# 6. Quark confinement from Color confinement

The Gribov-Zwanziger Scenario interrelates propagation properties of gluons and ghosts with the positivity properties of the correlation functions. it thus provides a test whether gluons/ghost could be part of the physical Hilbert space of states. As results from lattice measurements, DSEs or functional RG equations indicate, gluons & ghosts violate positivity and thus do not represent physical states. In essence, this signifies color confinement. However, it is not yet a signature for quark confinement. For this, we need equivalent information for the quark propagator (this has, however, an involved history of its own. It is fair to say that there is no consensus about what to look for in the quark propagator regarding confinement; the quark propagator however carries clear signatures of chiral symmetry breaking), or we may look at a quark-confinement criterion directly.

As discussed in the exercises, a confinement criterion can be formulated at finite temperature with the aid of the Polyakov loop

$$L(\vec{x}) = \frac{1}{N_c} \operatorname{tr} \mathcal{P} \exp\left(ig \int_0^\beta dx_0\, A_0(x_0,\vec{x})\right) \tag{6.1}$$

The expectation value of $L$ measures, whether center symmetry is realized by the thermal ensemble of field-configurations, with:

$$\begin{aligned} \langle L \rangle = 0 &: \quad \text{center symmetric (disordered), confining} \\ \langle L \rangle \neq 0 &: \quad \text{center breaking (ordered), deconfined} \end{aligned} \tag{6.2}$$

The relation to confinement becomes clear from the interpretation of $\langle L \rangle \sim e^{-\beta F}$ where $F$ is the free energy of a static quark:

$$\begin{aligned} \langle L \rangle = 0 \quad &\rightarrow \quad F \to \infty \quad \text{, i.e. single static quark cannot be put into the vacuum} \\ \langle L \rangle \neq 0 \quad &\rightarrow \quad F < \infty \quad \text{, single static quark is not energetically forbidden} \end{aligned} \tag{6.3}$$

In the Polyakov gauge,

$$\partial_0 A_0 = 0 \qquad \& \quad A_0 = A_0^{a_0} \tau^{a_0}, \; A_0^{\bar{a}} = 0 \tag{6.4}$$

we have

$$L[A_0](\vec{x}) = \frac{1}{N_c} \operatorname{tr} e^{ig\beta A_0(\vec{x})} . \tag{6.5}$$

It is possible to map the center properties also on the properties of $\langle A_0 \rangle$; e.g. owing to the Jensen inequality, we have

$$\langle L[A_0] \rangle \leq L[\langle A_0 \rangle] \tag{6.6}$$

such that $L[\langle A_0 \rangle] \neq 0$ in the center-broken deconfined phase. In the center-symmetric phase both sides are zero, so the equality in (6.6) holds.

For spatially homogeneous $A_0 = \text{const.}$ it is straight-forward to compute the one-loop effective potential for $A_0$:

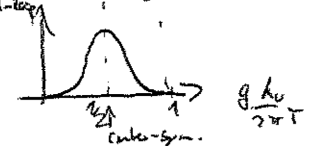
$$V^{1\text{-loop}}(A_0) = \Big\{ \underbrace{\frac{d-1}{2}}_{\perp \text{ gluons}} + \underbrace{\frac{1}{2}}_{\parallel \text{ gluons}} - \underbrace{1}_{\text{ghosts}} \Big\} \frac{1}{\Omega} \operatorname{Tr} \ln \left(-D^2[A_0]\right) \tag{6.7}$$

$$= -\frac{(d-2)\,\Gamma(d/2)}{\pi^{d/2}} T^d \sum_{\ell=1}^{N_c^2-1} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \nu_\ell |\varphi|}{n^d}$$

at finite temperature $T$; here we have $\varphi^a = \dfrac{g A_0^a}{2\pi T}$ (6.8) and $\nu_\ell$ denote the eigenvalues of $\varphi^a T^a$ in the adjoint color space. It is straightforward to verify that the minima of $V^{1\text{-loop}}$ are center-breaking, whereas the confining center-symmetric values of $A_0$ are maxima of the potential $V^{1\text{-loop}}$. (c.f. exercizes)

This is in agreement with the expectation that perturbation theory is blind to confinement. In particular, perturbation theory should be applicable to very high temperatures ($\hat{=}$ asymptotic freedom), where (6.7) then predicts that QCD has a deconfined high-temperature phase.

e.g. SU(2)




It can be verified that the final result (6.7) is dominated by those modes / eigenvalues of $-D^2$ which are near the temperature scale $\sim (2\pi T)^2$.

If we are now interested in the low-temperature region, we should use the low-momentum form of the propagators rather than the perturbative input: the operator $-D^2$ can be viewed as the covariant form of $\left(\frac{1}{p^2}\right)^{-1}$ arising from the gluon & ghost sectors.

In the deep IR, i.e. low-energy/temperature regime, we should rather use

$$\perp \text{ gluons}: \quad \left(\frac{1}{p^2}\right)^{-1} \to \left(p^2\right)^{1+\kappa_A} \to \left(-D^2\right)^{1+\kappa_A}$$

$$\parallel \text{ gluons}: \quad \left(\frac{1}{p^2}\right)^{-1} \to \left(p^2\right)^{1} \to \left(-D^2\right)^{1} \qquad (6.9)$$

$$\text{ghosts}: \quad \left(\frac{1}{p^2}\right)^{-1} \to \left(p^2\right)^{1+\kappa_c} \to \left(-D^2\right)^{1+\kappa_c}$$

Now performing the traces as in (6.7) yields an estimate for the effective $A_0$ potential in the deep IR:

$$V^{IR}(A_0) = \left\{ \frac{(d-1)}{2}(1+\kappa_A) + \frac{1}{2} - (1+\kappa_c) \right\} \frac{1}{\Omega} \operatorname{Tr} \ln\left(-D^2[A_0]\right)$$

$$= \left\{ 1 + \frac{(d-1)\kappa_A - 2\kappa_c}{d-2} \right\} V^{1\text{-loop}}(A_0) \qquad (6.10)$$

Since the confinement is associated with the maxima of $V^{1\text{-loop}}$, we obtain a new quark confinement criterion built from the IR exponents of the color sector:

$$d - 2 + (d-1)\kappa_A - 2\kappa_c < 0 \qquad (6.11)$$

(J. Braun, H.G., J. Pawlowski, 2006)

Specifically in $d=4$, we have

$$2 + 3\kappa_A - 2\kappa_c < 0 \qquad (6.12)$$

As a first check, we observe that (6.12) is compatible with Zwanziger's horizon conditions $\kappa_A < -1$, $\kappa_c > \frac{1}{2}$.

It is also compatible with the so-called "decoupling" solution observed in some gauges also on the lattice,

$\kappa_A = -1$ ("massive" gluon), $\kappa_c = 0$.

If the propagators satisfy the vertex sum rule (5.11), $-\kappa_A = 2\kappa_c$, the condition boils down to $\kappa_c > \frac{1}{4}$, which is satisfied by the "scaling solution".

The confinement criterion has by now been used to verify confinement in a variety of different approaches including functional methods, Hamiltonian quantization & semi-perturbative approaches to the Gribov problem. If the propagators are not only used asymptotically, but fully, an improved calculation of the type (6.10) can also be used to determine the critical temperature of the confinement-deconfinement phase transition, see the literature.

In summary, the mechanism presented above demonstrates how color confinement (absence of gluon/ghost degrees of freedom in the physical state space) can be related to quark confinement in a quantifyable manner. Of course, the difficult part still remains in the accurate and controlled determination of the Yang-Mills propagators which is a fully non-perturbative problem and has been and still is an active field of research during the past decades.