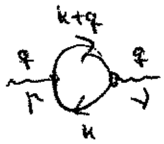
It is instructive to do the computation of $\Pi^{\mu\nu}(q)$ carefully. To lowest order, we need to compute

$$i\,\Pi^{\mu\nu}(q) =$$




$$= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\Big[\gamma^\mu \underbrace{\frac{i}{\not k - m}}_{=i\frac{\not k + m}{k^2-m^2}} \gamma^\nu \underbrace{\frac{i}{\not k + \not q - m}}_{=i\frac{\not k+\not q+m}{(k+q)^2-m^2}}\Big] \qquad (8.38)$$

Now we use some results from the exercises

$$\mathrm{tr}\,\gamma^\mu = 0\,, \quad \mathrm{tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}\,, \quad \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = 4\left(g^{\mu\nu}g^{\kappa\lambda} - g^{\mu\kappa}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\kappa}\right) \qquad (8.39)$$

$$\Rightarrow i\,\Pi^{\mu\nu}(q) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\left[k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\left(k\cdot(k+q) - m^2\right)\right]}{(k^2-m^2)\left((k+q)^2-m^2\right)} \qquad (8.40)$$

The denominator can be rewritten into a useful integral representation,

$$\frac{1}{(k^2-m^2)}\,\frac{1}{((k+q)^2-m^2)} = \int_0^1 dx\, \frac{1}{\left(k^2 + 2x\,k\cdot q + x^2q^2 - m^2\right)^2} \qquad (8.41)$$

which can straightforwardly be verified. The auxiliary integration parameter $x$ is called "Feynman Parameter". Using $\ell^\mu = k^\mu + x q^\mu$, we get

$$(8.41) = \int dx\, \frac{1}{\left(\ell^2 + x(1-x)q^2 - m^2\right)^2} \qquad (8.42)$$

Similarly for the numerator in (8.40), we have

$$[k^\mu (k+q)^\nu + \dots] = 2\ell^\mu \ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2) + \ell^\mu y_\mu \tag{8.43}$$

where $y_\mu$ depends on $q_\mu$, $x$ and $m$.

In order to perform the $\ell$ integral, let us recall that the propagators are defined with an $+i\varepsilon$ prescription: $\frac{1}{\ell^2 - m^2} \to \frac{1}{\ell^2 - m^2 + i\varepsilon}$.

This clarifies the position of the poles in the complex $\ell^2$ plane and justifies the transition to Euclidean loop momenta by performing a Wick rotation

$$\ell^0 = i\ell_E^0, \tag{8.44}$$

i.e., the $\ell^0$ contour is rotated onto the imaginary axis with $\ell_E^0 \in \mathbb{R}$, implying

$$\ell^2 = -\ell_E^2 = -(\ell_E^{0\,2} + \vec{\ell}^{\,2}) \tag{8.45}$$

$$\int d^4\ell = i\int d^4\ell_E. \tag{8.46}$$

In summary, we obtain

$$i\Pi^{\mu\nu}_{(q)} = -4ie^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{\left[-\frac{1}{2}g^{\mu\nu}\ell_E^2 + g^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)\right]}{(\ell_E^2 + m^2 - x(1-x)q^2)^2} \tag{8.47}$$

Here we used the fact that the terms linear in $\ell_{E\mu}$ vanish upon integration as the measure and the denominator are even in $\ell_E$. We also used the identity $\int d^4\ell_E\, f(\ell_E^2)\, \ell_\mu \ell_\nu = \frac{1}{4} g_{\mu\nu} \int d^4\ell_E\, f(\ell_E^2)\, \ell_E^2$. (8.48)

The integral is clearly divergent and thus ill defined. However we have to keep in mind that we will finally be interested in the difference $\hat{\Pi}(q) = \Pi(q) - \Pi(0)$, which will turn out to be finite. Instead of working with this finite difference, it is nevertheless useful to learn to work with $\Pi^{\mu\nu}(q)$ on its own.

For this, we have to introduce a regularization rendering (8.48) well defined. Various methods exist and are classified as regularization "schemes". In the present problem, it is important to choose a method that preserves gauge invariance.

For this, we use here the most common choice of

dimensional regularization ("dim. reg."). It has become standard in perturbative computations (but has to be handled with care beyond perturbation theory).

As an example, let us consider the integral

$$I(\Delta) := \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2+\Delta)^2} \xrightarrow{\text{general } D} \int \frac{d^D l_E}{(2\pi)^D} \frac{1}{(l_E^2+\Delta)^2} \qquad (8.49)$$

Introducing spherical coordinates in this generalized D-dimensional momentum space, we get

$$I(\Delta) = \int \frac{d\Omega_D}{(2\pi)^D} \int_0^\infty dl_E \, \frac{l_E^{D-1}}{(l_E^2+\Delta)^2}$$

$$= \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty dl_E \, \frac{l_E^{D-1}}{(l_E^2+\Delta)^2} \, , \qquad (8.50)$$

where we have used the result for the area of a D-dimensional unit sphere $\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} = \begin{cases} 2\pi & \text{for } D=2 \\ 4\pi & \text{for } D=3 \\ 2\pi^2 & \text{for } D=4 \end{cases}$.

In $D=4$, the integral (8.54) is, of course, still divergent; however, we can define and thus rigorously handle the integral for suitable values of $D$ ($D<4$ in this case):

$$I(\Delta) = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \, \frac{1}{2}\int_0 d(l_E^2) \, \frac{(l_E^2)^{\frac{D}{2}-1}}{(l_E^2+\Delta)^2}$$

$$u = \frac{\Delta}{(l_E^2+\Delta)}$$

$$= \frac{1}{2^D\pi^{D/2}\cancel{\Gamma(D/2)}} \left(\frac{1}{\Delta}\right)^{2-\frac{D}{2}} \underbrace{\int_0^1 du \, u^{1-\frac{D}{2}} (1-u)^{\frac{D}{2}-1}}$$

$$= B\left(2-\tfrac{D}{2}, \tfrac{D}{2}\right) \quad \text{Euler beta function}$$

$$= \frac{\Gamma(2-\frac{D}{2})\,\cancel{\Gamma(\frac{D}{2})}}{\Gamma(2-\frac{D}{2}+\frac{D}{2})}$$

$$= \frac{1}{2^D\pi^{D/2}} \, \frac{\Gamma(2-\frac{D}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-D/2}$$

$$= \frac{1}{(4\pi)^{D/2}} \, \Gamma\left(2-\tfrac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{2-D/2} \qquad (8.51)$$

As expected, the integral diverges for $D \to 4$. Choosing, e.g., $D = 4 - \varepsilon$, we get

$$\Gamma\left(2 - \tfrac{D}{2}\right) = \Gamma\left(\tfrac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \qquad (8.52)$$

$\gamma \simeq 0.5772$ (Euler-Mascheroni constant)

for small $\varepsilon$, exhibiting a $\frac{1}{\varepsilon}$ pole.

As the original integral had a log-like singularity in $D=4$, we conclude that $\frac{1}{\varepsilon}$ poles are indicative for such log-like divergencies.

(NB: As a particularity, we note that (8.50) would have stronger powerlike singularities for $D>4$, while (8.51) is well defined for all $D>4$ except for $\frac{1}{\varepsilon}$ poles at $D = 6, 8, 10, \ldots$; hence, (8.51) represents an analytic continuation of the original integral for general $D$. This analytic continuation is not sensitive to power-like divergencies but only to log-like divergencies, which occur in $D = 6, 8, 10, \ldots$ as subleading divergencies.)

Along the same lines, we can also work out the integral

$$I_2(\Delta) = \int \frac{d^D l_E}{(2\pi)^D} \frac{l_E^2}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{D/2}} \frac{D}{2} \Gamma\left(1 - \tfrac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{1 - \frac{D}{2}} \qquad (8.53)$$

With these results, we go back to Eq. (8.47). However, we should also recall that there was already one step, we we made explicit use of the dimensionality, namely in (8.48) which for general $D$ reads

$$\int d^D l_E \, f(l_E^2) \, l_\mu l_\nu = \frac{1}{D} g_{\mu\nu} \int d^4 l_E \, f(l_E^2) \, l_E^2 \quad (8.54)$$

Then, (8.47) can be written as

$$\Pi^{\mu\nu}(q) = -4e^2 \int_0^1 dx \Big[ \left(1-\frac{2}{D}\right) g^{\mu\nu} I_2(m^2 - x(1-x)q^2)$$
$$- 2x(1-x) q^\mu q^\nu I(m^2 - x(1-x)q^2)$$
$$+ g^{\mu\nu}(m^2 + x(1-x)q^2) I(m^2 - x(1-x)q^2) \Big]$$

$\Delta := m^2 - x(1-x)q^2$

$$= -\frac{4e^2}{(4\pi)^{D/2}} \int_0^1 dx \Big[ \left(\frac{D}{2}-1\right) g^{\mu\nu} \Gamma\left(1-\frac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{1-\frac{D}{2}}$$
$$- 2x(1-x) q^\mu q^\nu \cdot \Gamma\left(2-\frac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{2-D/2}$$
$$+ g^{\mu\nu}(m^2 + x(1-x)q^2) \Gamma\left(2-\frac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{2-D/2} \Big]$$

$$= -\frac{4e^2}{(4\pi)^{D/2}} \int_0^1 dx \Big[ - g^{\mu\nu} \underbrace{(\cancel{m^2} - x(1-x)q^2)}_{\Delta}$$
$$- 2x(1-x) q^\mu q^\nu$$
$$+ g^{\mu\nu}(\cancel{m^2} + x(1-x)q^2) \Big] \Gamma\left(2-\frac{D}{2}\right) \left(\frac{1}{\Delta}\right)^{2-D/2}$$

$$\Rightarrow \Pi^{\mu\nu}(q) = -\frac{8e^2}{(4\pi)^{D/2}}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\int_0^1 dx\; x(1-x)\,\Gamma\left(2-\tfrac{D}{2}\right)\frac{1}{(\Delta)^{2-\frac{D}{2}}} \tag{8.55}$$

Using $\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q)$, we can read off

$$\Pi(q) = -\frac{8e^2}{(4\pi)^{D/2}}\int_0^1 dx\; x(1-x)\,\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Delta^{2-\frac{D}{2}}} \tag{8.56}$$

As expected, the result diveges in the limit $\varepsilon \to 0$, with $D = 4-\varepsilon$:

$$\Pi(q) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\; x(1-x)\,(4\pi)^{\frac{\varepsilon}{2}}\,\Gamma\left(\tfrac{\varepsilon}{2}\right)\Delta^{-\varepsilon/2}$$

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\underset{+\mathcal{O}(\varepsilon^2)}{\left(1+\tfrac{\varepsilon}{2}\ln 4\pi\right)}\left(\tfrac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)\right)\cdot\underset{+\mathcal{O}(\varepsilon^2)}{\left(1-\tfrac{\varepsilon}{2}\ln\Delta\right)}\right]$$

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\frac{2}{\varepsilon} - \ln\Delta + \ln 4\pi - \gamma + \mathcal{O}(\varepsilon)\right] \tag{8.57}$$

This reflects the logarithmic singularity of this expression

However, for our physical observables, we are interested in

$$\hat{\Pi}(q) = \Pi(q) - \Pi(0)$$

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\Big[\cancel{\frac{2}{\varepsilon}} - \ln(m^2 - x(1-x)q^2) + \cancel{\ln 4\pi} - \cancel{\gamma}$$

$$- \cancel{\frac{2}{\varepsilon}} + \ln m^2 - \cancel{\ln 4\pi} + \cancel{\gamma} + \mathcal{O}(\varepsilon)\Big]$$

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\, \ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) + \mathcal{O}(\varepsilon), \qquad (8.58)$$

which is completely finite in the limit $\varepsilon \to 0$.

This is the desired result which we have already used and analyzed above.