4.4 Feynman diagrams in perturbation theory

Wick's theorem provides us with a technique to compute T-ordered products and their expectation values. This can be used to determine S matrix elements in a perturbative expansion. Let us discuss this in the following for ϕ^4 theory,

$$V_S = \frac{\lambda}{4!} \int d^4x \, \phi_S^4 \tag{4.48}$$

such that with $H_I = U_0^{-1} V_S U_0$, we have

$$\mathcal{H}_I = \frac{\lambda}{4!} U_0^{-1} \phi_S^4 U_0 = \frac{\lambda}{4!} \phi^4 \tag{4.49}$$

in the interaction picture. Here and in the following, we denote the field operator in this picture simply by

$$\phi \equiv \phi_I = U_0^{-1} \phi_S U_0 .$$

The S matrix and its perturbative expansion (Dyson series) then reads

$$S = T e^{-i \int d^D x \, \mathcal{H}_I} = T e^{-i \frac{\lambda}{4!} \int d^D x \, \phi^4(x)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i\lambda}{4!} \right)^n T \left(\int d^D x \, \phi^4 \right)^n =: \sum_{n=0}^{\infty} S^{(n)} \tag{4.50}$$

where $S^{(n)}$ denotes the n-th order in an expansion in λ around $\lambda = 0$.

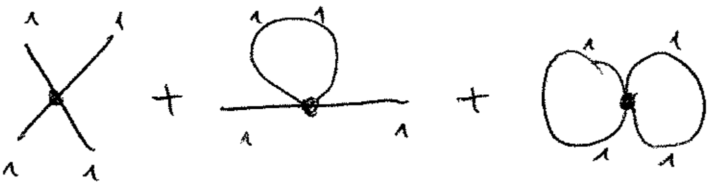
Let us compute the leading nontrivial order ($\varphi_i = \varphi(x_i)$):

$$n=0: \qquad S^{(0)} = \mathbb{1} \qquad \text{(trivially, non-interacting)} \qquad (4.51)$$

$$
\begin{aligned}
n=1: \qquad S^{(1)} &= -i\,\frac{\lambda}{4!}\int d^Dx_1\, T\left[\varphi_1^4\right] \\
&= -i\,\frac{\lambda}{4!}\int d^Dx_1 \left\{ N[\varphi_1^4] + 6\,\overbrace{\varphi_1\varphi_1}\,N[\varphi_1^2] + 3\,\overbrace{\varphi_1\varphi_1}\,\overbrace{\varphi_1\varphi_1} \right\} \\
&= -i\,\frac{\lambda}{4!}\int d^Dx_1\; N[\varphi_1^4] - i\frac{\lambda}{4}\, i\Delta_F(0)\int d^Dx_1\, N[\varphi_1^2] - i\,\frac{\lambda}{8}\left(i\Delta_F(0)\right)^2 V_D
\end{aligned}
$$

(4.52)

where we formally introduced the D-dimensional spacetime volume $V_D = \int d^Dx$. This result can also be graphically represented

$S^{(1)} =$  . (4.53)

Each external line ("leg") with index 1 denotes a field $\varphi_1 = \varphi(x_1)$. Internal lines 1 1 are self-contractions $i\Delta_F(0) = i\Delta_{F11} = i\Delta_F(x_1 - x_1)$. The dots $\bullet$ denote "vertices" carrying a factor $\sim \lambda$.

Each graph carries a symmetry factor, resulting from the prefactors $\sim \frac{1}{4!}$ and the number of combinatorial possibilities to form the graph.

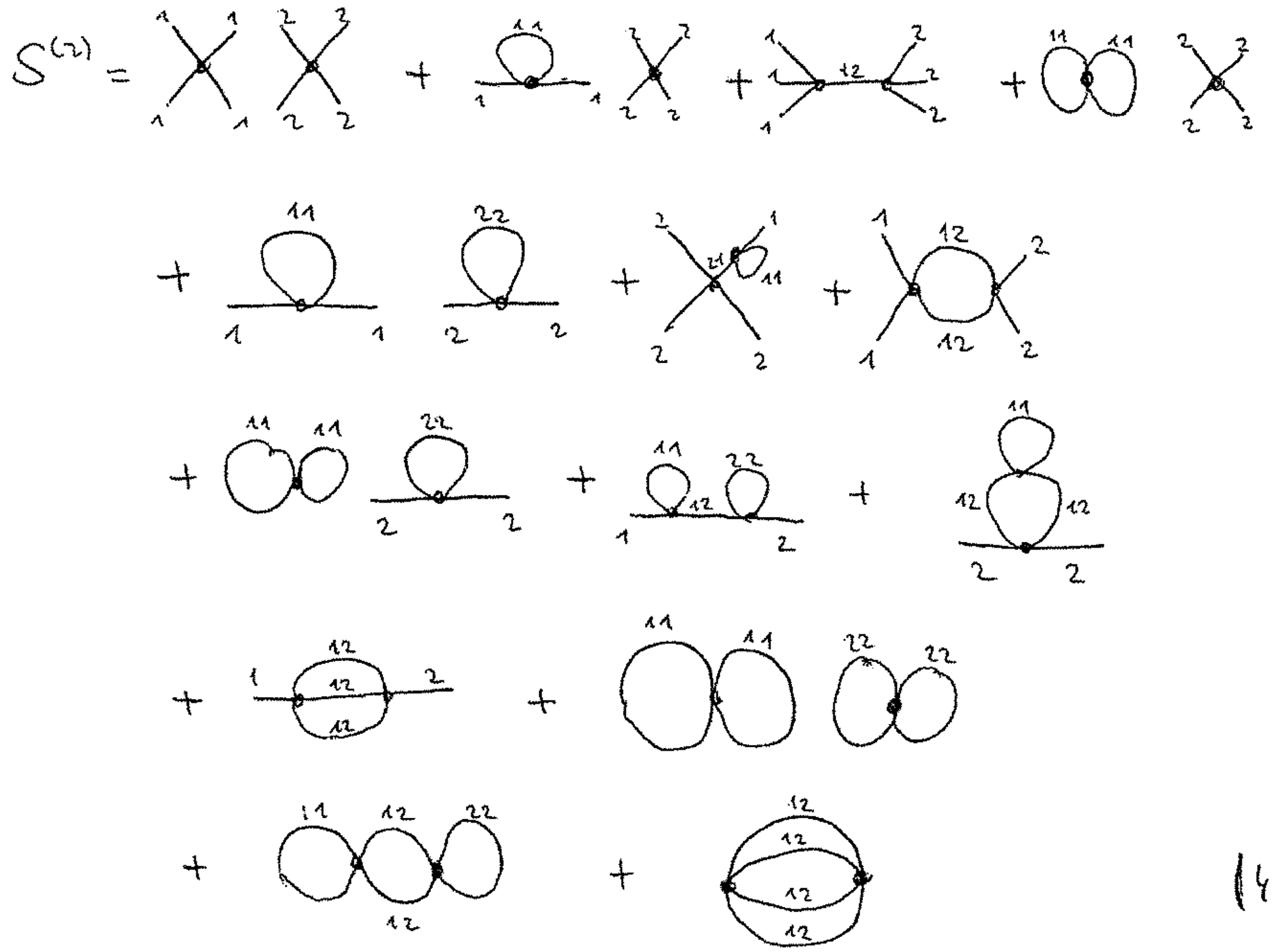
The 2. Order is already more extensive

$$S^{(2)} = \frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^Dx_1\, d^Dx_2\; T[\varphi_1^4 \varphi_2^4]$$

$$\begin{aligned}
= \frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^Dx_1\, d^Dx_2 \Big\{ & N[\varphi_1^4\varphi_2^4] + 2\cdot 6\, \overline{\varphi_1\varphi_1}\, N[\varphi_1^2\varphi_2^4] + 4\cdot 4\, \overline{\varphi_1\varphi_2}\, N[\varphi_1^3\varphi_2^3] \\
& + 2\cdot 3\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_1\varphi_1}\, N[\varphi_2^4] + 6\cdot 6\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_2\varphi_2}\, N[\varphi_1^3\varphi_2^3] \\
& + 2\cdot 6\cdot 2\cdot 4\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_1\varphi_2}\, N[\varphi_1\varphi_2^3] \\
& + \frac{1}{2!}\, 4\cdot 4\cdot 3\cdot 3\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, N[\varphi_1^2\varphi_2^2] \\
& + 2\cdot 6\cdot 3\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_2\varphi_2}\, N[\varphi_2^2] \\
& + 6\cdot 6\cdot 2\cdot 2\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_2\varphi_2}\, \overline{\varphi_1\varphi_2}\, N[\varphi_1\varphi_2] \\
& + 2\cdot 6\cdot 4\cdot 3\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, N[\varphi_2^2] \\
& + \frac{(4\cdot 3\cdot 2)^2}{3!}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, N[\varphi_1\varphi_2] \\
& + 3\cdot 3\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_2\varphi_2}\, \overline{\varphi_2\varphi_2} + 6\cdot 6\cdot 2\, \overline{\varphi_1\varphi_1}\, \overline{\varphi_2\varphi_2}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2} \\
& + \frac{(4\cdot 3\cdot 2\cdot 1)^2}{4!}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2}\, \overline{\varphi_1\varphi_2} \Big\}
\end{aligned} \qquad (4.54)$$

The symmetry factor can be worked out by combinatoric considerations or straightforwardly be computed by Wick's theorem.

The graphical representation of the 14 terms is more illustrative:



(4.55)

The building blocks of these Feynman diagrams are

– the Vertex $\bullet = -i\lambda \int d^Dx$. (4.56)

a vertex joins 4 lines in ϕ^4 theory

– the inner line represents a propagator (contraction),

$$\overline{\;i \qquad j\;} = \overline{\;\;}_{ij} = i\Delta_F(x_i - x_j) \quad . \qquad (4.57)$$

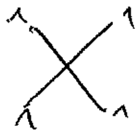
a self-contraction leads to a closed loop:

 $= i\Delta_F(0)$ (4.58)

- the external lines, which (in the present case) represent field operators occurring in an N-product

 $= N[\varphi_1 \ldots]$

 $= N[\varphi_1^2 \ldots]$ (4.59)

 $= N[\varphi_1^4 \ldots]$

- an implicit symmetry factor, taking the combinatorics into account.

In ϕ^4 theory, Wick's theorem tells us that the number of external legs is always even. In general, the number of external legs N_e allows for a classification of diagrams:

- $N_e = 0$: "vacuum bubbles"

, , , ... (4.60)

$\hat{=}$ fully contracted terms in the S matrix without N-product parts.

These graphs are proportional to $\sim \mathbb{1}$ and hence contribute as the only graphs to the vacuum expectation value of S,

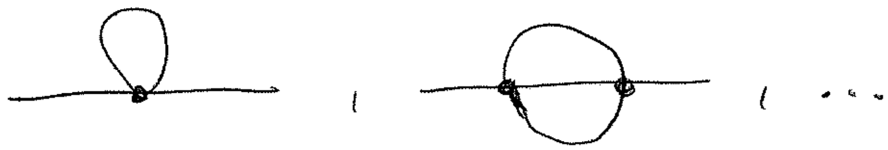
$$\langle 0 | S | 0 \rangle = \sum \text{vacuum bubbles} \; . \qquad (4.61)$$

This yields ~~the~~ probability amplitude for the vacuum to persist under the influence of interactions. Stability of the vacuum is equivalent to

$$| \langle 0 | S | 0 \rangle | = 1 , \qquad (4.62)$$

i.e. $\langle 0 | S | 0 \rangle$ and the sum of vacuum bubbles is just a phase.

- $N_e = 2$: "self-energy" diagrams



(4.63)

These diagrams describe the self-interaction of a particle by emitting and/or absorbing further "virtual" particles/anti-particles. As already mentioned above, interactions are never really switched off in QFT, they even occur for asymptotic states. This is precisely visible in these self-energy diagrams. Their sum corresponds therefore to the "real" asymptotic states even after the interaction with other particles or scattering potentials are switched off.

- $N_e \geq 4$: Vertex- or scattering diagrams

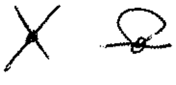
 (4.64)

These diagrams are the ones which are relevant for the description of scattering.

For a further classification of diagrams, let us define:

Def.: a diagram is called <u>connected</u>, if each pair of vertices is connected by one or more internal lines. (E.g. in the 1st line of (4.55), only the 3rd diagram is connected)

Def.: a diagram is 1-particle irreducible (1PI) if it remains connected after removing one internal line;

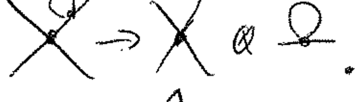
e.g.  is 1-particle reducible, $\overset{\text{cut}}{\rightsquigarrow}$ 

 is 1-particle irreducible (1PI)

4.5 Scattering amplitudes & Cross sections

As argued above, the nontrivial information in the S matrix is encoded in the deviation from the identity matrix given by the transition matrix, $S = \mathbb{1} + iT$.

As might be intuitively clear from the discussion of diagram classes, vacuum bubbles and self-energy corrections do not contribute to T. Hence, the description of the scattering process can be traced back to <u>connected</u> and <u>amputated</u> vertex diagrams. "Amputation" means that self-energy corrections on external lines are dismissed



amputated

Hence, we <u>tentatively</u> define the scattering amplitude as

$$i(2\pi)^D \delta^{(D)}\Big(\sum_f p_f - \sum_i p_i\Big)\, \mathcal{M}\big(\vec{p}_i \to \vec{p}_f\big) \qquad (4.65)$$

$$:= \langle \{\vec{p}_f\} |\, iT\, | \{\vec{p}_i\}\rangle_{\substack{\text{conn.}\\ \text{amput.}}}$$

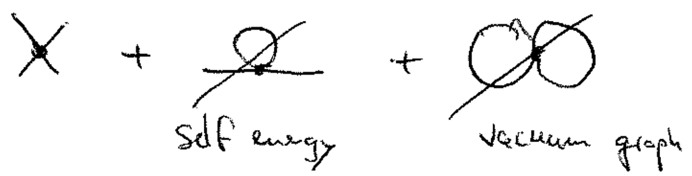
A global momentum conserving δ function has already been factored out here. Eq. (4.65) will be justified more comprehensively (and slightly corrected) later on with the aid of the LSZ formalism.

Let us try to understand (4.65) with the aid of a simple example: 2-to-2 scattering in ϕ^4 to leading order:

The right-hand side of (4.65) to leading order reads

$$\langle \vec{p}_1{}' \, \vec{p}_2{}' \,|\, i T^{(1)} \,|\, \vec{p}_1 \vec{p}_2 \rangle_{\substack{\text{com.}\\ \text{amp.}}}$$

$$= \langle \vec{p}_1{}' \, \vec{p}_2{}' \,|\, \underbrace{\left(-i \frac{\lambda}{4!}\right) \int d^D x_1 \, T \left[\varphi_1^4 \right]} \,|\, \vec{p}_1 \vec{p}_2 \rangle_{\substack{\text{com.}\\ \text{amp.}}}$$

$=$ 

$$= -i \frac{\lambda}{4!} \langle \vec{p}_1{}' \, \vec{p}_2{}' \,|\, \int d^D x_1 \, N \left[\varphi_1^4 \right] \,|\, \vec{p}_1 \vec{p}_2 \rangle \qquad (4.66)$$

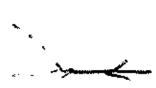
The normal-ordering helps to pick out the non-zero contributions,

e.g. $$\varphi(x_1) \,|\vec{p}_1\rangle \underset{\substack{\uparrow \\ \text{N ordering}}}{\sim} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \, a_{\vec{p}} \, e^{-i\bar{p}x_1} \sqrt{2E_{\vec{p}_1}} \, a^+_{\vec{p}_1} \,|0\rangle$$

$$= \int \frac{d^d p}{(2\pi)^d} \sqrt{\frac{E_{\vec{p}_1}}{E_{\vec{p}}}} \, e^{-i\bar{p}x_1} \underbrace{\left[a_{\vec{p}}, a^+_{\vec{p}_1} \right]}_{(2\pi)^d \delta^{(d)}(\vec{p}-\vec{p}_1)} |0\rangle$$

$$= e^{-i\bar{p}_1 x_1} \,|0\rangle \qquad (4.67)$$

This suggests to define also contractions for incoming states:

$$\varphi(x) \,|\vec{p}\rangle = e^{-i\bar{p}x} \equiv$$  $$(4.68)$$

Similarly for outgoing states, we have

$$\overbracket{\langle \vec{p} | \, \phi}(x) = e^{i \bar{p} x} \equiv$$  (4.69)

Note that the external states/lines automatically have to sit on the mass shell $\bar{p}^\mu = (E_{\vec{p}}, \vec{p})$, $E_{\vec{p}}^2 = \vec{p}^2 + m^2$.
In the following, we will hence drop the bar, as there is no ambiguity for external states.

Using (4.68) & (4.69) and accounting for all possible contractions, we find

$$\langle \vec{p}_1{}' \, \vec{p}_2{}' | \, i T^{(1)} \, | \vec{p}_1 \, \vec{p}_2 \rangle_{\text{con. amp.}} = \underset{\substack{\uparrow \\ \text{\# of possible} \\ \text{contractions}}}{4!} \left(- \frac{i\lambda}{4!} \right) \int d^D x_1 \, e^{-i(p_1 + p_2 - p_1' - p_2') \cdot x_1}$$

$$= - i \lambda \, (2\pi)^D \, \delta^{(D)} (p_1' + p_2' - p_1 - p_2) \qquad (4.70)$$

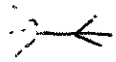
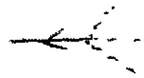
A comparison with our definition of the scattering amplitude gives

$$\mathcal{M}(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}') = - \lambda \qquad (4.71)$$

to lowest order. Hence, we find a constant amplitude to this order.

The computational steps performed above can also be summarized in terms of Feynman rules in momentum space:

The n-th order of the scattering amplitude $i\mathcal{M}$ is given by the sum of all connected amputated vertex diagrams to that order. These are constructed from:

- vertex  $= -i\lambda$
- inner line (propagator) $= \dfrac{i}{p^2-m^2+i\varepsilon} = i\Delta_F(p)$
- external leg (on-shell particle)  $= 1 =$ 

  plane-wave factors yield overall 4-momentum conservation $\sim \delta^{(D)}$

- 4-momentum conservation holds at each vertex

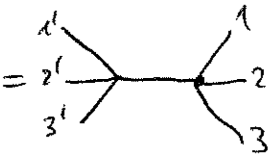
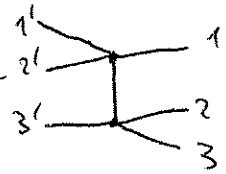
  $$(2\pi)^D\, \delta^{(D)}\left(\sum p_{OUT} - \sum p_{IN}\right)$$

- integrate over $\displaystyle\int \frac{d^D p}{(2\pi)^D}$ for each inner line with momentum p
- combinatorial symmetry factors.

(4.72)

Let us study these rules using a slightly less trivial example from ϕ^4-theory.

3-to-3 scattering to order λ^2

$$\langle 1'2'3' | iT^{(2)} | 123 \rangle =$$  $+$  + further diagrams (4.73)

$\uparrow$ $\vec{p}_1{}'$, etc.

For the present illustration, let us focus on the first diagram only:

$$= \frac{1}{2!} \frac{(-i\lambda)^2}{(4!)^2} 4! \cdot 4! \cdot 2\, i \int \frac{d^D p}{(2\pi)^D} (2\pi)^D \frac{\delta^{(D)}(p - \sum p_i)\, \delta^{(D)}(\sum p_f' - p)}{p^2 - m^2 + i\varepsilon}$$

$$= i(-i\lambda)^2 (2\pi)^{(D)} \delta^{(D)}\Big(\sum_f p_f' - \sum_i p_i\Big) \frac{1}{(\sum_i p_i)^2 - m^2 + i\varepsilon} \qquad (4.74)$$

Here we can read off the contribution of this diagram to the scattering amplitude

$$\mathcal{M}^{(2)}(\{\vec{p}_i\} \to \{\vec{p}_f\}) = \frac{(-i\lambda)^2}{p^2 - m^2 + i\varepsilon} + \ldots, \quad p^2 = \Big(\sum_i p_i\Big)^2 \qquad (4.75)$$

In this example, we obviously find a non-trivial momentum dependence of the scattering amplitude; this is actually the standard result: even though the interactions are local (similarly to classical point particles), the scattering amplitudes are momentum dependent. We also see in (4.75) that the scattering amplitude becomes large if the sum of the incoming momenta approaches the mass of the inner propagator, $(\sum_i p_i)^2 \to m^2$. (In this limit, the present diagram dominates all other contribution and the result (4.75) is a very good approximation). This enhancement of the scattering amplitude is characteristic for a resonance. In turn, experimental observations of such resonances can be indicative for the existence of an intermediate particle-like state.

Still, in both examples, no nontrivial momentum integration – possibly also inducing a momentum dependence beyond simple $\Delta_F(p)$ dependencies – was left over. Diagrams with this simplifying property, where all integrals can be resolved by Dirac δ's, are called <u>tree diagrams</u>. Tree diagrams do not contain closed <u>loops</u>.

For a diagram with V vertices and I inner lines, there are I momentum integrals and V δ functions. Total momentum conservation requires already one δ function, such that only $V-1$ δ functions remain to cancel integrations. The number of remaining momentum integrations L is thus given by

$$L = I - (V-1) = I - V + 1 \qquad (4.76)$$

This number is equal to the number of closed loops, cf. exercises.

A central observable in particle-physics phenomenology are (differential) <u>cross sections</u>. Heuristically, a cross section corresponds to a surface, denoting an interaction region faced by an incoming particle (beam) through a scattering center. Cross sections are already familiar from mechanics and quantum mechanics. Here, we adapt this notion to QFT:

<u>Def:</u> the <u>transition probability</u> for a transition $|i\rangle \to |f\rangle$ is given by ($|i\rangle \neq |f\rangle$)

$$P_{fi} := |T_{fi}|^2 = (2\pi)^D \delta^{(D)}\left(\sum p_f - \sum p_i\right) V_D \left|\mathcal{M}(i \to f)\right|^2 \tag{4.77}$$

Here, we have already made use of Fermi's Golden Rule, or the corresponding arguments that help to make physical sense out of the naively occurring product of two δ functions contained in the square of T_{fi}. A brief heuristic reminder is

$$(2\pi)^{2D}\left[\delta^{(D)}\left(\sum p_f - \sum p_i\right)\right]^2 \sim (2\pi)^D \delta^{(D)}\left(\sum p_f - \sum p_i\right) \int d^D x \, e^{-ix(\sum p_f - \sum p_i)}$$

$$\sim (2\pi)^D \underbrace{\int d^D x}_{= V_D} \; \delta^{(D)}\left(\sum p_f - \sum p_i\right) \tag{4.78}$$

where V_D is the D-dimensional spacetime volume. Division of (4.77) by this volume gives the

<u>Def.:</u> <u>transition rate</u>

$$R_{fi} := \frac{|T_{fi}|^2}{V_D} = (2\pi)^D \delta^{(D)}\left(\sum p_f - \sum p_i\right) |\mathcal{M}_{fi}|^2 . \tag{4.79}$$

In general, the states $|f\rangle$ are part of a continuum of states, depending continuously on momenta, angles, etc. A typical detector, however, resolves only finite elements in phase space, Δp, $\Delta\vartheta$, etc. The idealized limit of these finite elements already occurs in the Lorentz-invariant measure

$$d N_f := \frac{d^d p}{(2\pi)^d} \frac{1}{2 E_{\vec{p}}} \qquad (4.80)$$

for each outgoing particle.

The transition rate for processes ending up in such an infinitesimal element of momentum space is given by the

<u>Def.</u>: <u>differential transition rate</u>

$$dR := R_{fi}\, dN_f = (2\pi)^D \delta^{(D)}(\Sigma p_f - \Sigma p_i) \, |\mathcal{M}_{fi}|^2 \, dN_f \qquad (4.81)$$

We are aiming at a characteristic quantity of the scattering process, which is independent of the kinematic details of the incident beam (apart from the incident momenta). For this, we scale out a flux factor characterizing the relative influx of particles into the scattering region. For a 2-to-n particle scattering process, this factor is given by

<u>Def.</u> <u>flux factor</u>

$$F := \rho_1 \rho_2 \, v_{12} \qquad (4.82)$$

where ρ_1 and ρ_2 are the particle densities of the incoming beams, and v_{12} is their relative velocity. We determine the flux factor using the one-particle wave function,

$$\varphi_{\vec{p}}(x) = \langle 0 | \varphi(x) | \vec{p} \rangle = e^{-i\bar{p}x} \tag{4.83}$$

which is connected to the probability density

$$j^\mu = 2\,\mathrm{Im}\left(\varphi_{\vec{p}}\, \partial^\mu \varphi^*_{\vec{p}}\right) = 2\bar{p}^\mu \tag{4.84}$$

implying $\quad \rho \equiv j^0 = 2E_{\vec{p}} = 2\gamma m, \quad \gamma = \dfrac{1}{\sqrt{1-v^2}}$

$$\vec{j} = 2\vec{p} = 2\gamma m \vec{v} \tag{4.85}$$

Here, we can read off the velocity $\vec{v}$

$$\vec{v} = \frac{\vec{j}}{j^0} = \frac{\vec{j}}{\rho} = \frac{\vec{p}}{E_{\vec{p}}} \tag{4.86}$$

such that we can write the flux factor

$$F = 2E_{\vec{p}_1}\, 2E_{\vec{p}_2} \left| \frac{\vec{p}_1}{E_{\vec{p}_1}} - \frac{\vec{p}_2}{E_{\vec{p}_2}} \right| = 4 \left| E_{\vec{p}_2} \vec{p}_1 - E_{\vec{p}_1} \vec{p}_2 \right| \tag{4.87}$$

With all these definitions, we obtain the <u>differential</u>

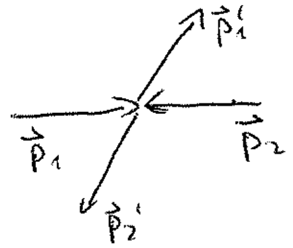
Def.: <u>cross-section</u> : $d\sigma := \dfrac{dR}{F}$ (4.88)

$$d\sigma = \frac{1}{2E_{\vec{p}_1} 2E_{\vec{p}_2} v_{12}} \left(\prod_f \frac{d^d p_f}{(2\pi)^d} \frac{1}{2E_{\vec{p}_f}} \right) (2\pi)^D \delta^{(D)}\left(\Sigma p_f - \Sigma p_i\right) |\mathcal{M}_{fi}|^2 \tag{4.89}$$

or schematically " $d\sigma = \dfrac{|\mathcal{M}|^2}{F} \cdot dN_f$ "

For the case that N particles in the final state are identical, we have to account for that by a factor $\frac{1}{N!}$.

As an example, we follow up on 2-to-2 scattering, c.f. p. 90, with the scattering amplitude being $\mathcal{M}_{(2\to2)} = -\lambda$, cf. Eq. (4.71). Let us consider a collider setup with a lab frame being identical to the center-of-mass frame (e.g. for identical particles in the incident beam at identical energies)



$$\vec{p}_1 = -\vec{p}_2 \equiv \vec{p} \qquad (4.90)$$

$$E_{\vec{p}_1} = E_{\vec{p}_2} = \sqrt{\vec{p}^{\,2} + m^2} \equiv \frac{E}{2}$$

For this configuration, the flux factor (4.87) reads

$$F = 4\left(E_{\vec{p}_2}\, |\vec{p}| + E_{\vec{p}_1}\, |\vec{p}| \right) = 4\, E\, |\vec{p}| \qquad (4.91)$$

and can be expressed in terms of the total energy in the center-of-mass frame.

Let the experiment be such that only particle 1' is detected within a detector with space-opening angle $d\Omega$. The momentum of particle 1' as well as all properties of particle 2' are not detected.

Using $d^3 p_1' = |\vec{p}_1|^2 \, dp_1 \, d\Omega \quad (d=3)$ (4.92)

the cross section (4.89) in this case reads

$$d\sigma = \frac{\lambda^2}{4E|\vec{p}|} \, d\Omega \int \frac{dp_1'}{(2\pi)^3} \frac{|p_1|^2}{2E_{\vec{p}_1}'} \int \frac{d^3 p_2'}{(2\pi)^3} \frac{1}{2E_{\vec{p}_2}'} (2\pi)^4 \delta(E_{\vec{p}_1}' + E_{\vec{p}_2}' - E) \, \delta^{(3)}(\vec{p}_1' + \vec{p}_2')$$

Due to the momentum δ-function, we have $\vec{p}' \equiv \vec{p}_1' = \vec{p}_2'$ and thus also $E_{\vec{p}_1'} = E_{\vec{p}_2'} = \sqrt{\vec{p}'^2 + m^2} \equiv \frac{E'}{2}$, implying

$$d\sigma = \frac{\lambda^2}{4E|\vec{p}|} \frac{d\Omega}{(2\pi)^2} \int dp' \frac{(p')^2}{E'^2} \, \delta(E' - E) \qquad (4.93)$$

We perform the variable substitution

$$p' = \sqrt{\frac{E'^2}{4} - m^2} \Rightarrow p' dp' = \frac{1}{4} E' dE',$$

yielding

$$d\sigma = \frac{\lambda^2}{4E|\vec{p}|} \frac{d\Omega}{(2\pi)^2} \int_0^\infty \frac{dE'}{4E'} \sqrt{\frac{E'^2}{4} - m^2} \, \delta(E' - E)$$

$$= \frac{\lambda^2}{4E|\vec{p}|} \frac{d\Omega}{(2\pi)^2} \frac{1}{4E} \underbrace{\sqrt{\frac{E^2}{4} - m^2}}_{= |\vec{p}|}$$

$$= \frac{\lambda^2}{64\pi^2 E^2} \, d\Omega \qquad (4.94)$$

We finally obtain the differential cross section in the centre-of-mass frame

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{\lambda^2}{64\pi^2 E^2}.} \qquad (4.95)$$