5.3 LSZ reduction formula

H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cimento 1 (1), 205 (1955)

"Zur Formulierung quantisierter Feldtheorien"

As discussed above, the Fourier transform of the 2-point function has a simple pole at the 1-particle mass

$$\int d^D x \; e^{ipx} \langle \Omega | T[\phi_H(x) \phi_H(y)] | \Omega \rangle \underset{p^2 \to m^2}{\sim} \frac{iZ}{p^2 - m^2 + i\varepsilon} \qquad (5.44)$$

The symbol "$\sim$" here and in the following indicates that the poles on both sides agree.
We wish to generalize this result in the following to the case of higher correlation functions. For this, we consider the Fourier transform of $G^{(n+2)}(x, \dots)$ with respect to one argument

$$\int d^D x \; e^{ipx} \langle \Omega | T[\phi_H(x) \phi_H(z_1) \phi_H(z_2) \dots] | \Omega \rangle . \qquad (5.45)$$

In order to identify the poles (e.g. in the complex plane of p^0), we split the time integral x^0 into 3 parts:

$$\int dx^0 = \underbrace{\int_{T_+}^{\infty} dx^0}_{\text{I}} + \underbrace{\int_{T_-}^{T_+} dx^0}_{\text{II}} + \underbrace{\int_{-\infty}^{T_-} dx^0}_{\text{III}} \qquad (5.46)$$

where $T_+ \gg z_i^0$, $T_- \ll z_i^0$.

As the integral II has finite boundaries with an integrand that is analytic in p^0, no singularity structure can arise from II. Hence, the singularities must arise from the unboundedness of the intervals I and III.

Consider I: x^0 is the latest time, hence $\phi(x)$ is on the left. The insertion of a complete set of states gives

$$\int_{T_+}^{\infty} dx^0 \int d^3x\, e^{ip x^0} e^{-i\vec{p}\cdot\vec{x}} \sum_\lambda \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} \langle \Omega | \phi_H(x) | \lambda_{\vec{q}} \rangle \cdot \langle \lambda_{\vec{q}} | T[\phi_H(z_1) \ldots] | \Omega \rangle \tag{5.47}$$

Using $\langle \Omega | \phi_H(x) | \lambda_{\vec{q}} \rangle = \langle \Omega | \phi(0) | \lambda_0 \rangle e^{-iqx} \Big|_{q^0 = E_{\vec{q}}(\lambda),\ \vec{q}}$ and inserting a factor $e^{-\varepsilon x^0}$ to actively monitor the convergence at large x^0 with a control parameter ε, we have

$$(5.47) = \sum_\lambda \int_{T_+}^{\infty} dx^0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} e^{ip^0x^0} e^{-iq^0x^0} e^{-\varepsilon x^0} \langle \Omega | \phi_H(0) | \lambda_0 \rangle (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \cdot \langle \lambda_{\vec{q}} | T[\phi_H(z_1) \ldots] | \Omega \rangle$$

$$= \sum_\lambda \frac{1}{2E_{\vec{p}}(\lambda)} \frac{i}{p^0 - E_{\vec{p}}(\lambda) + i\varepsilon} e^{i(p^0 - q^0)T_+} \langle \Omega | \phi_H(0) | \lambda_0 \rangle \langle \lambda_{\vec{p}} | T[\phi_H(z_1) \ldots] | \Omega \rangle \tag{5.48}$$

Here we see that a singularity at $p^0 \to E_{\vec{p}}(\lambda)$ arises. The nature of the singularity is decided by the nature of the distribution of eigenvalues of the Hamiltonian, parametrized by $m\lambda$. For

an isolated eigenvalue $m = m_\lambda$, we have a 1-particle state with a pole at $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$, therefore

$$\int d^D x\, e^{ipx} \langle \Omega | T[\phi_H(x)\, \phi_H(z_1) \ldots] | \Omega \rangle$$

$$\underset{p^0 \to E_{\vec{p}}}{\sim} \frac{i}{p^2 - m^2 + i\varepsilon} \sqrt{Z}\, \langle \vec{p} | T[\phi_H(z_1) \ldots] | \Omega \rangle \qquad (5.49)$$

where $Z = |\langle \Omega | \phi_H(0) | \lambda_0 \rangle|^2$ as before. We also wrote $|\lambda_{\vec{p}}) \to |\vec{p}\rangle$ to emphasize the one-particle nature of this state.

Analogously, we obtain from the integral III with x^0 being the earliest time

$$\int d^D x\, e^{ipx} \langle \Omega | T[\phi_H(x)\, \phi_H(z_1) \ldots] | \Omega \rangle$$

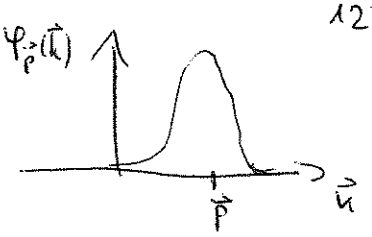
$$\underset{p^0 \to -E_{\vec{p}}}{\sim} \langle \Omega | T[\phi_H(z_1) \ldots] | -\vec{p} \rangle \sqrt{Z} \frac{i}{p^2 - m^2 + i\varepsilon} . \qquad (5.50)$$

Now, we would like to perform the Fourier transform with respect to the other coordinates z_i. For this, we have the situation in mind that the particle states have the character of asymptotic states in the intervals I and III. To formalize this and avoid interference, we consider isolated wave packets in (Fourier or coordinate) space. Hence, we repeat the calculation with the replacement

$$\int d^D x\, e^{ip^0 x^0} e^{-i\vec{p}\cdot\vec{x}} \to \int \frac{d^d k}{(2\pi)^d} \int d^D x\, e^{ip^0 x^0} e^{-i\vec{k}\cdot\vec{x}}\, \varphi_{\vec{p}}(\vec{k}) , \qquad (5.51)$$

where the wave function $\varphi_{\vec{p}}(\vec{k})$ should have a small distribution

in momentum space centered around $\vec{p} \approx \vec{k}$



In the limit $\varphi_{\vec{p}}(\vec{k}) \to (2\pi)^d \delta^{(d)}(\vec{k}-\vec{p})$, we recobtain our results so far. The "smearing" in momentum space $\delta^{(d)}(\vec{k}-\vec{p}) \to \varphi_{\vec{p}}(\vec{k})$ leads to the desired localization in coordinate space. The price we have to pay is that of a more involved singularity structure:

$$\text{I}: \sum_\lambda \int \frac{d^d k}{(2\pi)^d} \varphi_{\vec{p}}(\vec{k}) \frac{1}{2E_{\vec{k}}(\lambda)} \frac{i}{p^0 - E_{\vec{k}}(\lambda) + i\varepsilon} \langle \Omega | \phi(0) | \lambda_0 \rangle \langle \lambda_{\vec{k}} | T[\phi_H(z) \ldots] | \Omega \rangle$$

$$\underset{p^0 \to E_{\vec{p}}}{\sim} \int \frac{d^d k}{(2\pi)^d} \varphi_{\vec{p}}(\vec{k}) \frac{i}{\tilde{p}^2 - m^2 + i\varepsilon} \sqrt{Z} \langle \vec{k} | T[\phi_H(z) \ldots] | \Omega \rangle \qquad (5.52)$$

where $\tilde{p}^\mu = (p^0, \vec{k})$

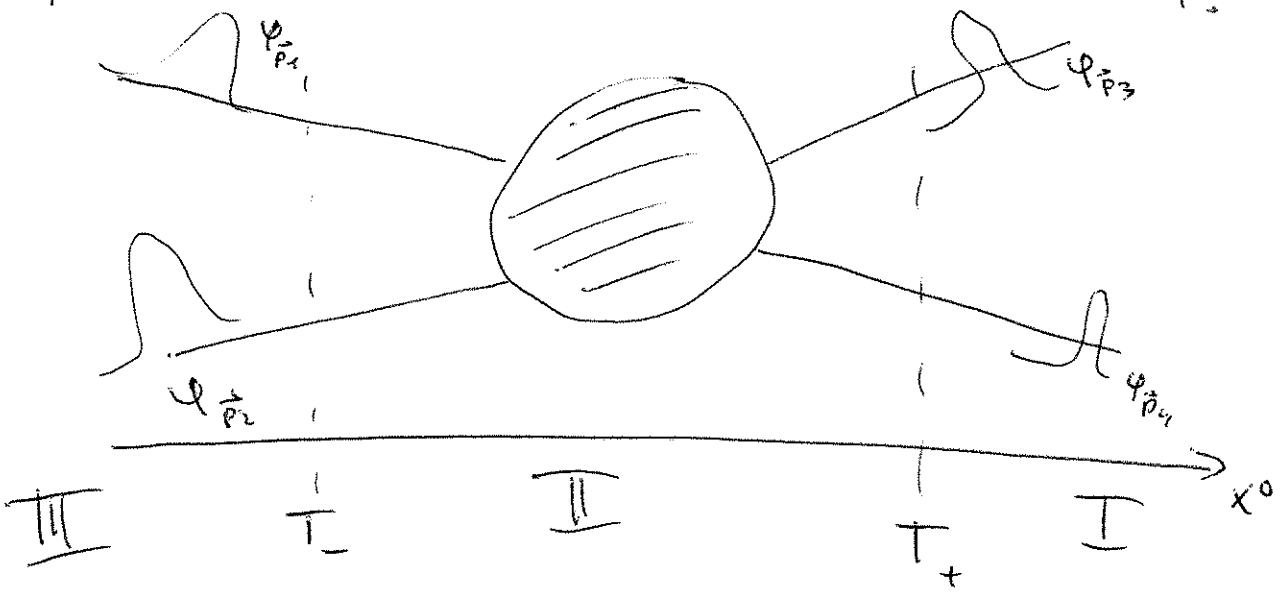
The 1-particle pole thus becomes a branch cut, the extent of which corresponds to the width of the wave packet in momentum space. In the limit $\varphi_{\vec{p}} \to \delta$, we recbtain the 1-particle pole.
The analysis for region III is analogous.

Now we apply this idea to all coordinates x_i of an $(n+2)$-point correlator

$$\left(\prod_i \int \frac{d^d k_i}{(2\pi)^d} \int d^D x_i \, e^{i\tilde{p}_i x_i} \varphi_{\vec{p}_i}(\vec{k}_i) \right) \langle \Omega | T[\phi_H(x_1) \phi_H(x_2) \ldots] | \Omega \rangle \qquad (5.53)$$

In coordinate space, the wave packets should overlap and interact mean $x=0$ in between the times T_- and T_+. However,

they should be well-separated for $x^0 < T_-$ and $x^0 > T_+$.



Again, no singularities can arise from region II. It suffices to consider x_i^0 values in I or II. As an example, we study the case, where $x_1^0, x_2^0 > T_+$ and all $x_{i\neq 1,2}^0 < T_-$.

The transformation with respect to x_1 and x_2 yields

$$\text{I}: \left(\prod_{i=1,2}\int \frac{d^d k_i}{(2\pi)^d}\int d^D x_i\, e^{i\tilde{p}_i x_i}\varphi_{p_i}^s(\vec{k}_i)\right)\langle\Omega|T\left[\varphi_H(x_1)\varphi_H(x_2)\varphi_H(x_3)\ldots\right]|\Omega\rangle$$

$$\overset{x_1^0, x_2^0 > x_{i\geq 3}^0}{=} \sum_\lambda \int \frac{d^d K}{(2\pi)^d}\frac{1}{2E_K(\lambda)}\left(\prod_{i=1,2}{}'\int\right)\langle\Omega|T\left[\varphi_H(x_1)\varphi_H(x_2)\right]|\lambda_K^{\vec{}}\rangle\langle\lambda_K^{\vec{}}|T\left[\varphi_H(x_3)\ldots\right]|\Omega\rangle \tag{5.54}$$

Since the wave packets are strongly localized in region I, only those states contribute dominantly to the λ-sum, which correspond to localized excitations at distinguishable spacetime points. Hence, only those states $|\lambda_K^{\vec{}}\rangle$ can contribute to $\langle\Omega|T[\varphi_H(x_1)\varphi_H(x_2)]|\lambda_K^{\vec{}}\rangle$ which are similar to two isolated one-particle states $|\lambda_K^{\vec{}}\rangle \sim |\lambda_{\vec{q}_1}, \lambda_{\vec{q}_2}\rangle$.

$$\Rightarrow \sum_{\lambda} \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{2E_{\vec{k}}(\lambda)} \langle \Omega | T[\varphi_H(x_1)\varphi_H(x_2)| \lambda_{\vec{k}}\rangle \langle \lambda_{\vec{k}}|$$

(5.55)

$$\simeq \sum_{\lambda_1,\lambda_2} \int \frac{dq_1}{(2\pi)^d} \frac{1}{2E_{\vec{q}_1}} \int \frac{dq_2}{(2\pi)^d} \frac{1}{2E_{\vec{q}_2}} \langle \Omega | \varphi_{(x_1)} | \lambda_{\vec{q}_1}\rangle \langle \Omega | \varphi_{(x_2)} | \lambda_{\vec{q}_2}\rangle$$

$$\langle \lambda_{\vec{q}_1}, \lambda_{\vec{q}_2}|$$

As we have seen before, only 1-particle states will contribute to the singularity structure we have been looking for. Analogously to (5.47-50), the integrals over x_i° and $\vec{q}_i$ generate a singularity in p_i°, which factorizes as in (5.55),

$$\Rightarrow \left(\prod_{i=1,2} \int \frac{d^dk_i}{(2\pi)^d} \varphi_{\vec{p}_i}(\vec{k}_i) \frac{i}{\tilde{p}_i^2 - m^2 + i\varepsilon} \sqrt{Z'} \right) \langle \vec{k}_1 \vec{k}_2 | T[\varphi_H(x_3) \dots] | \Omega \rangle$$

(5.56)

The limit of strong momentum space localization $\varphi_{\vec{p}_i}(\vec{k}_i) \to \delta^{(d)}(\vec{k} - \vec{p}_i)$ yields

$$\Rightarrow \left(\prod_{i=1,2} \frac{i}{p_i^2 - m^2 + i\varepsilon} \sqrt{Z'} \right) {}_f\langle \vec{p}_1 \vec{p}_2 | T[\varphi_H(x_3) \dots] | \Omega \rangle$$

(5.57)

where the subscript "f" shall remind us of the fact, that ${}_f\langle \vec{p}_1 \vec{p}_2 |$ corresponds to an outgoing 2-particle state.

This line of argument based on separated wave packets can analogously be applied to all $x_{i\geq 3}^\circ < T_-$, and we obtain

$$\left(\prod_{i=1,2}\frac{i}{p_i^2-m^2+i\varepsilon}\sqrt{Z}\right)\left(\prod_{j=3,\dots}\frac{i}{p_j^2-m^2+i\varepsilon}\sqrt{Z}\right)\ {}_F\langle \vec{p}_1\vec{p}_2|-\vec{p}_3,\dots\rangle_i \tag{5.58}$$

with incoming asymptotic states $|-\vec{p}_3,\dots\rangle_i$.

In principle, we have to be a bit more careful, as we have to make sure that the narrow-wave packet limit $\varphi\to\delta$ does not modify the singularity structure. This has in fact been discussed in the original work by LSZ (and we refer to higher authority in these notes), and we take over their result that this does not happen. Since

$$\langle \vec{p}_1,\dots\vec{p}_n|\,S\,|\vec{k}_1,\dots,k_m\rangle = {}_F\langle \vec{p}_1\dots\vec{p}_n|\vec{k}_1\dots k_m\rangle_i, \tag{5.59}$$

we therefore arrive at the LSZ reduction formula

$$\prod_{i=1}^{n}\int d^Dx_i\,e^{ip_ix_i}\prod_{j=1}^{m}\int d^Dx_j\,e^{-ik_jy_j}\langle\Omega|T\left[\phi_H(x_1)\dots\phi_H(x_n)\phi_H(y_1)\dots\phi_H(y_m)\right]|\Omega\rangle$$

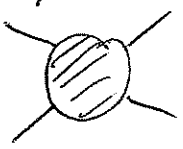
$$\sim \left(\prod_{i=1}^{n}\frac{\sqrt{Z}^{-1}}{p_i^2-m^2+i\varepsilon}\right)\left(\prod_{j=1}^{m}\frac{\sqrt{Z}^{-1}}{k_j^2-m^2+i\varepsilon}\right)\langle\vec{p}_1\dots\vec{p}_n|S|\vec{k}_1,\dots\vec{k}_m\rangle \tag{5.60}$$

$p_i^0\to E_{\vec{p}_i}$, $k_j^0\to E_{\vec{k}_j}$

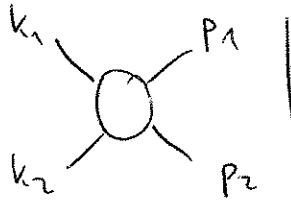
Though this looks messy at first sight, this yields a direct recipe for the computation of S-matrix elements from correlation functions: Compute the Fourier trafos of correlation functions $G^{(n+m)}$, consider that region in momentum space where the particles are close to their mass shell

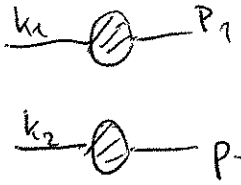
and identify the coefficients of the pole structure as S matrix elements.

The meaning of the LSZ reduction formula becomes more transparent in a diagrammatic analysis.

Consider a 2-to-2 process $G^{(4)}$ = 

From the analysis is Sect. 5.1, Eqs. (5.27) - (5.30), we already know that vacuum diagrams factorize and cancel. Still, there can be disconnected parts of the form



$\sim$ 

+ Similar

$$\sim \frac{\sqrt{Z}\, i}{k_1^2 - m^2 + i\varepsilon} \frac{\sqrt{Z}\, i}{k_2^2 - m^2 + i\varepsilon} \delta^{(D)}(k_1 - p_1)\, \delta^{(D)}(k_2 - p_2) \qquad (5.61)$$

+ Similar

Though pole structures are generated, they do not have the form of 4 different 1-particle-propagator poles as would be required to read off the S matrix element.

With hindsight, this justifies the prescription in (4.65), to consider only the connected parts of the scattering amplitude. Furthermore, we can decompose any diagram

contributing to $G^{(4)}$ into an amputated part (without self energy contribution on the external legs) and the self-energy contribution:

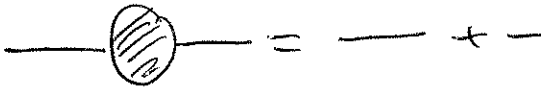
$G^{(4)}$ =

Let us now define the sum of all one-particle-irreducible (1PI) diagrams to the self-energy as

$-i\Sigma(p^2) =$

(5.62)

then, the full propagator, i.e., the sum of all possible (connected) diagrams arises as a geometric series:



$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}\left(-i\Sigma(p^2)\right)\frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-i\Sigma)\frac{i}{p^2-m_0^2}(-i\Sigma)\frac{i}{p^2-m_0^2} + \ldots$$

$$= \frac{i}{p^2-m_0^2-\Sigma(p^2)}, \qquad (5.63)$$

where the $i\varepsilon$ prescription is implicitly understood. Here, m_0 denotes the mass parameter occurring in the free part H_0 of the Hamiltonian. From now on, we will call it

the "bare" mass. In fact, we now observe here, that the physical mass defined as the pole of the propagator

$$m^2 = m_0^2 + \Sigma(p^2 = m^2) \qquad (5.64a)$$

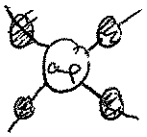
can (and in general does) differ from the bare mass as a consequence of the interactions. This is a direct consequence of the fact that interactions can never be really switched off – even in the absence of scattering with other real particles. A particle can always interact with itself by emission and absorption of virtual quanta. This is similar to the effective electron mass, i.e. the effective mass of electronic excitations in solid-state materials. Near the pole, the propagator hence acquires the form

$$\frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \underset{p^0 \to E_p}{\sim} \frac{iZ}{p^2 - m^2}, \qquad (5.64b)$$

where (see exercises)

$$Z = \frac{1}{1 - \frac{\partial \Sigma}{\partial p^2}(p^2 = m^2)}. \qquad (5.64c)$$

The self-energy hence determines the physical mass as well as the wave function renormalization.

The four legs in $G^{(4)}$ ~  hence

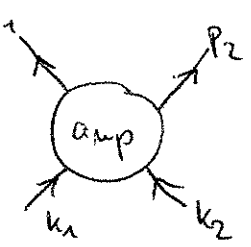
produce the proper pole structure

$$\frac{iZ}{p_1^2-m^2}\;\frac{iZ}{p_2^2-m^2}\;\frac{iZ}{k_1^2-m^2}\;\frac{iZ}{k_2^2-m^2} \tag{5.65}$$

as expected from the LSZ reduction formula.

The coefficient of the pole structure corresponds to the desired S matrix element

$$\langle \vec{p}_1 \vec{p}_2 \,|\, S \,|\, \vec{k}_1 \vec{k}_2 \rangle = \sqrt{Z}^4$$



. (5.66)

With hindsight, this justifies the second prescription in (4.65) to only take the amputated diagrams into account.

As a correction to (4.65), we see that also a factor of $\sqrt{Z}$ has to be taken into account for each external leg. (They do not contribute to leading order, but are important for higher orders) The correct version of (4.65) for 2-to-n processes hence reads

$$\langle \vec{p}_1 \dots \vec{p}_n \,|\, S \,|\, \vec{k}_1 \vec{k}_2 \rangle = \sqrt{Z}^{n+2}$$

. (5.67)