ERG Flow Equation

– a simple example –

▷ Euclidean anharmonic oscillator

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega^2 x^2 + \frac{\lambda}{24} m^5 x^4$$

▷ Effective average action (truncation)

$$\Gamma_k = \int d\tau \left[\frac{m}{2}\dot{x}^2 + \frac{m}{2}\omega_k^2 x^2 + \frac{\lambda_k}{24} m^5 x^4 + \underline{\underline{E_{Gk}}} \right]$$

▷ IR cutoff

$$R_k = m k^2$$

▷ Flow equation

$$\underline{\underline{\partial_t \Gamma_k}} \equiv k\frac{d}{dk}\Gamma_k = \frac{1}{2}\mathrm{Tr}\,\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k}$$

$$\overset{\text{trunc.}}{\simeq} \frac{1}{2}\mathrm{Tr}\,\frac{2k^2}{-\partial_\tau^2 + \omega_k^2 + k^2 + \frac{\lambda_k}{2} m^4 x^2}$$

▷ Project RHS onto truncation

$$\frac{1}{2}\,\mathrm{Tr}\left[\cdot/_\circ\right] = \mathrm{Tr}\left[\frac{k^2}{-\partial_\tau^2+\omega_k^2+k^2}\right] - \mathrm{Tr}\left[\frac{k^2}{-\partial_\tau^2+\omega_k^2+k^2}\,\frac{\lambda_k}{2}m^4x^2\,\frac{k^2}{-\partial_\tau^2+\omega_k^2+k^2}\right]$$

$$+\ \mathcal{O}(x^4) \qquad \text{(field expansion)}$$

▷ Compare with LHS

$$\partial_t \Gamma_k = \int d\tau \left[\ \underline{\underline{\partial_t E_{Gk}}} + \frac{m}{2}\,\underline{\underline{(\partial_t \omega_k^2)}}\,x^2 + \dots \right]$$

▷ Flows of the "couplings"

$$\frac{d}{dk} E_{Gk} = \frac{1}{2}\,\frac{k}{\sqrt{\omega_k^2+k^2}} - \frac{1}{2}$$

B.C. at $k = \Lambda_{UV}$

$$\frac{d}{dk}\,\omega_k^2 = -\frac{\lambda_k m^3 k}{4(\omega_k^2+k^2)^{3/2}}$$

$$\frac{d}{dk}\,\lambda_k = \frac{9}{8}\,\frac{\lambda_k^2 m^3 k^2}{(\omega_k^2+k^2)^{5/2}}$$

▷ 1st. order approximation: $\lambda_k \to \lambda = \text{const.}$

perturbative expansion:

$$E_G = \lim_{k\to 0} E_{Gk} = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{4!\,\omega^3}\right) - \frac{81}{40}\omega\left(\frac{\lambda}{4!\,\omega^3}\right)^2 + \dots$$

perturbation theory (BENDER & WU)

$$E_G^{PT} = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{4!\,\omega^3}\right) - \frac{105}{40}\omega\left(\frac{\lambda}{4!\,\omega^3}\right)^2 + \dots$$

$\Longrightarrow$ "1 loop" exact , "2 loop" 20% error

▷ strong-coupling limit ($m = 1 = \omega$)

$$E_G = \left(\frac{\lambda}{4!}\right)^{1/3}\left[\alpha_0 + \alpha_1\left(\frac{\lambda}{4!}\right)^{-2/3} + \dots\right]$$

(JANKE & KLEINERT)

$$\alpha_0^{JK} = 0.66798\dots$$

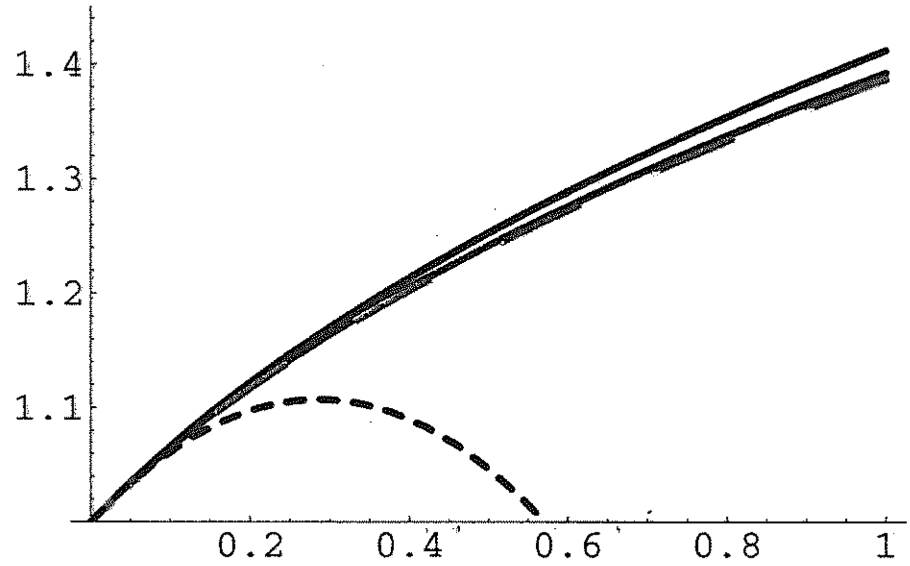
$$\alpha_0^{1st} = 0.69372\dots \qquad \text{(error 4\%)}$$

$$\alpha_0^{2nd} = 0.66007\dots \qquad \text{(error 1\%)}$$

(incl. λ_k)

Anharmonic Oscillator with ERG methods

▷ ground state energie $E_G(\frac{m^5}{4!}\lambda)$ $(m = 1/2,\ \omega = 2)$



——— exact solution

——— 1st order approximation (error $< 1.5\%$)

- - - - 2nd order approximation (incl. running of λ) (error $< 0.4\%$)

- - - - 2nd order perturbation theory