3 Quantization of the free Klein-Gordon field

In the following, we introduce field quantization by means of the standard canonical quantization structures known from quantum mechanics. The free non-interacting Klein-Gordon field is used as the simplest example

3.1 Spectrum of the free Klein-Gordon theory

The "free" Klein-Gordon field is characterized by a potential of the form $U = \frac{1}{2} m^2 \phi^2$, i.e. a potential which is at most quadratical in the fields such that the field equation is linear. (A linear term in $U(\phi)$ would correspond to a source term in the field equations.) The Hamiltonian density then reads

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 . \tag{3.1}$$

As a brief reminder, let us consider $D = 0+1$ dimensional field theory (zero space dimensions)

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2} m^2 \phi^2 \equiv H_{HO} \tag{3.2}$$

which is in fact identical to the Hamilton function of the harmonic oscillator (HO) with $\pi|_{QFT} = p|_{QM}$, $\phi|_{QFT} = x|_{QM}$ and $m^2|_{QFT} = \omega^2|_{QM}$; $1 = m|_{QM}$.

The quantization of the HO can be summarized very briefly: the classical phase space variables ϕ and π, satisfying the algebra

$$\{\phi, \pi\} = 1 \quad , \{\phi, \phi\} = 0 = \{\pi, \pi\} \tag{3.3}$$

with respect to the Poisson brackets are lifted to Hilbert space operators, being self-adjoint and satisfying the commutators

$$[\phi, \pi] = i \quad , [\phi, \phi] = 0 = [\pi, \pi] \tag{3.4}$$

The spectrum of energy eigenvalues can be determined algebraically by introducing ladder operators a and a^+,

$$\phi = \frac{1}{\sqrt{2m}} (a + a^+) \quad , \pi = -i \sqrt{\frac{m}{2}} (a - a^+) \tag{3.5}$$

satisfying

$$[a, a^+] = 1 \qquad [a, a] = 0 = [a^+, a^+] \tag{3.6}$$

The corresponding Hamilton operator can also be expressed in terms of ladder operators,

$$H_{HO} = m \left(a^+ a + \frac{1}{2} \right) \tag{3.7}$$

Non negative definiteness of the number operator $N = a^\dagger a$ implies that there is a ground state that satisfies

$$a\,|0\rangle = 0 \tag{3.8}$$

which at the same time is an eigenstate of the Hamiltonian

$$H_{HO}\,|0\rangle = \frac{1}{2}\,m\,|0\rangle \tag{3.9}$$

Verifying the relations

$$[H_{HO}, a^\dagger] = m\,a^\dagger \;,\; [H_{HO}, a] = -m\,a\,, \tag{3.10}$$

it can straightforwardly be shown that the other eigenstates are generated by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{3.11}$$

with eigenvalues

$$H_{HO}\,|n\rangle = \cancel{m}\,\left(n+\frac{1}{2}\right)|n\rangle\,. \tag{3.12}$$

This fully determines the spectrum of the quantum harmonic oscillator as well as of 0+1 dimensional free Klein-Gordon theory.

Let's go back to D dimensional field theory. As a first step, it is useful to consider the Hamiltonian (3.1) in momentum space

$$H = \int d^dx\, \mathcal{H}(\vec{x}) = \int d^dx\, \frac{1}{2}\left(\pi^2(\vec{x}) + \left(\vec{\nabla}\varphi(\vec{x})\right)^2 + m^2\varphi^2(\vec{x})\right). \tag{3.13}$$

Using

$$\varphi(\vec{x}) = \int \frac{d^dp}{(2\pi)^d}\, e^{i\vec{p}\cdot\vec{x}}\, \varphi(\vec{p})$$
$$\pi(\vec{x}) = \int \frac{d^dp}{(2\pi)^d}\, e^{i\vec{p}\cdot\vec{x}}\, \pi(\vec{p}) \tag{3.14}$$

(we do not use new symbols for the Fourier transforms $\varphi(\vec{p})$, $\pi(\vec{p})$; their meaning should be clear from the context.), we get

$$H = \underbrace{\int d^dx\, \frac{d^dp\, d^dq}{(2\pi)^{2d}}\, e^{i\vec{x}\cdot(\vec{p}+\vec{q})}}_{(2\pi)^d\,\delta^{(d)}(\vec{p}+\vec{q})}\, \frac{1}{2}\left(\pi(\vec{p})\,\pi(\vec{q}) + \varphi(\vec{p})(-\vec{p}\cdot\vec{q})\,\varphi(\vec{q}) + m^2\varphi(\vec{p})\,\varphi(\vec{q})\right)$$

$$= \int \frac{d^dp}{(2\pi)^d}\, \frac{1}{2}\left(\pi(-\vec{p})\,\pi(\vec{p}) + \varphi(-\vec{p})\,\vec{p}^{\,2}\,\varphi(\vec{p}) + m^2\varphi(-\vec{p})\,\varphi(\vec{p})\right). \tag{3.15}$$

Hence, the resulting Hamiltonian density has again the form of a harmonic oscillator

$$\mathcal{H}(\vec{p}) = \frac{1}{2}\left(\pi(-\vec{p})\,\pi(\vec{p}) + \omega_{\vec{p}}^2\,\varphi(-\vec{p})\,\varphi(\vec{p})\right) \tag{3.16}$$

with frequency

$$\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}. \tag{3.17}$$

In the classical case, the fact that $\mathcal{H}(\vec{x})$, $\mathcal{H}(\vec{p})$ are real, as are $\phi(\vec{x}), \pi(\vec{x}) \in \mathbb{R}$, it follows that $\phi(-\vec{p}) = \phi^*(\vec{p})$ and $\pi(-\vec{p}) = \pi^*(\vec{p})$. The reality of $\mathcal{H}(\vec{p}) \in \mathbb{R}$ then becomes obvious since $\pi(-\vec{p})\,\pi(\vec{p}) = \pi^*(\vec{p})\,\pi(\vec{p}) \in \mathbb{R}$, etc.

(In the quantized case, similar consequences follow from hermiticity, e.g. $\phi(-\vec{p}) = \phi^+(\vec{p})$, see below.)

From Eqs. (3.16), (3.17), we observe that we can view the Klein-Gordon field as an ensemble of ∞-many harmonic oscillators which are labeled by a d-dimensional continuous momentum variable $\vec{p}$.

As a straightforward generalization of (3.5), we now define field quantization by writing for each given $\vec{p}$:

$$\begin{aligned} \phi(\vec{p}) &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}} + a^+_{-\vec{p}}\right) \\ \pi(\vec{p}) &= -i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}} - a^+_{-\vec{p}}\right) \end{aligned} \qquad (3.18)$$

The momentum labels are chosen such that the hermiticity constraint, e.g. $\phi(-\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{-\vec{p}} + a^+_{\vec{p}}\right) = \phi^+(\vec{p})$, is evidently satisfied.

In analogy to the quantum mechanical HO, we postulate

$$\left[a_{\vec{p}} \,, a^{+}_{\vec{p}'} \right] = \underset{\substack{\uparrow \\ \text{due to our Fourier conventions}}}{(2\pi)^d} \delta^{(d)} (\vec{p} - \vec{p}') \tag{3.19}$$

and $$\left[a_{\vec{p}} \,, a_{\vec{p}'} \right] = 0 = \left[a^{+}_{\vec{p}} \,, a^{+}_{\vec{p}'} \right] . \tag{3.20}$$

Going back to coordinate space (3.14), the field operators then read

$$\phi(\vec{x}) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \Big(a_{\vec{p}} \, e^{i\vec{p}\cdot\vec{x}} + \underbrace{a^{+}_{-\vec{p}} \, e^{i\vec{p}\cdot\vec{x}}}_{\vec{p} \to -\vec{p} \ \longrightarrow \ a^{+}_{\vec{p}} \, e^{-i\vec{p}\cdot\vec{x}}} \Big) \tag{3.21}$$

$$\pi(\vec{x}) = \int \frac{d^d p}{(2\pi)^d} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} \, e^{i\vec{p}\cdot\vec{x}} - a^{+}_{\vec{p}} \, e^{-i\vec{p}\cdot\vec{x}} \right) .$$

It is straightforward to determine the resulting commutation relations in coordinate space:

$$\underline{\underline{\left[\phi(\vec{x}) \,, \pi(\vec{y}) \right]}} = \int \frac{d^d p \, d^d p'}{(2\pi)^{2d}} \frac{-i}{2} \sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}} \Big(\left[a^{+}_{\vec{p}} \,, a_{\vec{p}'} \right] e^{-i\vec{p}\cdot\vec{x} + i\vec{p}'\cdot\vec{y}} - \left[a_{\vec{p}} \,, a^{+}_{\vec{p}'} \right] e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{y}} \Big)$$

$$\overset{(3.19)}{=} \int \frac{d^d p}{(2\pi)^d} \left(+\frac{i}{2} \right) \left(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{i\vec{p}(\vec{x}-\vec{y})} \right)$$

$$= \underline{\underline{i \, \delta^{(d)} (\vec{x} - \vec{y})}} \tag{3.22}$$

This is the QFT analogue to $[q_i, p_j] = i\delta_{ij}$.

Similarly, we can show immediately that $[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$.

$$(3.23)$$

In the present construction, field quantization is imposed by lifting field amplitudes and conjugate canonical momenta to operators, and using the mapping to an ensemble of harmonic oscillators. Alternatively to (3.19), (3.20), we could have equally well postulated Eq. (3.22) & (3.23). Analogously to quantum mechanics, we have performed the quantization in the Schrödinger picture, where $\phi(\vec{x})$ and $\pi(\vec{x})$ are considered at a given fixed time. Of course, we can straightforwardly switch to the Heisenberg picture. In that case, it is important to note that (3.22) and (3.23) denote "equal-time commutation relations":

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(d)}(\vec{x} - \vec{y}) \quad \text{, etc.} \tag{3.24}$$

In order to determine the spectrum of the free Klein-Gordon theory, we express the Hamilton-operator in terms of ladder operators.

Insertion of (3.18) into (3.16) yields

$$H = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \Big(-\frac{\omega_{\vec p}}{2} \left(a_{-\vec p} - a_{\vec p}^{+}\right)\left(a_{\vec p} - a_{-\vec p}^{+}\right) + \frac{\omega_{\vec p}}{2} \left(a_{-\vec p} + a_{\vec p}^{+}\right)\left(a_{\vec p} + a_{-\vec p}^{+}\right)\Big)$$

$$= \int \frac{d^d \vec p}{(2\pi)^d} \frac{1}{4} \omega_{\vec p} \left(a_{-\vec p} a_{-\vec p}^{+} + a_{\vec p}^{+} a_{\vec p} + a_{-\vec p} a_{-\vec p}^{+} + a_{\vec p}^{+} a_{\vec p} \right)$$

$$= \int \frac{d^d \vec p}{(2\pi)^d} \frac{1}{2} \omega_p \Big(a_{\vec p}^{+} a_{\vec p} + \underbrace{a_{\vec p} a_{\vec p}^{+}}_{= [a_{\vec p}, a_{\vec p}^{+}] + a_{\vec p}^{+} a_{\vec p}} \Big)$$

$$= \int \frac{d^d p}{(2\pi)^d} \omega_{\vec p} \left(a_{\vec p}^{+} a_{\vec p} + \frac{1}{2} [a_{\vec p}, a_{\vec p}^{+}] \right) . \qquad (3.25)$$

The commutator piece naively is $\sim \delta^{(d)}(0)$ and thus an ill-defined divergent number. The occurrence of this divergence could have been expected, as each oscillator in the Klein-Gordon field consisting of an infinite ensemble of oscillators contributes an energy of $\frac{\omega_{\vec p}}{2}$ (ground state energy). However, since absolute energies are not measurable in particle physics experiments, we can eliminate this term by a correspondingly infinite shift of the zero point energy. This does not modify any experimental result that is sensitive to energy differences (see, however, the discussion of the "cosmological constant problem" below.)

After this subtraction, we obtain

$$H = \int \frac{d^d p}{(2\pi)^d} \, \omega_{\vec p} \, a^+_{\vec p} a_{\vec p} \quad . \tag{3.26}$$

In analogy to the quantum HO, it is straightforward to verify that

$$\begin{aligned}
\left[H , a^+_{\vec p} \right] &= \int \frac{d^d p'}{(2\pi)^d} \, \omega_{\vec p'} \underbrace{\left[a^+_{\vec p'} a_{\vec p'} , a^+_{\vec p} \right]}_{} \\
&= a^+_{\vec p'} \underbrace{\left[a_{\vec p'} , a^+_{\vec p} \right]}_{= (2\pi)^d \delta(\vec p' - \vec p)} + \underbrace{\left[a^+_{\vec p'} , a^+_{\vec p} \right]}_{=0} a_{\vec p'} \\
&= \omega_{\vec p} \, a^+_{\vec p}
\end{aligned} \tag{3.27}$$

and similarly $\left[H , a_{\vec p} \right] = - \omega_{\vec p} \, a_{\vec p}$

As a consequence, the algebraic properties of the Klein-Gordon ladder operators are identical to those of the quantum HO, such that the spectrum can straightforwardly be computed:

The ground state (vacuum) is given by

$$|0\rangle \quad \text{where} \quad a_{\vec p} |0\rangle = 0 \quad \text{for all } \vec p \tag{3.28}$$

After having subtracted the formally divergent zero-point energy, the energy of the ground state is

$$E_0 |0\rangle = H |0\rangle = \int \frac{d^d p}{(2\pi)^d} \, \omega_{\vec{p}} \, a^\dagger_{\vec{p}} \underbrace{a_{\vec{p}} |0\rangle}_{=0} = \underline{\underline{0}} . \qquad (3.29)$$

All other states can then be created by means of the ladder operators $a^\dagger_{\vec{p}}$ acting on $|0\rangle$. With (3.27), it follows that the state

$$a^\dagger_{\vec{p}} \, a_{\vec{q}}^\dagger \ldots |0\rangle \qquad (3.30)$$

is an eigenstate of the Hamiltonian H with energy

$$E = \omega_{\vec{p}} + \omega_{\vec{q}} + \ldots \qquad (3.31)$$

The complete spectrum ist described by eigenvectors of the type (3.30) (which we will suitably normalize shortly) and the corresponding eigen-energies (3.31).

It is instructive at this point to remember that H corresponds to the Noether charge (0-component of the field momentum 4-vector P^μ) related to the energy momentum tensor of the scalar field. We had in Eq. (2.22) that $P^0 = H$, $P^i = \int d^d x \, \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \, \partial^i \phi$.

So let us also study the spatial components

$$\vec{P}: \quad P^i = \int d^d x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^i \phi \equiv \int d^d x \, \pi(\vec{x}) \underbrace{\partial^i}_{\equiv -\vec{\nabla}_i} \phi(\vec{x}) = -\int d^d x \, \pi \, \vec{\nabla}_i \phi$$

$$\Rightarrow \quad \vec{P} = -\int \frac{d^d p}{(2\pi)^d} \pi(-\vec{p}) \, (\vec{\nabla}\phi)(\vec{p})$$

$$= -(-i) \int \frac{d^d p}{(2\pi)^d} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{-\vec{p}} - a_{\vec{p}}^\dagger \right) \frac{i\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a_{-\vec{p}}^\dagger \right)$$

$$= -\int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \underbrace{\vec{p}}_{\text{odd}} \Big(\underbrace{a_{-\vec{p}} a_{\vec{p}}}_{\substack{\text{even} \\ = a_{\vec{p}} a_{-\vec{p}}}} - \underbrace{a_{\vec{p}}^\dagger a_{-\vec{p}}^\dagger}_{\text{even}} + \underbrace{a_{-\vec{p}} a_{-\vec{p}}^\dagger}_{\substack{= a_{-\vec{p}}^\dagger a_{-\vec{p}} \\ + \underbrace{[a_{-\vec{p}}, a_{-\vec{p}}^\dagger]}_{\sim \delta^{(d)}(0) \text{ divergent but even}}}} - a_{\vec{p}}^\dagger a_{\vec{p}} \Big)$$

$$= \int \frac{d^d p}{(2\pi)^d} \vec{p} \, a_{\vec{p}}^\dagger a_{\vec{p}} \tag{3.32}$$

The total momentum of a field hence arises from the momenta in each harmonic oscillator of the ensemble.

To summarize: the operator $a_{\vec{p}}^\dagger$ creates a state with energy $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ and momentum $\vec{p}$.

(In the same way, $a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger \ldots$ creates a state with energy

$\omega_{\vec{p}} + \omega_{\vec{q}} + \ldots$ and momentum $\vec{p} + \vec{q} + \ldots$.)

Since this state satisfies the relativistic energy-momentum relation of a relativistic point particle, it is justified to identify this state with "a particle". This particle is localized in momentum space, i.e. carries a specific momentum, and an Energy $E_{\vec{p}} = \omega_{\vec{p}}$.

In this way, the somewhat mysteious (or mystified) particle-wave duality of quantum mechanics is put onto a solid ground (or resolved): the more comprehensive picture is neither the particle nor the wave but the (quantized) field: as a field, it naturally entails interference phenomena, its quantized excitations clearly have a particle character.

Let us now introduce a suitable normalization in the "one-particle" sector of the momentum eigenstates:

$$|\vec{p}\rangle := \sqrt{2E_{\vec{p}}}\; a^{\dagger}_{\vec{p}} |0\rangle \qquad (3.33)$$

With this normalization, we have

$$\langle\vec{q}|\vec{p}\rangle = 2\sqrt{E_{\vec{p}}E_{\vec{q}}}\;\langle 0|\underbrace{a_{\vec{q}}\, a^{\dagger}_{\vec{p}}}_{\underbrace{a^{\dagger}_{\vec{p}} a_{\vec{q}}}_{\to 0} + \underbrace{[a_{\vec{q}}, a^{\dagger}_{\vec{p}}]}_{(2\pi)^d \delta^{(d)}(\vec{q}-\vec{p})}}|0\rangle = 2E_{\vec{p}}(2\pi)^d \delta^{(3)}(\vec{p}-\vec{q}). \qquad (3.34)$$

Correspondingly, the normalization occurs in the completeness relation

$$\mathbb{1}_{1\,\text{particle}} = \int \frac{d^d p}{(2\pi)^d} \, |\vec{p}\rangle \frac{1}{2E_{\vec{p}}} \langle \vec{p}| \qquad (3.35)$$

check: $|\vec{q}\rangle = \mathbb{1}_{1\,\text{particle}} |\vec{q}\rangle = \int \frac{d^d \vec{p}}{(2\pi)^d} \, |\vec{p}\rangle \frac{1}{2E_{\vec{p}}} \underbrace{\langle \vec{p}\,|\,\vec{q}\rangle}_{2E_{\vec{p}}\,(2\pi)^d\,\delta^{(d)}(\vec{p}-\vec{q})}$

$= |\vec{q}\rangle \quad \checkmark$

The reason for this normalization is that the d-dimensional integral actually features a Lorentz invariant measure:

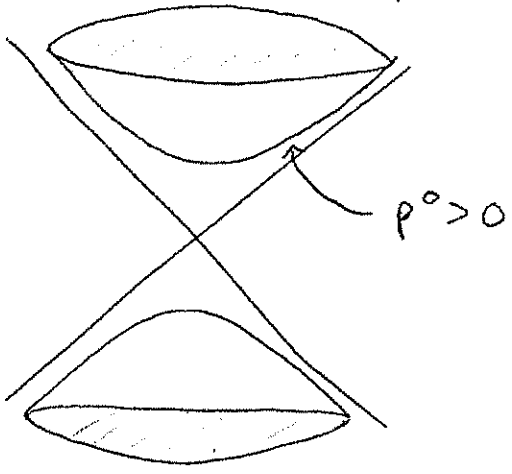
$$D = d+1 \quad \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2\sqrt{\vec{p}^2 + m^2}} = \int \frac{d^D p}{(2\pi)^D} (2\pi)\, \delta(\underbrace{p^2}_{=p_\mu p^\mu} - m^2)\Big|_{p^0>0} \qquad (3.36)$$

(which holds, since $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$ where $f(x_0) = 0$)

This implies that – for any Lorentz invariant function $F(p)$ – the integral $\int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} F(p)$ is also Lorentz invariant.

The meaning of the representation (3.36) is that the integral $\int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}}$ can be understood as an integral over the $p^0>0$ branch of the hyperboloid

$p^2 = m^2$ in Minkowski space (momentum space)



This ~~hyper~~hyperboloid is also called the "mass shell" and characterizes the energy-momentum relation of a classical particle with mass m (and positive energy $p^0>0$).

Using our definition of the momentum eigenstates (3.33), we can also determine how a field operator acts on the vacuum:

$$\phi(\vec{x})|0\rangle = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle . \qquad (3.37)$$

This we can interpret as a superposition of 1-particle momentum states $|\vec{p}\rangle$.

(It is instructive to compare this to coordinate space eigenstates in quantum mechanics: $\langle\vec{x}| = \langle\vec{x}|\mathbb{1}_{QM} = \int d^d p\, \langle\vec{x}|\vec{p}\rangle_{QM\,QM}\langle\vec{p}| = \int \frac{d^d p}{(2\pi)^{d/2}} e^{-i\vec{p}\cdot\vec{x}}\ {}_{QM}\langle\vec{p}|$)

Eq. (3.37) suggests to say that $\phi(\vec{x})$ is an operator that creates "a particle" at point $\vec{x}$.

3.2 Fock space of Klein-Gordon theory

So far, we have mainly considered excitations in the one-particle sector

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\; a^{\dagger}(\vec{p})\,|0\rangle \tag{3.38}$$

This concept can be generalized to n-particle states as already indicated above. We define the n-particle state in momentum space as

$$|\vec{p}_1, \dots, \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1}} \cdot \ldots \cdot \sqrt{2E_{\vec{p}_2}}\; a^{\dagger}(\vec{p}_1) \ldots a^{\dagger}(\vec{p}_2)\,|0\rangle \tag{3.39}$$

(where we sometimes use $a(\vec{p}) \equiv a_{\vec{p}}$ for better readability.)

The inner product of, e.g., two 2-particle states can directly be computed:

$$\langle \vec{p}_1\, \vec{p}_2 \,|\, \vec{q}_1\, \vec{q}_2\rangle = 2E_{\vec{p}_1} 2E_{\vec{p}_2}\, (2\pi)^{2d} \left[\delta^{(d)}(\vec{p}_1-\vec{q}_1)\,\delta^{(d)}(\vec{p}_2-\vec{q}_2) + \delta^{(d)}(\vec{p}_1-\vec{q}_2)\,\delta^{(d)}(\vec{p}_2-\vec{q}_1)\right] \tag{3.40}$$

This generalizes to the n-particle case.

For instance, the completeness relation for n-particle states reads

$$\mathbb{1}_{n\text{-particles}} = \frac{1}{n!} \int \frac{d^d p_1}{(2\pi)^d} \frac{1}{2E_{\vec{p}_1}} \cdots \int \frac{d^d p_n}{(2\pi)^d} \frac{1}{2E_{\vec{p}_n}} \, |\vec{p}_1, \ldots \vec{p}_n\rangle \langle \vec{p}_1, \ldots, \vec{p}_n|$$
$$\equiv |n\rangle\langle n|$$
$$\equiv \mathbb{P}_n \qquad (3.41)$$

Here, the notation should make clear that this is a unity operator on the n-particle subspace, but a null operator on the other subspaces. Hence, we are dealing with a projector onto the n-particle subspace. The completeness relation for the space of all states hence reads

$$\mathbb{1} = \sum_{n=0}^{\infty} \mathbb{P}_n . \qquad (3.42)$$

Mathematically, this implies that the states $|0\rangle, |\vec{p}_1\rangle, |\vec{p}_1 \vec{p}_2\rangle, \ldots$ form an orthogonal complete basis of the Hilbert space

$$\mathbb{H}_F = \bigoplus_{n=0}^{\infty} \mathbb{H}_n , \qquad (3.43)$$

which is a direct sum of the n-particle Hilbert spaces. This space is called <u>Fock space</u>.

The Fock space formulation allows to make direct contact with the Schrödinger wave function formulation of many-body states. We can represent a generic state vector $|\Psi\rangle \in \mathbb{H}_F$ as

$$|\Psi\rangle = \mathbb{1}|\Psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^d p_1}{(2\pi)^d} \frac{1}{2E_{\vec{p}_1}} \cdots \int \frac{d^d p_n}{(2\pi)^d} \frac{1}{2E_{\vec{p}_n}} \Psi(\vec{p}_1, \dots, \vec{p}_n) |\vec{p}_1, \dots, \vec{p}_n\rangle \tag{3.44}$$

where $$\Psi(\vec{p}_1, \dots \vec{p}_n) \equiv \langle \vec{p}_1, \dots, \vec{p}_n | \Psi \rangle \equiv \langle n | \Psi \rangle \equiv \Psi_n \tag{3.45}$$

For instance, applying a field operator on the vacuum as in (3.37), we have

$$|\Psi_1(\vec{x})\rangle := \phi(\vec{x})|0\rangle \overset{(3.37)}{=} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}} \, |\vec{p}\rangle, \tag{3.46}$$

and thus read off

$$\Psi_1(\vec{x};\vec{p}_1) = \langle \vec{p} | \Psi_1(\vec{x})\rangle = e^{-i\vec{p}\cdot\vec{x}}.$$

The Fock space formulation also allows for an instructive first glance at the interacting theory. For this, consider a Hamiltonian of the form

$$H = H_0 + V \tag{3.47}$$

where, e.g.,

$$H_0 = \int d^d\vec{x} \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \tag{3.48}$$

and $$V = \int d^dx \; U(\phi) \; , \quad U(\phi) \overset{\text{e.g.}}{=} \frac{\lambda}{4!}\phi^4 + \ldots$$

The theory would be solved if we determined the spectrum

$$(H_0 + V)\,|\Psi\rangle = E\,|\Psi\rangle \, . \tag{3.49}$$

The projection onto the n-particle sector yields

$$0 = \langle n | E - H_0 - V | \Psi\rangle = (E - E_n)\,\Psi_n - \sum_{k=0}^{\infty} \langle n|V|k\rangle\langle k|\Psi\rangle , \tag{3.50}$$

where
$$E_n = E_{\vec{p}_1} + \ldots + E_{\vec{p}_n} \quad \text{(free particle energies)}$$

$$\overset{E \neq E_n}{\Longrightarrow} \qquad \Psi_n = \sum_{k=0}^{\infty} \frac{\langle n|V|k\rangle}{E - E_n}\,\Psi_k \tag{3.51}$$

This equation shows that the interaction couples the n-particle sector $\mathbb{H}_n$ in general with many if not all other k-particle sectors. Taking into account that

$$|k\rangle\langle k| = \frac{1}{k!}\int \frac{d^dp_1}{(2\pi)^d}\frac{1}{2E_{\vec{p}_1}} \cdots \int \frac{d^dp_k}{(2\pi)^d}\frac{1}{2E_{\vec{p}_k}}\; |\vec{p}_1 \ldots \vec{p}_k\rangle\langle \vec{p}_1 \ldots \vec{p}_k| ,$$

we see that (3.51) corresponds to a system of infinitely many coupled integral equations for the wave functions $\Psi_n(\vec{p}_1, \ldots, \vec{p}_n)$. It is evident that exact solutions for $V \neq 0$ can only be expected in rather exceptional cases.

For $V = 0$, the integral equations decouple completely, such that we have been able to solve the free theory.

It is also interesting to compare to 1-particle QM in a potential. Then we have

$$\Psi_1(\vec{p}) \equiv {}_{QM}\langle \vec{p} \,|\, \Psi \rangle = \int \frac{d^d q}{(2\pi)^d} \, \frac{{}_{QM}\langle \vec{p} \,|\, V \,|\, q \rangle_{QM}}{E - \vec{p}^{\,2}/2m} \, \underbrace{{}_{QM}\langle \vec{q} \,|\, \Psi_1 \rangle_{QM}}_{\Psi_1(\vec{q})} \qquad (3.52)$$

In this case, we obviously have to solve only one integral equation. This illustrates the increase in complexity when going from the free theory or from quantum mechanics to quantum field theory.

Interlude: Cosmological Constant

Mass and energy are sources of the gravitational field, hence not only energy differences but also absolute energy densities become measurable once the coupling to gravity is taken into account. Here, we will not discuss the consequences for gravity, in particular cosmology, but elucidate a potential problem (or rather inconvenience) of the zero-point energies potentially contributing to the cosmological constant. The following discussion will be purely heuristic.

Let us rewrite the zero-point energy formally as

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^d \, \delta^{(d)}(p-p') = \int d^d x \; e^{i\vec{x}\cdot(\vec{p}-\vec{p}')} \tag{I.1}$$

$$\Rightarrow \quad H_0 \text{ "="} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{2} \omega_{\vec{p}} \, [a_{\vec{p}}, a^\dagger_{\vec{p}}]$$

$$\text{"="} \int d^d x \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{2} \omega_{\vec{p}} \tag{I.2}$$

Hence, we find a homogeneous energy density

$$\mathcal{E} = \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{2} \omega_{\vec{p}} \tag{I.3}$$

This density diverges for large momenta. From a physical perspective, we do not have any experimental information

about arbitrarily high energy or momentum scales.
Laboratory experiments go up to a few TeV at the moment. From high energy cosmic rays, we have measured a few events in excess of 10^{20} eV.
Hence it is legitimate to assume that our simple QFT description of nature may break down beyond a scale $p \geq \Lambda$, where a new theory sets in in which a finite value of ε can be computed. Λ is often called an ultraviolet (UV) cut off. It is technically a break-down scale of QFT, or a scale where QFT needs to be replaced (perhaps) by something else. Still the QFT description should be acceptable for all momenta $p \leq \Lambda$.
Hence, we can estimate the QFT contribution to the homogeneous energy density:

$$\begin{matrix} d=3 \\ D=3+1 \end{matrix} \qquad \varepsilon \sim \int_0^\Lambda p^2 dp \, \sqrt{p^2+m^2} \overset{\Lambda \gg m}{\sim} \int_0^\Lambda p^3 dp \sim \Lambda^4 . \qquad (\text{I}.4)$$

where we ignore factors of $\mathcal{O}(1 \cdot 10)$.
A possible scale of "new physics" is the Planck scale $\Lambda \sim 10^{19}$ GeV, where gravity is of a size comparable to the other interactions. If QFT in this simple form holds

up to the Planck scale, we get a contribution to the homogeneous energy density from the fluctuations of the quantum fields of the order of

$$\varepsilon \sim 10^{76}\,(\mathrm{GeV})^4 \qquad (\mathrm{I}.5)$$

In addition to this contribution, there can be another constant contribution ε_0, corresponding to the initial condition of the theory before we started integrating out the quantum fluctuations. The total energy density then is

$$\Lambda_{cc} := \varepsilon_0 + \varepsilon \qquad (\mathrm{I}.6)$$

(Λ_{cc} should not be confused with the UV cutoff Λ).

In fact, astrophysical observations of supernova as well as the cosmic microwave background indicate that Λ_{cc} has a finite value:

$$\Lambda_{cc} \simeq (2\,\mathrm{meV})^4 \simeq 10^{-47}\,(\mathrm{GeV}^4) \qquad (\mathrm{I}.7)$$

For (I.7) and (I.5) to be compatible with each other, the contributions of ε and ε_0 in (I.6) have to cancel one another with an accuracy of $1:10^{123}$.

This outrageously precise fine-tuning of ε_0 which seems necessary to reconcile QFT with the cosmological observation is called the "cosmological constant problem", which is presented here in its simplest and somewhat naive fashion.
Characteristic for any presentation of this problem, however, is that the c.c. problem is not a true inconsistency of the combination of QFT and gravity, but merely an inconvenience. Though it seems that such a necessary fine-tuned choice is completely unnatural, this choice is not impossible. There exists a suitable number ε_0 which is compatible with the cosmological observations. From the perspective of a natural scientist, this is all we can ask for. Of course, we may hope for a more natural perhaps dynamical explanation or mechanism. But there is no necessity for such thing to exist.

3.3 Antiparticles & Causality

The real scalar field does not carry a charge corresponding to an internal symmetry as there is no continuous symmetry giving rise to a Noether charge. Hence there is no concept of particles and antiparticles. (Or rather particles and antiparticles are identical.) The simplest example with particles and antiparticles is the complex scalar field featuring an internal $U(1)$ symmetry. The quantization will be performed in detail in the exercises. Let us summarize the essentials here in brief:

$$\varphi(\vec{x}) = \frac{1}{\sqrt{2}} \left(\varphi_1(\vec{x}) + i\, \varphi_2(\vec{x}) \right) \quad , \; \varphi_1, \varphi_2 \in \mathbb{R} \qquad (3.53)$$

The quantized field operator reads in terms of ladder operators

$$\varphi(\vec{x}) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right], \qquad (3.54)$$

where

$$a(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) + i\, a_2(\vec{p}) \right) \; , \; b(\vec{p}) = \frac{1}{\sqrt{2}} \left(a_1(\vec{p}) - i\, a_2(\vec{p}) \right) \qquad (3.55)$$

are combinations of the ladder operators for the real component fields φ_1, φ_2. a and b satisfy two independent ladder operator algebras. The Hamilton operator reads

$$H = \int d^dx \left(\pi^\dagger \pi + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + m^2 \phi^\dagger \phi \right) \tag{3.56}$$
$$= \int \frac{d^dp}{(2\pi)^d} E_{\vec{p}} \left[a^\dagger(\vec{p})\, a(\vec{p}) + b^\dagger(\vec{p})\, b(\vec{p}) \right]$$

Below, we interpret $a^\dagger$ and $b^\dagger$ as creation operators of a particle and of an antiparticle. Here, we see that both excitations contribute positively to the Hamiltonian.

For the Noether charge corresponding to $U(1)$ phase rotations, we get

$$Q = i \int d^dx \left[\phi^\dagger \partial^0 \phi - \phi \partial^0 \phi^\dagger \right]$$
$$= \int \frac{d^dp}{(2\pi)^d} \left(a^\dagger(\vec{p})\, a(\vec{p}) - b^\dagger(\vec{p})\, b(\vec{p}) \right) \tag{3.57}$$

From charge conservation, we can deduce $[Q, H] = 0$ as H is the generator of time evolution. This can directly be verified using the representations (3.56) & (3.57). It is also straightforward to verify

$$Q\, a^\dagger(\vec{p})\, |0\rangle = [Q, a^\dagger(\vec{p})]\, |0\rangle = a^\dagger(\vec{p})\, |0\rangle,$$
$$Q\, b^\dagger(\vec{p})\, |0\rangle = [Q, b^\dagger(\vec{p})]\, |0\rangle = -b^\dagger(\vec{p})\, |0\rangle. \tag{3.58}$$

This implies that the 1-particle states $a^\dagger|0\rangle$ and $b^\dagger|0\rangle$ are eigenstates of the charge operator with opposite eigenvalues ± 1.

We hence call $a^\dagger(\vec{p})\,|0\rangle$ a 1-particle and $b^\dagger(\vec{p})\,|0\rangle$ a 1-antiparticle-state. In terms of our relativistic normalization, we define

$$
\begin{aligned}
|\vec{p}\,;+\rangle &:= \sqrt{2E_{\vec{p}}}\; a^\dagger(\vec{p})\,|0\rangle \\
|\vec{p}\,;-\rangle &:= \sqrt{2E_{\vec{p}}}\; b^\dagger(\vec{p})\,|0\rangle \,.
\end{aligned}
\tag{3.59}
$$

This generalizes to an n-particle – m-antiparticle state with charge q:

$$
\begin{aligned}
|n;m;q\rangle &= |\vec{p}_1,\dots,\vec{p}_n\,|\,\vec{q}_1\dots\vec{q}_m\,;q\rangle \\
&:= \sqrt{2E_{\vec{p}_1}}\cdot\ldots\sqrt{2E_{\vec{p}_n}}\,\sqrt{2E_{\vec{q}_1}}\cdot\ldots\cdot\sqrt{E_{\vec{q}_m}}\; a^\dagger(\vec{p}_1)\ldots a^\dagger(\vec{p}_n)\,b^\dagger(\vec{q}_1)\ldots b^\dagger(\vec{q}_m)\,|0\rangle
\end{aligned}
\tag{3.60}
$$

which has the property

$$
Q\,|n,m;q\rangle = q\,|n,m;q\rangle \quad , \quad q = n-m \,. \tag{3.61}
$$

At this point, we can introduce an operator for charge conjugation that interchanges particle and anti-particle creation:

$$
\begin{aligned}
C\,a^\dagger\,C &= b^\dagger \quad , \quad C\,b^\dagger\,C = a^\dagger \\
C &= C^\dagger \quad , \quad C^2 = 1
\end{aligned}
\tag{3.62}
$$

such that

$$
C\,|n,m;q\rangle = |m,n;-q\rangle \tag{3.63}
$$

In many theories, charge conjugation is an important discrete symmetry property; in many (but not all) cases, it is an exact symmetry.

In the case of a real scalar field, we have $\phi = \phi^+$ such that we would have to choose $C a^+ C = a^+$ for charge conjugation; this justifies to say that the particle is its own antiparticle.

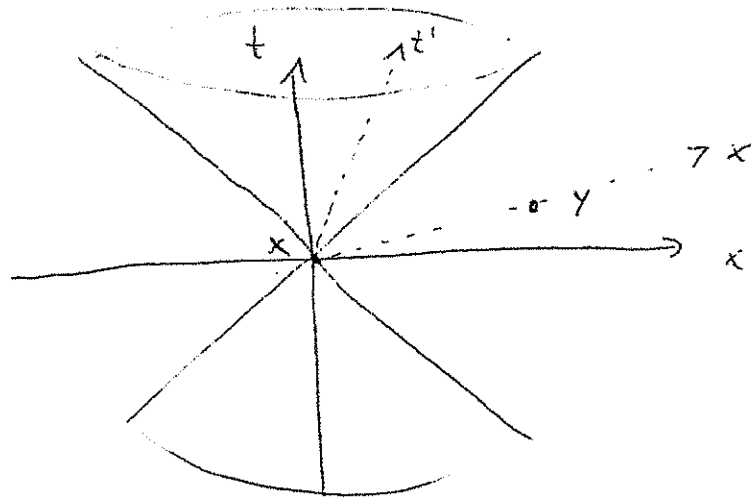
Let us now come to our motivating starting point for the necessity of quantum field theory: <u>causality</u>.

This implies that events in spacetime can only influence other events in their absolute future. The latter is formalized by its forward light cone and its interior.

In turn, spacelike separate points x and y with $(x-y)^2 < 0$ must not exert an influence onto each other.

In a quantum system, where physical observables are represented by operators, this is realized if operators associated to spacelike events x and y commute with one another.

From special relativity, we take over the result that – for two events x and y with spacelike separation – there always exists a coordinate system, in which x and y happen at <u>equal times</u>.



We have emphasized above that canonical quantization is imposed in one reference frame based on equal-time commutator relations, $[\varphi(\vec{x}), \varphi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$, the only non-trivial one being

$$\left[\varphi(x), \pi(y)\right]\Big|_{x^0=y^0=t} = \left[\varphi(\vec{x},t), \pi(\vec{y},t)\right] = \delta^{(3)}(\vec{x}-\vec{y}) \tag{3.64}$$

As the quantization can be performed in any reference frame, we could choose the primed-coordinate reference frame in the sketch above. In this case, it is obvious that for any spacelike separated pair of events,

$$(x-y)^2 < 0 \overset{x^0=y^0=0}{\Longrightarrow} -(\vec{x}-\vec{y})^2 < 0 \Rightarrow \left[\varphi(\vec{x},t), \pi(\vec{y},t)\right] = 0 \tag{3.65}$$

Since Lorentz transformations act linearly on the fields, this statement generalizes to any reference frame for events with spacelike separation. Quantum fields at spacelike separated points therefore do not influence each other. Causality is therefore preserved in QFT on the fundamental level. ("Micro-causality")

In order to elucidate this in more detail, let us consider the complex scalar field in space and time. So far, we have studied the field operators $\phi(\vec{x})$ at a given fixed time. This corresponds to the quantum mechanical Schrödinger picture.

The transition to the Heisenberg picture is performed by including the time evolution in the field operators

$$\phi(x) \equiv \phi(\vec{x},t) := e^{iHt}\,\phi(\vec{x})\,e^{-iHt} \tag{3.66}$$

where we have set the initial fixed time coordinate to $t=0$ for simplicity.

Next, we need some identities for the ladder operators:

$$[H, a_{\vec{p}}] = -E_{\vec{p}}\, a_{\vec{p}} \quad \Rightarrow \quad H a_{\vec{p}} = a_{\vec{p}} H - E_{\vec{p}}\, a_{\vec{p}}$$

$$\Rightarrow \quad H^n a_{\vec{p}} = a_{\vec{p}} \left(H - E_{\vec{p}}\right)^n$$

$$\Rightarrow \quad e^{iHt} a_{\vec{p}} = a_{\vec{p}}\, e^{i(H-E_{\vec{p}})t} \tag{3.67}$$

and similarly $\quad e^{iHt} a^\dagger_{\vec{p}} = a^\dagger_{\vec{p}}\, e^{i(H+E_{\vec{p}})t}$.

With this, we can derive the representation for the space-time dependent field operators, starting with the real scalar field,

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt} = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x} - iE_{\vec{p}}t} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x} + iE_{\vec{p}}t} \right)$$

$$= \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-i\bar{p}x} + a^\dagger_{\vec{p}}\, e^{i\bar{p}x} \right), \tag{3.68}$$

where $\bar{p}x = \bar{p}_\mu x^\mu = E_{\vec{p}}t - \vec{p}.\vec{x}$. Here, $\bar{p}$ denotes a 4-momentum which is on the $p^0 > 0$ mass shell discussed above. It is thus an "on-shell" four momentum.

Similar considerations apply to the complex scalar field,

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-i\bar{p}x} + b^\dagger_{\vec{p}}\, e^{i\bar{p}x} \right). \qquad (3.69)$$

We can now study the commutator beyond the equal-time prescription; a commutator of particular interest is

$$\left[\phi(x), \phi^\dagger(y) \right] = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\sqrt{2E_{\vec{q}}}} \left\{ \left[a_{\vec{p}}, a^\dagger_{\vec{q}} \right] e^{-i\bar{p}x + i\bar{q}y} + \left[b^\dagger_{\vec{p}}, b_{\vec{q}} \right] e^{i\bar{p}x - i\bar{q}y} \right\}$$

$$= \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}} \left(e^{-i\bar{p}(x-y)} - e^{i\bar{p}(x-y)} \right)$$

$$=: \quad i\Delta^+(x-y) - i\Delta^-(x-y), \qquad (3.70)$$

where we have defined the positive and negative energy/frequency parts of the commutator,

$$i\Delta^\pm(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_{\vec{p}}}\, e^{\mp i\bar{p}x}, \qquad (3.71)$$

which are related to the a and b modes, i.e. to the particles and anti-particles.

Using a rewriting of the integration measure similar to (3.36), we can put $\Delta^{\pm}$ into a manifestly Lorentz-invariant form,

$$i\,\Delta^{\pm}(x) = \int \frac{d^D p}{(2\pi)^D}\,(2\pi)\,\Theta(\pm p^0)\,\delta(p^2-m^2)\,e^{-ipx}, \qquad (3.72)$$

where Θ denotes the Heaviside function such that

$p^0\,\Theta(p^0)$ picks out $\bar{p}^0$ and $p^0\,\Theta(-p^0)$ picks out $-\bar{p}^0$.

$$\Rightarrow \quad \Delta^{+}(x) = \Delta^{-}(-x).$$

Using $\Theta(p^0) - \Theta(-p^0) = \text{sign}(p^0)$, the commutator also reads

$$\underline{\underline{i\,\Delta(x-y)}} := \left[\phi(x), \phi^{\dagger}(y)\right] = \int \frac{d^D p}{(2\pi)^D}\,2\pi\,\text{sign}(p^0)\,\delta(p^2-m^2)\,e^{-ip(x-y)}$$

$$\equiv i\left(\Delta^{+}(x-y) - \Delta^{-}(x-y)\right) \qquad (3.73)$$

Here, $\Delta(x-y)$ is also called Schwinger function or Pauli-Jordan function.

The commutator has the following important properties:

- Δ is invariant under (proper) Lorentz transformations
- Δ is odd $(\Delta(x) = -\Delta(-x))$, since it is a commutator
- Δ satisfies the Klein-Gordon equation (since both $\phi(x)$ and $\phi^{\dagger}(y)$ satisfy the K-G eq.).

- Δ is <u>causal</u>, i.e. $\Delta(x) = 0$ for $x^2 < 0$ spacelike (3.74)

Proof: $\Delta(x) = \Delta^+(x) - \Delta^-(x) = \Delta^+(x) - \Delta^+(-x)$

The RHS is a Lorentz-invariant decomposition, as $\Delta^\pm$ are Lorentz invariant.

For any $x^2 < 0$, there is always a Lorentz transformation that transforms $x \to -x$.

(NB. This is given by a boost that transforms $x^\mu \to (0, \vec{x})$, a spatial rotation $\vec{x} \to -\vec{x}$, and the inverse Lorentz boost $\Rightarrow$ $x^\mu \to -x^\mu$). This concludes the proof.

Since Δ^+ and Δ^- are associated with particles and antiparticles, causality in QFT is a consequence of cancelations between particles and anti-particles

3.4 The Feynman propagator

Though $\Delta(x)$ solves the Klein-Gordon equation and has "propagator-like" properties, it is not yet the propagator which we will need in order to describe the propagation of particles including "virtual" processes corresponding to quantum fluctuations. For this, we search for further Lorentz-invariant functions which solve the Klein-Gordon equation or its corresponding Green's function equation. For this, let us start from a representation of Dirac's δ function:

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \operatorname{Im} \frac{1}{x-i\varepsilon} = -\lim_{\varepsilon\to 0} \frac{1}{2\pi} \operatorname{Im} \frac{1}{x+i\varepsilon} \qquad (3.75)$$

Applying this to the mass shell δ function encountered above, we get

$$\delta(p^2-m^2) = \lim_{\varepsilon\to 0} \frac{i}{2\pi}\left(\frac{1}{p^2-m^2+i\varepsilon} - \frac{1}{p^2-m^2-i\varepsilon}\right)$$

$$\equiv \frac{i}{2\pi}\left(\Delta_F(p) - \Delta_F^*(p)\right) = -\frac{1}{\pi} \operatorname{Im} \Delta_F(p) . \qquad (3.76)$$

Here, we have defined a new invariant function

$$\underline{\underline{\Delta_F(p) = \frac{1}{p^2-m^2+i\varepsilon}}} \Rightarrow \underline{\underline{\Delta_F(x) = \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2-m^2+i\varepsilon}}} e^{-ipx} \qquad (3.77)$$

the <u>Feynman</u> <u>propagator</u>.

Here and in the following, the limit $\varepsilon \to 0$ is always implicitly understood.

Applying the Klein-Gordon operator to $\Delta_F(x)$ yields

$$\underline{(\partial^2 + m^2)\,\Delta_F(x)} = \int \frac{d^D p}{(2\pi)^D} \frac{-p^2+m^2}{p^2-m^2+i\varepsilon} e^{-ipx} = \underline{-\delta^{(D)}(x)}, \quad (3.78)$$

i.e. Δ_F is a Green's function of the Klein-Gordon equation. It can be used to solve the Klein-Gordon equation in the presence of a source:

$$(\partial^2+m^2)\,\phi = j(x) \quad \Rightarrow \quad \phi(x) = -\int d^D y\, \Delta_F(x-y)\, j(y) \qquad (3.79)$$

(NB: this solution is not unique; we can add arbitrary solutions of the homogeneous equation. The solution becomes unique, once we add suitable boundary or initial conditions.)

Equation (3.78) becomes fully algebraic in momentum space

$$(p^2-m^2)\,\Delta_F(p) = 1 = (p^2-m^2)\,\Delta_F^*(p) \qquad (3.80)$$

In momentum space, there is a simple relation between Δ_F and Δ:

$$\Delta(p) = -i2\pi\, \mathrm{sign}(p^0)\, \delta(p^2-m^2) = 2i\, \mathrm{sign}(p^0)\, \mathrm{Im}\, \Delta_F(p). \qquad (3.81)$$

There is also an interesting relation between Δ_F and $\Delta^\pm$ in coordinate space:

$$i\Delta_F(x) = \Theta(x^0)\, i\Delta^+(x) + \Theta(-x^0)\, i\Delta^-(x) \qquad (3.82)$$

as will be discussed in the exercises.

Recalling that $\Delta(x)$ was defined as a commutator, but actually is a number, we can understand $\Delta(x)$ as well as $\Delta^{\pm}$ as expectation values of field operators with respect to the vacuum, e.g.

$$i\,\Delta(x) = \langle 0|\,[\,\phi(x), \phi^{+}(0)\,]\,|0\rangle = \langle 0|\phi(x)\phi^{+}(0)|0\rangle - \langle 0|\phi^{+}(0)\,\phi(x)|0\rangle \qquad (3.83)$$

The corresponding representation of Δ_F is particularly interesting:

$$\begin{aligned} i\,\Delta_F(x) &= \Theta(x^0)\,\langle 0|\,\phi(x)\,\phi^{+}(0)\,|0\rangle + \Theta(-x^0)\,\langle 0|\,\phi^{+}(0)\,\phi(x)|0\rangle \\ &\equiv \langle 0|\,T\;\phi(x)\,\phi^{+}(0)\;|0\rangle . \end{aligned} \qquad (3.84)$$

Here, we have introduced the time-ordered product

$$\begin{aligned} T\,\phi(x)\,\phi(y) &= \Theta(x^0-y^0)\,\phi(x)\,\phi^{+}(y) + \Theta(y^0-x^0)\,\phi^{+}(y)\,\phi(x) \\ &= \begin{cases} \phi(x)\,\phi^{+}(y) & \text{for } x^0 > y^0 \\ \phi^{+}(y)\,\phi(x) & \text{for } x^0 < y^0 \end{cases} \end{aligned} \qquad (3.85)$$

Time-ordering hence is defined such that the time arguments of the operators increase from right to left.

There is a common (sometimes confusing) heuristic interpretation of (3.82): particles with positive frequencies in Δ^{+} contribute to the propagation forward in time, whereas anti-particles corresponding to negative frequencies propagate "backwards" in time. Read together, we have

$$\Theta(-x^0)\ e^{+iE_p x^0} \equiv \Theta(-x^0)\ e^{-iE_{\vec{p}}|x^0|} \qquad (3.86)$$

such that the negative frequency branch propagating backwards in time on the LHS can ~~be~~ re-interpreted as the anti-particle with positive frequency/energy propagating with $|x^0|$ forward in time.

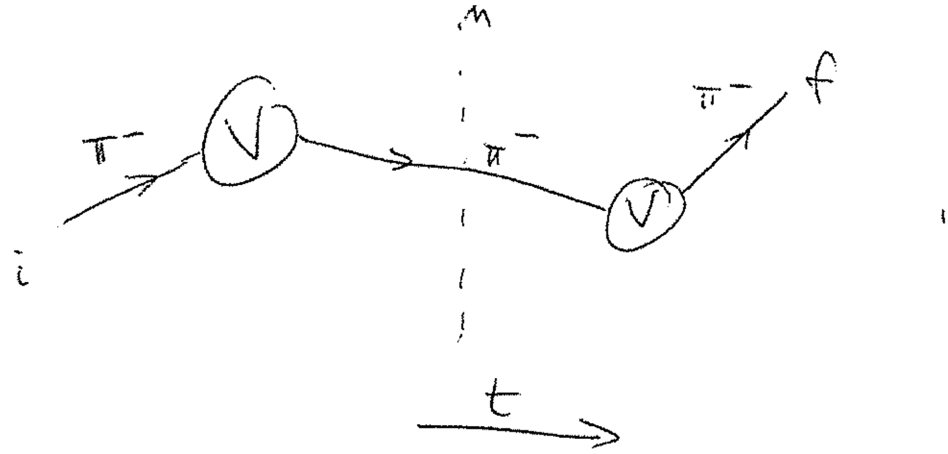
(This interpretation is confusing, because we have shown from the very beginning that both particles and anti-particles contribute positively to the Hamiltonian and thus to the energy spectrum.)

This interpretation can heuristically be illustrated with the following example: the complex scalar field can be considered as a simple model for a charged pion, say π^-. The pion can elastically scatter off a nuclear potential V. To 2nd order in perturbation theory, the quantum mechanical scattering amplitude has the form

$$\underset{\substack{\uparrow\\ \text{final}}}{\mathcal{M}_f}\!\underset{\substack{\uparrow\\ \text{initial}}}{{}_i} \sim \sum_{n\neq i} \frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_i - E_n}, \qquad \underset{\substack{\uparrow\\ \text{elastic}\\ \text{scattering}}}{E_i \equiv E_f}; \quad (3.87)$$

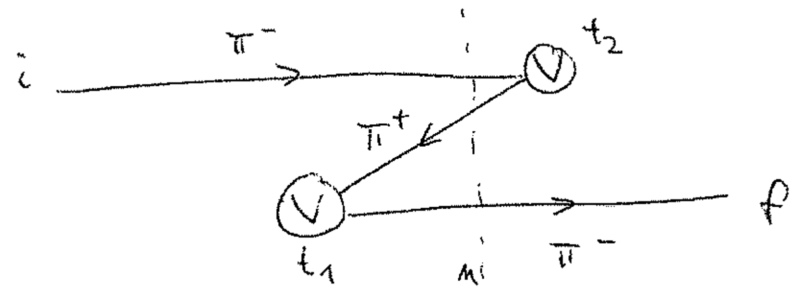
where the sum runs over all possible intermediate states.

In non-relativistic QM, we have the following process:



where the intermediate state is always a pion π^- with energy denominator $\sim \dfrac{1}{E_i - E_\pi}$.

In QFT, the intermediate state propagator receives "backward in time" contributions, which are interpreted as anti-particles with opposite charge π^+:



$$E_n = E_i + E_f + E_\pi$$

This process can be interpreted also in a different way:

At time t_1, a $\pi^+\pi^-$ pair is created, whereas at time $t_2 > t_1$ the π^+ innihilates the initial π^-. The outgoing final state is the newly created π^-.

Let us study the sum of the two processes:

$$\mathcal{M}_{fi} \sim \frac{1}{E_i - E_\pi} + \frac{1}{E_i - (\underbrace{E_i + E_f}_{= E_i} + E_\pi)} = \frac{1}{E_i - E_\pi} - \frac{1}{E_i + E_\pi}$$

$$= \frac{2E_\pi}{E_i^2 - E_\pi^2} = \frac{2E_{\vec{p}}}{p_0^2 - \underbrace{E_{\vec{p}}^2}_{\vec{p}^2 + m^2}} = \frac{2E_{\vec{p}}}{p^2 - m^2} \equiv 2E_{\vec{p}}\,\Delta_F(p) \qquad (3.88)$$

Hence, the contributions from the two time-orderings add up to the Feynman-Propagator Δ_F!

We finish this section by listing further propagators or propagator type quantities which have a relevance for certain types of problems. Let us start with the one studied above,

$$\Delta = -i\,[\varphi, \varphi^+] = \Delta^+ - \Delta^-$$ Commutator

$$\Delta_F = \Theta(x^0)\,\Delta^+ + \Theta(-x^0)\,\Delta^-$$ Feynman propagator (causal)

$$\Delta_D = \Theta(x^0)\,\Delta^- + \Theta(-x^0)\,\Delta^+$$ Dyson propagator (anti-causal)

$$\Delta_R = \Theta(x^0)\,\Delta$$ retarded propagator

$$\Delta_A = -\Theta(x^0)\,\Delta$$ advanced propagator

$$\Delta_1 = -i\,\{\varphi, \varphi^+\} = \Delta^+ + \Delta^-$$ anti commutator