2. Aspects of classical field theory

In classical physics, field theory occurs in the description of continuous media such as liquids or elastic materials as well as – prominently – in electrodynamics. From a more general viewpoint, also quantum mechanical equations for wave functions such as the Schrödinger, Klein-Gordon or Dirac equation can be viewed as classical field theories as they describe the evolution of a "field" $\psi(\vec{x},t)$, being an amplitude defined at every point in space and time. In the following, we summarize a few essentials of classical field theory; for more details, see my lecture notes on "particles and fields"

2.1 Hamilton's principle, Lagrangian dynamics

The Lagrangian formulation of field theory provides us with a simple means to encode all desired symmetries (Lorentz invariance, gauge invariance, internal symmetries) on the level of the action. The action itself is a dimensionless ($\hbar=1$) scalar under all symmetries and hence must be constructable from invariant building blocks. Let us start with a generic field $\Phi(x)$ being an amplitude defined at every space-time point x^μ. Analogously to classical mechanics, we assume that the action can be written in terms of a Lagrangian

$$S[\phi] = \int dt\, L \tag{2.1}$$

Since we are aiming at Lorentz invariant field theories, we assume that the time integration in (2.1) can be completed to a Lorentz invariant space time integration $dt \to d^D x$, where $D = d+1$ counts the number of spacetime dimensions, and d denotes the number of spatial dimensions

(it turns out that some parts of QFT can be formulated in arbitrary D, whereas others require a special dimensionality; it is useful to keep D arbitrary for identifying these points.)

Hence, we write the Lagrange function L in terms of a Lagrange density $\mathcal{L}$ by

$$L = \int d^d x\, \mathcal{L} \tag{2.2}$$

$$\Rightarrow \quad S[\phi] = \int d^D x\, \mathcal{L}\,.$$

Analogously to classical mechanics, we assume that

$\mathcal{L}$ depends on $\phi(x)$ and its derivatives $\partial_\mu \phi(x)$ (higher derivatives can also be treated analogously):

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi) \qquad (2.3)$$

The rules of analytical mechanics, working with a large set of generalized coordinates q_i, $i=1, \ldots, N$, can be generalized to continuous field variables, e.g. by means of functional differentiation

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij} \quad \rightarrow \quad \frac{\delta \phi(x)}{\delta \phi(y)} = \delta^{(D)}(x-y), \qquad (2.4)$$

which satisfies the standard rules of derivatives such as the Leibniz (product) and chain rule, etc.

Imposing Hamilton's principle that classical solutions extremize the action

$$\delta S[\phi] = 0, \qquad (2.5)$$

we find

$$0 = \frac{\delta S}{\delta \phi(x)} = \int d^D y \left[\frac{\partial \mathcal{L}}{\partial \phi(y)} \underbrace{\frac{\delta \phi(y)}{\delta \phi(x)}}_{=\delta^{(D)}(y-x)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(y))} \partial_\mu \underbrace{\frac{\delta \phi(y)}{\delta \phi(x)}}_{\delta^{(D)}(y-x)} \right]$$

$$= \int d^D y \left[\frac{\partial \mathcal{L}}{\partial \phi(y)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(y))} \right] \delta^{(D)}(y-x). \qquad (2.6)$$

In the last step, we integrated by parts and assumed (as usual) that x is not on the boundary (or, as usual, that the fields on the boundary are not varied).

In summary, we obtain

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))} . \qquad (2.7)$$

This is the classical field theory version of the Euler–Lagrange equations, forming the equations of motion of classical field theory.

As a simple example, let us study the dynamics of a real scalar field $\varphi(x) \in \mathbb{R}$ subject to the action

$$S[\varphi] = \int d^D x \left[\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - U(\varphi) \right] . \qquad (2.8)$$

In analogy to classical mechanics, we call the terms with derivatives the kinetic term, and $U(\varphi)$ the potential (note, however, that $U(\varphi)$ is not a potential in spacetime, but in field amplitude space). The equation of motion resulting from (2.7) is:

$$0 = -\frac{\partial U}{\partial \varphi(x)} - \frac{1}{2}\partial_\mu \left(2\partial^\mu \varphi(x)\right)$$

$$\Rightarrow \quad 0 = \underbrace{\partial^2}_{\equiv \Box}\varphi + U'(\varphi) \qquad (2.9)$$

(D'Alembert operator

Specializing to the simple case $U(\varphi) = \frac{1}{2} m^2 \varphi^2$ with some constant parameter m, we get

$$0 = \Box\varphi + m^2 \varphi \qquad (2.10)$$

This is the Klein-Gordon equation for the wave function of a relativistic scalar particle with mass m. In fact, the meaning of m as a mass also becomes visible by studying plane wave type solutions of (2.10) with an ansatz

$$\varphi = e^{-ip_\mu x^\mu}. \qquad (2.11)$$

Eq. (2.11) solves (2.10), provided that

$$-p^2 + m^2 = 0 \qquad \Rightarrow \qquad E_{\vec{p}}^2 = \vec{p}^{\,2} + m^2 \qquad (2.12)$$

with $p^\mu = (E_{\vec{p}}, \vec{p})$. Hence, the plane wave solutions with 4-momentum p^μ satisfy the relativistic dispersion relation of a particle with mass m.

Because of the close analogy to analytical mechanics, many features also translate to field theory; a prominent one being the relation between symmetries and conservation laws culminating in the Noether theorem:

Consider an infinitesimal deformation of the field

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) \,, \qquad (2.13)$$

where $\delta\phi(x)$ parametrizes an infinitesimal continuous deformation.

We call (2.13) a symmetry transformation if the equations of motion are left invariant.

The latter holds if the action is left invariant under (2.13). On the level of the Lagrangian, this means that $\mathcal{L}$ is allowed to change by a total derivative

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} \qquad (2.14)$$

where $\delta\mathcal{L} = \partial_\mu K^\mu$.

Provided that $K^\mu(x)$ vanishes sufficiently fast for $|x| \to \infty$, invariance of the action is guaranteed by Gauß's law.
With these prerequisites, we can formulate the Noether theorem:

Given a continuous infinitesimal symmetry transformation

$$\phi \to \phi + \delta\phi \qquad \text{with } \delta\mathcal{L} = \partial_\mu K^\mu , \tag{2.15}$$

there exists a 4-current called Noether current

$$J^\mu = \pi^\mu \delta\phi - K^\mu \qquad \text{where } \pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{2.16}$$

which is conserved

$$\Rightarrow \quad \partial_\mu J^\mu = 0 , \tag{2.17}$$

i.e. satisfies a continuity equation,
if ϕ satisfies the equation of motion.

Analogously to electrodynamics, we can define a charge (Noether charge),

$$Q = \int d^d x \, J^0 \tag{2.18}$$

which is a constant in time, $\dot{Q} = 0$, if $\vec{J}$ vanishes sufficiently fast for $|\vec{x}| \to \infty$.

The proof and the details of the following examples are dealt with in the exercises.

Examples:

1) Spacetime translations are part of the spacetime symmetries which – together with Lorentz transformations – form the Poincaré group. Translation-invariant systems do not have a distinguished point in spacetime. Hence, translating the fields by a constant spacetime vector a^μ,

$$\phi(x) \to \phi'(x) = \phi(x-a), \qquad (2.19)$$

must leave the equations of motion of such systems invariant. The corresponding Noether current is the energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \qquad (2.20)$$

which is conserved $\quad \partial_\mu T^{\mu\nu} = 0$

(The Noether current is a tensor of second rank, because there is a 4-current for each component of a^ν, $\nu = 0,1,2,3$.)

The corresponding Noether charge is

$$P^\nu := \int d^d x \, T^{0\nu} \qquad (2.21)$$

which is interpreted as the 4-momentum of the field with components

$$P^0 = \int d^d x \, T^{00} = H \quad \left(\begin{array}{l}\text{energy} \\ \hat{=} \text{ Hamiltonian}\end{array}\right)$$

$$P^i = \int d^d x \, \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} \partial^i \varphi \quad (\text{3-momentum}). \tag{2.22}$$

Noether charge conservation implies $\frac{d}{dt} P^\nu = 0$, i.e. momentum conservation of the field.

2) In addition to spacetime symmetries, we may have internal symmetries. A simple example is given by a complex scalar field formed out of two real scalar fields

$$\mathbb{C} \ni \phi = \frac{1}{\sqrt{2}} (\varphi^1 + i \varphi^2) \quad , \ \varphi^1, \varphi^2 \in \mathbb{R} \tag{2.23}$$

Adding up the Lagrangians for φ^1 and φ^2 with equal masses, we obtain

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi . \tag{2.24}$$

This Lagrangian is invariant under phase rotations

$$\phi_{(x)} \to e^{i\alpha} \phi_{(x)} \quad , \ \alpha \in \mathbb{R} \text{ const.}, \tag{2.25}$$

and so is the action.

Apart from irrelevant prefactors, the Noether current is given by

$$J^\mu = -2 \,\mathrm{Im}\left(\phi^* \partial_\mu \phi\right), \quad \partial_\mu J^\mu = 0 . \qquad (2.26)$$

As will be discussed later, the corresponding Noether charge

$$Q = \int d^d x \, J^0 = i \int d^d x \left(\phi^* \partial^0 \phi - \phi \partial^0 \phi^*\right) \qquad (2.27)$$

corresponds to the electric charge of field excitations, such that the complex scalar field can describe charged fields / particles, whereas the real scalar field does not carry electric charge.

2.2 Canonical Hamiltonian dynamics

Again in analogy to classical mechanics, we can formulate classical field theory also in a canonical phase space framework. As we have seen above, such a Hamiltonian formulation requires to choose a reference frame such that we have a notion of a preferred time. We write

$$\underset{\text{Minkowski}}{\mathbb{M}^D} \simeq \mathbb{R} \times \mathbb{R}^d , \quad x^\mu = (t, \vec{x}) \qquad (2.28)$$

Such a decomposition of spacetime is called a foliation and can also be generalized to curved spacetime forming the basis of the Hamiltonian ADM formalism.

Let us again use the example of a scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - U(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - U(\phi) \tag{2.29}$$

where we have used $\partial_\mu = (\partial_t, \vec{\nabla})$, $\partial^\mu = (\partial_t, -\vec{\nabla})$, and $\dot{\phi} = \partial_t \phi$.

We define the canonical momentum density

$$\pi(\vec{x}) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})} = \dot{\phi}(\vec{x}) \tag{2.30}$$

Since all quantities here and in the following are considered at a given Galilei-Newton time, we only display the space dependencies (dictionary: $q_i, \dot{q}_i \Leftrightarrow \phi(\vec{x}), \dot{\phi}(\vec{x})$)

Using (2.30), we can define the Hamiltonian density in terms of the standard Legendre transform

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + U(\phi) \tag{2.31}$$

These terms can be interpreted as the required energy density for the time evolution of the field ($\sim \pi^2$), for the spatial variations ($\sim (\vec{\nabla}\phi)^2$) and for the displacement of the amplitude in a potential $U(\varphi)$.

In complete analogy to classical mechanics, we can now formulate the equations of motion in phase space (φ, π).

For this, we define the Poisson brackets for two arbitrary phase space functionals $A[\varphi, \pi]$, $B[\varphi, \pi]$ bei

$$\{A, B\} = \int d^d z \left(\frac{\delta A}{\delta \varphi(\vec{z})} \frac{\delta B}{\delta \pi(\vec{z})} - \frac{\delta A}{\delta \pi(\vec{z})} \frac{\delta B}{\delta \varphi(\vec{z})} \right) . \qquad (2.32)$$

It is straightforward to verify the fundamental Poisson brackets

$$\{ \varphi(\vec{x}), \pi(\vec{y}) \} = \delta^{(d)}(\vec{x} - \vec{y}) \qquad (2.33)$$

$$\{ \varphi(\vec{x}), \varphi(\vec{y}) \} = 0 = \{ \pi(\vec{x}), \pi(\vec{y}) \} ,$$

treating $\varphi(\vec{x})$ and $\pi(\vec{x})$ as independent field variables.

This is in direct analogy to classical mechanics ($\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = 0 = \{p_i, p_j\}$)

Introducing the Hamilton functional

$$H[\varphi, \pi] = \int d^d y \, \mathcal{H}(\varphi, \pi; y) , \qquad (2.34)$$

the canonical equations of motion read

$$\dot{\phi}(\vec{x}) = \{\phi(\vec{x}), H\} \quad , \quad \dot{\pi}(\vec{x}) = \{\pi(\vec{x}), H\} \qquad (2.35)$$

Applying this to the Hamiltonian in Eq. (2.31),
we rediscover the Klein - Gordon equation with
a non-trivial nonlinear interaction potential

$$0 = \ddot{\phi} - \vec{\nabla}^2 \phi - U'(\phi) \equiv \Box \phi + U'(\phi) . \qquad (2.36)$$

Most importantly, we obtain the fully Lorentz invariant
equation of motion from this non manifestly Lorentz
invariant Hamiltonian formalism.