6. The Quantized Dirac Field

Almost all known visible matter in the universe is made up of fermions. All experimental results so far are compatible with these fundamental building blocks being Dirac spinor degrees of freedom (only neutrinos could also still be Majorana spinors; the smoking gun for the latter would be the measurement of a neutrinoless double β decay which has not been observed so far – but is still actively searched for.)

In fact, the requirement of Lorentz invariance imposed by the symmetries of spacetime implies that the fields must be in one-to-one relation with representations of the Lorentz group. For a classification of these representation and thus the classification of all possible quantum fields, see the lecture course on Particles & Fields. Here, we take over the known results for the Dirac field; most of this should be familiar from the course on Advanced Quantum Mechanics (QM II). We start from

6.1 The Dirac equation

Classical Dirac theory is defined in terms of the (free) Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi \equiv (i\not{\partial} - m)\psi = 0 \qquad (6.1)$$

where γ^μ are 4×4 matrices in 4-dimensional spacetime* that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \tag{6.2}$$

where $g^{\mu\nu}$ is the Minkowski metric. Various equivalent representations of the γ^μ can be found in the literature. In this lecture, we use the so-called "chiral" representation

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad , \text{where } \sigma^\mu = (\mathbb{1}_{2\times2}\ \ -\vec{\sigma}) \tag{6.3}$$
$$\bar{\sigma}^\mu = (\mathbb{1}_{2\times2}\ \ \vec{\sigma}) \,,$$

and $\vec{\sigma}$ are the Pauli matrices. Physical observables are independent of the representation such as (6.3), but only rely on the underlying structure defined by (6.2).
Consequently, ψ is a 4-tuple (spinor) of complex numbers. Multiplying the Dirac equation by $(-i\gamma^\nu\partial_\nu - m)$, yields

$$0 = (\underbrace{\gamma^\mu\gamma^\nu}_{=\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]} \overbrace{\partial_\mu\partial_\nu}^{\text{sym.}} + m^2)\psi(x) = (\partial^2 + m^2)\psi(x) \,, \tag{6.4}$$

$$\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{=2g^{\mu\nu}} + \frac{1}{2}\underbrace{[\gamma^\mu,\gamma^\nu]}_{\text{antisym.}}$$

hence solutions of the Dirac equation also satisfy the Klein-Gordon equation componentwise.

---

* For brevity, we confine ourselves to $D = 4 = 3+1$ in this section. Still, generalizations to other dimensions are possible though less straightforward than for scalar fields.

The discussion of the properties of the Dirac equation (phase, chiral & Lorentz symmetries) and its solution is a rich topic (part of which has been covered in "QM II" or "Particles & Fields"), here we merely summarize the essentials as they are needed for the following.

To solve the free Dirac equation (6.1) we start with the ansatz:

$$\psi(x) = u(p)\, e^{-ipx} \quad , \text{ where } p^2 = m^2 \qquad (6.5a)$$

$$\Rightarrow \quad (\gamma^\mu p_\mu - m)\, u(p) \equiv (\not{p} - m)\, u(p) = 0 \,, \qquad (6.5b)$$

and $u(p)$ denotes a 4-component spinor that solves the matrix equation (6.5b). With the representation (6.3), the solution can be written as

$$u(p) = \begin{pmatrix} \sqrt{p_\mu \bar{\sigma}^\mu}\ \xi \\ \sqrt{p_\mu \sigma^\mu}\ \xi \end{pmatrix} \qquad (6.6)$$

with ξ being an arbitrary 2-component spinor (e.g. think of $\xi^{1,2} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, i.e., spin-up, spin-down with respect to the axis defined by the $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ direction).

Hence, (6.5a) together with (6.6) parametrizes 2 solutions, This solution is normalized to

$$u^{\dagger r}(p)\, u^s(p) = 2\,E_{\vec p}\,\delta^{rs} \quad , \; r,s=1,2 \qquad (6.7a)$$

as can be verified straightforwardly. Introducing the Dirac conjugate of a spinor,

$$\overline{\psi} := \psi^\dagger \gamma^0 \,, \qquad (6.7b)$$

The normalization can be written in a covariant form:

$$\bar{u}^r(p)\, u^s(p) = 2\,m\,\delta^{rs} \qquad (6.7c)$$

(Note that the RHS of (6.7a) transforms as a 0-component of a 4-vector, while the RHS of (6.7c) is a Lorentz scalar).

The γ^0 in (6.7b) acts like a "spin metric" such that $\overline{\psi}\psi$ transforms as a scalar under Lorentz transformations.

In addition to the "positive frequency" solutions (6.5a), there are also negative frequency solutions

$$\psi(x) = v(p)\, e^{+ipx} \quad , \; p^2 = m^2 \; , \; p^0 > 0 \qquad (6.8a)$$

with $$v(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\; \eta^s \\ -\sqrt{p_\mu \bar\sigma^\mu}\; \eta^s \end{pmatrix}$$ with spinor η^s, $s=1,2$ (6.8b)

and normalization $\bar{v}^r v^s = -2m\,\delta^{rs}$, $v^{\dagger r} v^s = +2E_{\vec p}\,\delta^{rs}$. (6.8c)

The spinors $u(p)$ and $v(p)$ are orthogonal

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0 \,. \tag{6.9}$$

Also useful are the spin sums

$$\begin{aligned} \sum_s u^s(p)\, \bar{u}^s(p) &= \gamma^\mu p_\mu + m \,, \\ \sum_s v^s(p)\, \bar{v}^s(p) &= \gamma^\mu p_\mu - m \,. \end{aligned} \tag{6.10}$$

All these identities can straightforwardly be checked.

The Dirac equation also follows from the action principle applied to the action

$$S_D = \int d^4x \left(i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right) \tag{6.11}$$

Similarly to the complex Klein-Gordon field, the action is invariant under phase rotations,

$$\psi \to e^{i\alpha} \psi \quad \overset{(6.7b)}{\Longrightarrow} \quad \bar{\psi} \to \bar{\psi} e^{-i\alpha} . \tag{6.12}$$

Analogously to the complex Klein-Gordon field, ψ and $\bar{\psi}$ can formally be treated independently for the action principle. E.g. the Dirac equation (6.1) follows from

$$\frac{\delta}{\delta \bar{\psi}} S_D = 0 \qquad \left(\begin{array}{l} \text{alternatively, } \frac{\delta}{\delta \psi} S_D \text{ would lead to the} \\ \text{Dirac conjugate of (6.1)} \end{array} \right) . \tag{6.13}$$

6.2 Dirac field operators

Quite a number of steps of the quantization of the free Dirac field are in direct analogy to that of the complex Klein-Gordon field. We will concentrate hence on the differences and skip the details of those parts which are completely analogous.

Similarly to the complex scalar field, we expect that two kinds of ladder operators are needed, $a_{\vec{p}}, a_{\vec{p}}^{+}$ and $b_{\vec{p}}, b_{\vec{p}}^{+}$, to annihilate and create particle and anti-particle contributions. As for the scalar field, they go along with the positive and negative frequency contributions of the field operator. At the same time, the latter correspond to two spinors $u^s(p)$ and $v^s(p)$, each carrying two spin states, $s=1,2$. These considerations motivate the following ansatz for a quantized Dirac spinor operator,

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a_{\vec{p}}^s u^s(p)\, e^{-ipx} + b_{\vec{p}}^{s+} v^s(p)\, e^{ipx} \right), \qquad (6.14a)$$

and correspondingly

$$\overline{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(b_{\vec{p}}^s \bar{v}^s(p)\, e^{-ipx} + a_{\vec{p}}^{s+} \bar{u}^s(p)\, e^{ipx} \right). \qquad (6.14b)$$

From experiment, we know that two electrons cannot occupy the same quantum state (Pauli exclusion principle).

If $\psi(x)$ is meant to describe an electron, it must respect this exclusion principle. This implies: if

$$a^{s+}_{\vec{p}} |0\rangle =: |\vec{p}, s\rangle \tag{6.15}$$

defines a one-particle state with momentum $\vec{p}$ and spin s, the quantity

$$a^{s+}_{\vec{p}} \, a^{s+}_{\vec{p}} \, |0\rangle \tag{6.16a}$$

must not describe an existing state. Hence, we require

$$a^{s+}_{\vec{p}} \, a^{s+}_{\vec{p}} \, |0\rangle \overset{!}{=} 0 . \tag{6.16b}$$

We know that this requirement cannot be realized with $a^{s}_{\vec{p}}, a^{s+}_{\vec{p}}$ satisfying a standard ladder operator algebra. (6.16b) is immediately satisfied, if a^{s+}_{p} squares to zero $(a^{s+}_{\vec{p}})^2 = 0$. Then, also $a^{s}_{\vec{p}} \, a^{s}_{\vec{p}} = 0$. Under these conditions, a non-trivial algebra can still be formed using the anti-commutator structure:

$$\{ a^{r}_{\vec{p}}, a^{s+}_{\vec{q}} \} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \delta^{rs} = \{ b^{r}_{\vec{p}}, b^{s+}_{\vec{q}} \} \tag{6.17}$$

with all others being zero, schematically $\{a,a\} = \{a^+, a^+\} = 0$
$\{b,b\} = \{b^+, b^+\} = 0$.

Based on (6.17), we straightforwardly obtain from (6.14a,b):

$$\{ \psi_\alpha(\vec{x}), \psi^+_\beta(\vec{y}) \} = \delta^{(3)}(\vec{x}-\vec{y}) \, \delta_{\alpha\beta} \tag{6.18a}$$

where $\alpha, \beta = 1,2,3,4$ denote spin components.

all other equal-time anti-commutators vanish,

$$\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = 0 = \{\bar{\psi}_\alpha(\vec{x}), \bar{\psi}_\beta(\vec{y})\}. \qquad (6.18b)$$

In fact ψ^+ corresponds to the canonical momentum of ψ, because

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(\vec{x})} = \frac{\partial \mathcal{L}}{\partial \partial^0 \psi(\vec{x})} = \frac{\partial(i\bar{\psi}\gamma_0 \partial^0 \psi)}{\partial \partial^0 \psi(\vec{x})}$$

$$= -i\bar{\psi}\gamma_0 = -i\psi^+ \gamma_0^2 = -i\psi^+(\vec{x}) \qquad (6.19)$$

The minus i here "explains" the seemingly missing i in (6.18a). The minus sign occurring first in the second line takes care of the fact that $\bar{\psi}$ and ψ, as well as $\bar{\psi}$ and $\frac{\partial}{\partial \psi}$ anti-commute. This is natural on the quantum level in view of the defining ladder algebra given above. If, however, we wish to understand (6.19) purely on the classical level, we have to understand $\psi(\vec{x})$ as a Grassmann-valued (anti-commuting) n-tuple of numbers, c.f. lecture on Particles & Fields.

Based on these defining equations, we can determine the spectrum of the free theory; starting from the vacuum

$$a^s_{\vec{p}}\,|0\rangle = 0 = b^s_{\vec{p}}\,|0\rangle, \qquad (6.20)$$

the spectrum can be read off from diagonalizing the Hamiltonian,

$$H = \int d^3x \, (\dot{\psi}\pi - \mathcal{L})$$

$$= \int d^3x \, (i\psi^\dagger \partial^0 \psi - \bar{\psi}(i\partial\!\!\!/ - m)\psi)$$

$$= \int d^3x \; \bar{\psi}\,(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = \int d^3x \, \psi^\dagger(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0)\psi \qquad (6.21a)$$

analogously to the scalar field, we obtain from a straight-forward computation

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right) \qquad (6.21b)$$

(of course, after having subtracted the zero-point energy).

Similarly, the momentum operator yields,

$$\vec{P} = \int d^3x \quad \psi^\dagger (-i\vec{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right). \qquad (6.22)$$

We can read off that $a^{s\dagger}_{\vec{p}}$ and $b^{s\dagger}_{\vec{p}}$ create particles with energy $E_{\vec{p}}$ and momentum $\vec{p}$. Similarly to the complex scalar field, the $U(1)$ phase symmetry (6.12) induces a corresponding conserved Noether charge. The Noether current (up to irrelevant constant factors) is

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad (6.23a)$$

implying a charge

$$Q = \int d^3x \, \psi^\dagger(\vec{x}) \, \psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right) \quad (6.23b)$$

(again ignoring an infinite constant, i.e. shifting the absolute zero point of charge.)

We conclude that $a^{s\dagger}_{\vec{p}}$ creates particles with charge $+1$ (fermions) and $b^{s\dagger}_{\vec{p}}$ creates charge -1 particles (anti-fermions).

The new aspect compared to complex scalars is the spinorial degree of freedom carried by $\psi(x)$. The relation to angular momentum becomes visible if we consider the invariance of the theory under space time rotations.

For this, we first have to recall how Dirac spinors behave under Lorentz transformations, parametrized by a matrix

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (6.24a)$$

(c.f. exercises and Lecture notes "Particles & Fields"). While scalars transform trivially, $\phi'(x') = \phi(x)$, a Lorentz transformation of the Dirac spinor does not only affect the coordinates, but also the spinor components (from a "standard" viewpoint; an alternative viewpoint exists in the spin-base invariant formulation):

$$\psi(x) \to \psi'(x') = A\,\psi(x) \quad (6.24b)$$

or $$\psi'(x) = A\,\psi(\Lambda^{-1}x) \quad \text{using (6.24a)}, \qquad (6.24c)$$

where A is the representation of the Lorentz-transformation (6.24a) in the space of Dirac spinors. A Lorentz transformation is parametrized by 6 parameters: 3 boost parameters summarized in the three velocity components $\vec{v}$ corresponding to the relative velocity between two Lorentz frames, and 3 angle parameters corresponding to the relative rotation between the two frames. These 6 parameters can be summarized in an anti-symmetric matrix $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. For instance, infinitesimally, we have $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu + \mathcal{O}(\varepsilon^2)$.

For a finite transformation, the spinorial representation reads

$$A = D_{\text{spin}}(\Lambda) = e^{-\frac{i}{4}\varepsilon_{\mu\nu}\sigma^{\mu\nu}} \qquad (6.24d)$$

where $$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad (6.24e)$$

All we need here for the present analysis is an infinitesimal rotation about the z axis with an angle $\varepsilon_{12} = -\varepsilon_{21} = \theta \ll 1$. For this special case, we have (ignoring terms of order ε^2)

$$A = \mathbb{1} - \frac{i}{4}\varepsilon_{\mu\nu}\sigma^{\mu\nu} = \mathbb{1} - \frac{i}{2}\theta\,\Sigma^3 \qquad (6.25a)$$

where $$\Sigma^3 \equiv \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \qquad (6.25b)$$

and $$\Lambda^{-1}x = (t,\, x+\theta y,\, y-\theta x,\, z) \qquad (6.25c)$$

The corresponding infinitesimal transformation of the field is

$$\delta\Psi(x) = A\Psi(\Lambda^{-1}x) - \Psi(x) \tag{6.25d}$$

$$= -\theta\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\Psi(x). \tag{6.25e}$$

Using the invariance of the free Dirac field under rotations, the corresponding conserved Noether current is

$$j^0 = -i\,\bar{\Psi}\gamma^0\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\Psi \tag{6.26a}$$

and $\vec{j} = \bar{\Psi}\left(\vec{x}\times(-\vec{\nabla}) + \frac{1}{2}\vec{\Sigma}\right)\Psi$ (6.26b)

where for the second line, we have permitted analogous rotations about the x or y axis. Here we see that angular orbital momentum structures $\sim \vec{x}\times\vec{p}$ pair up with spinorial angular momentum $\sim \vec{\Sigma}$.

A general decomposition of (6.26) into ladder operators is more involved. Here, we are satisfied with one-particle states at rest. Using $\vec{J} = \int d^3x\,\vec{j}$, we find e.g.

$$J_z\, a_0^{s\dagger}|0\rangle = \pm\frac{1}{2}\,a_0^{s\dagger}|0\rangle, \quad J_z\, b_0^{s\dagger}|0\rangle = \mp\, b_0^{s\dagger}|0\rangle \tag{6.27}$$

where the upper/lower sign apply to $\xi^s = \binom{1}{0}$, $\xi^s\binom{0}{1}$, respectively.

We conclude that $a_{\vec{p}}^{s\dagger}$ creates electrons with say spin-up $\frac{1}{2}$ (for $\xi^s = \binom{1}{0}$), while $b_{\vec{p}}^{s\dagger}$ creates positrons with an oppositely polarized spin $\frac{1}{2}$.