7 Functional Integrals

In Quantum Mechanics, the Lagrangian formalism complementing the canonical quantization leads to Feynman path integrals. The analogous construction in QFT yields functional-integral formulations of generating functionals. In fact, the conceptual flexibility of this approach has triggered a large part of the modern development (~ last 50 years) of QFT. Let us go back to the generating functional of correlation functions introduced in Sect. 5:

$$T[J] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^D x_1 \dots d^D x_n \; G^{(n)}(x_1, \dots x_n) J(x_1) \dots J(x_n), \tag{7.1}$$

where $J(x)$ denotes an auxiliary source variable. We obtain the correlation functions by functional differentiation

$$G^{(n)}(x_1 \dots x_n) = i^n \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} T[J]. \tag{7.2}$$

If $T[J]$ was known, we could compute all correlators. The theory would then be "solved".
We already found two further representations of $T[J]$,

$$T[J] = \langle \Omega | T \, e^{-i \int d^D x \, J(x) \phi_H(x)} | \Omega \rangle \tag{7.3}$$

and
$$T[J] = \frac{\langle 0|\, S[J]\, |0\rangle}{\langle 0|S[0]\,|0\rangle} \qquad \left(\equiv \frac{Z[J]}{Z[0]}\right) \qquad (7.4a)$$

where
$$S[J] = T\left(e^{-i\int d^Dx\, \mathcal{H}_I(x) \; - \; i\int d^Dx\, J(x)\, \varphi(x)}\right) \qquad (7.4b)$$

is the S matrix in the presence of a source $J(x)$.
Eq. (7.4a) served as a basis for perturbative computations of correlators by expanding $S[J]$ in powers of $\mathcal{H}_I$. (Eq. (7.3) had so far been rather useless for concrete computations.)

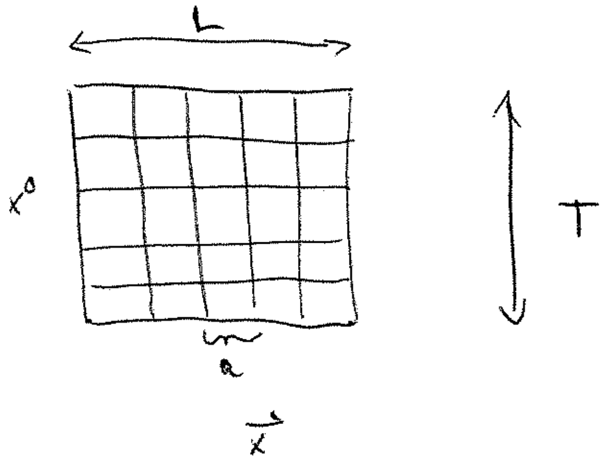
7.1 Functional integrals as a continuum limit of lattice theories

Leaving continuum field theory aside for the moment, let us consider a scalar "field" on a discrete spacetime lattice Λ^D in D dimensions:

$$x \in \Lambda^D := a\,\mathbb{Z}^D = \{\, x \in \underset{\substack{\uparrow \\ \text{Minkowski space}}}{\mathbb{M}} : \tfrac{x^\mu}{a} \in \mathbb{Z}\,;\ \mu = 0,1,\dots,D-1 \} \qquad (7.5a)$$

Here, a is the lattice spacing. In addition, we may consider a finite extent L in each spatial direction as well as in time direction

$$L = N\cdot a \quad , \quad T = N\cdot a \qquad (7.5b)$$



For simplicity, we choose the lattice to have the same number of sites N in each spacetime direction. Also for reasons of simplicity, let us choose periodic boundary conditions.

Let a scalar (Heisenberg) field operator be defined on each site x^μ: $\hat{\phi}_H(x)$. From now on, we use hats ^ to distinguish between operators and numbers. The corresponding Schrödinger operator is

$$\hat{\phi}_{\vec{x}} = e^{-i\hat{H}t}\, \hat{\phi}_{H(x)}\, e^{i\hat{H}t}, \tag{7.6}$$

with respect to a reference time which is put to zero.

Consider a basis of the Hilbert space of states that diagonalizes the Schrödinger field operator:

$$\hat{\phi}_{\vec{x}} \,|\phi_{\vec{x}}\rangle = \phi_{\vec{x}}\,|\phi_{\vec{x}}\rangle \tag{7.7}$$

(analogously to QM: $\hat{q}|q\rangle = q\,|q\rangle$)

As $\varphi_{\vec{x}}$ can acquire continuous values, we choose the normalization

$$\langle \varphi_{\vec{x}} | \varphi'_{\vec{x}} \rangle = \prod_{\vec{x}} \delta(\varphi_{\vec{x}} - \varphi'_{\vec{x}}) . \qquad (7.8)$$

For a complete set of states, the completeness relation reads

$$\mathbb{1} = \int \prod_{x} d\varphi_{\vec{x}} \, |\varphi_{\vec{x}}\rangle \langle \varphi_{\vec{x}}| =: \int \mathcal{D}\varphi_x \, |\varphi\rangle\langle\varphi| , \qquad (7.9)$$

where we have introduced a short hand notation in the last step. Arbitrary wave functions can be spanned by this basis in terms of the "coordinates"

$$\Psi[\varphi_{\vec{x}}] = \langle \varphi_{\vec{x}} | \Psi \rangle . \qquad (7.10)$$

The canonically conjugate momentum $\hat{\pi}_{\vec{x}}$ can also be represented in this basis:

$$\underbrace{\hat{\pi}_{\vec{x}} = -i \frac{\delta}{\delta \varphi_{\vec{x}}}}_{\text{continuum notation}} \equiv \underbrace{-\frac{i}{a^{D-1}} \frac{\partial}{\partial \varphi_{\vec{x}}}}_{\text{lattice notation / realization}} \qquad (7.11)$$

(c.f. QM: $\hat{p} \to -i \frac{\partial}{\partial q}$).

The momentum operator is diagonal in a corresponding momentum basis

$$\hat{\Pi}_{\vec{x}} \, | \Pi_{\vec{x}} \rangle = \Pi_{\vec{x}} \, | \Pi_{\vec{x}} \rangle \; , \quad \mathbb{1} = \int \mathcal{D}\Pi_{\vec{x}} \; | \Pi \rangle \langle \Pi | \, . \tag{7.12}$$

The $\varphi_{\vec{x}} \leftrightarrow \Pi_{\vec{x}}$ transition amplitudes are generalized plane waves

$$\langle \varphi_{\vec{x}} | \Pi_{\vec{x}} \rangle = e^{i a^{D-1} \sum_{\vec{x}} \varphi_{\vec{x}} \Pi_{\vec{x}}} =: e^{i (\varphi_{\vec{x}}, \Pi_{\vec{x}})} \tag{7.13}$$

In the limit $a \to 0$ (with L and $T = \text{const}$), the sum turns into a Riemann integral $(\varphi_{\vec{x}}, \Pi_{\vec{x}}) \to \int d^{D-1}x \; \varphi(\vec{x}) \Pi(\vec{x})$.

Consider the time evolution of a wave function,

$$| \Psi(t) \rangle = e^{-i \hat{H} (t - t_0)} \, | \Psi(t_0) \rangle \, . \tag{7.14}$$

A projection onto the $|\varphi_{\vec{x}}\rangle$ basis yields

$$\begin{aligned} \langle \varphi_{\vec{x}} | \Psi(t) \rangle &= \langle \varphi_{\vec{x}} | \, e^{-i \hat{H}(t-t_0)} \, | \Psi(t_0) \rangle \\ &= \int \mathcal{D}\varphi'_{\vec{x}} \; \langle \varphi_{\vec{x}} | \, e^{-i\hat{H}(t-t_0)} \, | \varphi'_{\vec{x}} \rangle \langle \varphi'_{\vec{x}} | \Psi(t_0) \rangle \, . \end{aligned} \tag{7.15}$$

Here, we encounter the transition amplitude

$$\begin{aligned} \langle \varphi_{\vec{x}} | \, e^{-i\hat{H}(t-t_0)} \, | \varphi'_{\vec{x}} \rangle &= \langle \varphi_{\vec{x}} | \, e^{-i\hat{H}t} \, e^{i\hat{H}t_0} \, | \varphi'_{\vec{x}} \rangle \\ &\equiv \langle \varphi_{\vec{x}}, t \, | \, \varphi'_{\vec{x}}, t_0 \rangle \end{aligned} \tag{7.16}$$

for the transition of a field configuration $\varphi'_{\vec{x}}$ at time t_0

to a configuration $\varphi_{\vec{x}}$ at time t. This is equivalent to the matrix element of the time evolution operator with respect to the $\varphi_{\vec{x}}$ basis.

The states introduced in the last step are actually the eigenstates $|\varphi_{\vec{x}}, t\rangle$ of the Heisenberg field operator

$$\hat{\varphi}_H(x)|\varphi_{\vec{x}}, t\rangle = \left(e^{i\hat{H}t}\hat{\varphi}_{\vec{x}}e^{-i\hat{H}t}\right)e^{i\hat{H}t}|\varphi_{\vec{x}}\rangle$$

4-vector

$$= e^{i\hat{H}t}\hat{\varphi}_{\vec{x}}|\varphi_{\vec{x}}\rangle = \varphi_{\vec{x}}\,e^{i\hat{H}t}|\varphi_{\vec{x}}\rangle$$

$$= \varphi_{\vec{x}}|\varphi_{\vec{x}}, t\rangle . \qquad (7.17)$$

This demonstrates that the eigenvalues of $\varphi_H(x)$ are actually time independent. Completeness of $|\varphi_{\vec{x}}\rangle$ induces the completeness of the states $|\varphi_{\vec{x}}, t\rangle$:

$$\underline{\mathbb{1}} = \int \mathcal{D}\varphi_{\vec{x}}\,|\varphi_{\vec{x}}, t\rangle\langle\varphi_{\vec{x}}, t| \qquad (7.18)$$

Let us evaluate the transition amplitude (7.16) for $t_0 = 0$ and $t = T$ by inserting a complete set of states (7.18) at each intermediary time step of the time lattice,

$$t_0 = 0,\ t_1 = a,\ t_2 = 2a, \dots t_i = ia, \dots, t_N = T .$$

This yields

$$\langle \varphi_{\vec{x}}, T | \varphi'_{\vec{x}}, 0 \rangle = \langle \varphi_{\vec{x}} | e^{-i\hat{H}T} | \varphi'_{x} \rangle$$

$$= \langle \varphi_{\vec{x}} | e^{-i\hat{H}a} e^{-i\hat{H}a} \cdots e^{-i\hat{H}a} | \varphi'_{\vec{x}} \rangle$$

$$= \int \mathcal{D}\varphi_{1,\vec{x}} \cdots \mathcal{D}\varphi_{N-1,\vec{x}} \langle \varphi_{\vec{x}} | e^{-i\hat{H}a} | \varphi_{N-1,\vec{x}} \rangle \langle \varphi_{N-1,\vec{x}} | e^{-i\hat{H}a} | \varphi_{N-2,\vec{x}} \rangle \cdot$$

$$\cdots \cdot \langle \varphi_{1,\vec{x}} | e^{-i\hat{H}a} | \varphi'_{\vec{x}} \rangle \qquad (7.19)$$

The subscripts $1, \dots, N-1$ label the integration variables on each time slice. The transition amplitude thus becomes a convolution of the single amplitudes mediating between the time slices

$$\mathcal{A}_k := \langle \varphi_{k+1,\vec{x}} | e^{-i\hat{H}a} | \varphi_{k,\vec{x}} \rangle , \quad t_{k+1} - t_k = a , \quad k = 0, 1, \dots, N-1 \qquad (7.20a)$$

where $\varphi_{0,\vec{x}} \equiv \varphi'_{\vec{x}}$ and $\varphi_{N,\vec{x}} \equiv \varphi_{\vec{x}}$.

Let us assume that the Hamiltonian is of the standard form

$$\hat{H} = \hat{T}[\hat{\pi}] + \hat{U}[\hat{\varphi}] , \qquad (7.20b)$$

with a kinetic term depending on momenta and a potential $\hat{U}$ depending on the amplitudes.

We compute $\mathcal{A}_k$ to first order in a:

(we suppress the coordinate index $\vec{x}$ from now on; it is implicitly understood.)

$$\mathcal{A}_k = \langle \varphi_{k+1} | 1 - i \underbrace{\hat{H}}_{=\hat{T}+\hat{U}} a | \varphi_k \rangle + \mathcal{O}(a^2)$$

$$= \langle \varphi_{k+1} | \varphi_k - ia \langle \varphi_{k+1} | \hat{T}[\hat{\pi}] + \hat{U}[\hat{\varphi}] | \varphi_k \rangle + \mathcal{O}(a^2)$$

$$= \int \mathcal{D}\pi_k \langle \varphi_{k+1} | \pi_k \rangle \langle \pi_k | \varphi_k \rangle \left(1 - ia\left(T[\pi_k] + U[\bar{\varphi}_k]\right)\right) + \ldots$$

$$\xrightarrow{a \to 0} \int \mathcal{D}\pi_k \, e^{i(\pi_k, \varphi_{k+1} - \varphi_k)} \, e^{-ia H(\pi_k, \bar{\varphi}_k)} \qquad (7.21)$$

In the last step, we have exponentiated the infinitesimal terms again (This computation / derivation can be made more rigorous using the Trotter product formula; the result is that of (7.21). One important assumption is, that the Hamiltonian as well as T and U are bounded from below.)

In the third line, we have used the mid-point prescription

$$\bar{\varphi}_k := \frac{1}{2}(\varphi_{k+1} + \varphi_k). \qquad (7.22)$$

Since

$$\langle \varphi_{k+1} | \hat{U}[\hat{\varphi}] | \varphi_k \rangle = U[\varphi_{k+1}] \langle \varphi_{k+1} | \varphi_k \rangle = \langle \varphi_{k+1} | \varphi_k \rangle U[\varphi_k]$$
$$= \ldots \quad (\text{various possibilities}), \qquad (7.23)$$

different prescriptions are conceivable and legitimate at this point. The midpoint prescription has the advantage to preserve gauge invariance in the case of gauge theories.

Looking back at (7.21), we have arrived at an important point: the right-hand side does not contain any operator! $H(\pi_k, \overline{\varphi}_k)$ denotes a classical Hamiltonian evaluated on the eigenvalues π_k and $\overline{\varphi}_k$.

For the case that H has a standard form

$$\hat{H} = \hat{T}[\hat{\pi}] + \hat{U}[\hat{\varphi}] = \frac{1}{2}(\hat{\pi}, \hat{\pi}) + \hat{U}[\hat{\varphi}], \quad (7.24a)$$

We have

$$\mathcal{A}_k = e^{-ia\,U(\overline{\varphi}_k)} \int \mathcal{D}\pi_k \, e^{i\left\{(\pi_k, \varphi_{k+1} - \varphi_k) - \frac{a}{2}(\pi_k, \pi_k)\right\}} \, . \quad (7.24b)$$

At each site, the integral is of Fresnel-type. By a completion of the square, we can bring it to standard form

$$\mathcal{A}_k = e^{-ia\,U(\overline{\varphi}_k)} \int \mathcal{D}\pi_k \, e^{-i\frac{a}{2}\left(\pi_k - \frac{\varphi_{k+1}-\varphi_k}{a}, \overbrace{\pi_k - \frac{\varphi_{k+1}-\varphi_k}{a}}^{=: \tilde{\pi}_k}\right)} \cdot e^{+\frac{a}{2}\left(\frac{\varphi_{k+1}-\varphi_k}{a}, \frac{\varphi_{k+1}-\varphi_k}{a}\right)}$$

$$= e^{-ia\,U(\overline{\varphi}_k)} \, e^{i\frac{a}{2}\left(\frac{\varphi_{k+1}-\varphi_k}{a}, \frac{\varphi_{k+1}-\varphi_k}{a}\right)} \int \mathcal{D}\tilde{\pi}_k \, e^{-i\frac{a}{2}(\tilde{\pi}_k, \tilde{\pi}_k)}$$

(7.25)

The $\tilde{\pi}$ integral is a product of Fresnel integrals at each of the N^{D-1} spatial lattice sites per time slice

$$\int \mathcal{D}\tilde{\pi}_k \, e^{-i\frac{a}{2}(\tilde{\pi}_k, \tilde{\pi}_k)} = \sqrt{\frac{2\pi}{ia}}^{N^{D-1}}, \qquad (7.26)$$

yielding an unimportant field-independent constant.
In our result (7.25), we can identify the lattice derivative in time direction

$$\nabla_t \varphi_k = \frac{\varphi_{k+1} - \varphi_k}{a} \qquad (7.27)$$

$$\Rightarrow \mathcal{A}_k = \sqrt{\frac{2\pi}{ia}}^{N^{D-1}} e^{i\frac{a}{2}(\nabla_t \varphi_k, \nabla_t \varphi_k) - i a \mathcal{U}(\varphi)}$$

$$\equiv \sqrt{\frac{2\pi}{ia}}^{N^{D-1}} e^{ia L[\varphi_k]}, \qquad (7.28)$$

where we have introduced the lattice Lagrangian $L[\varphi_k]$ at each time slice t_k.
With this representation of the amplitude $\mathcal{A}_k$ characterizing the transition between the time slices, we can now compute the full amplitude of (7.19):

$$\langle \varphi_{\vec{x}}, T | \varphi'_{\vec{x}}, 0 \rangle = \left(\sqrt{\frac{2\pi}{ia}}^{N^{D-1}} \right)^N \int \mathcal{D}\varphi_1 \ldots \mathcal{D}\varphi_{N-1} \, e^{ia \sum_{k=0}^{N-1} L[\varphi_k]}$$

(7.29)

The exponent defines the lattice action

$$S[\varphi_{\vec{x}}; T, 0] = a \sum_{k=0}^{N-1} L[\varphi_k, \vec{x}] \quad , \tag{7.30}$$

such that we can also write

$$\langle \varphi_{\vec{x}}, T | \varphi'_{\vec{x}}, 0 \rangle = \sqrt{\frac{2\pi}{ia}}^{N} \int \prod_{t=a}^{T-a} \prod_{\vec{x}} d\varphi_{t,\vec{x}} \, e^{iS[\varphi_{\vec{x}}; T, 0]} \quad . \tag{7.31}$$

This implies that the transition amplitude between two $d\,(=D-1)$ dimensional field amplitudes $\varphi'_{\vec{x}}$ at time $t=0$ and $\varphi_{\vec{x}}$ at time $t=T$ is given by an integral over <u>all</u> D dimensional configurations with boundary conditions

$$\varphi_{\vec{x}, t=0} = \varphi'_{\vec{x}} \quad , \quad \varphi_{\vec{x}, t=T} = \varphi_{\vec{x}} \quad . \tag{7.32}$$

For this, we also use the shorthand notation

$$\langle \varphi_{\vec{x}}, T | \varphi'_{\vec{x}}, 0 \rangle = \int_{\varphi_{t=0} = \varphi'_{\vec{x}}}^{\varphi_{t=T} = \varphi_{\vec{x}}} \mathcal{D}\varphi \, e^{iS[\varphi; 0, T]} \tag{7.33}$$

where the constant factor $\sqrt{\frac{2\pi}{ia}}^{N}$ has been included in the measure $\mathcal{D}\varphi$.

The notation (7.33) is also used for the limit of a continuous spacetime $a \to 0$ $(N \to \infty)$ as well as the thermodynamic limit $L \to \infty$ – provided they exist.

While the lattice formulation used above is indeed well-defined. The existence of a continuum formulation is difficult to proof rigorously.

(NB: proofs exist for some special cases, e.g. $D=1$ and "Euclidean" signature / imaginary time: "Wiener measure").

Let us emphasize some important points:

- the transition amplitude ($\hat{=}$ matrix element of time evolution operator) is expressed as a high-dimensional integral over ordinary functions (no operators!)
- the integrand is a phase with the angle being the classical action. This is similar to partition functions in statistical physics where the integrand can be a Boltzmann factor $\sim e^{-\frac{E}{kT}}$
- Since $\Phi_{\vec{x}}$ is a function defined for all (admissible $\vec{x}$), we are integrating over functions $\Rightarrow$ "functional integral".

7.3 Correlation Functions

The representation of the transition amplitude as a functional integral is just a first step on our way to find a corresponding integral representation for the generating functional of correlation functions.
As a first step, we consider the matrix element of a time-ordered product of field operators in the Heisenberg picture. We use a new notation:

$$\varphi_{\vec{x}}^{i} \to \varphi_{i,\vec{x}} \quad , \quad \varphi_{\vec{x}} \to \varphi_{f,\vec{x}} \quad , \quad \begin{matrix} T = t_f \, , \\ 0 \to t_i \end{matrix}$$

Let us first assume that $t_f > t_2 > t_1 > t_i$

$$\langle \varphi_{f,\vec{x}}, t_f | T[\hat{\varphi}_{t_1,\vec{x}_1} \hat{\varphi}_{t_2,\vec{x}_2}] | \varphi_{i,\vec{x}}, t_i \rangle$$

$$\overset{t_2 > t_1}{=} \langle \varphi_f | \hat{\varphi}_2 \hat{\varphi}_1 | \varphi_i \rangle$$

$$= \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \, \langle \varphi_f | \hat{\varphi}_2 | \varphi_2 \rangle \langle \varphi_2 | \hat{\varphi}_1 | \varphi_1 \rangle \langle \varphi_1 | \varphi_i \rangle$$

$$= \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \; \varphi_1 \varphi_2 \; \langle \varphi_f | \varphi_2 \rangle \langle \varphi_2 | \varphi_1 \rangle \langle \varphi_1 | \varphi_i \rangle \tag{7.34}$$

The integrand contains 3 transition amplitudes, $\varphi_i \to \varphi_1$, $\varphi_1 \to \varphi_2$, $\varphi_2 \to \varphi_f$. Each one can be represented by a functional integral

$$\langle \varphi_f | \hat{\varphi}_2 \hat{\varphi}_1 | \varphi_i \rangle = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \int_{\varphi_2}^{\varphi_f} \mathcal{D}\varphi \, e^{iS_{t_f,t_2}} \int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi \, e^{iS_{t_2,t_1}} \int_{\varphi_i}^{\varphi_1} \mathcal{D}\varphi \, e^{iS_{t_1,t_i}} \tag{7.35}$$

(Some care is in order: $\int \mathcal{D}\varphi_1$ or $\int \mathcal{D}\varphi_2$ denote integrals over all $\varphi_{\vec{x}}$ at a fixed time t_1 or t_2, whereas $\int_{\varphi_2}^{\varphi_f} \mathcal{D}\varphi$ is an integral over all $\varphi_{\vec{x}}$ at all times in between φ_2 and φ_f)

Since φ_1 and φ_2 serving as boundary conditions for some of the integrals are integrated over as well, all integrations can be summarized in one overarching integration over all fields in between t_i and t_f:

$$\langle \varphi_f | \underline{\underline{\hat{\varphi}_2 \hat{\varphi}_1 | \varphi_i \rangle}} = \int_{\varphi_i}^{\varphi_f} \mathcal{D}\varphi \, \underline{\underline{\varphi_1 \varphi_2 \, e^{iS_{t_f,t_i}[\varphi]}}} \tag{7.36}$$

While we have derived (7.36) with the assumption that $t_2 > t_1$, it is easy to show that the identical result also follows for $t_1 > t_2$, i.e.:

$$\langle \varphi_f, t_f | \underline{\underline{T[\hat{\varphi}_1 \hat{\varphi}_2]}} | \varphi_i, t_i \rangle = \int_{\varphi_i}^{\varphi_f} \mathcal{D}\varphi \, \underline{\underline{\varphi_1 \varphi_2}} \, e^{iS_{t_f,t_i}[\varphi]} \tag{7.37}$$

The functional integral representation of the transition amplitude hence automatically account for the time ordering!

This result holds true for matrix elements of higher operator products as well:

$$\langle \varphi_f | T[\hat{\varphi}_1 \dots \hat{\varphi}_n] | \varphi_i \rangle = \int_{\varphi_i}^{\varphi_f} \mathcal{D}\varphi \, \varphi_1 \dots \varphi_n \, e^{iS_{t_f,t_i}[\varphi]} \tag{7.38}$$

For making the relation to correlation functions, we have to "replace" $\langle \varphi_f|$ and $|\varphi_i\rangle$ by the ground state $|\Omega\rangle$, $\langle \Omega|$. For this, we can multiply by $\langle \Omega | \varphi_f \rangle$ and $\langle \varphi_i | \Omega \rangle$ and integrate over φ_f and φ_i using the completeness of these states. In other words:

$$\begin{aligned} G^{(n)}_{(x_1, \dots x_n)} &= \langle \Omega | T[\varphi_{H1} \cdots \varphi_{Hn}] | \Omega \rangle \\ &= \int \mathcal{D}\varphi_f \mathcal{D}\varphi_i \, \langle \Omega | \varphi_f t_f \rangle_H \, {}_H\langle \varphi_f t_f | T[\varphi_1 \dots \varphi_n] | \varphi_i t_i \rangle_H \, {}_H\langle \varphi_i t_i | \Omega \rangle \\ &= \int \prod_{\vec{x}} \prod_{t=t_i}^{t_f} d\varphi_{\vec{x},t} \, \varphi_1 \dots \varphi_n \, e^{iS_{t_f,t_i}} \langle \Omega | \varphi_f t_f \rangle_H \, {}_H\langle \varphi_i t_i | \Omega \rangle , \end{aligned} \tag{7.39}$$

where we have dropped the subscript "H" at the operators from the second line on for brevity; we are nevertheless working in the Heisenberg picture here. It is useful to now make the transition to the interaction picture using

$$\hat{\Omega}_t = e^{i\hat{H}t} e^{-i\hat{H}_0 t}, \quad |\varphi, t\rangle_H = \hat{\Omega}_t |\varphi, t\rangle_I . \tag{7.40a}$$

Also recall the relation between $|\Omega\rangle$ and the 0-particle vacuum in the interaction picture:

$$|\Omega\rangle = \hat{\Omega}_t\,|0,t\rangle_I = \hat{\Omega}_t\,\hat{U}_I^\dagger(+\infty,t)\,|0\rangle\,, \qquad (7.40b)$$

where $|0\rangle$ denotes the asymptotic OUT vacuum for $t\to+\infty$, and $\hat{U}_I(t_2,t_1) = T\,e^{-i\int_{t_1}^{t_2} dt\,\hat{H}_I(t)}$ is the time evolution operator in the interaction picture, as before.

This yields for the last factors in the integral of (7.39):

$$\langle\Omega|\varphi_f t_f\rangle_H\;{}_H\langle\varphi_i t_i|\Omega\rangle = \langle 0|\hat{U}_I(\infty,t_f)|\varphi_f t_f\rangle_I\;{}_I\langle\varphi_i t_i|\hat{U}_I^\dagger(\infty,t_i)|0\rangle \qquad (7.41)$$

In the asymptotic limit, we get

$$\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} \langle\Omega|\varphi_f t_f\rangle_H\;{}_H\langle\varphi_i t_i|\Omega\rangle = \langle 0|\varphi_f\,\infty\rangle_I\;{}_I\langle\varphi_i\,-\infty|\hat{U}_I^\dagger(\infty,-\infty)\,|0\rangle$$

$$= \langle 0|\varphi_f,\infty\rangle_I\,\underbrace{\sum_n}_{\text{sum over } n\text{-particle states}}\;{}_I\langle\varphi_i\,-\infty|n\rangle\langle n|\hat{S}^\dagger|0\rangle\,, \qquad (7.42)$$

where we have used the definition of the S matrix $\hat{S} = \hat{U}_I(\infty,-\infty)$. We had already shown in (5.13), that the stability of the vacuum implies $\langle n|\hat{S}|0\rangle = 0$ for $n\neq 0$, hence:

$$\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} \langle\Omega|\varphi_f t_f\rangle_H\;{}_H\langle\varphi_i t_i|\Omega\rangle = \langle 0|\hat{S}^\dagger|0\rangle\,\langle 0|\varphi_f\,\infty\rangle_I\;{}_I\langle\varphi_i\,-\infty|0\rangle\,. \qquad (7.43)$$

Below (5.13), we also noted that $\langle 0|\hat{S}^\dagger|0\rangle$ is a pure phase, i.e. $\langle 0|\hat{S}^\dagger|0\rangle = \langle 0|\hat{S}|0\rangle^{-1}$, such that

$$G^{(n)}(x_1 \dots x_n) = \frac{1}{\langle 0|\hat{S}|0\rangle} \int \mathcal{D}\varphi\, \varphi_1 \dots \varphi_n\, e^{iS} \langle 0|\varphi_f\rangle\langle\varphi_i|0\rangle \Big|_{\substack{t_f\to\infty \\ t_i\to-\infty}} \tag{7.44}$$

It is instructive to study the case $n=0$:

$$\begin{aligned} G^{(0)} &= \langle\Omega|\Omega\rangle = 1 \\ &= \frac{1}{\langle 0|\hat{S}|0\rangle} \int \mathcal{D}\varphi\, e^{iS} \langle 0|\varphi_f\rangle\langle\varphi_i|0\rangle \Big|_{\substack{t_f\to\infty \\ t_i\to-\infty}}, \end{aligned}$$

implying $$\langle 0|\hat{S}|0\rangle = \int \mathcal{D}\varphi\, e^{iS} \langle 0|\varphi_f\rangle\langle\varphi_i|0\rangle \Big|_{\substack{t_f\to\infty \\ t_i\to-\infty}} \tag{7.45}$$

$G^{(0)}$ is obviously independent of the asymptotic wave functions $\langle 0|\varphi_f\rangle\langle\varphi_i|0\rangle$. For $|t_f|, |t_i|\ (\to\infty) \ggg |t_1| \dots |t_n|$, we expect the same to hold for $G^{(n)}(x_1 \dots x_n)$. Hence, these asymptotic wave functions can be absorbed in the measure definition as well. We therefore end up with

$$G^{(n)}(x_1 \dots x_n) = \frac{\int \mathcal{D}\varphi\, \varphi_1 \dots \varphi_n\, e^{iS}}{\int \mathcal{D}\varphi\, e^{iS}} \tag{7.46}$$

This is the desired functional integral representation of n-point correlation functions. Along the same lines, we can also find an explicit expression for the generating functional by generalizing the action by a source term

$$S_J[\Phi] = S[\Phi] - \int d^D x \, \Phi(x) J(x) \equiv S - (\Phi, J) . \tag{7.47}$$

We then define

$$Z[J] = \int \mathcal{D}\Phi \, e^{i S_J[\Phi]} , \tag{7.48}$$

which gives us

$$T[J] = \frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}\Phi \, e^{i S_J[\Phi]}}{\int \mathcal{D}\Phi \, e^{i S[\Phi]}} . \tag{7.49}$$

Here, it is easy to verify that

$$\begin{aligned} G^{(n)}(x_1, \dots x_n) &= i^n \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} T[J] \Big|_{J=0} \\ &= \frac{1}{Z[0]} i^n \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} Z[J] \Big|_{J=0} \end{aligned} \tag{7.50}$$

Eq. (7.49) gives us a fully non-perturbative definition of the generating functional for correlation functions (at least on a latticized spacetime; if the continuum limit exists, also for the continuum field theory). In addition to perturbation theory, the lattice version gives us another way of approximating the correlation functions by numerically carrying out the high-dimensional integrals. This is typically done (if doable) by stochastic ("Monte Carlo") methods.