^2 + \bar{\psi}\partial_t \psi - \frac{1}{2}F^2 + \bar{\psi}\,\frac{\partial W}{\partial \chi}\,\psi - F W(\chi) \Big] \,. \tag{5.39}$$

In these conventions one has the following supersymmetry transformations

$$\delta\chi = -\bar{\epsilon}\psi + \epsilon\bar{\psi} \,, \quad \delta F = -\bar{\epsilon}\partial_t\psi - \epsilon\partial_t\bar{\psi} \,, \quad \delta\psi = -\epsilon\partial_t\chi - \epsilon F \,, \quad \delta\bar{\psi} = \bar{\epsilon}\partial_t\chi - \bar{\epsilon}F \,. \tag{5.40}$$

These are respresented in matrix form by

$$\delta\varphi = (\epsilon\mathcal{M} + \bar{\epsilon}\bar{\mathcal{M}})\varphi \tag{5.41}$$

where

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\partial_t \\ -\partial_t & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \bar{\mathcal{M}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 0 \\ \partial_t & -1 & 0 & 0 \end{pmatrix} \,, \tag{5.42}$$

and the multiplet is arranged as a vector according to (5.38).

Each component of the continuum multiplet gets a lattice counterpart,

$$\phi_n = (\chi_n, F_n, \psi_n, \bar{\psi}_n)^T \,, \tag{5.43}$$

and according to the results of section 5.3 the naive transformations are the same as the continuum transformations apart from a replacement of the derivative operator with the SLAC derivative on the lattice. Hence the naive lattice transformations are

$$\delta\phi_n^i = (\epsilon M_{nm}^{ij} + \bar{\epsilon}\bar{M}_{nm}^{ij})\phi_m^j\,, \tag{5.44}$$

with M and $\bar{M}$ defined as

$$M_{nm}^{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\nabla \\ -\nabla & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{nm} \qquad \bar{M}_{nm}^{ij} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\nabla & 0 \\ 0 & 0 & 0 & 0 \\ \nabla & -1 & 0 & 0 \end{pmatrix}_{nm} . \tag{5.45}$$

Finally, the relation (5.11) for the lattice action must be solved to find a symmetric lattice action. Since the trace part does not contribute in the case of supersymmetry this is the same problem as to solve (5.15) for K. Since the lattice action should be hermitian and translational invariant I start with the following ansatz for the lattice action:

$$\frac{S_{\mathrm{w}}[\phi]}{a} = -\frac{1}{2}\chi\Box\chi + \bar{\psi}(\hat{\nabla} + m_f)\psi - \frac{1}{2}F\mathcal{I}F - Fm_b\chi\,, \tag{5.46}$$

which implies

$$\frac{K_{mn}^{ij}}{a} = \begin{pmatrix} -\Box & -m_b & 0 & 0 \\ -m_b & -\mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & \hat{\nabla} - m_f \\ 0 & 0 & \hat{\nabla} + m_f & 0 \end{pmatrix}_{mn} . \tag{5.47}$$

$(\Box)_{mn}$, $(m_b)_{mn}$, and $(m_f)_{mn}$ are so far undetermined symmetric (real) matrices. $(\hat{\nabla})_{mn}$ is antisymmetric. All of theses matices are assumed to be circulant because of translational invariance.[11]

The last thing that must be specified in order to find a solution of the symmetry

[11]Note that circulant matrices form a commutative algebra, which makes it much simpler to solve the symmetry relation.

relation is the blocking matrix α. Here it has the same form as K,

$$\frac{\alpha^{ij}_{mn}}{a} = \begin{pmatrix} -\alpha_{\Box} & -\alpha_{m_b} & 0 & 0 \\ -\alpha_{m_b} & -\alpha_{\mathcal{I}} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{\hat{\nabla}} - \alpha_{m_f} \\ 0 & 0 & \alpha_{\hat{\nabla}} + \alpha_{m_f} & 0 \end{pmatrix}_{mn} , \tag{5.48}$$

where $(\alpha_{\Box})_{mn}$, $(\alpha_{m_b})_{mn}$, and $(\alpha_{m_f})_{mn}$ are symmetric and $(\alpha_{\hat{\nabla}})_{mn}$ antisymmetric circulant matices.

5.4.2 Solutions for a quadratic action

It is instructive to start the discussion with possible symmetric blocking matrices, according to equation (5.13). Since they are of the same form, one can similarly search for quadratic actions invariant under the naive symmetry transformations. For simplicity I will consider first only M and not $\bar{M}$. Note, however, that these solutions also apply to the general case. For the kernel of a symmetric quadratic action K_S is of the form (5.47). In addition, it is invariant under the naive transformations, $M^T K_S + (M^T K_S)^T = 0$, which leads to

$$\Box = \nabla\hat{\nabla} , \quad m_b = m_f , \quad \text{and} \quad \nabla\mathcal{I} = \hat{\nabla} . \tag{5.49}$$

Thus the symmetric kernel depends only on the matrices m_b and $\mathcal{I}$ in the bosonic part and the fermionic part follows from the symmetry conditions, (5.49).[12] From this solution the general form of a symmetric blocking matrix α_S is deduced by a replacement of m_b with α_{m_b} and $\mathcal{I}$ with $\alpha_{\mathcal{I}}$.

The most obvious ansatz for a blocking matrix in a supersymmetric theory is a supersymmetric mass term: $(\alpha_{m_b})_{mn} = (\alpha_{m_f})_{mn} \propto \delta_{mn}$ and all other entries of α are zero. In this case the deformed symmetry is, however, reduced to the nonlocal naive symmetry since the right hand side of equation (5.11) vanishes. The invariant action follows then from (5.49). In this action the product of ∇ and $\mathcal{I}$ and ∇^2 and $\mathcal{I}$ appears. The nonlocality of ∇ is caused by a discontinuity at the boundary of the first BZ. This can only be amended if $\mathcal{I}(p)$ vanishes with all its derivatives at this boundary or else $\mathcal{I}$ itself must have a discontinuity at this point. Then $\mathcal{I}(p)$ can only be differentiable for all degrees of differentiation, but it can not be analytic. It follows that in real space the entries of $\mathcal{I}_{0n}$ do not show a strict exponential decay with increasing n; but a stronger than polynomial decay can be achieved. Thus with $(\alpha_{m_b})_{mn} = (\alpha_{m_f})_{mn} \propto \delta_{mn}$ the symmetry transformations are nonlocal and the action can only be chosen according to that sightly relaxed

[12] Note that this condition also determines $\Box$ in the bosonic part.

version of locality.[13]

On the other hand, one can start instead of α with a specific form for K. In (5.18) α is specified by the inverse of K,[14]

$$(K^{-1})^{ij}_{mn} = \begin{pmatrix} \frac{\mathcal{I}}{-\Box\mathcal{I}+m_b^2} & \frac{-m_b}{-\Box\mathcal{I}+m_b^2} & 0 & 0 \\ \frac{-m_b}{-\Box\mathcal{I}+m_b^2} & \frac{\Box}{-\Box\mathcal{I}+m_b^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-\hat{\nabla}+m_f}{-\hat{\nabla}^2+m_f^2} \\ 0 & 0 & \frac{-\hat{\nabla}-m_f}{-\hat{\nabla}^2+m_f^2} & 0 \end{pmatrix}_{mn}, \tag{5.50}$$

up to a symmetric part. Thus for a given K one gets $\alpha(\alpha_S) = (K^{-1} + \alpha_S^{-1})^{-1}$. The addition of a nonzero symmetric part α_S^{-1} is crucial since otherwise M_{def} would be zero (cf. (5.21)) and the lattice counterpart of the symmetry would not be well defined.

The main task is to find a nonzero α_S that renders $M_{\text{def}} = -M\alpha_S^{-1}K$ local for a given local K (and the nonlocal M of (5.45)). In addition one has to investigate if the resulting $\alpha(\alpha_S)$ is a suitable blocking kernel. Let us start with the inverse of the general symmetric blocking kernel and make some redefinitions,[15]

$$\begin{aligned}(\alpha_S^{-1})^{ij}_{mn} &= \frac{1}{-(\nabla\alpha_{\mathcal{I}})^2 + \alpha_{m_b}^2} \begin{pmatrix} \alpha_{\mathcal{I}} & -\alpha_{m_b} & 0 & 0 \\ -\alpha_{m_b} & \nabla^2\alpha_{\mathcal{I}} & 0 & 0 \\ 0 & 0 & 0 & -\nabla\alpha_{\mathcal{I}} + \alpha_{m_b} \\ 0 & 0 & -\nabla\alpha_{\mathcal{I}} - \alpha_{m_b} & 0 \end{pmatrix}_{mn} \\ &=: \begin{pmatrix} -\frac{1}{2}\nabla^{-1}(R+R') & \frac{1}{2}(R-R') & 0 & 0 \\ \frac{1}{2}(R-R') & -\frac{1}{2}\nabla(R+R') & 0 & 0 \\ 0 & 0 & 0 & R' \\ 0 & 0 & R & 0 \end{pmatrix}_{mn}. \end{aligned} \tag{5.51}$$

[13]The conditions for this locality and a specific example for $\mathcal{I}$can be found in appendix A.3.2.

[14]The appearing fractions are assumed to be performed in momentum space and Fourier transformed back into real space.

[15]The factor ∇^{-1} should be seen as a division by p in Fourier Space.

With such a symmetric blocking kernel the deformed lattice supersymmetries are

$$M_{\text{def}} = -M\alpha_S^{-1}K = \begin{pmatrix} 0 & 0 & 0 & -(\hat{\nabla} - m_f)R \\ 0 & 0 & 0 & \nabla(\hat{\nabla} - m_f)R \\ (\Box + \nabla m_b)R' & (\nabla\mathcal{I} + m_b)R' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.52}$$

$$\bar{M}_{\text{def}} = -\bar{M}\alpha_S^{-1}K = \begin{pmatrix} 0 & 0 & (m_f + \hat{\nabla})R' & 0 \\ 0 & 0 & \nabla(m_f + \hat{\nabla})R' & 0 \\ 0 & 0 & 0 & 0 \\ -(\Box - \nabla m_b)R & (\nabla\mathcal{I} - m_b)R & 0 & 0 \end{pmatrix}. \tag{5.53}$$

In these expressions a product of R and R' with local and nonlocal operators appear. Thus these matrices must be local and render at the same time the product ∇R (or $\nabla R'$) local (with ∇ nonlocal). This is similar to the problem concerning the locality of $\mathcal{I}$ and $\nabla\mathcal{I}$ that is discussed above. Hence only the relaxed version of locality can be achieved for ∇R and R. M and K can at the same time be local only in this relaxed sense.

A special solution with such a property is constructed as follows. A good starting point is $m_f = m_b$, $\Box = \nabla^2\mathcal{I}$ with a $\hat{\nabla} = \nabla\mathcal{I}$ that fulfils the condition of the relaxed locality. For the continuum limit it is clear the $\mathcal{I}$ must approach 1 at the centre of the BZ whereas the function and all derivatives of it vanish at the boundary. Then one can use $R = \frac{-\mathcal{I}}{\nabla\mathcal{I} - m_b}$ and $R' = \frac{-\mathcal{I}}{\nabla\mathcal{I} + m_b}$ to get the deformed lattice symmetry $M_{\text{def}} = \mathcal{I}M$ ($\bar{M}_{\text{def}} = \mathcal{I}\bar{M}$) that is also local in the relaxed sense. The corresponding $\alpha(\alpha_S)$ can be deduced from equation (5.18) . The inverse of this matrix is simply $\alpha^{-1} = (1\!\!1 - \mathcal{I})K^{-1}$. This matrix fulfils all the conditions for a suitable blocking kernel since its inverse vanishes in the continuum limit. A possible doubling problem in this solution is easily amended. It is enough to add a Wilson mass to m_b ($= m_f$) to remove it. A more serious problem appears when the transformation to an on-shell theory is done before the continuum limit. Since $\mathcal{I}(p)$ vanishes at the boundary of the BZ divergences appear in $1/\mathcal{I}(p)$ and, consequently, in the on-shell action.[16]

[16]Note that this solution is the same as the naive symmetry (5.49). This is because the additional matrix $\mathcal{I}$ in M_{def} drops out in the symmetry condition. Hence a possible nonsymmetric part of α^{-1} is not needed to get the relaxed locality.

5.5 Polynomial order of the solutions for a general symmetry

The main goal of the Ginsparg-Wilson approach to a supersymmetric theory is to find a local lattice action compatible with the symmetries. Thus we have to consider also the interacting, non-quadratic theory. The main advantage of the relation (5.11) is that it is applicable to a general, and not only quadratic, theory.

In the interaction case we are, however, lead to an additional problem of the Ginsparg-Wilson approach. The action as it emerges from the solution of the relation appears generically in a non-polynomial way. This is not only the case in supersymmetric theories. Hence the following general discussion is not restriced to supersymmetry. The example of solution in the zero-mode sector of interacting supersymmetric quantum mechanics (appendix F) may illustrate these findings.

To consider the general case, let us start with a lattice action consisting of polynomials up to order R in the fields, represented as

$$S_{\mathrm{w}}[\phi] = \sum_{r=1}^{R} s^{(r)}[\phi]\,, \quad s^{(r)}[\phi] = K^{i_1\ldots i_r}_{n_1\ldots n_r}\phi^{i_1}_{n_1}\ldots\phi^{i_r}_{n_r}\,, \tag{5.54}$$

where $s^{(r)}$ contains the rth order in the fields (cf. (F.1)). The coefficients K are so far not further specified, they can imply a simple multiplication of fields at the same lattice point, but are also allowed to contain lattice derivatives or to smear the powers of the fields over several lattice sites, as long as they obey the correct continuum limit. With this ansatz relation (5.11) becomes a complicated nonlinear differential equation coupling derivatives with respect to the fields at different lattice points. In the zero modes sector it is represented as (F.3). An expansion in the order of the fields yields[17]

$$O(\phi^0): \qquad 0 = M\alpha^{-1}\Big(\frac{\delta s^{(1)}}{\delta\phi}\frac{\delta s^{(1)}}{\delta\phi} - \frac{\delta^2 s^{(2)}}{\delta\phi\delta\phi}\Big) + (\mathrm{Str}M - \langle\mathrm{Str}\mathcal{M}\rangle) \tag{5.55}$$

$$O(\phi^{r=1\ldots R-2}): \quad M\phi\,\frac{\delta s^{(r)}}{\delta\phi} = M\alpha^{-1}\sum_{s+t=r+2}\Big(\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi} - \frac{\delta^2 s^{(r+2)}}{\delta\phi\delta\phi}\Big) \tag{5.56}$$

$$O(\phi^{r=R-1,R}): \quad M\phi\,\frac{\delta s^{(r)}}{\delta\phi} = M\alpha^{-1}\sum_{s+t=r+2}\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi} \tag{5.57}$$

$$O(\phi^{r=R+1\ldots 2R-2}): \qquad 0 = M\alpha^{-1}\sum_{s+t=r+2}\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi}\,. \tag{5.58}$$

These coupled equations can be read as restrictions for the $K^{i_1\ldots i_r}_{n_1\ldots n_r}$ parametrising $s^{(r)}$ imposed by the symmetry. In the case of R=2 only the conditions (5.55),(5.56) are relevant resulting in (5.15).

[17] This is written in a short hand notation, and a summation of the indices is understood.

A necessary condition for a truncation is that (5.58) is fulfilled for a certain order R. Otherwise higher orders of the fields must be added to the action to get a closed set of equations. Consider the highest order of these equations. It is

$$0 = (M\alpha^{-1})^{ij}_{nm}\Big(\frac{\delta s^{(R)}}{\delta\phi^j_m}\frac{\delta s^{(R)}}{\delta\phi^i_n}\Big)\,, \tag{5.59}$$

or equivalently

$$0 = v^T(M\alpha^{-1})\,v \quad \text{with} \quad v^i_n = \frac{\partial s^{(R)}}{\partial\phi^i_n}\,. \tag{5.60}$$

This relation implies that

$$M\alpha^{-1} + (M\alpha^{-1})^T = 0 \tag{5.61}$$

within the subspace of lattice fields spanned by the v^i_n. If the v^i_n span the whole space of ϕ^i_n and equation (5.61) holds, the relation is reduced to the naive symmetry.

On the other hand, if the v^i_n do not span the whole space of the fields they must be linearly dependent. Then some linear combinations of the v^i_n vanish and from the definition (5.60) it is clear that the highest part of the action $s^{(R)}$ does not depend on some particular combinations of fields. On this subspace there is no constraint like (5.61).

After all, it is only possible to get a truncation of the action, if (5.61) is fulfilled on that subspace of ϕ's, on which the highest term of the action, $s^{(R)}$, depends. Keeping translational invariance, it is impossible to have $s^{(R)}$ independent of fields at particular lattice points n, but $s^{(R)}$ may be independent of a whole field component (ϕ^i) of the multiplet. Such a case appears for constant fields (cf. (F.6)) when a_1 is set to zero. Then the highest term of the action, χ^4, depends only on χ, and $M\alpha^{-1}+(M\alpha^{-1})^T$ has no matrix entries for this field component. In this way a polynomial solution can be achieved.

To get the full solution in the interacting case, also the relations for the action that involve the terms of lower order in the fields must be solved. Note also that it is hard to investigate the continuum limit of such a theory since in perturbation theory certain vertices are introduced that vanish in the continuum limit.[18]

5.6 Conclusions for this approach

In the application of the Ginsparg-Wilson approach to supersymmetric theories there appear two kind of difficulties. The first is the nonlocal (SLAC) operator that emerges from the additional constraint (5.7). Equation (2.51) shows us the origin of the additional constraint. A similar equation holds for a general Wilsonian effective action. More precisely, a Wilsonian effective action is, like in (5.2), defined by

$$S_{\mathrm{w}}[\phi] = -\log\langle e^{-S_\alpha[\varphi,\phi]}\rangle\,. \tag{5.62}$$

[18] This is, however, comparable to the case of lattice gauge theories.

S_α is similar to the regulator S_k and in for the lattice theory $S_\alpha = \frac{1}{2}(\phi_f - \phi)\alpha(\phi_f - \phi)$. In an analogous way as in (2.47) for the Ward-identities one obtains (neglecting the anomaly part)[19]

$$\langle(\varepsilon\delta S + \varepsilon\delta S_\alpha)e^{-S_\alpha[\varphi,\phi]}\rangle = 0\,. \tag{5.63}$$

The additional constraint requires that this can be reformulated as a "symmetry" of the S_{w} provided $\delta S = 0$. Thus one needs

$$0 = \langle\varepsilon\delta S_\alpha e^{-S_\alpha[\varphi,\phi]}\rangle \stackrel{!}{=} (\varepsilon M_{\mathrm{def}}\phi)\langle e^{-S_\alpha[\varphi,\phi]}\rangle = (\varepsilon M_{\mathrm{def}}\phi)e^{-S_{\mathrm{w}}[\phi]}\,. \tag{5.64}$$

For a general continuum theory the additional condition must be fullfiled for this requirement. The locality of M_{def} can be increased, but not fully established, in a quadratic supersymmetric theory.

The other problem is the non-polynomial solution in the interacting case. It applies not only for supersymmetric theories, but is a general feature of the solutions of generalised Ginsparg-Wilson relations. The appearance of these non-polynomial solutions is, on the other hand, not quite unexpected. The perfect action is comparable to the full quantum effective action. For such an action one cannot expect to get polynomial solutions unless one considers only a free theory. A solution of this problem can be the truncation according to the conditions found in 5.5. A different possibility is to find out which operators are relevant in the continuum limit and which one can safely be ignored for the symmetry. Further investigations of this problem are still on the way.

[19] δS_R means the part of $S_R[\varepsilon\delta\varphi, \phi]$ linear in ε.

6 Different approaches to the one-loop approximation

The loop expansion, introduced in section 2.3.2, yields an approximation of the effective action of a given theory. It can be compared with the lattice results, the weak coupling expansion, and the FRG approximations. The aim of the following investigations is also to examine different expansions of the effective action. An appropriate expansion is necessary for the FRG calculations.

Furthermore, the simple model of supersymmetric quantum mechanics is well suited for the examination of a certain "puzzle" that can be found in the literature in connection with the loop expansions of a supersymmetric theory. In [63] and other works the loop expansion was employed in the off-shell theory, as usual, to approximate the effective potential u. u comes from the zeroth order term of a derivative expansion and contains all orders of the auxiliary field, as detailed in section 2.2.1. To arrive at an approximation of the on-shell effective potential u_{on}, the nonlinear equations of motion for this field were solved. In this way a phase transition to a phase with spontaneously broken supersymmetry could be determined.[1] Alvarez-Gaume, Freedman, and Grisaru found in [94] that the result is different from the one-loop approximation in the on-shell theory $u_{\text{on}}^{(1)}$.[2] They called this an unresolved "puzzle". They also found that $u_{\text{on}}^{(1)}$ is negative for some values of the field, which is incompatible with supersymmetry. Murphy and O'Raifeartaigh [95] attributed this "puzzling feature" to the formal (unphysical) character of the auxiliary (there called "dummy") field. As found in their examination of $u_{\text{on}}^{(1)}$ the classical spontaneous broken or unbroken supersymmetry was stable under the quantum corrections. Furthermore, they could successfully explain the negative values of $u_{\text{on}}^{(1)}$ with the breakdown of the loop expansion for a nonconvex potential.

Here I present a more careful analysis. In fact, what was considered in the mentioned literature as different calculations of the same one-loop on-shell potential are in fact two different approximations of the effective potential. In supersymmetric quantum mechanics one has the additional possibility to compare these potentials with the exact effective potential obtained according to appendix D.

6.1 Definitions of different approximations for the effective potential

As I have explained there are two different expansions for the off-shell theory. The first one is done in terms of derivatives, the second one in terms of covariant derivatives.

[1] This was done for an $\mathcal{N} = 1$; $D = 2$ Wess Zumino model.

[2] As explained the number denotes the level of the loop-expansion, $u = u^{(0)} + \hbar u^{(1)} + \hbar^2 u^{(2)} + \ldots$. The same applies for the loop expansion of u_{on} and u_S.

The first term of the derivative expansion is U. The first term of the covariant derivative expansion, obtained from the effective superpotential after the integration of the Grassman coordinates, is U_S. In both potentials no derivative terms appear. In addition U_S contains, in contrast to U , only terms of at most linear order in the auxiliary field. Thus U_S is the linear term of an expansion of U in powers of the auxiliary field. U and U_S will be obtained here in a loop expansion. Note that in the computation of U_S in superspace all effects of the covariant derivatives can be neglected. Therefore the computation of U_S, or equivalently the effective superpotential, in superspace is much easier than the one of U.

In the following only the part of the effective action with zero fermionic fields is considered. This corresponds to a formal minimisation of the effective potential with respect to these fields. U is not invariant under the full supersymmetry, but only under its zero-mode part of them.[3] This can be used to construct the full effective potential from the one with zero fermionic fields. In supersymmetric quantum mechanics the relation between the fermion part and the bosonic part by the zero-mode supersymmetry leads to[4]

$$u(\varphi, \psi, \bar{\psi}, F) = u(\varphi, F) - \bar{\psi}\psi \frac{1}{F}\frac{\partial}{\partial \varphi} u(\varphi, F)\,. \tag{6.1}$$

Thus the relevant information is contained in $u(\varphi, F) = \lim_{(\bar{\psi},\psi)\to(0,0)} u(\varphi, \psi, \bar{\psi}, F)$.

The final result should be compared to the on-shell effective potential u_{on}. Therefore, the auxiliary field must be eliminated. To do this in case of U_S, the terms with covariant derivatives are added in their classical form $S_{S,k}$[5] to U_S and the equation of motion resulting from this combination is solved. These equations are linear in the auxiliary field. The result $U_{S,\text{on}}$ is for constant arguments (as the classical kinetic part contains derivative operators) an approximation of the on-shell effective potential U_{on}.

The loop expansion of U_S follows from the loop expansion of U, as the projection onto the linear order in the auxiliary field does not change the order of $\hbar$. $U_S^{(n)}$ is the same as the term of $U^{(n)}$ linear in F. Similarly the linear equation of motion does not mix the orders of $\hbar$. The expansion of the solution is hence (for a constant F and φ)

$$F = -\frac{\partial}{\partial F}\sum_{l=0}\hbar^l u_S^{(l)}(\varphi, F) = -i\sum_{l=0}\hbar^l W_{\text{eff}}^{(l)\prime}(\varphi) = -\sum_{l=0}\hbar^l \frac{\partial}{\partial F} u^{(l)}(\varphi, F)\Big|_{F=0}\,. \tag{6.2}$$

(The second term denotes the loop expansion of W'_{eff}.) The auxiliary field is then inserted into $S_{S,k} + U_S$ to obtain $U_{S,\text{on}}$. There the auxiliary field appears quadratically and the

[3] As explained the zero-mode supersymmetry is the part of the transformations without derivative operators.

[4] Small letters indicate the division by a volume factor and constant fields as arguments, $u = \lim_{(\varphi,F)\to\text{const}} U/\Omega$.

[5] Usually $S_{S,k}$ consists of the kinetic terms for fermions and bosons and the quadratic term, $\frac{1}{2}F^2$, of the auxiliary field.

orders of $\hbar$ mixed. For supersymmetric quantum mechanics one obtains

$$u_{S,\mathrm{on}} = \frac{1}{2}\left(\sum_l \hbar^l W_{\mathrm{eff}}^{(l)\prime}(\varphi)\right)^2 = \frac{1}{2}W'^2 + \hbar W_{\mathrm{eff}}^{(1)\prime} W_{\mathrm{eff}}^{(0)\prime} + \hbar^2\left(\frac{1}{2}(W_{\mathrm{eff}}^{(1)\prime})^2 + W_{\mathrm{eff}}^{(2)\prime} W_{\mathrm{eff}}^{(0)\prime}\right) + \hbar^3\left(W_{\mathrm{eff}}^{(0)\prime} W_{\mathrm{eff}}^{(3)\prime} + W_{\mathrm{eff}}^{(1)\prime} W_{\mathrm{eff}}^{(2)\prime}\right) + O(\hbar^4)\,. \quad (6.3)$$

This is the loop expansion of $u_{S,\mathrm{on}}$.

Thus we have obtained first the loop expansion of W_{eff} and then of $u_{S\mathrm{on}}$ from the loop expansion of u. The result is an approximation of the on-shell effective potential, u_{on}, where contributions of higher orders of F are neglected in the corresponding off-shell theory. At each order of $\hbar$ a comparison with the exact result for u_{on} tells us how important these contributions are.

A zero of the classical potential $\frac{1}{2}W'(\varphi_0)^2 = 0$ implies $W'(\varphi_0) = 0$. Equation (6.3) shows that such a zero is not lifted by one-loop corrections; but can be changed by the first contribution at the two-loop level. When all loop orders of W_{eff} are set to zero, except the zeroth and first, $u_{S,\mathrm{on}}$ receives nonzero contributions only up to the second order in $\hbar$.

Now let us take also the higher orders of the auxiliary field into account. Instead of the linear part U_S, we consider U with contributions of all orders of F. To arrive at U_{on} from the off-shell effective potential U , nonlinear equations of motion for F must be solved. With u given in a loop expansion, these equations of motion are (again constant fields)[6]

$$F = -iW' - \sum_{l=1} \hbar^l \frac{\partial}{\partial F} u^{(l)}(\varphi, F) = -\sum_l \hbar^l F^{(l)}(\varphi)\,. \quad (6.4)$$

In contrast to (6.2) $F^{(l)}$ is not only determined by $u^{(l)}$ with the same l. Instead one finds the expansion

$$\begin{aligned}
F^{(0)}(\varphi) &= -iW', \quad F^{(1)}(\varphi) = -\big(\partial_F u^{(1)}\big)(\varphi, F^{(0)}) = -\big(\partial_F u^{(1)}\big)(\varphi, -iW'),\\
F^{(2)}(\varphi) &= -\big(\partial_F^2 u^{(1)}\big)(\varphi, F^{(0)})\; F^{(1)} \;-\; \big(\partial_F u^{(2)}\big)(\varphi, F^{(0)})\\
&= \big(\partial_F^2 u^{(1)}\big)(\varphi, -iW')\; \big(\partial_F u^{(1)}\big)(\varphi, -iW') \;-\; \big(\partial_F u^{(2)}\big)(\varphi, -iW')\,,\\
F^{(3)}(\varphi) &= -\frac{1}{2}\big(\partial_F^3 u^{(1)}\big)(\varphi, F^{(0)})\; \big(F^{(1)}\big)^2 \;-\; \big(\partial_F^2 u^{(1)}\big)(\varphi, F^{(0)})\; F^{(2)}\\
&\quad - \big(\partial_F^2 u^{(2)}\big)(\varphi, F^{(0)})\; F^{(1)} \;-\; \big(\partial_F u^{(3)}\big)(\varphi, F^{(0)})\,, \quad \text{etc.}
\end{aligned} \quad (6.5)$$

This expansion has to be inserted in u to arrive at the on-shell effective action u_{on}. In

[6] Note that $u^{(0)} = \frac{1}{2}F^2 + iW'F$.

this way one arrives at the loop expansion of the on-shell action

$$\begin{aligned} u_{\rm on} &= \frac{1}{2}F^2 + iW'F + \hbar\, u^{(1)}(\varphi, F) + \hbar^2 u^{(2)}(\varphi, F) + O(\hbar^3) \\ &= \frac{1}{2}W'^2 + \hbar\, u^{(1)}(\varphi, -iW') - \frac{\hbar^2}{2}\left(\left(\partial_F u^{(1)}\right)(\varphi, -iW')\right)^2 + \hbar^2 u^{(2)}(\varphi, -iW') \\ &\qquad + O(\hbar^3)\,. \end{aligned} \tag{6.6}$$

Integrating out the auxiliary field in the beginning and a loop expansion of the on-shell theory leads to the same result. This is what Murphy and O'Rafaertaigh did in [95]. Thus their result can be obtained also from the off-shell loop expansion, consistently including the auxiliary ("dummy") field. To arrive at this result, it is important to take care of the $\hbar$-order in each step.

What was done in [94] is to start with the off-shell effective potential, calculated up to the one-loop order ($u \approx u^{(0)} + \hbar u^{(1)}$), forget the factor $\hbar$ of the expansion (as it is one in natural units), and solve the nonlinear equation

$$F = -iW' + \partial_F u^{(1)}\,. \tag{6.7}$$

Instead of an expansion (6.5) one obtains a set of solutions of this nonlinear equation. The equations of motion should minimise the effective action. Hence, of these solutions one chooses the one with the lowest real effective potential u. This solution is then inserted into the approximation of u to obtain the on-shell counterpart $\tilde{u}_{\rm on}$. From the above detailed derivation with the careful consideration of the expansion parameter $\hbar$ it is clear that the result cannot be the same as the on-shell loop expansion. The "puzzle" is resolved in this way.

Nevertheless, the approach of [94] should not be completely disregarded with reference to the unsystematic treatment of the expansion parameter $\hbar$. It should be understood as an approximation method different from the loop expansion, or a (asymptotic) re-summation of it. To explain this in further detail, let us return to (6.5) that solves the equations of motion order by order in $\hbar$. At each order $F^{(l)}$ contributions appear that solely depend on $u^{(0)}$ and $u^{(1)}$. In fact, when one sets all of the $u^{(l)}$ with $l > 2$ to zero $F = \sum_l \hbar^l F^{(l)}$ represents nothing but a perturbative solution of (6.7). The perturbative series does not always converge to the exact result, in some cases it represents only an asymptotic expansion of it.

In a similar way the contributions of $u^{(l)}$ with $l > 2$ in (6.6) can be set to zero. Then $\tilde{u}_{\rm on}$ represents a (asymptotic) re-summation of all loop contributions. *Thus the result of [94] is a re-summation of certain loop contributions and yields an approximation different from the loop expansion, although one starts with a loop expansion of the off-shell theory.*

It hence carries non-perturbative information.

In the loop expansion of the on-shell theory (6.6) one can immediately find out which contributions were neglected at each order $\hbar$ in this approximation. These are all contributions containing $u^{(l)}$ with $l > 2$. The relevance of these neglected contributions can so far not be estimated in such a conclusive way as for the loop expansion, which works well whenever the classical configurations are dominant. To investigate this in further detail, let us consider the simple case of supersymmetric quantum mechanics.

6.2 Supersymmetric quantum mechanics in the loop approximation

The supersymmetric model with the action (2.12) and the superpotential

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3 \tag{6.8}$$

is considered here in the context of the loop expansion. Further details of this calculation and the weak coupling expansion of the model are given in appendix E.

The staring point is the derivative action (2.37) in superspace with a constant Φ_s

$$S_{\Phi_s}[\Phi] = \int dz \left[\frac{1}{2}\Phi\left(K + i(m + 3g\Phi_s^2)\right)\Phi + ig\Phi_s\Phi^3 + i\frac{g}{4}\Phi^4\right]. \tag{6.9}$$

The loop expansion of $\Gamma[\Phi]$ is read from the corresponding 1PI vacuum graphs at $\Phi_s = \Phi$, as explained in section 6. Here constant Φ are considered and an additional volume factor is included to arrive at the effective potential u. The first contribution comes from the superdeterminant and is given by

$$u^{(1)}(\varphi, F) = \frac{1}{2}\left((W''^2 + iFW''')^{1/2} - |W''|\right) = \frac{i}{4}\frac{FW'''}{|W''|} + O(F^2)\,. \tag{6.10}$$

The first term of the expansion in F is $u_S^{(1)}$.[7] In superspace the calculation of u_S is quite simple because in the inversion of the field-dependent propagator, $K + W''(\Phi_s)$, all the commutators of K and Φ_s are neglected. With this propagator at hand the superspace Feynman rules can be used.

To arrive at the loop expansion of u, the contributions of K acting on Φ_s cannot be neglected. The fermion fields are set to zero and all other fields to a constant value. Thus

[7] In superspace, neglecting all terms that arise from a commutator of K and the superfield Φ, one arrives at the effective superpotential $W_{\text{eff}}^{(1)} = \log W''$. This is consistent with this solution.

the $\Phi_s = \varphi_s + \theta\bar{\theta}F_s$ dependent propagator for the calculation of the vacuum diagrams is

$$[K(k) + iW''(\Phi_s)]^{-1}(\theta, \theta', \bar{\theta}, \bar{\theta}')|_{\Phi_s=\varphi_s+\bar{\theta}\theta F_s} = \frac{1 - iW''(\varphi_s)(\bar{\theta}\theta + \bar{\theta}'\theta') + (k^2 + iFW'''(\varphi_s))\bar{\theta}\theta\bar{\theta}'\theta'}{k^2 + W''^2(\varphi_s) + iF_sW'''(\varphi_s)} + \frac{k(\bar{\theta}'\theta - \bar{\theta}\theta') + iW'(\varphi_s)'(\bar{\theta}'\theta + \bar{\theta}\theta')}{k^2 + W''^2(\varphi_s)} . \quad (6.11)$$

For the effective potential the vacuum diagrams of the derivative action must be calculated at $\Phi_s = \Phi$. The lines ( ) correspond to the Φ_s-dependent propagator (6.11) and the vertices can be read from (6.9) (denotes the field dependent vertex $ig\Phi_s$). With these Feynman rules the two-loop contribution of u is obtained as

$u^{((2)}(F, \varphi) =$  $\Big|_{\Phi_s=\varphi+\bar{\theta}\theta F}$ +  $\Big|_{\Phi_s=\varphi+\bar{\theta}\theta F}$

$$= \left(\frac{3g}{4}\left(\frac{W''}{W''^2 + iFW'''} - (W''^2 + iFW''')^{-1/2}\right) + \frac{Fg^2(F - i\varphi(6W'' - 9\varphi W'''))}{4(W''^2 + iFW''')^2}\right) \quad (6.12)$$

$$= -\frac{3iFgW'''}{8W''^3} - \frac{i3g^2}{4}F\left(\frac{2\varphi}{W''(\varphi)^3} - \frac{3\varphi^2 W'''(\varphi)}{W''(\varphi)^4}\right) + O(F^2) .$$

The expansion of the result in terms of F and φ yields the mass ($O(F\varphi)$) and the wave function renormalisation ($O(F^2)$) as obtained in the weak coupling expansion, cf. equation (E.6), at $p = 0$. The two-loop contribution for the supersymmetric effective potential can be derived, as detailed above, in a much simpler way. Again the result agrees with the part of $u^{(2)}$ that is linear in the auxiliary field.

Since it starts with a linear term the whole effective potential vanishes when F approaches zero. This is expected since in the present case supersymmetry is unbroken and hence $\langle F \rangle = 0$. This fact can also be read from the weak coupling expansion since there is no tadpole contribution that leads to a nonvanishing $\langle F \rangle$.

As explained in (6.6), the loop expansion of the on-shell effective potential u_{on} is derived from u. According to (6.3) the approximation without the nonlinear contributions of the auxiliary field $u_{S,\text{on}}$ is obtained. These approximations are compared to $\tilde{u}$ that results from the solution of the nonlinear equations of motion (6.6). The numerical calculations, cf. chapter D, show how well u_{on}, $u_{S,\text{on}}$ at one and two-loop order or the re-summation $\tilde{u}$ approximate the exact u_{on}.

Assuming an imaginary value of F, the one and two loop effective potential acquires still a nonzero imaginary part if $W''^2 + iFW''' < 0$, even though the classical action is

real. After the elimination of F with its linear classical equations of motion this expression determines exactly the region where the on-shell potential is nonconvex, since then

$$\frac{\partial^2}{\partial\varphi^2}(W')^2 = W''^2 + W'W''' < 0\,. \tag{6.13}$$

Consequently, $u_{S,\mathrm{on}}$ and u_{on} are in the one and two-loop approximation complex in the nonconvex region of the classical potential. This agrees with the well-known breakdown of the loop expansion in such a nonconvex region. As found in [95] this breakdown also explains the negative values of the one and two-loop result.

It is interesting to note that the complex region does not appear if the nonlinear equations of motion for the auxiliary field are solved. $\tilde{u}$ stays real, even in this region.

For large fields all approximations approach the classical potential since also the difference between classical and effective potential decreases in this limit. As the whole effective potential vanishes for $F = 0$ ($\langle F\rangle = 0$) the one-loop approximation of $u_{S,o}$ and u_o vanishes always at the minima of the classical on-shell potential.[8] This observation agrees with the result of [95]. There it was found that a zero of the classical potential is not changed by (one-loop) quantum corrections.

In figure 6.1(a) the weak ($g/m^2 \ll 1$) and strong ($g/m^2 \approx 1$) coupling case with a convex potential is shown. For weak coupling all approximations agree very well with the exact result. The situation changes for strong couplings. The best approximation of the exact results is then the one-loop approximation $u^{(1)}_{\mathrm{on}}$. Already in the weak coupling expansion an artificial phase transition in the two-loop approximation for large g/m^2 can be observed, cf. figure 7.1(c), since the two-loop mass becomes negative. This fact is reflected also in the loop expansion of the effective potential, where for large coupling constants the two-loop potential develops a local maximum at $\varphi = 0$.

An example of a nonconvex classical potential is shown in figure 6.1(b). The classical potential for $W' = m\varphi + g\varphi^3$ has three minima for negative m. The only minimum of the exact effective potential is still at $\varphi = 0$ because no spontaneous breaking of the Z_2 symmetry of the action appears in quantum mechanics. Thus the other two minima must be lifted by the quantum corrections.[9] The one-loop approximation of u_{on} and $u_{S,\mathrm{on}}$ cannot show this effect. The two loop approximation of these potentials is much closer to the exact value. Remarkably, the best approximation is obtained with the unconventional approximation $\tilde{u}$. This approximation seems to incorporate the relevant contributions, even in the vicinity of the lifted classical minima. In the region of nonconvex classical on-shell potential the loop approximations of u_o and $u_{S,o}$ are, as expected, complex. $\tilde{u}$ is

[8] This is true only for unbroken classical supersymmetry, where these minima correspond to zeros of $W'(\varphi)$.

[9] A similar effect appears for spontaneously broken supersymmetry.

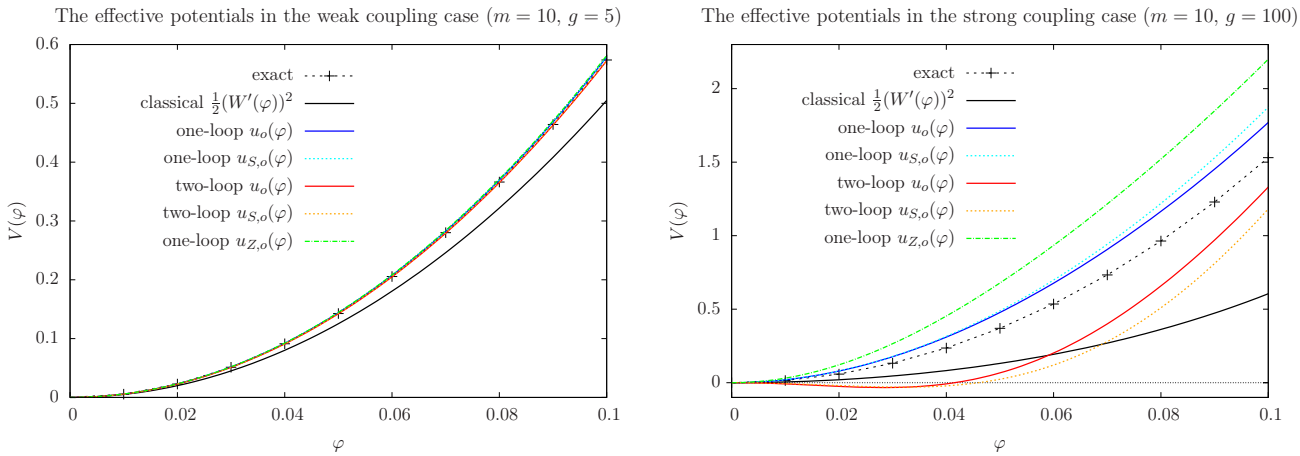
a real function, but also fails to reproduce the exact behaviour in the nonconvex region and at the exact minimum.

Let us further investigate these findings. As explained, solving the nonlinear equations of motion for the auxiliary field corresponds to a re-summation of loop contributions. In the expansion (6.6) these are the contributions at each loop order that solely depend on $u^{(0)}$, $u^{(1)}$, and their derivatives. I have calculated these contributions up to the twelfth loop order. For a convex classical potential their sum converges rapidly towards $\tilde{u}$. On the other hand, for the nonconvex case this sum forms an asymptotic expansion of the potential. This is shown in figure 6.2(a), where the sum of these contributions up to the order 1 to 12 is depicted in comparison to $\tilde{u}$. Especially at the lifted classical minimum the sum fails to converge to $\tilde{u}$. This indicates that the contribution neglected in the re-summation is dominated by the part contained in $\tilde{u}$. Hence $\tilde{u}$ is a better approximation than the loop expansion. A different way to estimate this fact is obtained from the two-loop level of (6.6). There $\big(\big(\partial_F u^{(1)}\big)(\varphi, -iW')\big)^2$ is considered in $\tilde{u}$ and $u^{(2)}\big)(\varphi, -iW')$ is neglected. Assume now $u^{(2)}$ is smaller than $u^{(1)}$. Then a dominance of the part considered in $\tilde{u}$ is indicated by $|\big(\partial_F u^{(1)}\big)(\varphi, -iW')|^2 \gg |u^{(1)}|^2$.

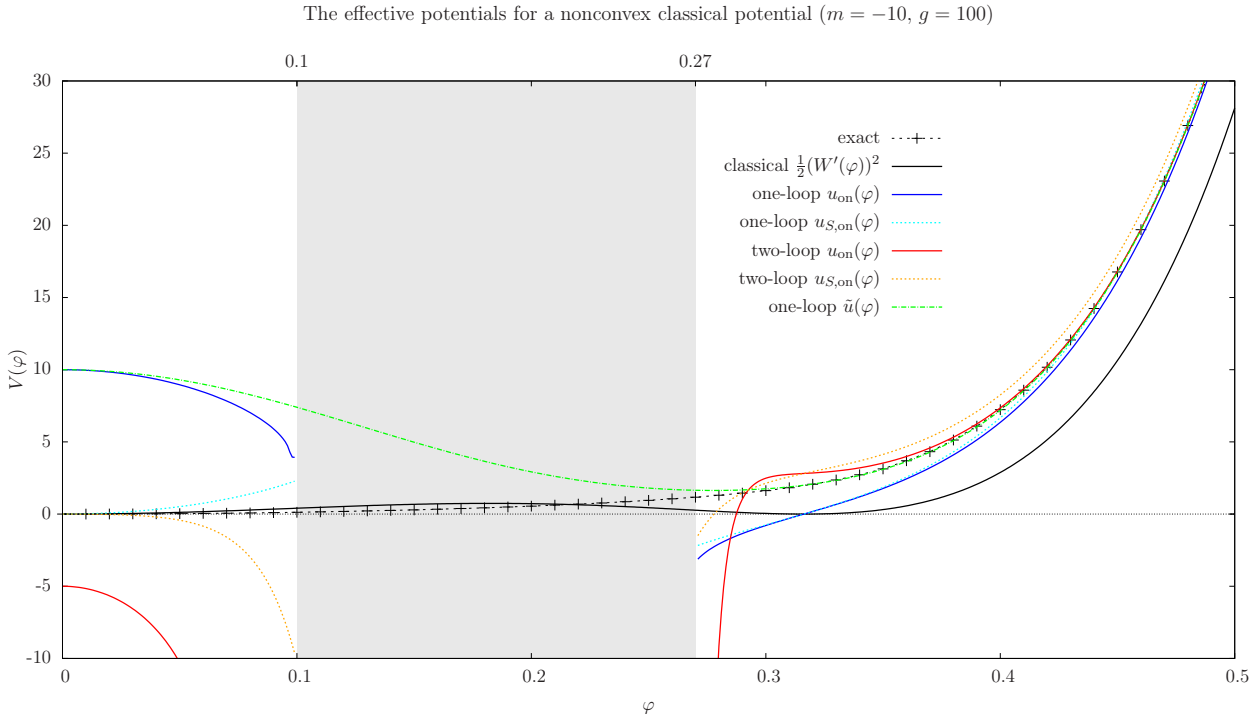
A different situation is shown in figure 6.2(b). Here $(\partial_F u^{(1)})^2$ is always much smaller than $u^{(1)}$. As expected from this rough estimate the one and two-loop on-shell effective potentials are better approximations of the exact result than $\tilde{u}$.

A comparison of one and two-loop approximation of u_o and $u_{S,o}$ in the present example shows only marginal difference in the one-loop case. This is in agreement with the weak coupling expansion, equation (E.6), where a wave function renormalisation Z different from one appears at first at the two-loop level. Hence the $O(F^2)$ term of the effective action does not differ from its classical form at one loop and possible corrections must be of higher order in F or in $\hbar$. A considerable difference between u_o and $u_{S,o}$ appears at two-loop in particular in the strong coupling regime and in the nonconvex case. There the inclusion of higher orders in F leads to a better result and cannot be neglected.

Further investigation of these approximations can, unfortunately, not be shown here. They were also calculated in the $\mathcal{N} = 2$ two-dimensional Wess-Zumino model with the superpotential mentioned in 2.1.4. The result is less conclusive since the zeros of the classical potential are not lifted by all of the approximations. This is in accordance with the nonrenormalisation theorem. The result has also been compared with the constraint effective potential, as obtained from the lattice simulations. Investigations of the case with spontaneously broken supersymmetry (supersymmetric quantum mechanics and $\mathcal{N} = 1$ D = 2 Wess-Zumino model) are still ongoing. Note, furthermore, that the fermionic fields are zero in the approximation. Whether the introduction of a general "dummy" field in bosonic theories can lead to a better approximation is hence a relevant question.

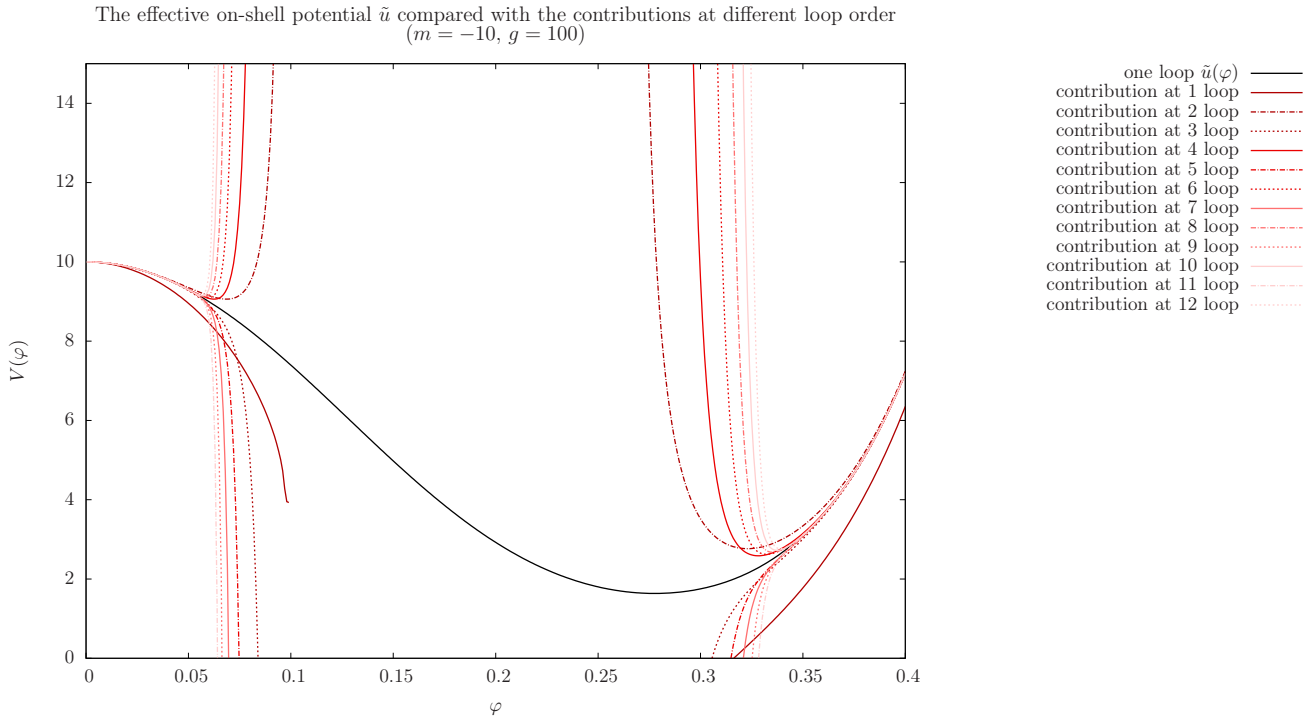


(a) The different approximations of the effective potential are compared with the exact (high precision numerical) results in the region of convex potential for weak and strong coupling. For weak coupling all approximations lead to a good agreement with the predictions.

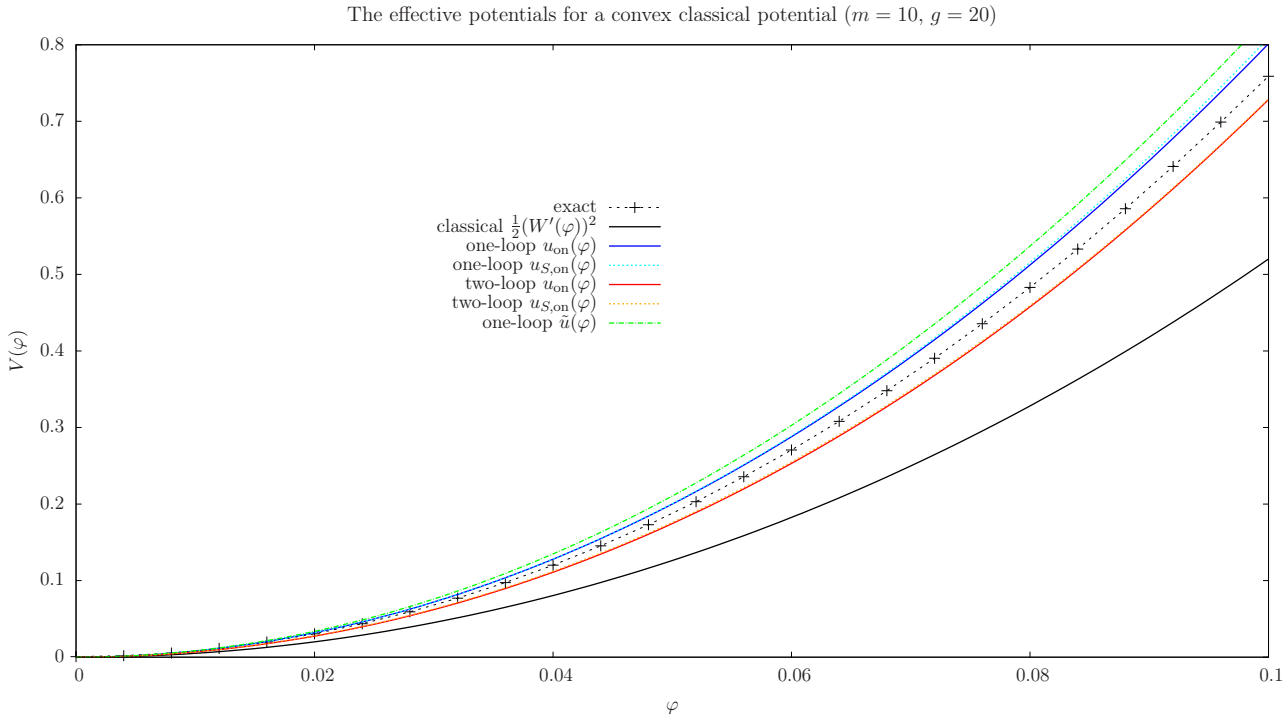


(b) The different approximations of the effective potential are compared with the exact (high precision numerical) predictions for a nonconvex classical potential ($m = -10$, $g = 100$). As expected the loop approximation fails where the classical potential is nonconvex ($V(\varphi)'' < 0$; $0.1 < \varphi < 0.27$).

Figure 6.1: The different approximations of the effective potential are compared with the exact (high precision numerical) predictions for a convex (a) and nonconvex (b) potential.



(a) An illustration how the contributions of $u^{(1)}$ and its derivatives at the different orders of the loop expansion form an asymptotic expansion of $\tilde{u}$.



(b) An example where $\tilde{u}$ does not lead to a better approximation of the exact on-shell effective potential than the one and two-loop approximations of it.

Figure 6.2: The different approximations of the effective potential are compared with the exact (high precision numerical) predictions in the region of convex potential and weak coupling ($m = 10$, $g = 5$). In this case all approximations lead to a good agreement with the predictions.

7 Functional renormalisation group flow and supersymmetry

As we have seen in the last chapter the general realisation of a supersymmetric lattice theory according to the approach of Ginsparg and Wilson is so far not completely satisfying. A prescription for the construction of a local lattice action as in the chiral case can hence not be given. The lattice regulator beaks supersymmetry and the effect of the breaking must be investigated for each theory separately. In most cases only the results of numerical simulations can show if a supersymmetric theory can be approached in the continuum limit. In view of these problems it is difficult to obtain completely reliable results from the supersymmetric lattice simulations. To obtain reliable information about the nonperturbative properties of supersymmetric theories, one should hence consider also alternative methods. For these considerations a method without an explicit supersymmetry breaking is preferable. Such a method can be provided by the (truncated) functional renormalisation group flow (FRG) introduced in section 2.3.4.

The idea of the renormalisation group flow is related the lattice simulations. In a lattice simulation the finiteness of the considered volume provides an infrared cutoff for the theory. The finite lattice spacing, on the other hand, introduces also an ultraviolet cutoff. Fluctuations on scales between these two cutoffs are integrated out in the lattice path integral. So the lattice is a regulator that introduces a sharp cutoff for the momentum of the modes. The result of a breaking of the continuum symmetry by the regulator is exemplified by the investigations of the last chapter. It corresponds to the modified Slavnov-Taylor identities (section 2.2.3). This leads to a rather complicated set of equations and that demands for non-polynomial solutions.

In this chapter I consider the FRG for symmetric regulators. The approximation method is a restriction to a certain truncation of the effective action. All operators that will be generated beside this truncation during the flow are neglected. These results were published in [96].

7.1 Supersymmetric quantum mechanics

The starting point of the flow is the classical supersymmetric action of equation (2.12). The easiest way to construct possible quadratic regulators compatible with supersymmetry is to define R_k as a function of covariant derivatives. This leads to the following expression

$$\begin{aligned} S_k &= \frac{1}{2}\int \frac{dp}{2\pi} d\theta d\bar{\theta}\ \Phi(-p,\theta,\bar{\theta}) R_k(D,\bar{D})\Phi(p,\theta,\bar{\theta}) \\ &= \frac{1}{2}\int \frac{dp}{2\pi} d\theta d\bar{\theta}\ \Phi(-p,\theta,\bar{\theta})(ir_1(p,k)+r_2(p,k)K)\Phi(p,\theta,\bar{\theta})\,. \end{aligned} \tag{7.1}$$

The anticommutation relation of the covariant derivatives (2.11) is used to express higher powers of the covariant derivatives in terms of spacetime derivatives and K. In addition, a truncation of the effective action must be defined. I choose here the classical action with a fixed kinetic part (F^2 inclusive) and an arbitrary function W_k as superpotential that can be changed during the flow

$$\Gamma_k[\phi, F, \bar{\psi}, \psi] = \int dt \int d\theta d\bar{\theta} \left[\frac{1}{2}\Phi K \Phi + i W_k(\Phi)\right] . \tag{7.2}$$

W_k interpolates between an approximation of the effective superpotential (W_{eff}) and the classical superpotential. Thus at $k = \Lambda$ the classical potential defines a starting point of the equation,

$$\lim_{k\to\Lambda} W_k'(\varphi) = W'(\varphi) . \tag{7.3}$$

Strictly speaking the cutoff Λ should be infinite. However, in the model considered here the result does not depend on the value of Λ as long as it is large enough compared to the other scales of the potential. Thus, effectively, the starting point in terms of the regularised effective action is

$$\Gamma[\varphi, k] \approx S_R[\varphi, k] := S[\varphi] + S_k[\varphi] . \tag{7.4}$$

This corresponds to the first term in a saddle point approximation for large S_k in the path integral of $W[k, j]$, (2.39). It means that for large Λ a good approximation of the starting point is $\Gamma_k[\varphi] = S[\varphi]$. The endpoint of the flow yields now a approximation of u_S defined in section 2.2.1, $u_S(\varphi, F) \approx \lim_{k\to 0} F W_k'(\varphi)$.

It is clear that the inverse on the right hand side of the flow equation (2.43) generates not only the terms that appear in the current ansatz of the effective action. These additional terms are not considered in the approximation. The derivation for the flow of W_k can be done in superspace, details about the connection between the superspace and real space expressions are reviewed in appendix H. The general flow equation reads in superspace[1]

$$\partial_k \Gamma_k = \frac{1}{2} \int dz\, dz'\, \partial_k R_k(z, z') G_k(z', z) , \qquad G_k = (\Gamma_k^{(2)} + R_k)^{-1} . \tag{7.5}$$

In the present truncation, all terms generated on the right hand side by an application of the covariant derivatives on a field are neglected. Thus all terms containing covariant derivatives in the action maintain their classical form. In this way the full renormalisation

[1] The coordinate z denotes here $(t, \theta, \bar{\theta})$.

group flow is projected onto the part of the superpotential. One arrives at

$$\partial_k W_k(\phi) = \frac{1}{2}\int \frac{dp}{2\pi} \frac{h(p,k)\partial_k r_1(p,k) - \partial_k r_2(p,k)\mathcal{W}''(p,k,\varphi)}{h^2 p^2 + \mathcal{W}''(p,k,\varphi)^2}\,, \tag{7.6}$$

with $h = 1 + r_2$ and $\mathcal{W}'' = r_1 + W_k''$. For simplicity I specify the regulator to be of a Callen-Symanzik type: $r_1(p,k) = k$ and $r_2(p,k) = 0$. Then the differential equation for the flow of the effective superpotential is

$$\partial_k W_k(\varphi) = \frac{1}{4}\frac{1}{k + W_k''(\varphi)}\,. \tag{7.7}$$

Numerical results

Equation (7.7) is used for the numerical investigations of the flow equations. As an additional boundary condition, the effective potential should approach the classical potential for large φ. (7.7) can hence be solved for a given starting point (7.3) using numerical methods.[2] Apart from the effective superpotential ($W_{\text{eff}} = \lim_{k\to 0} W_k(\varphi)$) the solution yields also a value for the effective mass in the on-shell theory. In the current approximation m^2_{eff} is the curvature of the effective on-shell potential at its minimum[3]

$$m_{\text{eff}} = \lim_{k\to 0} W_k''(\varphi)\Big|_{\varphi=\varphi_{\min}}\,. \tag{7.8}$$

For the convex classical potential (with positive m) considered in section 6 (cf. (6.8)) the obtained effective potential ($u_{S,\text{on}}$) agrees in the weak coupling sector with the loop approximation (cf. figure 7.1(a)). Moreover, it also yields a very good approximation in the strong coupling sector (cf. figure 7.1(b)). The approximation of the effective masses is reasonable and close to the exact and lattice results, as shown in figure 7.1(c). In figure 7.1(d) one observes that for certain nonconvex potentials the approximation works also quite well.

On the other hand, the truncation of the flow contains only the first term of a covariant derivative expansion. The terms that are changed from their classical form are hence only linear terms in the auxiliary field and carry no momentum dependence. Indeed, the deviation from the exact result in figure 7.1(c), although the effective potential is well approximated (7.1(b)), can be attributed to the truncated momentum dependence. For a better result the pole of the effective propagator has to be determined with renormalised higher momentum contributions. As we have observed from the one and two loop calculations, the higher order contributions of the auxiliary field can be important,

[2] In this case Mathematica routines were applied.

[3] This holds for unbroken supersymmetry. The classical kinetic part is added to the effective potential and the pole of the effective on-shell propagator $1/\Gamma''(\varphi)|_{\varphi=\varphi_{\min}}$ determines this result.

especially for nonconvex potentials. Therefore the linear – and even the quadratic – truncation in the auxiliary field may not be enough. This effect can be seen in figure 7.1(d). For the superpotential (6.8) with negative m it leads to an even larger deviation from the exact result.

7.2 Some general statements about supersymmetric flow equations

For a supersymmetric theory the truncation can, however, be adjusted in such a way, that the symmetry is not broken. Together with the supersymmetric regulator, this allows for a nonperturbative treatment without the breaking of supersymmetry. As a supersymmetric regulator, one can also in higher dimensions introduce R as a function of the covariant derivative.

The limitation of the flow equations is the truncation of the considered effective action. This error has to be compared with the discretisation error of the lattice simulations. When an agreement of the flow equations with the lattice simulations can be achieved, it is a strong indication that truncation and discretisation errors are under control.

What is now the real difference between the lattice calculations and the functional renormalisation group flow? $r_1(p,k)$ represents a momentum dependent mass. If it is very large for the high momentum modes with $p^2 > \Lambda_c(k)^2$ these modes are effectively removed from the theory. Thus only the modes below this Lorentz covariant cutoff contribute and $S_R[\varphi,k]$ defines a theory regularised by $\Lambda_c(k)$. S_R is the counterpart of the lattice action in this approach. When, instead of the Lorentz covariant cutoff, a lattice cutoff ($\Lambda_c(k) = \Lambda_L$) is introduced for the modes one arrives at the nonlocal lattice theory derived in section 3.1.1. A periodic continuation of the delta does not appear after the removal of the higher modes. This regulator the result is similar to the nonlocal lattice realisation. The cutoff leads, in terms of the Gibbs phenomenon, to a deviation from the continuum exponential decay of the correlation functions. The starting point of the flow ($S_R[\varphi, k \to \Lambda]$) is nonlocal when it contains such a sharp cutoff. The locality is reobtained for $\Lambda \to \infty$ which is a safe procedure in the (finite) theories considered here.

The second interesting finding is the role of the nonlinearity in the on-shell supersymmetry transformations. The regularised classical action that defines the starting point of the flow is

$$S_R = \int dt \left(-\frac{1}{2}(\varphi h \partial_t^2 \varphi) - i\bar{\psi} h \partial_t \psi + \frac{1}{2} F h F + i F \mathcal{W}'(\varphi) - i\bar{\psi}\mathcal{W}''(\varphi)\psi \right) . \tag{7.9}$$

The linear equations of motion for the auxiliary ($F = -ih^{-1}\mathcal{W}'$) lead to the corresponding

on-shell regularised action:

$$S_R = \int d\tau \left(-\frac{1}{2}\varphi h \partial_t^2 \varphi - i\bar{\psi} h \partial_t \psi - i\bar{\psi}\mathcal{W}''(\varphi)\psi + \frac{1}{2}\mathcal{W}'(\varphi) h^{-1} \mathcal{W}'(\varphi) \right). \tag{7.10}$$

The supersymmetric off-shell regulator S_k leads, due to the nonlinear supersymmetry transformations, to a rather complicated form of the regularised on shell action. As for the Wilson mass the introduction of a modified mass term leads to regulator dependent modified vertices. The rather simple regulator in the off-shell theory corresponds thus to a nontrivial choice in the on-shell theory. The flow equations would involve higher loop terms and become much more complicated in the on shell theory. Furthermore h^{-1} introduces an additional nonlocality in the theory. With $h = 1$ the equations of motion for the auxiliary field are $F(p) + ir_1(p)\varphi(p,k) = iW_k'(\varphi)$. The sharp cutoff (divergent $r_1(p,k)$ above the cutoff) prevents here the generation of momentum modes above the cutoff on the right hand side.

After all the renormalisation group equation provide an attractive tool for the investigation of the nonperturbative sector of a supersymmetric theory. For a sharp cutoff there are many similarities to the nonlocal lattice realisation. There is, however, much more freedom in the choice of the regulator than for a lattice theory. In addition, for simple truncations, the result can be obtained with less numerical effort. The disadvantage of this approach is the diffculty in controling of the truncation error. This must be carefully investigated for each considered model and set of parameters.

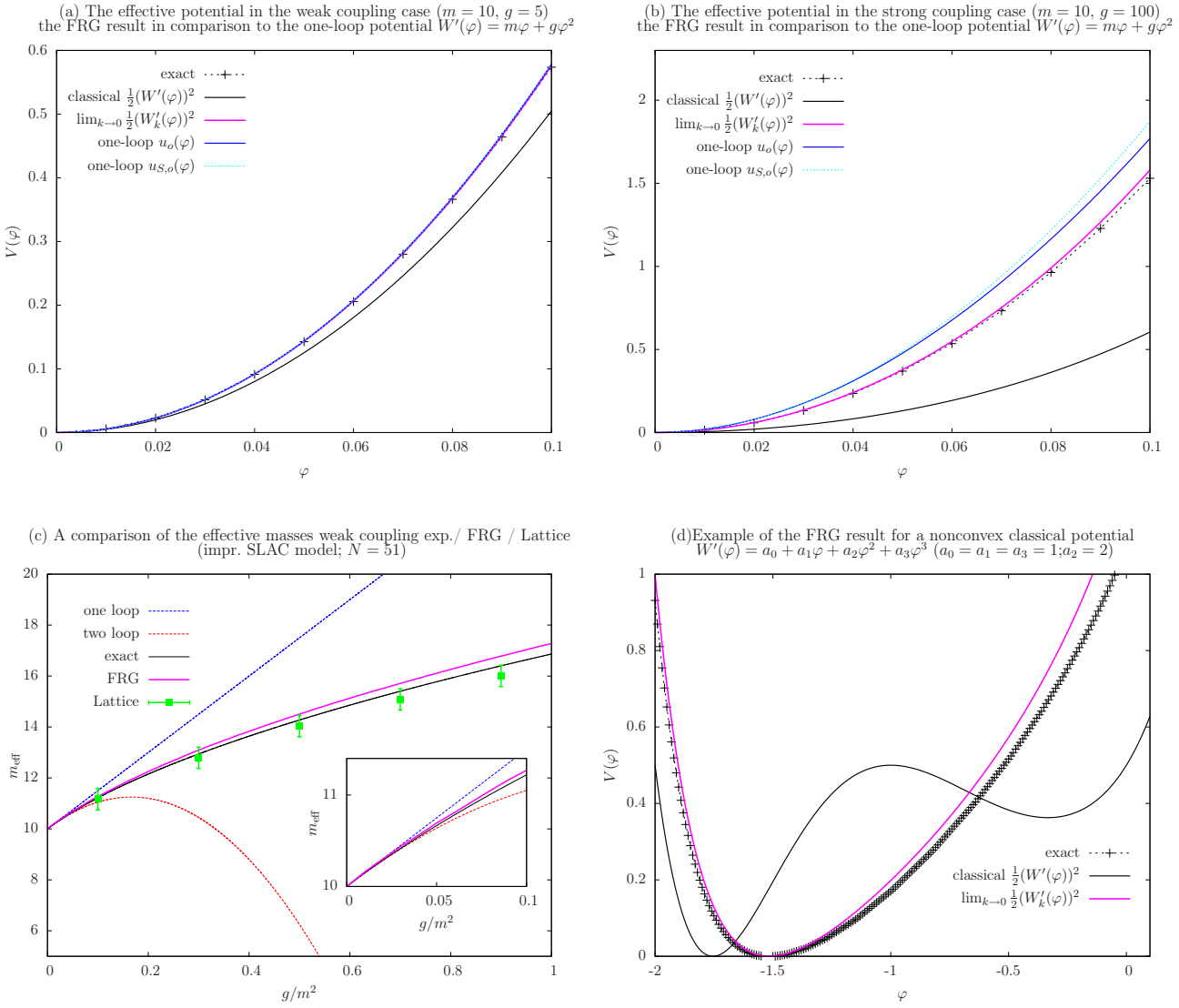


Figure 7.1: In (a) and (b) the results of the functional renormalisation group flow are compared to the exact and one loop effective potential. This is done in the case of the superpotential (6.8). In the weak coupling as well as the strong coupling regime the functional renormalisation group approximation achieves a good approximation of the exact effective potential. Figure (c) shows a comparison of the effective masses obtained in such an approximation in terms of the curvature of the effective potential at its minimum. This is compared to the weak coupling result, the exact mass gap (cf. section D), and the effective lattice mass. The lattice mass was obtained with the improved SLAC model on a $N = 51$ lattice without an extrapolation to the continuum (5×10^5 indep. config.). The error indicates the deviation of the $g = 100$ result at the same lattice size from the extrapolated continuum value, cf. fig. 4.1. (d) shows an example of a nonconvex potential. In this case a deviation between the functional renormalisation group approximation is still close to the exact result but a larger deviation from it is observed.

8 Summary and conclusions

In the present thesis I have investigated various basic consequences of the application of quantum field-theoretical methods to supersymmetric theories.

The main focus was put on the well-established method of lattice simulations. In this approach the discretisation of spacetime breaks supersymmetry. The main cause of this breaking is the inevitable violation of the Leibniz rule by any discretised derivative operator. I have shown that the symmetry can only be restored by means of a non-local derivative operator, such as the SLAC derivative, and a non-local interaction term.

Another source for the breaking of supersymmetry arises in the application of standard methods to the construction of a local lattice action. These procedures involve the introduction of an additional mass term (the Wilson mass) for the fermions in order to avoid an effective doubling of the degrees of freedom. The corresponding contributions to the action do not appear in the bosonic sector. Consequently, supersymmetry is not restored in the continuum limit, as becomes clear already in the one-loop lattice perturbation theory. This happens even though the classical contribution of the Wilson term vanishes. For this reason the additional mass term must be consistently represented in the bosonic sector. Only in this way may one still achieve a supersymmetric continuum limit. Another advantage of the non-local formulations is that such a problem does not occur. Altogether the non-local lattice formulations can guarantee supersymmetry. In the context of lattice perturbation theory I have found that the locality in low-dimensional models is restored in the continuum limit.

It must be stressed that the above considerations about the lattice formulations were scrutinised and confirmed in this thesis by means of numerical simulations. Towards this end, various different lattice formulations were compared and the relevant supersymmetry operators were measured with high precision. I could successfully apply non-local lattice operators in the simulations. This also allowed the application of realisations with completely intact supersymmetry.

The simulation were done in the one-dimensional theory of supersymmetric quantum mechanics and in the $\mathcal{N} = 2$ twodimensional Wess-Zumino model.

To check for supersymmetry in the simulations, I measured the masses of fermions and bosons as well as Ward identities. The implementation according to the standard methods shows supersymmetric properties neither at a finite lattice spacing nor in the continuum limit. However, a number of other discretisations could be found where supersymmetry is restored in the continuum limit.

Discretisations that employ a "Nicolai improvement" realise a part of supersymmetry on the lattice. This is reflected in the Ward identities I have measured. Nevertheless, for the remaining part the symmetry breaking at a finite lattice spacing is amplified. Moreover, in applying this method to the two-dimensional Wess-Zumino model an unphysical

phase can appear with amplified high-momentum modes. This does not mean in general that the partial realisation of supersymmetry implies no improvement. It only raises the question whether the benefits of partially preserved symmetry properties outweigh other potential problems introduced in the lattice theory. This must be answered separately for each considered model and measurement. In any case, all models with partially preserved supersymmetry showed the complete restoration of the symmetry in the continuum limit. This is, however, also achieved in similar formulations without partial realised lattice supersymmetry.

In this thesis I have performed the first simulation with a non-local discretisation that preserves the full supersymmetry even at finite lattice spacing. This proves that supersymmetry can, in fact, be conserved as long as one is willing to accept a violation of locality.

Thus, supersymmetry or locality must be violated in a controlled way, such that both are present in the continuum limit. Similarly, chiral symmetry must be broken on the lattice in a controlled way. This is required by the Nielsen-Ninmoya theorem. The Ginsparg-Wilson relation guarantees such a controlled breaking of chiral symmetry. It corresponds to the symmetry of a perfect lattice action. In the present thesis I have extended this symmetry relation to general global linear symmetries. However, when applied to supersymmetry, two difficulties arise in this approach:

The first one is that as a translation of the derivative operator in the supersymmetry transformations the non-local SLAC-derivative appears on the lattice. This follows from the transition from continuum fields to averaged lattice fields. The locality can be improved with the help of the blocking matrix.

The second problem occurs when the approach of Ginsparg and Wilson is applied to a non-quadratic action. Then the solutions are in general non-polynomial. This is not unexpected since the perfect action has a generically non-polynomial form. I could identify possible solutions of this problem, and further investigations will follow.

A good alternative to the lattice simulations is the method of the functional renormalisation group flow. This approach can also provide information about the non-perturbative sector of a supersymmetric theory. One can use regulators that do not break supersymmetry. However, one challenge of this method is that an appropriate truncation of the effective action must be found. In order to obtain a completely reliable result one should, therefore, compare it with the one obtained in the lattice simulations. With such a comparison one can ensure that the errors of both methods are not relevant.

I have exemplified this in the case of supersymmetric quantum mechanics, where a good approximation of the effective action could be achieved with this method. In fact, the obtained masses are comparable to the lattice results.

As a third method the loop expansion in a supersymmetric theory was investigated.

In this case there appear certain inconsistencies and discussions in the literature on the subject about the correct treatment of the auxiliary field. As shown in this thesis, these inconsistencies correspond to two different approximations. One of them is an effective re-summation of certain loop contributions and allows a different insight into the exact properties of the theory than the ordinary loop expansion. The other corresponds to the ordinary loop expansion, where a lifting of the classical minima can not be found at one-loop order.

All these investigations show that it is possible to get information about the non-perturbative sector of a supersymmetric theory. This can be done without an additional supersymmetry breaking by the applied approximation method. For the lattice simulations one has to ensure that loality and supersymmetry are recovered in the continuum limit. This has to be investigated for each considered model separately. Further investigations towards a Ginsparg-Wilson relation for supersymmetry can help to find a solution that does not depend on a specific model. In any case, it is instructive to compare the obtained results with other approximation schemes, as they are indicated in this thesis.

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A Rules and conventions

A.1 Indices and summations

The Euclidian spacetime indices are always labeled by μ and ν whereas spinor indices are denoted as α, β. A summation of the spacetime indices is only implied when they appear in the upstairs downstairs combination $\partial_\mu \partial^\mu = \sum_\mu \partial_\mu \partial_\mu$. The summation over μ runs always from zero to $\mathrm{D}-1$ where D is the dimension of the space-time.

The indices i and j label different field species. When no specific model is considered the fields φ and ϕ stand for a general multiplet of fields. The components of such a general multiplet are labelled by the index i and j (i_1, i_2, etc.). In that case these components can also contain the spinor components of a spinor field or the vector components of a field A^μ.

A.2 The lattice

The lattice is a hypercubic discretisation of the space time. The lattice points are x_n . In the μ direction there are N_μ lattice points separated by the spacing a_μ. Here the number of lattice points is assumed to be odd in each direction. If the index of N and a is not specified the same number of points and lattice spacing is assumed for all directions. Each lattice point is labelled by a vector n with D components. These components run from 0 to $N_\mu - 1$. The lattice point is just defined as $x_n = \sum_\mu n_\mu a_\mu$ The vector e_μ has zero components except a 1 in its μ direction. So $n + e_\mu$ labelled the next neighbouring lattic point of x_n in μ direction. φ_n is the field associated with the lattice points x_n.

The size of the lattice in μ direction is, consequently, $L_\mu = N_\mu a_\mu$ and its volume $\Omega_L = \prod_\mu L_\mu$. Periodic boundary conditions are assumed for this volume. These are implemented by setting $n + N_\mu e_\mu = n$ in every direction μ. If not further specified the sum $\sum_n$ of a lattice index runs over the whole lattice and contains an additional factor $\prod_\mu a_\mu$:

$$\sum_n = \prod_\mu \left(a_\mu \sum_{n_\mu=0}^{n_\mu=N_\mu-1} \right) \tag{A.1}$$

The lattice index is labeled by n and m. n_i or n_j are treated as individual lattice indices; only the n_μ or n_ν stands for a component of an index. So $(n_1)_\mu$ is the component of n_1 in μ direction.

A.3 Fourier transformation on the lattice

The first step for the formulation of a lattice theory is to consider a finite volume with periodic boundary conditions. This represents the lattice in the continuum (but not thermodynamic) limit. The hypercubic volume has the length L_μ in μ direction and a

volume Ω_L. In this case a general continuum field has the Fourier series representation

$$\varphi(x) = \frac{1}{\Omega_L} \sum_{q \in \mathbb{Z}^D} \varphi(p_q) e^{ip_q x} \tag{A.2}$$

$$\varphi(p_q) = \int_{\Omega_L} d^D x \, \varphi(x) e^{-ip_q x} \,, \tag{A.3}$$

with dimensionless wave numbers $q \in \mathbb{Z}$ and $(p_q)_\mu = \frac{2\pi q_\mu}{L_\mu}$. ($p_q x$ here stands for a scalar product of the two vectors.) This representation is applied to the averaging function $f(an - x)$ in section 5.3. In the thermodynamic limit ($L \to \infty$) the Fourier modes p_q become continuous and the usual continuum Fourier representation is obtained.

Functions on the lattice can be parametrised by N independent waves,

$$\phi_n = \sum_k \phi(p_k) e^{ip_k x_n} \tag{A.4}$$

$$\phi(p_k) = \sum_n \phi_n e^{-ip_k x_n} \,. \tag{A.5}$$

If not further specified the above sum over k represents

$$\sum_k = \frac{1}{\Omega_L} \prod_\mu \sum_{k_\mu = -(N-1)/2}^{k_\mu = (N-1)/2} \,. \tag{A.6}$$

(Here $k \in Z^D$ represents a vector with components $k_\mu \in \mathbb{Z}$ for each direction of the space time μ.) From this relation it is clear that $\phi(p_k)$ is periodic in p_k, $\phi(p_k) = \phi(p_k + e_\mu l 2\pi/a_\mu)$ $\forall l \in \mathbb{Z}$ and all directions μ. The same transformation is used for the $\phi_f(an)$ in section 5.3. The momentum of the modes is inside the Brillouin zone (BZ) defined by BZ $= \{(p_\mu) | \, |p_\mu| \leq (\Lambda_L)_\mu = \frac{\pi}{a_\mu}\}$.

This implies also a representation of the delta on the lattice

$$\begin{aligned} \delta(x_n - x_m) &= \delta_{nm} := \sum_k e^{ip_k(x_n - x_m)} = \prod_\mu (a_\mu)^{-1} \bar{\delta}_{n_\mu m_\mu} \\ \delta(p_{k_1} - p_{k_2}) &:= \sum_n e^{-i(p_{k_1} - p_{k_2})x_n} = \prod_\mu (N_\mu a_\mu) \bar{\delta}_{(k_1)_\mu (k_2)_\mu} \,, \end{aligned} \tag{A.7}$$

where $\bar{\delta}_{n_\mu m_\mu}$ is one for $n_\mu = m_\mu \mod N_\mu$ and zero otherwise. These two delta functions are periodic: $\delta(p_k) = \delta(p_k + e_\mu l 2\pi/a_\mu)$; $\delta(x_k) = \delta(x_k + e_\mu l N_\mu)$ $\forall l \in \mathbb{Z}$ and μ.

For the lattice perturbation theory the thermodynamic of the lattice expressions is performed. Then one gets the following Fourier representation with the now continuous

momentum p:

$$\phi_n = \int_p \phi(p) e^{ipx_n} := \int_{\mathrm{BZ}} \frac{d^D p}{(2\pi)^D}\, \phi(p) e^{ipx_n} \tag{A.8}$$

$$\phi(p) = \sum_n \phi_n e^{-ipx_n}\,. \tag{A.9}$$

(n now runs over an infinite number of lattice points.) One can easily change between the Fourier representation on a finite lattice and in the thermodynamic limit. The expressions in Fourierspace remain the same, only the discrete momentum p_k has to be replaced by the continuous p. This continuous momentum is, because of the finite lattice spacing still restricted to the BZ. Consequently the delta in Fourier space is still periodic as on the lattice.

In a similar way the Fourier representation of translational invariant operators with two indices is derived on the lattice:

$$\nabla^\mu_{nm} = \sum_k \nabla^\mu(p_k)\, e^{ip_k(x_n - x_m)}\,. \tag{A.10}$$

(Translational invariant means in one dimension a circulant matrix.) This operator has the same periodicity as the Fourier space representation of a field. On the lattice the matrix entries of ∇ should be real. Therefore, the imaginary part of $\nabla(p)$ is antisymmetric ($\Im\nabla(-p) = -\Im\nabla(p)$) and the real part is symmetric ($\Re\nabla(-p) = \Re\nabla(p)$).

An antisymmetric derivative operator can always be represented as

$$\nabla^\mu_{nm} = \sum_{r=1}^{N-1} c_r (\nabla^{(r)})^\mu_{mn}\,, \quad \text{with} \quad (\nabla^{(r)})^\mu_{nm} = \delta_{n+r,m} - \delta_{n-r,m}\,, \tag{A.11}$$

and some constants c_r.

For the operators defined in the main text the Fourier representation is thus

$$\begin{aligned}
\nabla^{(s)}_\mu(p) &= \frac{i}{a_\mu}\sin(p_\mu a_\mu) \\
(\nabla^{(+)}_\mu \nabla^{(-)}_\mu)(p) &= \frac{4}{a_\mu^2}\sin^2(p_\mu a_\mu/2) \\
m^{(W)}(p) &= \sum_\mu \frac{2r}{a_\mu}\sin^2(p_\mu a_\mu/2) \\
\nabla^{\mathrm{SLAC}}_\mu(p) &= ip_\mu\,.
\end{aligned} \tag{A.12}$$

Apart from the SLAC derivative these are all functions of the type $a^d F(ap)$ where d is determined by the dimension of the operator. Obviously the behaviour of F in the vicinity

of $p = 0$ is the important part in the continuum limit unless there are other points with $F = 0$.

Now consider a more complicated operator $\tilde{C}_{m_1,\ldots m_{n_\mathrm{f}}}$. Translational invariance implies that a shift of all lattice points by x_m is irrelevant. One way for the representation in Fourier space is

$$\begin{aligned} \tilde{C}_{m_1,\ldots,m_{n_\mathrm{f}}} &= \sum_{k_1,\ldots,k_{n_\mathrm{f}}} \tilde{C}(p_{k_1},\ldots,p_{k_{n_\mathrm{f}}}) e^{i(p_{k_1}x_{m_1}+\ldots+p_{k_{n_\mathrm{f}}}x_{m_{n_\mathrm{f}}})} \\ \tilde{C}(p_{k_1},\ldots,p_{k_{n_\mathrm{f}}}) &= \sum_{m_1,\ldots,m_{n_\mathrm{f}}} \tilde{C}_{m_1,\ldots,m_{n_\mathrm{f}}} e^{-i(p_{k_1}x_{m_1}+\ldots+p_{k_{n_\mathrm{f}}}x_{m_{n_\mathrm{f}}})} . \end{aligned} \tag{A.13}$$

The translational invariance means for the Fourier space representation

$$\tilde{C}(p_{k_1},\ldots,p_{k_{n_\mathrm{f}}}) e^{i(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}})x_m} = \tilde{C}(p_{k_1},\ldots,p_{k_{n_\mathrm{f}}}) . \tag{A.14}$$

This implies $p^\mu_{k_{n_\mathrm{f}}} = -p^\mu_{k_1} - \ldots - p^\mu_{k_{n_\mathrm{f}-1}} \mod \Lambda^\mu_L$ for all μ. A representation that explicitly implies the translational invariance is obtained by going from $\tilde{C}$ with n_f indices to a matrix C with $n_\mathrm{f} - 1$ indices according to

$$\tilde{C}_{m_1,\ldots,m_{n_\mathrm{f}}} = C_{(m_{n_\mathrm{f}}-m_1),\ldots,(m_{n_\mathrm{f}}-m_{n_\mathrm{f}-1})} . \tag{A.15}$$

A representation in Fourier space of this matrix is

$$C_{m_1,\ldots,m_{n_\mathrm{f}-1}} = \sum_{k_1,\ldots,k_{n_\mathrm{f}-1}} C(p_{k_1},\ldots,p_{k_{n_\mathrm{f}-1}}) e^{ip_{k_1}x_{m_1}+\ldots+ip_{k_{n_\mathrm{f}-1}}x_{m_{n_\mathrm{f}-1}}} . \tag{A.16}$$

Thus the two representations are in Fourierspace related by

$$\tilde{C}(p_{k_1},\ldots,p_{k_{n_\mathrm{f}}}) = \delta(p_{k_1}+\ldots+p_{k_{n_\mathrm{f}}}) C(p_{k_1},\ldots,p_{k_{n_\mathrm{f}-1}}) . \tag{A.17}$$

A.3.1 The nonlocal interaction term

In this section the one dimensional formulation of the non local interaction term is considered. These investigations can be generalised to higher dimensions. The matrix $\mathcal{F}$ in

section 3.1.1 follows from

$$\begin{aligned}\mathcal{F}_{nm} &= \frac{1}{aN}\sum_{k=-(N-1)/2}^{(N-1)/2}\exp\left(i\frac{2\pi k}{aN}\left(am-\frac{a}{(n_f-1)}n\right)\right)\\ &= \frac{e^{-i\frac{2\pi(N-1)}{2N}(m-\frac{n}{n_f-1})}}{aN}\frac{1-e^{i2\pi(m-n/(n_f-1))}}{1-e^{i2\pi/N(m-n/(n_f-1))}}\\ &= \frac{\sin(\pi(m-n/(n_f-1))}{aN\sin(\pi/N(m-n/(n_f-1))}\,. \end{aligned} \tag{A.18}$$

Note that the additional factor of $1/a$ cancels the factor a in front of the summation over the lattice ($\sum_n$) according to our conventions. It is clear that for this kind of matrix the following summation rule holds

$$\frac{a}{(n_f-1)}\sum_{n=0}^{n_fN-1}\mathcal{F}_{nm_1}\mathcal{F}_{nm_2}=\delta(x_{m_1}-x_{m_2})\,. \tag{A.19}$$

$\mathcal{F}$ maps the fields $\varphi^{(i)}$ with a lattice size N on the fields $\tilde{\varphi}^{(i)}_n=\sum_n\mathcal{F}_{nm}\varphi^{(i)}_n$ with a lattice size $(n_f-1)N$. The fields on the larger lattice have, however, a momentum constraint that is constraint below $\frac{\pi}{a}$ instead the larger lattice cutoff $\frac{\pi(n_f-1)}{a}$. So $\mathcal{F}$ generates a one to one map of $\varphi^{(i)}(p_k)$ onto the modes of the larger lattice below the cutoff $\frac{\pi}{a}$.

A.3.2 A Fourier space representation of locality

It is a well-known fact that the smoothness of a function in Fourier space is related to the locality or the "broadness" of a function in real space. All of the following one dimensional considerations can be extended easily to higher dimensions. In [97] it is shown that the exponential decay needs analyticity in Fourier space. In the discussion of section 5.2 this could not be achieved, but a "modified" version of locality was possible. Let us therefore find an explicit example of such a solution.The considered lattice operators O are, because of translational invariance, circulant matrices,

$$O_{mn}=O_{n-m}=F(a(n-m))\,. \tag{A.20}$$

The slightly modified condition for locality demands that F decays faster than any polynomial. That means

$$|x^rF(x)|<\infty\quad\forall\, r\in\mathbb{N},\, x,y\in a\mathbb{N}\,. \tag{A.21}$$

If the Fourier transform of $F(x)$, $f(p)$, and its derivatives have no singularities and fulfil periodic boundary conditions at edge of the BZ the following estimation can be made

$$\begin{aligned} |x^r F(x)| &= |\int_{\mathrm{BZ}} (\partial_p^r f(p)) e^{ipx}| \\ &\leq \int_{\mathrm{BZ}} |\partial_p^r f(p)| \leq C_r < \infty \,. \end{aligned} \tag{A.22}$$

Consider now a non-local operator similar to the SLAC derivative. This non-local operator should have no singularities within the BZ for all of its derivatives. The boundary conditions are, however, not periodic. According to the discussion of the locality of K and M_{def} in section 5.2 it should support the modified locality after a multiplication with a local operator. In view of the above argument the boundary conditions must hence be enforced by this local operator without spoiling the differentiability of f. Its representation in Fourier space, $I(p)$, must therefore be a function that vanishes together with all its derivatives at the edge of the BZ. In addition no singularities should appear within the BZ for any of its derivatives. One function that fulfils these requirements is

$$I(p) = \begin{cases} \exp\left(-\frac{\epsilon^2}{\epsilon^2 - p^2}\right) & |p| < \epsilon \\ 0 & |p| \geq \epsilon \end{cases} \text{ with } \epsilon \leq \frac{\pi}{a}. \tag{A.23}$$

It is clear that $I(p)$ cannot be analytic since any analytic function that vanishes with all its derivatives at a specific point must be identical to zero. So the common definition of locality in terms of analyticity in momentum space cannot be satisfied.

A.3.3 The Fourier representation of the additional constraint

Here only the one dimensional case (N lattice points separated by a) was considered and it can easily be generalised to the higherdimensional case. In Fourier space the convolution in the averaged field of eq. (5.1) becomes a product,

$$\begin{aligned} \phi_f(p_k) &= \frac{1}{\Omega_L} \sum_{q \in \mathbb{Z}} \sum_n e^{i(p_q - p_k)x_n} f(p_q)\varphi(p_q) \\ &= \sum_{l=-\infty}^{\infty} f(p_k + l\frac{2\pi}{a})\varphi(p_k + l\frac{2\pi}{a}) \,. \end{aligned} \tag{A.24}$$

The last line comes from the periodicity of the delta (A.7). This shows how the averaging projects the Fourier components of φ onto the first Brillouin zone. In addition one easily observes that the Fourier components of ϕ_f and the lattice fields are determined by f, which means that f introduces a cutoff for the lattice momentum if $f(p_q)$ vanishes for all p_q greater than the cutoff.

The additional constraint (5.36) reads after partial integration

$$\sum_m \nabla_{nm} f(am - x) + \partial_x f(an - x) = 0 \quad \forall\, n, x\,. \tag{A.25}$$

With the Fourier representation of the derivative operator the constraint becomes

$$\sum_{q=-\infty}^{\infty} f(p_q)\Big[\nabla(p_q) - ip_q\Big] e^{ip_q(an-x)} = 0\,, \tag{A.26}$$

which for every individual component p_q gives the constraint (5.37).

A.4 Supertraces and Determinants

In the present investigations the superdeterminant is defined by

$$(\mathrm{Sdet}\, M)^{-1/2} = \int \mathcal{D}\varphi \exp(-\frac{1}{2}\varphi_i M_{ij} \varphi_j) \tag{A.27}$$

For real bosonic fields this means $\det^{-1/2}$, for Majorana fermions the Pfaffian.

The same expression $\mathrm{Sdet}\, M^{-1}$ is consequently the Jacobi factor when a transformation $\varphi \to M\varphi$ is applied. For the transformation $\varphi \to e^{i\varepsilon M}\varphi$ it can be reformulated as

$$\exp\left(-\mathrm{STr}\log e^{i\varepsilon M}\right) := \mathrm{Sdet}\left(e^{i\varepsilon M}\right)^{-1} \approx 1 - i\varepsilon \mathrm{STr} M \tag{A.28}$$

This is the definition of the supertrace applied here.

B Proof of renormalisability for the SLAC derivative is two dimensions

The purpose of this chapter is an explicit investigation of the one-loop diagrams of a twodimensional theory with a Yukawa type interaction of fermions and bosons. Supersymmetry is not needed in the derivation. Gauge theories are excluded in this discussion. It is shown that the SLAC derivative leads to a correct continuum limit of perturbation theory. The counterterms are local and Lorentz covariant. Furthermore they are all of the same form as the continuum counterterms.

For the SLAC derivative, the momentum space representation of the propagators

$$\frac{1}{P(k)^2 + m^2} \quad \text{and} \quad \frac{-i\not{P}(k) + m}{P(k)^2 + m^2} \tag{B.1}$$

for bosons and fermions contains the saw tooth function

$$P_\mu(k) = k_\mu - 2l\Lambda_L \quad \text{where} \quad (2l-1)\Lambda_L \leq k_\mu \leq (2l+1)\Lambda_L. \tag{B.2}$$

In addition arbitrary (local) vertices are allowed. The momentum integration is always restricted to the first BZ. The internal lines in the one-loop diagrams carry either the internal momentum k or a sum $k+q$ of internal and external momenta where q denotes a linear combination of the external momenta (using momentum conservation, there are $n-1$ such linear combinations q_j in a diagram with n vertices).

Integrations over loop momenta k_μ can be split into integrations over a square $D = \{(k_\mu)|\, |k_\mu| \leq \frac{\pi\varepsilon}{a}\}$ for an arbitrary $0 < \varepsilon < \frac{1}{2}$ and the rest of the Brillouin zone, BZ$\backslash D$. In the following, an upper bound for the boson propagator in momentum space is determined which will be used later on to argue that parts of the integrals in lattice perturbation theory are going to vanish in the continuum limit. For a given set of external momenta $\{q_j\}$, one may choose $\eta = \max_{\mu,j}\{\frac{a_0}{\pi}|q_{j\mu}|\}$ with a_0 small enough such that $0 < \eta < \varepsilon < \frac{1}{2}$. For $(k_\mu) \in D$, one can then read off from

$$a|k_\mu \pm q_\mu| \leq a(|k_\mu| + |q_\mu|) < \pi(\varepsilon + \eta) \quad \text{for all} \quad a < a_0 \tag{B.3}$$

that $|k_\mu \pm q_\mu| \leq \Lambda_L \varepsilon'$ with $\varepsilon' := \varepsilon + \eta < 1$, i. e. $(k_\mu \pm q_\mu)$ is also inside the first Brillouin zone and $P(k_\mu \pm q_\mu) = (k_\mu \pm q_\mu)$. On the other hand, if $(k_\mu) \in$ BZ$\backslash D$,

$$\pi(\varepsilon - \eta) \leq a(|k_\mu| - |q_\mu|) \leq a|k_\mu + q_\mu| \leq a(|k_\mu| + |q_\mu|) \leq \pi(1 + \eta) \tag{B.4}$$

for such lattice spacings a. The latter inequality may be used in order to find an upper

bound for the propagator,

$$\frac{1}{P(k\pm q)^2+m^2} < \frac{1}{P(k\pm q)^2} < Ca^2 \tag{B.5}$$

with $C = \big((\varepsilon-\eta)\sqrt{2}\pi\big)^{-2}$.

It can be easily seen that in the considered models, only two different types of integrals contribute at one-loop level. The first type of integrals resembles a diagram with bosonic lines and has the form

$$\begin{aligned}
I_\varepsilon + I_\pi &= \int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{1}{(P(k)^2+m^2)(P(k+q_1)^2+m^2)\dots(P(k+q_{n-1})^2+m^2)}, \qquad \text{(B.6)}\\
I_\varepsilon &= \int_{D} \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+m^2)((k+q_1)^2+m^2)\dots((k+q_{n-1})^2+m^2)},\\
I_\pi &= \int_{\mathrm{BZ}\setminus D} \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2+m^2)(P(k+q_1)^2+m^2)\dots(P(k+q_{n-1})^2+m^2)}\\
&\le (Ca^2)^{n-1} \int_{|k|\le\sqrt{2}\Lambda_L} \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m^2)} = \frac{(Ca^2)^{n-1}}{4\pi}\log\Big(1+\frac{2\pi^2}{a^2m^2}\Big).
\end{aligned}$$

Here, we have applied (B.4) in order to find an upper bound for the integrand in I_π and then enlarged the integration domain to a full disk including the first Brillouin zone. Thus, I_π vanishes in the continuum limit if $n > 1$. Therefore, the integral I_ε tends to the continuum value of the integral as a goes to zero (and the corresponding continuum integral is convergent by power counting), so as long as we are considering diagrams with more than one vertex, this type of integrals does not spoil renormalisability.

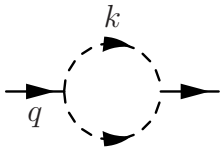
An addition class of integrals arises from diagrams with fermionic lines. This class of integrals is

$$\begin{aligned}
I'_\varepsilon + I'_\pi &= \int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{P_\mu(k)P_\nu(k+\tilde q_1)\dots P_\varrho(k+\tilde q_l)}{(P(k)^2+m^2)(P(k+q_1)^2+m^2)\dots(P(k+q_{n-1})^2+m^2)}, \qquad \text{(B.7)}\\
I'_\varepsilon &= \int_{D} \frac{d^2k}{(2\pi)^2} \frac{k_\mu\dots(k+\tilde q_l)_\varrho}{(k^2+m^2)((k+q_1)^2+m^2)\dots((k+q_{n-1})^2+m^2)},\\
I'_\pi &= \int_{\mathrm{BZ}\setminus D} \frac{d^2k}{(2\pi)^2} \frac{P_\mu(k)\dots P_\varrho(k+\tilde q_l)}{(k^2+m^2)(P(k+q_1)^2+m^2)\dots(P(k+q_{n-1})^2+m^2)}\\
&\le \int_{\mathrm{BZ}\setminus D} \frac{d^2k}{(2\pi)^2} \frac{|P_\mu(k)|\dots|P_\varrho(k+\tilde q_l)|}{(k^2+m^2)(P(k+q_1)^2+m^2)\dots(P(k+q_{n-1})^2+m^2)}\\
&\le \Big(\frac{\pi}{a}\Big)^{l+1}(Ca^2)^{n-1} \int_{|k|\le\sqrt{2}\pi/a} \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m^2)} = \frac{C^{n-1}a^{2n-l-3}}{4\pi}\log\Big(1+\frac{2\pi^2}{a^2m^2}\Big).
\end{aligned}$$

The $\tilde q_i$ are taken from the q_j, so $l \le n-1$. The same arguments as above show that the continuum limit is correct for any $n > 2$ (again, all continuum integrals are convergent by power counting).

Therefore, renormalisability only remains to be shown for two kinds of integrals. The first consists of diagrams with $n = 1$, e. g., tadpole diagrams. In this case, the loop momentum is independent of the (vanishing) exterior momentum so that the argument of $P_\mu(k)$ is restricted to the first Brillouin zone (where $P_\mu(k) = k_\mu$). In the continuum the same diagram appears without this restriction. This continuum integral is divergent and can be regularised with a covariant cutoff Λ_c. In the BPHZ renormalisation scheme it is subtracted. This means that the counterterm is the same as the integral itself. For $\Lambda_c = \Lambda_L$ the difference between the continuum integral and its lattice counterpart is nonzero even in the continuum limit. However, the lattice integral does not need a new type of counterterm. A lattice version of the BPHZ renormalisation scheme is hence to subtract the lattice regularised integral.

The second kind of integrals, $n = 2$ and $l = 1$ in (B.7), requires a more careful investigation. This kind of integral arises from a fermion loop with two internal lines such as



$$\propto \int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{P_\mu(k)P^\mu(k-q) + m^2}{(P^2(k) + m^2)(P^2(k-q) + m^2)}. \tag{B.8}$$

Depending on the vertices of the theory more than two external lines may appear; in this case q denotes the sum of all incoming external momenta. Such a diagram can be split into a finite part of the class (B.6) and the considered $n = 2$ and $l = 1$ of (B.7). The result does, obviously, not depend on whether or not the additional finite part is present.

As above we try to apply the same kind of renormalisation procedure as in the continuum BPHZ scheme. This means here to subtract the (as yet finite) value of the lattice integral with vanishing exterior momenta. The counterterm is hence of the same form as in the continuum. Thus, we arrive at

$$\begin{aligned}
&\int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{P_\mu(k)P^\mu(k-q)}{(P^2(k) + m^2)(P^2(k-q) + m^2)} + \int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{m^2}{(P^2(k) + m^2)(P^2(k-q) + m^2)} \\
&\quad - (\text{value at } q = 0) \\
= &\int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{P^\mu(k-q)\big(k_\mu - P_\mu(k-q)\big)}{(P^2(k) + m^2)(P^2(k-q) + m^2)} \\
= &\int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \frac{q^\mu P_\mu(k-q)}{(k^2 + m^2)(P(k-q)^2 + m^2)} \\
&\qquad - 2\Lambda_L \sum_\mu \int_{-\Lambda_L}^{-\Lambda_L + q_\mu} dk_\mu \int_{-\Lambda_L}^{\Lambda_L} \frac{dk_{\nu\neq\mu}}{(2\pi)^2} \frac{P^\mu(k-q)}{(k^2 + m^2)(P(k-q)^2 + m^2)}
\end{aligned} \tag{B.9}$$

In the last step, a_0 was chosen in such a way that for all $a = \pi/\Lambda_L < a_0$, shifting $k_\mu \in$ BZ

by $-q_\mu$, one winds up either in the same or in an adjacent Brillouin zone, i. e.,

$$P_\mu(k-q) = k_\mu - q_\mu + 2\Lambda_L\big(\Theta(-\Lambda_L - k_\mu + q_\mu) - \Theta(k_\mu - q_\mu - \Lambda_L)\big). \tag{B.10}$$

The first term on the right-hand side of B.9 can be easily seen to converge to the value of its continuum counterpart by similar arguments as in (B.6) and (B.7). In order to prove that the second term does not give rise to any corrections in the continuum limit, we make use of (B.3) and B.5 and observe that an upper bound for its modulus is given by

$$\begin{aligned}
& 2\Lambda_L \sum_\mu \int_{-\Lambda_L}^{-\Lambda_L+q_\mu} dk_\mu \int_{-\Lambda_L}^{\Lambda_L} \frac{dk_{\nu\neq\mu}}{(2\pi)^2} \frac{|P^\mu(k-q)|}{(k^2+m^2)(P(k-q)^2+m^2)} \\
\leq\ & 2C\pi^2 \int_{-\Lambda_L}^{-\Lambda_L+q_1} \frac{dk_1}{2\pi} \int_{-\Lambda_L}^{\Lambda_L} \frac{dk_2}{2\pi} \frac{1}{k^2+m^2} + (q_1 \leftrightarrow q_2, k_1 \leftrightarrow k_2) \\
=\ & \frac{C}{2} \left| \int_{-\Lambda_L}^{\Lambda_L} dk_2 \arctan\Big(\frac{q_1}{\omega(k_2) - q_1\Lambda_L\omega(k_2)^{-1} + \Lambda_L^2\omega(k_2)^{-1}}\Big)\omega(k_2)^{-1} \right| + (q_1 \leftrightarrow q_2, k_1 \leftrightarrow k_2) \\
\leq\ & \frac{C}{2} \int_{-\Lambda_L}^{\Lambda_L} dk_2 \Big| \frac{q_1}{m^2 + k_2^2 - \Lambda_L q_1 + \Lambda_L^2} \Big| + (q_1 \leftrightarrow q_2, k_1 \leftrightarrow k_2)
\end{aligned} \tag{B.11}$$

with $\omega(k) = \sqrt{m^2+k^2}$; here, we have also used that $|\arctan(x)| \leq |x|$. It is obvious that this upper bound converges to zero in the limit where the lattice cutoff is removed.

So the for the discretisation of models like the $N = 2$ and $N = 1$ Wess-Zumino model in two dimensions the renormalisation of lattice integrals in the continuum limit needs no other type of counterterms than the continuum theory. There appear no counterterms that are in contradiction with the space-time symmetries. After the renormalisation the lattice perturbation theory reproduces its continuum counterpart in the continuum limit.

C Nicolai improvement

The Nicolai map can guide the construction of lattice action with partial realisation of supersymmetry. This transfers the bosonic part of the action into a Gaussian measure and the Jacobi determinant of this transformation is cancelled by the fermion determinant (Matthews-Salam-Seiler determinant). In supersymmetric quantum mechanics it can be found as

$$\int \mathcal{D}\varphi \; e^{-\int dt \left(\frac{1}{2}(\partial_t\varphi)^2+\frac{1}{2}W'(\varphi)^2+\bar{\psi}(\partial_t+W''(\varphi))\psi\right)} = \int \mathcal{D}\varphi \; e^{-\int dt \left(\frac{1}{2}\xi(\varphi)^2+\bar{\psi}\frac{\delta\xi(\varphi)}{\delta\varphi}\psi\right)} = \int \mathcal{D}\xi \; e^{-\frac{1}{2}\int dt\, \xi^2}\,, \quad \text{(C.1)}$$

with the Nicolai variable $\xi(\varphi) = \partial_t\varphi + W'(\varphi)$ (or $\tilde{\xi}(\varphi) = -\partial_t\varphi + W'(\varphi)$ with $\left(\frac{\delta\tilde{\xi}(\varphi)}{\delta\varphi}\right)^T$ as fermion matrix). In the second step the periodic boundary conditions were used to cancel the arising surface term. Thus an improved action was found

$$\frac{1}{2}\int dt \xi(\varphi(t))^2 + \int dt dt' \bar{\psi}(t)\frac{\delta\xi(\varphi(t))}{\delta\varphi(t')}\psi(t')\,. \quad \text{(C.2)}$$

A supersymmetry that is now fulfilled without the application of the Leibniz rule is

$$\delta\varphi = \bar{\varepsilon}\psi; \quad \delta\bar{\psi} = -\bar{\varepsilon}\xi(\varphi) \qquad (\delta\varphi = \bar{\psi}\varepsilon; \quad \delta\psi = \bar{\varepsilon}\tilde{\xi}(\varphi)) \quad \text{(C.3)}$$

It is hence preserved for arbitrary boundary conditions. Thus one can discretise $\xi(\varphi) \to \xi(\varphi)_n = (\nabla\varphi)_n + W'_L(\varphi)_n$ and construct the action

$$\sum_n \frac{1}{2}\xi(\varphi)_n\xi(\varphi)_n + \sum_{nm}\bar{\psi}_n\frac{\delta\xi(\varphi)_n}{\delta\varphi_m}\psi_m\,, \quad \text{(C.4)}$$

out of it. It preserves the discretised supersymmetry on the lattice. In this way the improved action, (4.8), is obtained.

Whenever a local Nicolai map one can apply this procedure. In fact, the result of a Nicolai map has always a form similar to (C.1) with a quadratic part and a fermion part the contains the Jacobian matrix.

For the two-dimensional $N = 2$ Wess-Zumino model a Nicolai map with complex Nicolai variables is

$$\xi = 2(\bar{\partial}\bar{\phi}) + W(\phi) \quad \text{and} \quad \xi = 2(\partial\phi) + \bar{W}(\bar{\phi})\,. \quad \text{(C.5)}$$

The action in terms of these variables is after the discretisation

$$\sum_n \left(\frac{1}{2} \bar{\xi}_n \xi_n + \bar{\psi}_n (K_f)_{nm} \psi_m \right) , \tag{C.6}$$

with

$$(K_f)^{\alpha\beta}_{nm} = \begin{pmatrix} \partial\xi_n/\partial\phi_m & \partial\xi_n/\partial\bar{\phi}_m \\ \partial\bar{\xi}_n/\partial\phi_m & \partial\bar{\xi}_n/\partial\bar{\phi}_m \end{pmatrix} = \begin{pmatrix} \partial W'_L(\phi)_n/\partial\phi_m & \nabla_{nm} \\ \nabla_{nm} & \partial\bar{W}'_L(\phi)_n/\partial\bar{\phi}_m \end{pmatrix} \tag{C.7}$$

in the complex formulation of the theory. This leads us to the improved formulation (4.36).

D Hamiltonian formulation of supersymmetric quantum mechanics

For of the model introduced in section 2.1.4 the Hamilton-operator of the quantised theory is

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + W'(\hat{x})^2) + \frac{1}{2}W''(\hat{x})\left[\hat{\psi}^\dagger\,,\,\hat{\psi}\right] \tag{D.1}$$

In the quantisation the fermionic creation and annihilation operators $\hat{\psi}^\dagger$ and $\hat{\psi}$ are introduced. The ordering in the last term is done in such a way that the zero point energy vanishes. One can introduce two supercharges,

$$\hat{Q} = (\hat{p} + iW'(\hat{x}))\hat{\psi} \quad \text{and} \quad \hat{Q}^\dagger = (\hat{p} - iW'(\hat{x}))\hat{\psi}^\dagger\,, \tag{D.2}$$

commutating with $\hat{H}$ and leading to

$$\left\{\hat{Q}\,,\,\hat{Q}^\dagger\right\} = 2\hat{H}\,. \tag{D.3}$$

The states can be divided into those with fermion number one and zero. The corresponding number operator is $\hat{n_f} = \hat{\psi}^\dagger\hat{\psi} = \frac{1}{2}(1 - \left[\hat{\psi}\,,\,\hat{\psi}^\dagger\right])$, i.e. $\hat{\psi}^\dagger$ increases – $\hat{\psi}$ decreases – n_f by one. According to the subspaces with $n_f = 0$ and $n_f = 1$ the above operators can represented in a matrix form

$$\hat{H} = \frac{1}{2}\begin{pmatrix} \hat{A}\hat{A}^\dagger & 0 \\ 0 & \hat{A}^\dagger\hat{A} \end{pmatrix} \;;\quad \hat{Q} = \begin{pmatrix} 0 & \hat{A} \\ 0 & 0 \end{pmatrix} \;;\quad \hat{Q}^\dagger = \begin{pmatrix} 0 & 0 \\ \hat{A}^\dagger & 0 \end{pmatrix}\,, \tag{D.4}$$

with $\hat{A} = \hat{p} + iW'(\hat{x})$.

The whole spectrum of the Hamiltonian is, appart from the ground states, doubly degenerate. The rationale of this observation is that the supercharges $\hat{Q}$ and $\hat{Q}^\dagger$ commute with the Hamilton operator. Each bosonic state has a corresponding fermionic counterpart with the same energy, and the supercharges generate the transition between these two states. From equation (D.3) one can easily derive that the ground state of the model must have a positive or zero energy.

$$\hat{Q}|0\rangle = 0 \Rightarrow \quad |0\rangle(x) = C\exp\left(\int^x dy W'(y)\right) \qquad (n_f = 1)\,, \tag{D.5}$$

$$\hat{Q}^\dagger|0\rangle = 0 \Rightarrow \quad |0\rangle(x) = C\exp\left(-\int^x dy W'(y)\right) \qquad (n_f = 0)\,. \tag{D.6}$$

In case of an odd W' (even W) none of these two states can be normalised. This means supersymmetry is spontaneously broken, $\hat{Q}|0\rangle \neq 0$ ($\hat{Q}^\dagger|0\rangle \neq 0$) is a massless fermionic

state, the goldstino, and the vacuum energy is different from zero.

Since the one-dimensional theory corresponds to a quantum mechanical system well-known numerical techniques can be used to get reasonable results for the observables. The methods rely on the numerical solution of the Schrödinger equation. The spectrum of the Hamiltonian can be calculated numerically and the mass gap between the ground state and the first exited state yields the effective mass (correlation length) of the theory. In our case the diagonalisation of a discretised Hamilton operator was used to determine the low lying energy eigenvalues to a high precision. This method was already employed in [34], where a good applicability of this approach was found. For comparison also the Numerov algorithm with the shooting method was used.

The result of these numerical calculations are the low lying eigenvalues of the Hamilton operator. The difference between the two lowest eigenvalues ($m_{\text{eff}} = E_1 - E_0$) is the mass gap of the theory and determines the correlation length in the twopoint functions, $\langle x(t)x(t')\rangle \sim e^{-|m_{\text{eff}}|(t-t')}$ for large positive $t - t'$.

The effective potential can also be calculated with these numerical methods to a high precision starting with the Schwinger functional calculated from

$$e^{W[j]} = \lim_{\beta\to\infty} \mathrm{Tr}(e^{-\beta(\hat{H}+j\hat{x})}) = e^{E_0[j]} \,, \tag{D.7}$$

where $E_0[j]$ is the lowest eigenvalue of the source dependent Hamiltonian $\hat{H} + j\hat{x}$. From $W[j]$ with constant j a numerical Legendre-transformation yields then the effective potential $u(x)$.

E Details about the perturbative calculations in superspace

This chapter contains some futher details of the loop expansion in supersymmetric quantum mechanics. It contains first the calculation of the Feynman diagrams of the weak coupling expansion in superspace. These can then be compared with the on-loop result presented later on.

E.1 The effective mass in the weak coupling expansion

The most efficient way to perform perturbative calculations in a supersymmetric theory is the perturbation theory in superspace. The basics about this method can be found in [29, 98]. This method will be applied here for a model considered later on in the lattice simulations. Since I want to consider the restoration of supersymmetry in the continuum limit, the even superpotential,

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{4}\Phi^4\,, \tag{E.1}$$

is used first.

I give only a short summary of the Feynman rules (lines and vertices) here more details can be found in the next section:

lines:  $[K+im]^{-1}(z,z') = \dfrac{K-im}{K^2+m^2}\delta(z-z')$

vertices:  $\dfrac{ig}{4}\displaystyle\int d\theta d\bar{\theta}\,,$

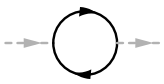
where $K^2 = -\partial_t^2$. Obviously no divergences can appear in the Feyman diagrams of this theory, but one gets, nevertheless, a finite renormalisation of the parameters. For the comparison with the numerical calculations the mass and wave function renormalisation are the intresting quantities.

The only one loop contribution to the 1PI two point function, $\Sigma(p,\bar{\theta},\bar{\theta}',\theta,\theta')$, is

 $= -i\dfrac{3g}{2m}\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')\,,$ (E.2)

and gives a contribution to the mass renormalisation. From the two loop contribution I

get also a wave function renormalisation. The corresponding Feynam graphs are



$$= -\frac{9g^2}{2(9m^4+m^2p^2)}(K(p)-3im)\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')\,, \tag{E.3}$$

and

$$= i\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')\frac{9g^2}{4m^3}\,. \tag{E.4}$$

The contributions to the mass and wave function renormalisation of these graphs are obtained from

$$(Z_r(p)K(p)+im)\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta') = (K(p)+im)\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta') - \Sigma(p,\bar{\theta},\bar{\theta}',\theta,\theta')\,. \tag{E.5}$$

Since the two renormalised quantities appear with different products of the fermionic coordinates, one can easily separate the corresponding renormalisation. In this way one gets

$$\begin{aligned} Z_r(p) &= 1+\frac{9g^2}{2(9m^4+m^2p^2)} = 1+g^2Z_2(p) \\ m_r(p) &= 1+\frac{3g}{2m}-\frac{27g^2m}{2(9m^4+m^2p^2)}-\frac{9g^2}{4m^3} = m+gm_1+g^2m_2(p)\,. \end{aligned} \tag{E.6}$$

For a comparison with the lattice results the on shell mass renormalisation must be calculated. In a generic supersymmetric field theory the modification of the nonlinear equations of motions for the auxiliary field do not appear until the third order of the weak coupling expansion since every derivative with respect to j_F carries a factor of the coupling constant in $S_{\text{int}}[\frac{\delta}{\delta j}]$. That means this field is maximally quadratic in Γ as long as only the first and second order of perturbation theory is considered[1]. So the on shell bosonic propagator that follows from the solutions of the linear equation is the inverse of

$$Z_r(p)p^2+\frac{m_r(p)^2}{Z_r(p)}\,. \tag{E.7}$$

The pole of this propagator that appears for imaginary p is the effective mass of the theory. A solution can be found order by order in the coupling constant g:

$$ip_{\text{pole}} = m+gm_1+g^2(m_2(p=im)-mZ_2(p=im)) = m+g\frac{3}{2m}-g^2\frac{9}{2m^3}\,. \tag{E.8}$$

In this way an effective mass is derived that can be compared with the results of other

[1] In the present case this is also true for the fourth order of the expansion.

calculation methods.

As expected a good agreement with the high precision numerical results is obtained for small g. The pole of the propagator, equation (E.8), can be directly compared with the mass gap, m_{eff}. A quadratic fit of the data $m_{\text{eff}}(g)$ in the range $g/m^2 < 0.002$ yields the same coefficients as predicted from (E.8). Increasing the values of the coupling constant to $g > 0.1m^2$ the deviation between the one loop and the two loop result gets large and both deviate from the exact result that stays in between them. For $g/m^2 \to 1$ the two loop renormalised mass gets even negative whereas the exact result is positive. This is a sign of an artificial phase transition in the loop expansion that has no coincidence with the physical properties of the theory. So one can trust the weak coupling expansion only in the region of very small couplings. A good estimate for its validity is the agreement between the one and two

E.2 Calculation of the diagrams in superspace

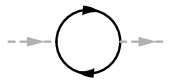
A basic relation for the weak coupling expansion is

$$K^2 = \frac{1}{4}(D\bar{D}D\bar{D} + \bar{D}D\bar{D}D) = \frac{1}{4}(-2i\partial_t)(\left\{D\,,\,\bar{D}\right\} = -\partial_t^2\,. \tag{E.9}$$

It transforms a products of the K operator into ordinary derivatives. This relation leads to the representation of the superspace propagator as shown in section E.1. As usual for the momentum space the momentum of each line has a momentum integration and each vertex a momentum conservation. For the additional $\theta, \bar{\theta}$ components each vertex carries an additional integration. The usual combinatorial prefactors are calculated as in ordinary perturbation theory. So the one loop contribution to the 1PI two-point function is

$$\begin{aligned} &= -\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')4\cdot 3\frac{ig}{4}\int\frac{dp}{2\pi}\frac{K(p)-im}{p^2+m^2}\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')\Big|_{\theta=\theta',\bar{\theta}=\bar{\theta}'} \\ &= -\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')3ig\int\frac{dp}{2\pi}\frac{1}{p^2+m^2} = -i\frac{3g}{2m}\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta')\,, \end{aligned} \tag{E.10}$$

where z has to be set to z' after the application of the differential operator K. Similarly a first two-loop contribution to the 1PI two point function can be calculated:



$$= -\frac{2\cdot 4\cdot 4\cdot 3\cdot 2g^2}{2\cdot 16}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\frac{(K(p-k_1-k_2)-im)\delta(\bar\theta-\bar\theta')\delta(\theta-\theta'))}{((p-k_1-k_2)^2+m^2)}$$

$$\times\frac{(K(k_1)-im)\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')(K(k_2)-im)\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')}{(k_1^2+m^2)(k_2^2+m^2)}$$

$$= -(1-3im\bar\theta\theta-3im\bar\theta'\theta'+(p-3im)\bar\theta'\theta-(p+3im)\bar\theta\theta'+p^2\bar\theta\theta\bar\theta'\theta')$$

$$\times 6g^2\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\frac{1}{((p-k_1-k_2)^2+m^2)(k_2^2+m^2)(k_2^2+m^2)}$$

$$= -\frac{18g^2(1-3im\bar\theta\theta-3im\bar\theta'\theta'+(p-3im)\bar\theta'\theta-(p+3im)\bar\theta\theta'+p^2\bar\theta\theta\bar\theta'\theta')}{4(9m^4+m^2p^2)}$$

$$= -\frac{9g^2}{2(9m^4+m^2p^2)}(K(p)-3im)\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')\,, \quad \text{(E.11)}$$

A second contribution arises from

$$= -\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')\frac{2\cdot 4\cdot 3\cdot 4\cdot 3g^2}{2\cdot 16}$$

$$\times\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int d\theta''d\bar\theta''\frac{(K(k_1)-im)\delta(\bar\theta''-\bar\theta''')\delta(\theta''-\theta'''))}{(k_1^2+m^2)}\Big|_{\theta''=\theta''',\bar\theta''=\bar\theta'''}$$

$$\times\frac{(K(k_2)-im)\delta(\bar\theta-\bar\theta'')\delta(\theta-\theta'')}{(k_2^2+m^2)}\frac{(K(k_2)-im)\delta(\bar\theta''-\bar\theta)\delta(\theta''-\theta)}{(k_2^2+m^2)}$$

$$= -\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')\frac{2\cdot 4\cdot 3\cdot 4\cdot 3g^2}{2\cdot 2\cdot 16m}\int\frac{dk_2}{2\pi}\int d\theta'd\bar\theta'\left(\frac{(K(k_2)-im)\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')}{(k_2^2+m^2)}\right)^2$$

$$= -\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')\frac{9g^2}{2m}\int\frac{dk_2}{2\pi}\frac{-2im}{(k_2^2+m^2)^2} = i\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')\frac{9g^2}{4m^3}\,. \quad \text{(E.12)}$$

With these results the mass renormalisation can be computed up to two loop.

E.3 The loop expansion for supersymmetric quantum mechanics

The one loop effective potential is calculated for the logarithm of the superdeterminant. This is just the quotient of the fermionic and the bosonic determinant, in the present

case[2]

$$u^{(1)}(\varphi,F) = \frac{1}{2}\log \mathrm{Sdet}\, S''[\phi] = \frac{1}{2}\mathrm{Str}\log S''[\phi] = \frac{1}{2}\int \frac{dp}{2\pi}\log\left(\frac{p^2+W''(\phi)^2+iFW'''(\phi)}{p^2+W''^2(\phi)}\right)$$
$$= \frac{1}{2}\left((W''^2+iFW''')^{1/2}-|W''|\right) = \frac{i}{4}\frac{FW'''}{W''}+O(F^2)\,. \qquad \text{(E.13)}$$

The first term of an expansion in terms of the auxilliary field is the one loop contribution to the supersymmetric effective Potential, that can also be calculated from using the Schwinger proper time representation:

$$\frac{1}{2}\mathrm{Str}\log\frac{K+iW''(\Phi)}{K+im} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Str}\left(e^{it(K+iW'')}-e^{it(K+im)}\right)\delta(\theta-\theta')\delta(\bar\theta-\bar\theta')|_{\theta=\theta';\,\bar\theta=\bar\theta'}$$
$$= -\frac{1}{2}\int d\theta d\bar\theta\int_0^\infty\frac{dt}{t}\int\frac{dp}{2\pi}\left(e^{-tW''}-e^{-tm}\right)\frac{\sinh(-itp)}{p}$$
$$= \frac{i}{4}\int d\theta d\bar\theta\log W''(\Phi)+\text{const.} = \frac{i}{4}\frac{FW'''(\phi)}{W''(\phi)}+\text{const.}\,. \qquad \text{(E.14)}$$

Here the trace has been replaced by a θ, $\bar\theta$, and p integration. The terms for the commutation of K and W'' correspond to higher orders in the expansion in covariant derivatives and are neglected. Only odd powers of K in e^{itK} contribute since the Grassman delta must be cancelled.

For the two-loop contributions the inverse of a field dependend propagator with a full dependence on the auxiliary field

$$[K(k)+iW''(\Phi_s)](\theta,\theta',\bar\theta,\bar\theta')|_{\Phi_s=\varphi+\bar\theta\theta F} = [K(k)+iW''(\Phi_s)]\delta(\bar\theta-\bar\theta')\delta(\theta-\theta')|_{\Phi_s=\varphi+\bar\theta\theta F} \qquad \text{(E.15)}$$

is needed. It is

$$[K(k)+iW''(\Phi_s)]^{-1}(\theta,\theta',\bar\theta,\bar\theta')|_{\Phi_s=\varphi+\bar\theta\theta F} = \frac{1-iW''(\varphi)(\bar\theta\theta+\bar\theta'\theta')+(k^2+iFW'''(\varphi))\bar\theta\theta\bar\theta'\theta'}{k^2+W''^2(\varphi)+iFW'''(\varphi)}$$
$$+\frac{k(\bar\theta'\theta-\bar\theta\theta')+iW'(\varphi)'(\bar\theta'\theta+\bar\theta\theta')}{k^2+W''^2(\varphi)}\,, \qquad \text{(E.16)}$$

[2] A possible sign change of the determinant was neglected.

which is proven by

$$\begin{aligned}
&\int d\theta'\bar{\theta}'[K(k)+iW''(\Phi_s)]^{-1}(\theta,\theta',\bar{\theta},\bar{\theta}')|_{\Phi_s=\varphi+\bar{\theta}\theta F}[K(k)+iW''(\Phi_s)](\theta',\theta'',\bar{\theta}',\bar{\theta}'')|_{\Phi_s=\varphi+\bar{\theta}\theta F}\\
&=\frac{iW''+(p^2-iW'''F)\bar{\theta}''\theta''-iW''+W''^2(\bar{\theta}\theta+\bar{\theta}''\theta'')+(p^2-iW'''F)\bar{\theta}\theta}{p^2+iW'''F+W''^2}\\
&+\frac{iW''p(\bar{\theta}''\theta-\bar{\theta}\theta'')+p^2(-\bar{\theta}''\theta-\bar{\theta}\theta'')-W''^2(\bar{\theta}\theta''+\bar{\theta}''\theta)+iW''p(\bar{\theta}\theta''-\bar{\theta}''\theta)}{p^2+W''^2}\\
&=\bar{\theta}\theta+\bar{\theta}''\theta''-\bar{\theta}''\theta-\bar{\theta}\theta''=\delta(\bar{\theta}''-\bar{\theta})\delta(\theta''-\theta)\,.
\end{aligned}$$

Two diagrams must be calculated to obtain the two loop effective potential. One of them is

$$\begin{aligned}
|_{\Phi_s=\varphi+\bar{\theta}\theta F}&=-\frac{3\cdot 2g^2}{2}\int d\theta d\bar{\theta}d\theta' d\bar{\theta}'\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}dk_3\delta(k_1+k_2+k_3)\Phi_s(\theta,\bar{\theta})\\
&[K(k_1)+iW''(\Phi_s)]^{-1}(\theta,\theta',\bar{\theta},\bar{\theta}')[K(k_2)+iW''(\Phi_s)]^{-1}(\theta,\theta',\bar{\theta},\bar{\theta}')\\
&[K(k_3)+iW''(\Phi_s)]^{-1}(\theta,\theta',\bar{\theta},\bar{\theta}')\Phi_s(\theta',\bar{\theta}')|_{\Phi_s(\theta,\bar{\theta})=\varphi+\bar{\theta}\theta F;\,\Phi_s(\theta',\bar{\theta}')=\varphi+\bar{\theta}'\theta' F}\\
&=-3g^2\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}dk_3\delta(k_1+k_2+k_3)\\
&\frac{F^2-6iFW''\varphi+(k_1^2+k_2^2+k_3^2+3iFW'''-6W''^2)\varphi^2}{(k_1^2+W''^2+iFW''')(k_2^2+W''^2+iFW''')(k_3^2+W''^2+iFW''')}\\
&+\left(\frac{2(k_2k_3+W''^2)\varphi^2}{(k_2^2+W''^2)(k_3^2+W''^2)((k_1+k_2)^2+W''^2+iFW'')}+k_3\leftrightarrow k_1+k_2\leftrightarrow k_1\right)\,.
\end{aligned}\tag{E.17}$$

The integration of the momentum yields the first contribution presented in equation (6.12).

The second contribution is obtained from

$$|_{\Phi_s=\varphi+\bar{\theta}\theta F} =$$

$$\begin{aligned}
&-\frac{3ig}{4}\int d\theta d\bar{\theta}\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}[K(k_1)+iW'']^{-1}(\theta,\bar{\theta},\theta,\bar{\theta})[K(k_2)+iW'']^{-1}(\theta,\bar{\theta},\theta,\bar{\theta})|_{\Phi_s=\varphi+\bar{\theta}\theta F}\\
&\qquad=-\frac{3ig}{4}\int d\theta d\bar{\theta}\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\left(\frac{1-2iW''\theta\bar{\theta}}{k_1^2+W''^2+iFW'''}+\frac{2iW''\bar{\theta}\theta}{k_1^2+W''^2}\right)\\
&\qquad\qquad\times\left(\frac{1-2iW''\theta\bar{\theta}}{k_2^2+W''^2+iFW'''}+\frac{2iW''\bar{\theta}\theta}{k_2^2+W''^2}\right)\\
&\qquad=-\frac{3ig}{4}\int d\theta d\bar{\theta}\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\left(\frac{4iW''\bar{\theta}\theta}{(k_1^2+W''^2+iFW''')(k_2^2+W''^2)}\right.\\
&\qquad\qquad\left.-\frac{4iW''\bar{\theta}\theta}{(k_1^2+W''^2+iFW''')(k_2^2+W''^2+iFW''')}\right)\\
&=-\frac{3ig}{4}\int d\theta d\bar{\theta}\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\left(\frac{4iW''(k_1^2+W''^2-k_1^2-W''^2+iFW''')\bar{\theta}\theta}{(k_1^2+W''^2+iFW''')(k_1^2+W''^2)(k_2^2+W''^2+iFW''')}\right)\\
&\qquad=3ig\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\left(\frac{FW''W'''}{(k_1^2+W''^2+iFW''')(k_1^2+W''^2)(k_2^2+W''^2+iFW''')}\right). \qquad \text{(E.18)}
\end{aligned}$$

This is the calculation of the loop diagrams as they are represented in the main text of the thesis.

F The solution in the zero momentum sector

The sero mode sector of the theory contains only the constant fields. The supersymmetry transformations in this sector contain no derivative terms and the auxilliary field is itself invariant under the transformations. If all fields are constant the lattice counterpart of action (5.39) in supersymmetric quantum mechnics has the following form

$$\frac{S}{aN} = \bar{\psi}\psi\, g(\chi) - h(\chi, F)\,. \tag{F.1}$$

aN stands here for the onedimensional volume and a possible F dependence of the fermionic part was neglected. The from of the the continuum action, equation (5.39), implies that the undetermined functions h and g approach $h(\chi, F) = FW'(\chi)$ and $Fg = \partial/\partial\chi$ in the continuum limit. In the present example I choose

$$a(\alpha^{-1})^{ij}_{mn} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & -a_1 & 0 \end{pmatrix}_{mn} \tag{F.2}$$

as the blocking matrix. In the continuum limit of the zero mode sector all entries of this matrix vanish. In this case the relation (5.11) becomes a partial differential equation in g and h:

$$Fg - \frac{\partial h}{\partial \chi} = -Na_1 g\frac{\partial h}{\partial \chi} - Na_0 g\frac{\partial h}{\partial F} - \frac{a_1}{a}\frac{\partial g}{\partial \chi}\,. \tag{F.3}$$

To minic the continuum Yukawa interaction the following from of g is assumed:

$$g(\chi) = \lambda\chi\,. \tag{F.4}$$

The general solution of (F.3) is restricted by the requirement that for vanishing constants a_i the term h should resemble the continuum result $F^2/2 + \lambda F\chi^2/2$. One obtains the non-polynomial solution

$$\begin{aligned} h(\chi, F) = \frac{1}{2}F^2 - \frac{1 + a_0N}{a_1N}\chi F + \frac{a_0(1 + a_0N)}{2a_1^2N}\chi^2 \\ - \left(\frac{1}{aN} + \frac{1 + a_0N}{a_1^2\lambda N^2}F - \frac{a_0(1 + a_0N)}{a_1^3\lambda N^2}\chi\right)\log(1 - a_1\lambda N\chi) \\ + \frac{a_0(1 + a_0N)}{2a_1^4\lambda^2N^3}(\log(1 - a_1\lambda N\chi))^2\,. \end{aligned} \tag{F.5}$$

Although this is a rather complicated expression it becomes polynomial in the limit of $a_1 \to 0$

$$h(\chi, F) = \frac{1}{2}F^2 + \frac{\lambda}{2}(1 + a_0 N)F\chi^2 + \frac{a_0}{8}\lambda^2 N(1 + a_0 N)\chi^4 \,. \tag{F.6}$$

A vanishing of elements of α is possible since a symmetric blocking matirx α_S can be added that does not change the relation.

So what we learn from this example is that generically the relation leads in the interacting (nonquadratic) case to nonpolynomial solutions. Under certain circumstances that involve the disappearace of certain nonsymmetric contributions in the blocking matrix a truncation at a finite order of the fields can be achieved.

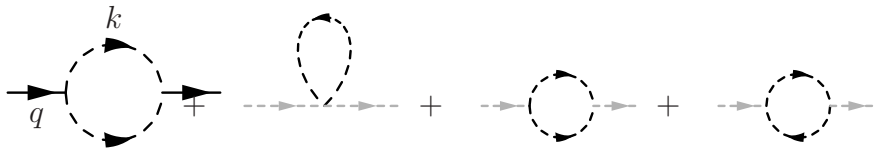
G The effective mass of the two-dimensional Wess-Zumino modell in the weak coupling approximation

The weak coupling analysis is in this model used for a comparison with the lattice results. In the lattice calculations the effective masses will be of great interest. The 1PI two-point function determines the renormalised mass in the on-shell theory. These calculations are done in a one loop approximation. The Feynman-rules can be read from the complex formulation of the theory,

$$S = \int d^2x \left(\frac{1}{2}(\partial_\mu \bar{\phi} \partial^\mu \phi + m^2 \bar{\phi}\phi) + \bar{\psi}(\not{\partial} + m)\psi \right.$$
$$\left. + \frac{mg}{2}(\bar{\phi}\phi^2 + \phi\bar{\phi}^2) + \frac{g^2}{2}\bar{\phi}^2\phi^2 + 2g\bar{\psi}(P_+\phi + P_-\bar{\phi})\psi \right) . \quad \text{(G.1)}$$

As propagators the usual propagator of a Euclidian bosonic theory, represented by a solid line, and the corresponding fermionic counterpart, denoted by a dashed line, appears. For convenience, I introduce an additional arrow on the lines indicating the direction of the contraction from a $\bar{\phi}$ to ϕ (or from $\bar{\psi}$ to ψ). The theory has three bosonic vertices with coupling constants mg and g^2. In addition, there are also two vertices of the Yukawa interaction that come with a coupling constant g and the projectors P_+ and P_-.

Tadpole diagrams can be constructed from the three point vertex and the Yukawa interaction. Both diagrams would be logarithmically divergent. These two contributions cancel each other so the theory remains, nevertheless, finite. The 1PI bosonic two-point function is obtained in terms of one loop diagrams:

$g^2\Sigma^{(1)}(m,p^2) =$ 

$$= -g^2 \frac{m^2 - p^2}{\pi\, p \sqrt{m^2 + \frac{p^2}{4}}} \log\left(\frac{\sqrt{m^2 + \frac{p^2}{4}} + \frac{p}{2}}{\sqrt{m^2 + \frac{p^2}{4}} - \frac{p}{2}} \right) , \quad \text{(G.2)}$$

where $p = \sqrt{p^2}$ stands for the modulus of the two-dimensional momentum vector $\vec{p}$.[1] The effective mass is obtained from the imaginary pole of the propagator. It is in the one loop approximation

$$(ip_{\text{pol}})^2 = m^2 - g^2\Sigma(g, m, p^2) + O(g^4) . \quad \text{(G.3)}$$

[1] Obviously the expression is invariant under $p \to -p$. So it is a function of p^2.

So up to the second order in g it is

$$m_{\text{eff}} = (ip_{\text{pol}}) \approx m - g^2 \frac{\Sigma(m, (-im)^2)}{m} = m - \frac{g^2}{m} \frac{4 \arctan(\sqrt{3})}{\sqrt{3}\pi} = m \left(1 - \frac{4g^2}{3\sqrt{3}m^2}\right) . \quad \text{(G.4)}$$

This is the effective mass relevant for the decay of the two-point functions.

H The flow equations in superspace

In this appendix I derive the flow equation in superspace. The superspace-coordinates $(x, \theta, \bar{\theta})$ are denoted by z.

The supertrace that defines the flow of the effective action translates into a superspace integral:

$$\partial_k \Gamma_k = \frac{1}{2} \int dz\, dz'\, \partial_k R_k(z, z') G_k(z', z)\,, \qquad G_k = (\Gamma_k^{(2)} + R_k)^{-1} \tag{H.1}$$

As in the component formulation the fields are taken to be constant to calculate the Green's function $G_k(z', z)$. In addition the expression is expanded in terms of the covariant derivatives D and $\bar{D}$. To zeroth order in the covariant derivatives one finds

$$\begin{aligned} i \int d\theta d\bar{\theta}\, \partial_t W(\Phi) = \frac{1}{2} \int \frac{dp}{2\pi}\, d\theta d\bar{\theta}\, d\theta' d\bar{\theta}'\ (i\partial_t r_1(p) + \partial_t r_2(p) K(p)) \times \\ \times \delta(\bar{\theta}' - \bar{\theta}) \delta(\theta' - \theta) \frac{hK(p) - i\mathcal{W}''(\Phi)}{hp^2 + (\mathcal{W}''(\Phi))^2} \delta(\bar{\theta}' - \bar{\theta}) \delta(\theta' - \theta)\,. \end{aligned} \tag{H.2}$$

Note that in momentum space the operator $K = \frac{1}{2}(D\bar{D} - \bar{D}D)$ still contains derivatives with respect to the Grassmann-coordinates. These derivatives act on the first entry of the adjacent delta-functions. The only two contributions that remain after an integration over θ' and $\bar{\theta}'$ are the ones where the highest Grassmann derivative acts on one and only one of the delta functions inside the integral. Therefore one obtains

$$\int d\theta d\bar{\theta}\, \partial_t W(\Phi) = \frac{1}{2} \int \frac{dp}{2\pi}\, d\theta d\bar{\theta}\, \left(\frac{h\partial_t r_1(p) - \mathcal{W}''(\Phi) \partial_t r_2(p)}{hp^2 + (\mathcal{W}''(\Phi))^2} \right). \tag{H.3}$$

For the lowest component of the superfield this is exactly the flow equation (7.6).

I German summary (Zusammenfassung)

In der vorliegenden Arbeit sind die grundlegenden Konsequenzen, die sich durch die Anwendung verschiedener Methoden der Quantenfeldtheorie auf supersymmetrische Modelle ergeben, untersucht worden.

Dabei ging es vor allem um die etablierte Methode der Gittersimulationen. Die dafür nötige Diskretisierung bricht die Supersymmetrie. Die wichtigste Ursache dieser Brechung ist die Versetzung der Leibnizregel durch einen beliebigen diskretisierten Ableitungsoperator. Es konnte gezeigt werden, daß diese Brechung nur mit einem nicht-lokalen Ableitungsoperator wie der SLAC-Ableitung und einen nicht-lokalen Wechselwirkungsterm beseitigt werden kann.

Eine weitere Ursache für die Brechung der Supersymmetrie entsteht bei der Anwendung von Standardmethoden zur Konstruktion einer lokalen Gitterwirkung. Im Rahmen dieser Verfahren wird, um eine effektive Verdopplung der Freiheitsgrade zu verhindern, ein zusätzlicher Massenterm (Wilson-Terms) für die Fermionen eingeführt. Die Beiträge dieses Massenterms fehlen im bosonischen Bereich. Schon in der ein-loop Gitter-Störungstheorie sieht man, daß diese Brechung der Supersymmetrie nicht wieder hergestellt werden kann. Dies geschieht obwohl der klassische Beitrag des Wilson-Terms verschwindet. Deshalb muß dieser zusätzliche Massenterm auch konsistent auf den bosonischen Sektor übertragen werden. Nur so kann man einen supersymmetrischen Kontinuumslimes erreichen.

In nicht-lokalen Formulierungen tritt auch dieses Problem nicht auf. Insgesamt ermöglichen die nicht-lokalen Gitterformulierungen eine Sicherstellung der Supersymmetrie. Im Rahmen der Gitterstörungstheorie zeigt sich, daß die Lokalität in niedrigdimensionalen Modellen im Kontinuumslimes wiederhergestellt wird.

Es ist wichtig, daß diese Überlegungen in der Arbeit durch numerische Simulationen bestätigt und vertieft werden. Dazu mußten viele verschiedene Gitterformulierungen verglichen und für Supersymmetrie relevante Operatoren mit hoher Präzision gemessen werden. In den behandelten Modellen sind sogar Simulationen mit nicht-lokalen Gitter-Operatoren gelungen. Dies ermöglichte die Verwendung von Formulierungen, die die Supersymmetrie vollständig erhalten. In den Gittersimulationen wurde die Supersymmetrische Quantenmechanik und das zweidimensionale $\mathcal{N} = 2$ Wess-Zumino-Modell untersucht.

Zur Überprüfung der Supersymmetrie in den Simulationen wurden die Massen der Fermionen und Bosonen, sowie die Ward-Identitäten gemessen. Die Realisierung nach den Standardmethoden zeigte weder bei endlichem Gitterabstand noch im Kontinuumslimes supersymmetrische Eigenschaften. Eine ganze Reihe anderer Diskretisierungen konnte aber gefunden werden, bei denen die Supersymmetrie im Kontinuumslimes wiederhergestellt wird.

Mit der Methode des Nicolai-Improvements konstruierte Diskretisierungen erhalten

einen Teil der Supersymmetrie auf dem Gitter. Dies zeigt sich auch in den gemessenen Ward-Identitäten. Für den verbleibenden Anteil wird die Brechung der Symmetrie bei endlichem Gitterabstand aber verstärkt. Außerdem trat bei diesem Verfahren im zweidimensionalen Wess-Zumino-Modell eine unphysikalische Phase mit einer Verstärkung von hohen Impulsmoden auf. Dies bedeutet nicht, daß diese Realisierung eines Teils der Supersymmetrie keine Verbesserung bedeutet. Es wirft aber die Frage auf, ob der Nutzen der teilweise erhaltenen Symmetrie die möglichen problematischen Eigenschaften der Gittertheorie aufwiegt. Dies muß für das jeweilige Modell und die behandelten Fragestellungen geklärt werden. Auf jeden Fall zeigt sich bei allen Modellen mit teilweise erhaltener Supersymmetrie ein Kontinuumslimes mit vollständiger Supersymmetrie. Dieser konnte aber auch in vergleichbaren Formulierungen ohne teilweise erhaltene Supersymmetrie erreicht werden.

Erstmals gelang in dieser Arbeit eine Simulation mit einer nicht-lokalen Diskretisierung, die die vollständige Supersymmetrie auch bei endlichem Gitterabstand erhält.

Supersymmetrie oder Lokalität müssen in den Gittermodellen in kontrollierter Weise verletzt werden, so daß beide Eigenschaften im Kontinuumslimes wieder hergestellt sind. Auch die chirale Symmetrie muß in kontrollierter Weise gebrochen werden. Dies wird durch das Nielsen-Ninmoya-Theorem verlangt. Die Ginsparg-Wilson-Relation stellt eine kontrollierte Brechung der chiralen Symmetrie sicher. Sie entspricht der Symmetrie einer perfekten Gitterwirkung. Eine Verallgemeinerung dieser Symmetrie-Relation auf allgemeine globale lineare Symmetrien sind Bestandteil dieser Arbeit. Bei der Anwendung auf Supersymmetrie treten allerdings zwei Schwierigkeiten auf. Zum einen ist die Übersetzung der Ableitungen in den Supersymmetrietransformationen die nicht-lokale SLAC-Ableitung. Dies folgt aus dem Übergang von Kontinuumsfeldern zu gemittelten Gitterfeldern. Die Lokalität kann mit Hilfe der Blocking-Matrix verbessert werden. Das zweite Problem tritt bei der Anwendung des Ansatzes von Ginsparg und Wilson auf eine nicht-quadratische Wirkung auf. Die Lösungen sind dann im allgemeinen nicht-polynomial. Dies ist verständlich, wenn man bedenkt, daß auch die perfekte Wirkung nicht-polynomialen Charakter hat. Auswege konnten aufgezeigt werden, und weitere Untersuchungen werden folgen.

Eine gute Alternative zu den Gittersimulationen ist die Methode des exakten Renormierungsgruppenflusses. Auch sie kann Einblick in den nicht-perturbativen Sektor einer supersymmetrischen Theorie geben. Man kann Regulatoren verwenden, die zu keiner Brechung der Supersymmetrie führen. Ein Problem dieser Methode ist aber, die passende Trunkierung der effektiven Wirkung zu finden. Um ein verläßliches Resultat zu erhalten, sollte man die Ergebnisse deshalb mit den Gitterrechnungen vergleichen. Mit einem solchen Vergleich kann man sicherstellen, daß die Fehler, die in beiden Verfahren auftreten, nicht relevant sind.

Als drittes Verfahren wurde auch die Schleifenentwicklung in einer supersymmetrischen Theorie untersucht. Dabei stößt man in der Literatur auf Unstimmigkeiten im Umgang mit dem Hilfsfeld. Wie diese Arbeit zeigt, entsprechen diese Unstimmigkeiten zwei unterschiedliche Näherungsverfahren. Eines davon entspricht einer Resummation von Schleifenbeiträgen. Es ermöglicht einen anderen Einblick in die exakten Eigenschaten der Theorie.

All diese Untersuchungen zeigen, daß eine Analyse einer supersymmetrischen Theorie auch jenseits der Störungstheorie möglich ist. Man kann die Methoden der Quantenfeldtheorie anwenden, ohne Supersymmetrie zu verletzen. Im Falle der Gittersimulationen muß man bei jedem gegebenen Modell überprüfen, daß Lokalität und Supersymmetrie im Kontinuumslimes wiederhergestellt werden. Weiter Untersuchungen einer Ginsparg-Wilson-Relation für Supersymmetrie können zu einer Lösung führen, die nicht von den spezifischen Modellen abhängt. Ein Vergleich mit den Ergebnissen anderer Verfahren ist in jedem Falle vorteilhaft. Möglichkeiten dafür konnten aufgezeigt werden.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, daß ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderern Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet. Auch die Ergebnisse, die in Zusammenarbeit mit den Mitgliedern des Lehrstuhles für Quantenfeldtheorie in Jena und anderen Kooperationen entstanden sind, sind in der Arbeit entsprechend benannt.

Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

Ich versichere ehrenwörtlich, daß ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

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