8 Classical field theory for particle physics — an example —

In this course, we have mainly discussed the classical field theory aspects which are relevant for particle physics. This included mainly the aspects of possible degrees of freedom (scalars, spinors, vectors, ...), their symmetries (external spacetime symmetries and internal symmetries) and the construction of interactions on the level of the classical action. However, a thorough discussion of particle physics applications typically involves quantization, as it is the quantized excitations of these fields which are relevant for computing observables. Also, some aspects which could, in principle, be discussed on the classical level ("tree-level processes"), follow much more elegantly within the quantized formalism making it less worthwhile to deal with the classical equations of motion.

Still, the language of classical field theory does become even more useful than the quantum notion of Fock spaces etc., as soon as the corresponding experimental situation involves coherent classical fields. In the following, we want to illustrate this with an example from experimental searches for new particles.

8.1 Photon-axion-conversion

The standard model of particles has various short comings, a prominent one being the rather large number of parameters such as fermion masses which do not seem to follow a natural pattern. Even more serious is the fact that some parameters which, in principle, are allowed to be sizable seem to be zero or at least unnaturally small. Most prominently, there is an angle type of parameter θ (a combination of a QCD parameter and the phase of the determinant of the quark mass matrix) which would physically induce CP violation in the strong interactions. If so, QCD bound states would be expected to show CP-violating properties. An example would be given by an electric dipole moment of the neutron $\vec{d}_n$. Measurements so far have only found an upper bound on a possibly nonzero value: $|d_n| < 3 \cdot 10^{-26}\, e\,\mathrm{cm}$ (2015)

The precise relation between $|d_n|$ and θ is difficult to compute as any bound-state property of QCD from first principles. However, simple estimates translate the value as follows into θ:

given the diameter of the neutron $\sim 10^{-15}$ m and assuming a linear dependence on θ, we may estimate

$$|d_n| \simeq c\,\theta \cdot e\, 10^{-15}\,\mathrm{m} = c\,\theta \cdot 10^{-13}\, e\,\mathrm{cm} \qquad (8.1)$$

where c is a constant to be determined from a full calculation. Generic field theory computations often yield factors inversely proportional to the phase space and thus to the volume of a 4-sphere. So the smallest number, one typically gets is $c \simeq \frac{1}{32\pi^2} \simeq 10^{-3}$

and hence we conclude that $\theta \lesssim 10^{-10}$.

As θ is an angle $\in [0, 2\pi]$, we would naturally expect it to be of $\mathcal{O}(1)$, rendering $\theta = 10^{-10}$ or smaller rather unnatural. This is the "strong-CP" problem.

One possibility to "explain" $\theta \simeq 0$ is to impose a suitable symmetry. This is not completely trivial as θ receives contributions from two different origins (QCD + quark mass matrix).

All requirements are ultimately satisfied by models that lift θ to be the expectation value of a dynamical field that acquires a suitable potential in a dynamical fashion. Ultimately, these models do not only predict (post-dict) $\theta = 0$ but also feature the possibility of having excitations on top of the vacuum, corresponding to a pseudo-scalar field: the "axion".

To cut a long story short: the so-far only valid solution to the strong-CP problem predicts another pseudo-scalar particle ϕ which in many respects behaves like the neutral pion π^0, in particular, it has a nonzero mass m and can couple to two photons $\phi \Leftrightarrow 2\gamma$. The corresponding effective classical field theory is:

$$\mathcal{L}_{axED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} g \, \phi F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (8.2)$$

"Axion Electrodynamics"

which involves a coupling between the axion and the pseudoscalar invariant $-\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \vec{E} \cdot \vec{B}$ (8.3)

This effective field theory involves two parameters m and g. Dimensional analysis reveals that g must have an inverse mass dimension, so g^{-1} corresponds to a mass scale.

In order to solve the strong-CP problem, g and m are related:

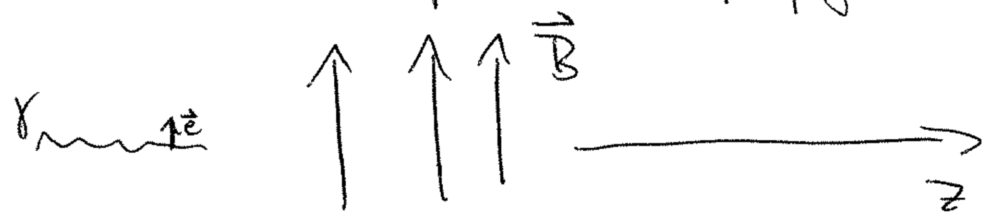
$$\frac{m}{[1\,\mathrm{meV}]} \sim \frac{g}{[10^{13}\,\mathrm{GeV}]^{-1}} \tag{8.4}$$

The fact that we haven't observed any direct signature of the axion puts severe constraints on the coupling. Hence, the axion can be expected to be rather light (if it exists).

Now, the coupling $\sim \phi\, \vec{E}\cdot\vec{B}$ inspires to look at the following process: consider a plain wave with electric field component $\vec{e}$ propagating across a magnetic field $\vec{B}$ with $\vec{e} \parallel \vec{B}$. Then, this interaction allows for a mixing of the plane wave $\vec{e}$ with the axion field ϕ. So, even if we initially start with a pure plane wave, the axion field will acquire a nonzero amplitude after some distance of propagation inside the magnetic field.

A quantitative analysis follows from the field equations. Using a Weyl-Coulomb gauge ($A_0=0$, $\vec{\nabla}\cdot\vec{A}=0$), the plane wave field can be parametrized by a pure vector potential $\vec{a}$, $\vec{e}=-\dot{\vec{a}}$.

Considering only the relevant case, where $\vec{e} \parallel \vec{B}$, with $\vec{B}$ being a constant field pointing perpendicular to the direction of plane wave propagation,



we can write the interaction term as

$$
\begin{aligned}
-\int d^4x \ \frac{1}{4} \ g \ \phi \ F_{\mu\nu}\tilde{F}^{\mu\nu} &= \int d^4x \ g \ \phi \ \vec{E}\cdot\vec{B} \\
&= \int d^4x \ g \ \phi \ e B = \int d^4x \ g \ a \ \dot{\phi} B
\end{aligned}
\qquad (8.5)
$$

where $e=|\vec{e}|$, $B=|\vec{B}|$, $a=|\vec{a}|$.

The interaction term hence contributes to both, the Maxwell as well as the Klein-Gordon equation for $\vec{a}$ and ϕ, respectively

We find

$$\Box \varphi + m^2 \varphi - g e \vec{B} = 0$$
$$\Box a - g \dot{\varphi} B = 0 \qquad (8.6)$$

We are interested in solutions that propagate along the z direction, hence $a = a(z,t)$, $\varphi = \varphi(z,t)$

$$\Rightarrow \Box \rightarrow \partial_t^2 - \partial_z^2 \qquad (8.7)$$

Though both fields a and φ are real, it is useful to formally complexify the fields and perform a Fourier transformation to frequency space :

$$a(z,t) = \int d\omega\, e^{-i\omega t}\, a(\omega, z)$$
$$\varphi(z,t) = -i \int d\omega\, e^{-i\omega t}\, \chi(\omega, z) \qquad (8.8)$$

Then (8.6) turns into equations for the frequency modes $a(\omega,z)$ and $\chi(\omega,z)$:

$$\left(-\omega^2 - \partial_z^2 + m^2\right)\left(-i\chi(\omega,z)\right) - i g \omega\, a(\omega,z) B = 0$$
$$\left(-\omega^2 - \partial_z^2 \right) a(\omega,z) + g\, \omega\, \chi(\omega,z) B = 0 \qquad (8.9)$$

or in matrix notation

$$\left[\mathbb{1}\left(\omega^2 + \partial_z^2\right) - M\right]\begin{pmatrix}\chi \\ a\end{pmatrix} = 0 \qquad (8.10)$$

where
$$M = \begin{pmatrix} +m^2 & g\omega B \\ g\omega B & 0 \end{pmatrix} \qquad (8.11)$$

Assuming a plane wave form in wave number space

$$\{a,\chi\}(\omega,z) = \{a,\chi\}(\omega)\, e^{ikz} \qquad (8.12)$$

leads to the algebraic equation

$$\left(\mathbb{1}\left(\omega^2 - k^2\right) - M\right)\begin{pmatrix}\chi \\ a\end{pmatrix} = 0 \qquad (8.13)$$

Solutions exist if $\det\left(\mathbb{1}(\omega^2 - k^2) - M\right) = 0$

$$\Rightarrow \quad (\omega^2 - k^2 - m^2)(\omega^2 - k^2) = (g\omega B)^2, \qquad (8.14)$$

the roots of which define the dispersion relations

$$k_\pm^2 = \omega^2 - \left(m^2 - (gB)^2\right)\left(\frac{\cos 2\theta \pm 1}{2\cos 2\theta}\right) \qquad (8.15a)$$

where
$$\tan 2\theta = \frac{2\omega g B}{(m^2 - (gB)^2)}. \qquad (8.15b)$$

θ can be interpreted as a mixing angle between axion and photon.

In the limit of vanishing coupling or vanishing magnetic field $gB \to 0$, we have $\theta \to 0$ and hence $k_-^2 = \omega^2$, $k_+^2 = \omega^2 - m^2$.

In this limit, k_- corresponds to the wave number of a free photon, and k_+ to that of the massive axion.

In a real experiment, a fixed scale is set by the frequency ω of the propagating laser, and the wave numbers follow from the dispersion relation.

The general solution of the equations of motion for a propagating mode along the positive z direction reads

$$a(\omega,z) = a^-(\omega)\, e^{ik_-z} + \tan^2\theta\, a^+(\omega)\, e^{ik_+z}$$

$$\chi(\omega,z) = \frac{\omega}{k_-}\tan\theta\, a^-(\omega)\, e^{ik_-z} - \frac{\omega}{k_+}\tan\theta\, a^+(\omega)\, e^{ik_+z} \qquad (8.16)$$

Let us consider a monochromatic wave, $a^-(\omega) = a^+(\omega) = \text{const.}$ for one fixed ω, and an axion mass much smaller than the optical laser frequency $m^2 << \omega^2$. We also confine ourselves to a small mixing angle $\theta << 1$. Then, the induced axion amplitude reads ($a^- = a^+ = a_{IN}$

$$\chi(\omega, z) = a_{IN}\, \theta \left(e^{ik_- z} - e^{ik_+ z} \right) \tag{8.17}$$

where we keep $k_\pm$ in the phases as the wave numbers can be multiplied by large values of z, but approximate $k_\pm \simeq \omega$ in the prefactor.

Now, we use the fact that the classical field equations lead to amplitudes that can be interpreted as quantum mechanical probability amplitudes. Hence, we arrive at the probability that an initial photon amplitude is converted into an axion as a function of the length L of propagation inside B:

$$P(\gamma \to \phi; L) = \frac{|\chi|^2}{|a_{IN}|^2} = |\theta|^2 \left| e^{ik_- z} - e^{ik_+ z} \right|^2 \Big|_{z=L}$$

$$= |\theta|^2 \left(2 - 2\cos\left((k_+ - k_-) L \right) \right) \tag{8.18}$$

In the above-mentioned limits, the occurring quantities yield

$$|\Theta|^2 = \left(\frac{\omega g B}{m^2}\right)^2 ,$$

$$\begin{aligned} k_+ - k_- &= \sqrt{\omega^2 - m^2} - \omega = \omega\left(\sqrt{1-\frac{m^2}{\omega^2}} - 1\right) \\ &\simeq \omega\left(1 - \frac{m^2}{2\omega^2} - 1\right) = \frac{m^2}{2\omega} . \end{aligned} \tag{8.19}$$

Using $2 - 2\cos x = 2(1-\cos x) = 2\ 2\sin^2\frac{x}{2}$,

we get

$$P(\gamma \to \phi ; L) = 4\left(\frac{\omega g B}{m^2}\right)^2 \sin^2\frac{m^2 L}{4\omega} . \tag{8.20}$$

For a given length of the magnetic field, the probability in the small mass limit becomes

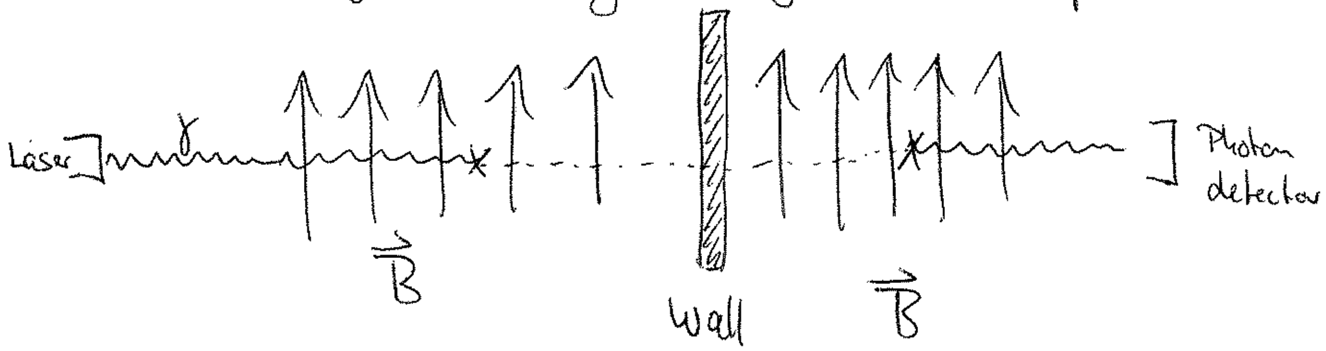
$$P(\gamma \to \phi, L)\Big|_{m\to 0} \simeq \frac{1}{4}(gBL)^2 , \tag{8.21}$$

and thus independent of the mass.

When it comes to discovery experiments, it is not sufficient to convert photons to axions, because

We have no "axionmeter" that could measure the axion amplitude. Instead one uses the following idea (Sikivie '83, van Bibber '87):

a "light-shining-through-wall" experiment



Shine a laser onto a wall and try to observe photons behind the wall. Use a strong magnetic field to convert part of the photon wave (function) into an axion in front of the wall and back into a photon behind the wall. Since the axion is weakly interacting, it can traverse the wall in contrast to photons.

This type of experiment has a couple of attractive features: the interaction regions (size of the $\vec{B}$ field) can be macroscopic (in contrast to small collision points in colliders), and can even be enhanced by the use of cavities.

The number of incoming photons can be very large $\gtrsim 10^{20}$, whereas the detection of a single photon can already constitute a signal of "new physics". Apart from exceedingly small processes from photon-neutrino-pair processes or photon-graviton conversion, the experiment is essentially background free.

A number of experiments (BFRT, BMV, GammeV, LISPS and ALPS) have been performed. The non observation of a signal constitute the currently best laboratory bounds on axions, complementing astrophysical bounds. Currently, a major upgrade of ALPS at DESY is in preparation.

To get a rough estimate on the sensitivity, we first note that the back-conversion $\varphi \to \gamma$ features the same probability as in (8.21). Assuming that the magnetic field behind and in front of the wall have the same length L, we have

$$P(\gamma \to \varphi \to \gamma, L)\big|_{m \to 0} \simeq \frac{1}{16} C \, (gBL)^4 \qquad (8.22)$$

where C is an enhancement factor if cavities are used in order to enhance the photonic input power. For one cavity in front of the wall $C \sim (N/2)^4$ the finesse of the cavity which can be of order $N \sim 10^3$. The current upgrade of ALPS even plans to put a locked cavity behind the wall, which would give an $C \sim \left(\frac{N}{2}\right)^8$ improvement. Converting the units into GeV, we have

$$P \simeq \frac{1}{16} C \left(\frac{g}{[1/\text{GeV}]} \frac{B}{[1\,\text{Tesla}]} \frac{L}{[1\,\text{m}]} \right)^4 \qquad (8.23)$$

with N_γ being the number of incoming photons per second, the number of reconverted photons per second behind the wall is $N_{obs} = N_\gamma \cdot P$.
Having N_γ in excess of 10^{20}, experiments with $C = 1$ already become sensitive to values of

$$g^{-1} \sim 10^5\,\text{GeV} = 10^2\,\text{TeV}$$

for meter size fields and Tesla strong fields.
In fact, ALPS has reached a sensitivity of $g^{-1} \gtrsim 10^7\,\text{GeV}$ which is a factor of 1000 larger than current collider energy scales.