8. Aspects of Quantumelectrodynamics (QED)

In chapter 6, we have already discussed / guessed the Feynman rules of QED. A number of tree-level processes are computed in the exercises. In the following, we concentrate on aspects of QED that are paradigmatic for general properties of QFTs in general, and QFTs with gauge symmetry in particular.

The Lagrangian of QED reads

$$\mathcal{L}_{QED} = \bar{\Psi}\,(i\not{D} - m)\,\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{8.1}$$

where

$$D_\mu = D_\mu[A] = \partial_\mu + ie\,A_\mu(x) \tag{8.2}$$

denotes the gauge covariant derivative, describing the "minimal coupling" between electrons and photons. We encountered this coupling already in Eq. (6.53). In the preceding section, we have discussed that the kinetic term for the photon $\sim F_{\mu\nu}F^{\mu\nu}$ is invariant under local gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x). \tag{8.3}$$

This local invariance also holds for the full action, provided the Dirac fields transform locally as

$$\psi(x) \to e^{-ie\lambda(x)}\psi(x) \,, \quad \bar{\psi} \to \bar{\psi}\, e^{ie\lambda(x)} . \qquad (8.4)$$

It is straightforward to check that the covariant derivative transforms as

$$D_\mu[A']\psi'(x) = e^{-ie\lambda(x)} D_\mu[A]\psi(x) \,, \qquad (8.5)$$

such that the invariance of (8.1) becomes obvious.

Since (8.4) denotes phase transformations, the corresponding symmetry group is a $U(1)$ gauge group, $e^{-ie\lambda(x)} \in U(1)$.

The local symmetry entails a global $U(1)$ symmetry for the special case of $\lambda = \text{const}$. The corresponding Noether charge is the fermion number which in QED is also directly linked with the electric charge. For $\lambda = \text{const.}$, the photon transforms trivially, $A_\mu \overset{(8.3)}{\to} A_\mu$, hence it is uncharged with respect to the Noether charge and thus electrically neutral.

The gauge transformations (8.3) & (8.4) can be summarized into a generator of infinitesimal gauge transformations,

$$\mathcal{G}(x) = -\partial_\mu \frac{\delta}{\delta A_\mu(x)} - ie\psi(x)\frac{\delta}{\delta\psi(x)} + ie\bar{\psi}(x)\frac{\delta}{\delta\bar{\psi}(x)} . \qquad (8.6)$$

Then, an infinitesimal gauge transform of a functional of the fields reads

$$F[A,\psi,\bar\psi] \to F[A,\psi,\bar\psi] + \delta F[A,\psi,\bar\psi], \qquad (8.7)$$
$$\delta F[A,\psi,\bar\psi] = \int d^Dx \, \lambda(x)\, \mathcal{G}(x)\, F[A,\psi,\bar\psi].$$

According to our functional integral formalism, the generating functional of QED is given by

$$Z_{QED}[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{iS_{QED}[A,\psi,\bar\psi] + iS_{gf} - i\int J_\mu A^\mu - i\int \bar\eta\psi + i\int \bar\psi\eta}, \qquad (8.8)$$

where we also have introduced the necessary gauge fixing term $S_{gf} \overset{e.g.}{=} -\frac{1}{2\alpha}\int (\partial_\mu A^\mu)^2$. (8.9)

The functional integral is invariant under substitution / transformation of the integration variables $A_\mu, \psi, \bar\psi$. Since a gauge transformation can also be understood as such a variable change, the functional integral must be invariant. Therefore,

$$0 = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\; \mathcal{G}(x)\, e^{iS_{QED} + iS_{gf} - i\int J_\mu A^\mu - i\int \bar\eta\psi + i\int\bar\psi\eta}. \qquad (8.10)$$

Other than S_{QED}, S_{gf} and the source terms are not explicitly invariant. For $\eta, \bar\eta = 0$ we have

$$0 = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi \Big[\underbrace{\mathcal{G}(x) S_{gf} - \mathcal{G}(x) \int J_\mu A_\mu}_{} \Big] e^{iS_{QED} + iS_{gf} - i\int JA}$$

$$= - \frac{1}{\alpha} \partial^2 \partial_\mu A^\mu(x) - \partial_\mu J^\mu(x) \qquad (8.11)$$

Taking the derivative $i\dfrac{\delta}{\delta J_\nu(y)}$ at $J=0$ yields

$$0 = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi \Big[\frac{1}{\alpha} \left(\partial^2 \partial_\mu A^\mu(x)\right) A^\nu(y) - i\partial^\nu \delta^{(4)}(x-y) \Big] e^{iS_{QED} + iS_{gf}}$$

$$\Rightarrow \quad 0 = \frac{1}{\alpha} \partial^2 \partial_\nu G_A^{\mu\nu}(x-y) - i\partial^\nu \delta^{(4)}(x-y) , \qquad (8.12)$$

where G_A denotes the full photon 2-point function. To lowest order in $e \to 0$, we have, cf. (7.94)

$$G_A^{\mu\nu}(x-y) = iD^{\mu\nu}(x-y) + \mathcal{O}(e) \qquad (8.13)$$

Eq. (8.12) becomes more transparent in Fourier space,

$$0 = \frac{1}{\alpha} p^2 p_\mu G_A^{\mu\nu}(p) + ip^\nu$$

$$\Rightarrow \quad p_\mu G_A^{\mu\nu} = -i\alpha \frac{p^\nu}{p^2} . \qquad (8.14)$$

Comparing this to the free propagator

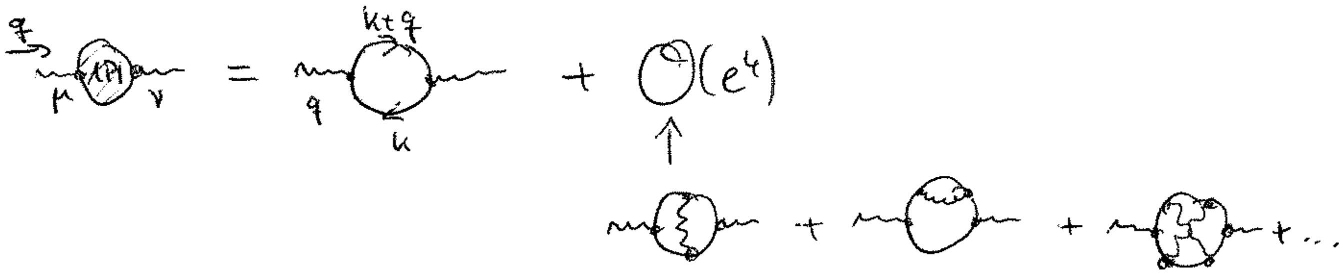
$$iD^{\mu\nu} = -\frac{i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + \alpha \frac{p^\mu p^\nu}{p^2} \right) \qquad (8.15)$$

We obtain

$$p_\mu G_A^{\mu\nu}(p) \equiv p_\mu i D^{\mu\nu} = -i\alpha \frac{p^\nu}{p^2} . \qquad (8.16)$$

This implies that the longitudinal part of the full propagator is not modified by quantum fluctuations. This is a fully non perturbative statement that holds to all orders in the perturbative expansion. Obviously, this property is a consequence of gauge invariance. The identity (8.16) (or variants thereof) is called Ward identity.

The Ward identity can also be studied diagrammatically: Consider all one-particle-irreducible (1PI) diagrams that contribute to the full photon propagator:



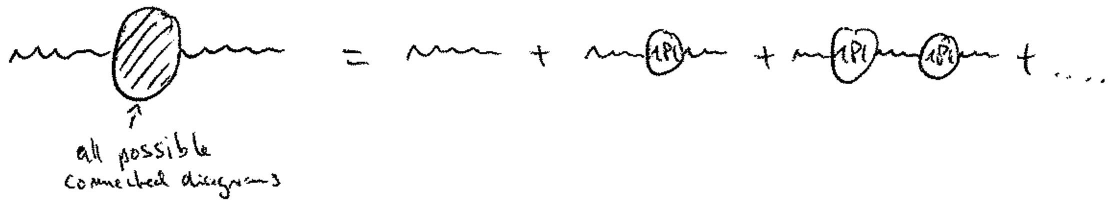
$$=: \; i\,\Pi^{\mu\nu}(q) \qquad (8.17)$$

This object is also called the polarization tensor. (We will see that it contains lots of physical information). The Ward identity tells us that the tensor structure of $\Pi^{\mu\nu}(q)$ must be transversal

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q) \,, \qquad (8.18)$$

hence the information is carried by a scalar function $\Pi(q)$.

The relation between $\Pi^{\mu\nu}$ and the full propagator $G_A^{\mu\nu}$ is given by a geometric series:

$G_A^{\mu\nu}(p) =$ 

$$= iD^{\mu\nu} + iD^\mu{}_\kappa \, i\Pi^{\kappa\lambda} \, iD_\lambda{}^\nu + iD^\mu{}_\kappa \, i\Pi^{\kappa\lambda} \, iD_{\lambda\rho} \, i\Pi^{\rho\sigma} \, iD_\sigma{}^\nu + \dots$$

$P_T^{\mu\nu} := g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$

$$= -\frac{i\alpha}{p^2} p^\mu p^\nu - \frac{i}{p^2} P_T^{\mu\nu} + \left(-\frac{i}{p^2}\right) P_T{}^\mu{}_\kappa P_T^{\kappa\lambda} \, ip^2\Pi(p) \left(-\frac{i}{p^2}\right) P_{T\lambda}{}^\nu$$

$P_T^{\mu\nu} P_{T\nu}{}^\lambda = P_T^{\mu\lambda}$

Projector

$$+ \left(-\frac{i}{p^2}\right) P_T{}^\mu{}_\kappa P_T^{\kappa\lambda} \, ip^2\Pi(p) \left(-\frac{i}{p^2}\right) P_{T\lambda\rho} \, P_T^{\rho\sigma} \, ip^2\Pi(p) \left(-\frac{i}{p^2}\right) P_{T\sigma}{}^\nu + \dots$$

$$= -\frac{i\alpha}{p^2} p^\mu p^\nu - \frac{i}{p^2} P_T^{\mu\nu} - \frac{i}{p^2} \Pi(p) P_T^{\mu\nu} - \frac{i}{p^2} \Pi^2(p) P_T^{\mu\nu}$$

$$= -\frac{i\alpha}{p^2} p^\mu p^\nu - \frac{i}{p^2} \frac{1}{(1-\Pi(p))} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \qquad (8.19)$$

Here, we have used the transversal projector $P_T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$ and its properties $P_T^2 = P_T$ and $P_T^{\mu\nu} p_\nu = 0$. In fact, G_A satisfies the Ward identity (8.16), showing that (8.18) is consistent with this identity. In turn, if (8.18) had contained an independent longitudinal part $\sim \frac{q^\mu q^\nu}{q^2}$, the corresponding resummed result in (8.19) would have violated the Ward identity. The Ward identity (8.18) is also often written as

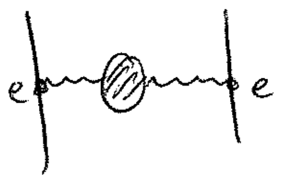
$$p_\mu \Pi^{\mu\nu}(p) = 0 . \tag{8.20}$$

The structure of the full propagator (8.19) tells us already two important facts: first, the photon propagator always exhibits a pole at $p^2=0$. In this manner, gauge invariance guarantees that the photon remains massless also in the fully interacting theory (recall that the physical mass typically differs from the bare mass parameter as a consequence of interacting fluctuations). Hence the masslessness of the photon is protected by symmetry against "radiative corrections". A second important consequence of (8.19) is that the residue of the pole is shifted by $\Pi(p\to 0)$:

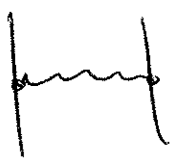
$$Z_3 := \frac{1}{1-\Pi(0)} . \tag{8.21}$$

This is the wave function renormalization of the photon.

Let us, for instance, consider a scattering process of charged particles, e.g. in the low energy limit. Taking the full propagator into account, we have diagrammatically



as compared to



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This implies that the effect of fluctuations is characterized by the replacement

$$\ldots \frac{e^2}{p^2} \ldots \longrightarrow \ldots \frac{Z_3 e^2}{p^2} \qquad (8.22)$$

in the corresponding scattering amplitudes. In other words, the coupling e always occurs in combination with a factor of $\sqrt{Z_3}$. In an actual experiment, we can thus only measure the product $\sqrt{Z_3}\, e$. This suggests to introduce the "physical coupling"

$$e_R := \sqrt{Z_3}\, e \qquad (8.23)$$

or, more precisely, the <u>renormalized</u> coupling e_R which differs from the bare coupling e (being a parameter in the Lagrangian) by a multiplicative factor.

In fact, it is not useful to speak of THE physical coupling, as the coupling strength turns out to be scale dependent. To see this, let us extend the above line of argument to scattering processes with a finite momentum transfer q^2.

The amplitude of such a process contains the quantity

$$-\frac{i}{q^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{e^2}{1-\Pi(q)} = -\frac{i}{q^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{e_R^2}{\underbrace{Z_3}_{=\frac{1}{1-\Pi(0)}}(1-\Pi(q))}$$

$$\simeq -\frac{i}{q^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{e_R^2}{1-\Pi(q^2)+\Pi(0)} \quad . \qquad (8.24)$$

Analogously to (8.23), we can thus define a momentum-dependent effective coupling strength

$$\alpha_{eff}(q^2) := \frac{e_R^2/4\pi}{1-\left(\Pi(q)-\Pi(0)\right)} \qquad (8.25)$$

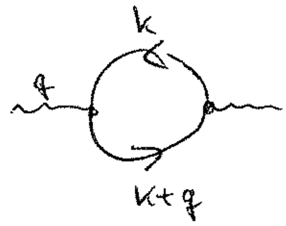
(NB: Though we have motivated this form here by an expansion for small $\Pi(0)$, the form of (8.25) actually holds to all orders in the full theory.)

We have to conclude that couplings in quantum field theory do not correspond to fixed constants, but depend on the momentum scale, at which they are measured.

In QFT, we talk about "running couplings".

For the quantitative discussion of further physical consequences, we need the explicit form of $\Pi(q)$. The full form is not known, but already the lowest-order result in perturbation theory is interesting and relevant. Diagrammatically, we find

$$i\Pi^{\mu\nu}(q) = (-ie)^2 \underset{\substack{\uparrow \\ \text{closed fermion loop}}}{(-1)} \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[\gamma^\mu \frac{i}{\not{k}-m} \gamma^\nu \frac{i}{\not{k}+\not{q}-m}\right]$$

$\hat{=}$ 

(8.26)

according to the Feynman rules for QED given in Sect. 6 and justified and in agreement with the functional integral formalism. The computation of (8.26) goes along with a number of steps and techniques that are interesting in their own right. We will discuss all details in the second half of this section. But first let us discuss the physics that is contained in the result.

In view of the combination found in (8.25), we are interested in

$$\hat{\Pi}(q) := \Pi(q) - \Pi(0) = -\frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \qquad (8.27)$$

computation, see later

For large spatial momentum transfers $-q^2 \gg m^2$, we get

$$\hat{\Pi}(q) \simeq -\frac{2\alpha}{\pi}\int_0^1 dx \, \underbrace{x(1-x)}_{\downarrow \, \frac{1}{6}} \left[-\ln\left(\frac{-q^2}{m^2}\right) + \ln\frac{1}{x(1-x)} + \mathcal{O}\left(\frac{m^2}{-q^2}\right)\right]$$

$$= \frac{\alpha}{3\pi}\left[\ln\frac{-q^2}{m^2} - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{-q^2}\right)\right] \qquad (8.28)$$

Using (8.25), we find as the effective coupling constant

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha_R}{1-\frac{\alpha}{3\pi}\ln\left(\frac{-q^2}{e^{5/3}\cdot m^2}\right)} = \frac{\alpha_R}{1-\frac{\alpha_R}{3\pi}\ln\left(\frac{-q^2}{e^{5/3}m^2}\right)} + \mathcal{O}(\alpha_R^2)$$

(8.29)

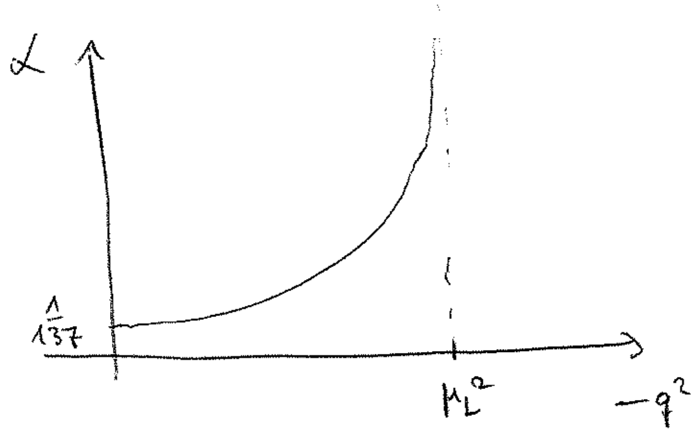
We observe that the effective coupling constant grows with an increasing momentum transfer $-q^2$!

This property of QFT is in fact confirmed by experiments: while the coupling can be measured in low-energy scattering experiments, e.g. Thomson scattering, yielding $\alpha \simeq \frac{1}{137}$, high-energy experiments find different values: e.g. an increased coupling by an amount of 5% is found in Bhabba scattering ($e^+e^- \to e^+e^-$) at $(-q^2) = (30\,\text{GeV})^2$. In LEP2 experiments at CERN, the precision measurements have typically been fixed at the mass of the Z boson $\simeq 90$ GeV, yielding

$\alpha \simeq \frac{1}{128}$ (NB: it is important to note that at these energy scales not only e^+e^- fluctuations are relevant but all charged particles with masses smaller than $-q^2$ contribute).

Taking (8.29) at face value, the coupling seems to grow without bound and even diverge for

$$-q^2 = \mu_L^2 := m^2 e^{\frac{3\pi}{\alpha_R} + \frac{5}{3}} \qquad (8.30)$$



This singularity goes under the name "Landau pole". For QED it occurs at extremely large scales $\mu_L \simeq 10^{270}$ GeV which is way beyond any reasonable physical energy scale. Since we have derived (8.29) under the assumption of a small coupling, the divergence (8.30) tells us in the first place that perturbation theory breaks down at highest energy scales. A resolution of this puzzling divergence therefore needs methods that go beyond perturbation theory.

Assuming for a moment that (8.29) would at least qualitatively reflect the behavior of the theory at highest scales, the conclusion would be that QED predicts its own breakdown at the scale μ_L. Physics of scattering processes with $-q^2 > \mu_L^2$ would not be analytically connected to QED as we observe it. Our QED description would only be valid for $-q^2 \lesssim \mu_L^2$.

QFTs thus appear to have a new quality compared to other physical theories: they can predict their own break down in terms of a maximal validity scale such as μ_L.

In fact the question as to whether QFTs may or may not be valid on arbitrary scales, i.e. whether QFTs can be truly fundamental, is an active field of research.

Let us now turn to the other extreme of small momentum transfers. We have already discussed in the exercises that the Coulomb potential can also be understood in terms of a one-photon exchange with the free propagator $iD^{\mu\nu}$.

How is this result modified if we instead use the full photon propagator $G_A^{\mu\nu}$?

Using the mapping between the propagator and the Fourier transform of the effective one-particle potential e.g. Eq. (6.49), we conclude that

$$\tilde{V}(\vec{q}) = \frac{-e^2}{|\vec{q}|^2 \left(1 - \hat{\Pi}(-|\vec{q}|^2)\right)} \tag{8.31}$$

is the Fourier transform of the (electron-positron) attractive Coulomb potential including the QFT effect of fluctuations.

For $q^2 \simeq -|\vec{q}|^2 \ll m^2$, we have

$$\underline{\hat{\Pi}(q^2)} \simeq +\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left(1 - x(1-x)\frac{q^2}{m^2}\right)$$

$$\simeq -\frac{2\alpha}{\pi}\frac{q^2}{m^2} \underbrace{\int_0^1 dx \left[x(1-x)\right]^2}_{=1/30}$$

$$= -\frac{\alpha}{15\pi}\frac{(-|\vec{q}|^2)}{m^2}, \tag{8.32}$$

and thus

$$\underline{\tilde{V}(\vec{q})} \overset{(8.31)}{\simeq} -\frac{e^2}{|\vec{q}|^2 (1 - \hat{\Pi}(q))} \simeq -\frac{e^2}{|\vec{q}|^2} - \frac{e^2 \alpha}{15\pi\, m^2}$$

$$= -\frac{4\pi\alpha}{|\vec{q}|^2} - \frac{4\alpha^2}{15 m^2} \tag{8.33}$$

The back transformation to coordinate space results in

$$V(\vec{x}) = \int \frac{d^3 q}{(2\pi)^3} \, e^{i\vec{q}\cdot\vec{x}} \, \tilde{V}(\vec{q})$$

$$= -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15m^2} \delta^{(3)}(\vec{x}) \qquad (8.34)$$

This implies that QED predicts corrections to the Coulomb potential which affects the force law at very short distances. Using (8.34) as a correction to the Coulomb law for the computation of atomic states tells us that only s-states with angular orbital momentum $l=0$ are affected as their wave function is non zero at the origin.
E.g. to first order in QM perturbation theory, the shift of the bound-state energy in atomic physics is

$$\Delta E = \int d^3x \, |\psi(\vec{x})|^2 \cdot \left(-\frac{4\alpha^2}{15m^2} \delta^{(3)}(\vec{x}) \right) = -\frac{4\alpha^2}{15m^2} |\psi(0)|^2 . \qquad (8.35)$$

For instance, for the 2s state, we have $|\psi_{2s}(0)|^2 = \frac{\alpha^3 m^3}{8\pi}$ (in our units)

$$\Rightarrow \quad \Delta E = -\frac{\alpha^5 m}{30\pi} = -1.123 \cdot 10^{-7} \, eV \qquad (8.36)$$

This yields a small but well measurable and well confirmed contribution to the Lamb shift.

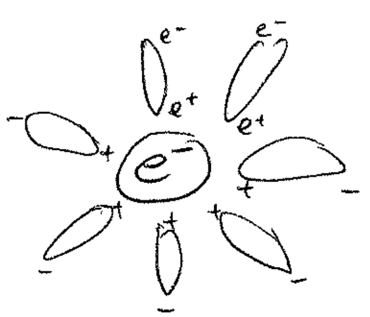
The correction to the Coulomb potential in (8.34) is called Uehling potential.

In fact, the $\delta^{(3)}(\vec{x})$ function is only an approximation in the strict limit $|\vec{q}|^2 \ll m^2$. It is straightforward to keep more details of the momentum dependence in the Fourier transformation improving the validity range of the approximation a bit. The result is

$$V(\vec{x}) = -\frac{\alpha}{|\vec{x}|}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\,\frac{e^{-2m|\vec{x}|}}{(m|\vec{x}|)^{3/2}} + \dots\right). \quad (8.37)$$

The second term turns into a representation of the δ-function for $|\vec{x}| \gg \frac{1}{m}$, being the Compton wave-length of the electron.

This correction can be interpreted as the screening of the bare charge of the Coulomb central potential by virtual e^+e^--fluctuations,



much in the same way as polarization changes screen charges in a dielectric medium. Hence, the QFT vacuum to some extent shows properties similar to medium properties by virtue of ubiquitous quantum fluctuations.

In the same picture, the increase of the coupling can be understood: at higher momentum transfers, the scattering particle resolves more and more of the inner structure of the cloud of polarization charges, thus "seeing" more and more of the bare charge. This illustrates the increase of the effective coupling with $|\vec{q}|$.