Spectral sums for Dirac operator and Polyakov loops

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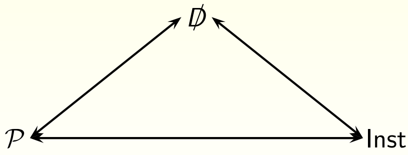
Finite temperature gauge theories

- confinement
  order parameter: Polyakov loop $\langle \mathrm{Tr}\,\mathcal{P}\rangle,\ \mathcal{P} = \mathcal{P}\exp\left(i\int A_0 d\tau\right)$
  induced by topological defects (monopoles,. . .)
- chiral symmetry breaking
  order parameter: chiral condensate $\langle\bar\psi\psi\rangle$
  induced by instantons
- deconfining and chiral phase transitions at same T_c determined by low lying eigenvalues of $\not{D}$ (Banks-Casher; Gattringer)



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Interrelations:

- winding of Polyakov loops $\leftrightarrow$ instanton number
  Reinhard; Jahn, Lenz; Ford, Mitreuter, Tok, Pawlowski, AW; ...
- spectrum of $\not{D}$ $\leftrightarrow$ Polyakov loop
  Bruckmann, Gattringer; Soldner; Synatschke, Wozar, AW; ...

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Physics

- finite $T \Rightarrow$ euclidean functional integral
  (anti)periodic fields on $[0,\beta]\times\mathbb{R}^3,\ \beta = 1/T$
- gauge transformation:

$$\begin{aligned}
\psi(x) &\rightarrow \psi^V(x) = V(x)\psi(x)\\
A_\mu(x) &\rightarrow A_\mu^V(x) = V(x)A_\mu(x)V^{-1}(x) + iV\partial_\mu V^{-1}\\
\mathcal{P}(x) &\rightarrow \mathcal{P}^V(x) = V(x^0,\boldsymbol{x})\mathcal{P}(x)V^{-1}(0,\boldsymbol{x})
\end{aligned}$$

Path ordered integral

$$\mathcal{P}(x^0,\boldsymbol{x}) = \mathcal{P}\exp\left[i\int_0^{x^0} d\tau A_0(\tau,\boldsymbol{x})\right] \in G$$

loop integral: $\mathcal{P}(\beta,\boldsymbol{x}) \equiv \mathcal{P}(\boldsymbol{x})$

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- $V(x^0, \boldsymbol{x})$ periodic in euclidean time

$$\mathcal{P}^V(\boldsymbol{x}) = V(0,\boldsymbol{x})\mathcal{P}(\boldsymbol{x})V^{-1}(0,\boldsymbol{x})$$

$\Rightarrow$ Polyakov loops

$$P(\boldsymbol{x}) = \mathrm{Tr}\,\mathcal{P}(\boldsymbol{x})$$

are gauge invariant!

- free energies F and $F_{q\bar{q}}$:

$$\langle P(\boldsymbol{x})\rangle_\beta = e^{-\beta F(\boldsymbol{x})} \quad , \quad \langle P(\boldsymbol{x})P^\dagger(\boldsymbol{y})\rangle_\beta = e^{-\beta F_{q\bar{q}}(\boldsymbol{x}-\boldsymbol{y})}$$

$\langle P\rangle$ order parameter for confinement:

$$\langle P\rangle = \begin{cases} 0 & T < T_c \\ \neq 0 & T > T_c \end{cases}$$

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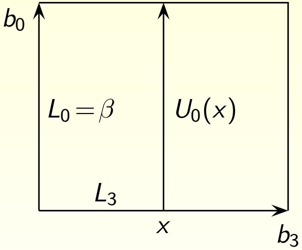
Mathematics

- euclidean spacetime $\mathcal{M} = \mathbb{T}^4 = [0,\beta] \times \mathbb{T}^3$
- only gauge invariant fields (anti)periodic:



$$\psi(x + b_\nu) = \pm\psi^{U_\nu}(x)$$
$$A_\mu(x + b_\nu) = A_\mu^{U_\nu}(x)$$

Cocycle condition (no twists):

$$U_\nu(x + b_\mu)U_\mu(x) = U_\mu(x + b_\nu)U_\nu(x)$$
$$U_\nu^V(x) = V(x + b_\nu)U_\nu(x)V^{-1}(x)$$

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- bundle classified by instanton number

$$q = q(U_\nu) = \frac{1}{16\pi^2}\int_{\mathbb{T}^4} \mathrm{Tr}(F \wedge F) \in \mathbb{Z}, \quad \left(\frac{\mathbb{Z}}{|\mathcal{Z}|}\right)$$

- Periodicity properties:

$$\mathcal{P}(x^0, \boldsymbol{x} + b_i) = U_i(x^0, \boldsymbol{x})\mathcal{P}(x^0, \boldsymbol{x})U_i^{-1}(0, \boldsymbol{x})$$

(differential equation and initial condition)
exist transition functions with (Annals Phys. 269 (1998) 26)

$$U_0 = \mathbb{1},\ U_i(0, \boldsymbol{x}) = \mathbb{1} \Longrightarrow U_i(x + b_0) = U_i(x)$$

- $\Rightarrow \mathcal{P}(\beta, \boldsymbol{x}) \equiv \mathcal{P}(\boldsymbol{x})$ periodic and

$$q = \frac{1}{24\pi^2}\int_{\mathbb{T}^3} \mathrm{Tr}\,\left(\mathcal{P}^{-1} d\mathcal{P}\right)^3, \quad \mathbb{T}^3 = \{(0, \boldsymbol{x})\}.$$

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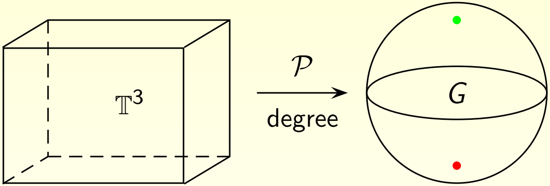
proof: gauge trafo with $\mathcal{P} \Longrightarrow U_0 = \mathcal{P}$ and $U_i = \mathbb{1}$, use $q = q(U_\nu)$

instanton number = degree of map $\mathcal{P} : \mathbb{T}^3 \longrightarrow G, \;\; \pi_3(G) = \mathbb{Z}$



localize winding number: diagonalize $\mathcal{P}$

$$\mathcal{P}(\boldsymbol{x}) = W(\boldsymbol{x}) D(\boldsymbol{x}) W^{-1}(\boldsymbol{x}), \quad D \text{ diagonal}$$

W and D not unique; pick unique D in conjugacy class of $\mathcal{P}$

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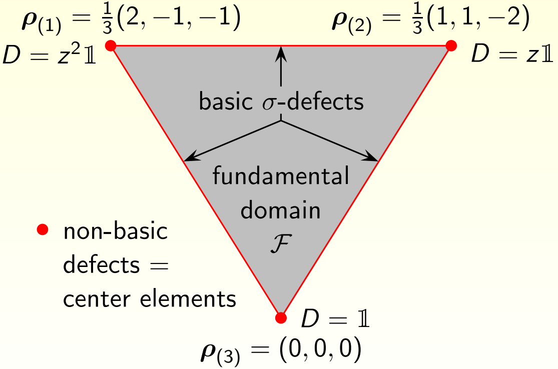
- for $SU(3)$:

$$D(\boldsymbol{\rho}) = \text{diag}\left(e^{2\pi i \rho_1}, e^{2\pi \rho_2}, e^{2\pi \rho_3}\right), \quad \boldsymbol{\rho} \in \mathcal{F}$$
$$\mathcal{F} = \left\{\boldsymbol{\rho} \in \mathbb{R}^3 \middle| \sum \rho_i = 1,\ \rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_1 - 1\right\}$$

- singularities at points, where $\mathcal{P}$ has coinciding eigenvalues



- defects (monopoles, strings, sheets) localized in

$$\mathcal{D} = \{\boldsymbol{x} \in \mathbb{T}^3 | \, \mathcal{C}_{D(\boldsymbol{x})} \text{ non-Abelian}\}$$

- $\boldsymbol{x}_0$ isolated point in $\mathcal{D}$: Higgs-field ϕ near $\boldsymbol{x}_0$

$$\mathcal{P}(\boldsymbol{x}) = e^{i\phi(\boldsymbol{x})}\mathcal{P}(\boldsymbol{x}_0), \quad \phi(\boldsymbol{x}_0) = 0$$

  small sphere S^2 around defect at $\boldsymbol{x}_0$; consider $\phi : S^2 \to \text{Lie}(G)$

$$\Rightarrow \text{winding number } w(\phi, \boldsymbol{x})$$

- magnetic charge of defect: Abelian gauge field

$$A_{\text{mag}} = i\left(W^{-1}dW\right)_{\text{Cartan}}, \quad F_{\text{mag}} = dA_{\text{mag}}$$

$$\Rightarrow \text{magnetic charge } q_{\text{mag}}(\boldsymbol{x}_0) = \int_{S^2} F_{\text{mag}}$$

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- cp. 't Hooft-Polyakov monopole: for every defect

$$w(\phi, x_0) = q_{\rm mag}(x_0)$$

- general group: rank+1 types of magnetically charged defects
  winding of $\mathcal{P}$ = number of times a given effect occurs
  magnetic charge of all defects of type σ

$$\sum_{\sigma-\text{defects}} w(\phi, x_i) = \sum_{\sigma-\text{defects}} q_{\rm mag}(x_i) = \pm q, \quad 1 \le \sigma \le N$$

- $q \neq 0 \Rightarrow$ there are magnetic monopoles
- magnetic monopoles sit on Gribov horizon for Polyakov gauge
- simulations $\Rightarrow$ magnetic monopoles relevant for confinement

Spectrum of $\not{D}$ and chiral symmetry breaking

- chiral boundary conditions or $\mathbb{T}^4, S^4, \ldots$

$$i\not{D}_A \psi_n = \lambda_n \psi_n \Longrightarrow i\not{D}_A(\gamma_5\psi_n) = -\lambda_n(\gamma_5\psi_n)$$

- massive quark propagator

$$\langle q(x)\bar{q}(y)\rangle_A = \langle x|\frac{1}{i\not{D}_A + im}|y\rangle = \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{\lambda_n + im}$$

current quark mass m, external gauge potential A
assuming a regularization (e.g. lattice):

$$\int_V d^4x \left\langle \bar{q}(x)q(x)\right\rangle_A = \sum_{\lambda_n > 0} \frac{2im}{\lambda_n^2 + m^2}$$

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- quark condensate

$$\langle\bar{q}q\rangle = \lim_{V\to\infty}\frac{1}{V}\Big\langle \int d^4x \big\langle \bar{q}(x)q(x)\big\rangle_A \Big\rangle$$

relation to spectral density $\rho(\lambda)$ of Dirac operator

$$\langle\bar{q}q\rangle = 2im\int d\lambda\,\frac{\langle\rho_A(\lambda)\rangle}{\lambda^2+m^2}\qquad \rho_A(\lambda)=\frac{1}{V}\sum_{\lambda_n>0}\delta\big(\lambda-\lambda_n\big),$$

- main contribution $\lambda \le m \ll \Lambda_{\rm QCD}$
  first $V\to\infty$ then chiral limit $m\to 0$

$$\Sigma = \lim_{m\to 0}|\langle\bar{q}q\rangle| = \pi\rho(0),\qquad \rho(\lambda)=\langle\rho_A(\lambda)\rangle,$$ Banks-Casher

free fermions: $\rho\sim\lambda^3,\ \langle\bar{q}q\rangle\sim m^3\to 0$ in chiral limit $(\Lambda<\infty)$

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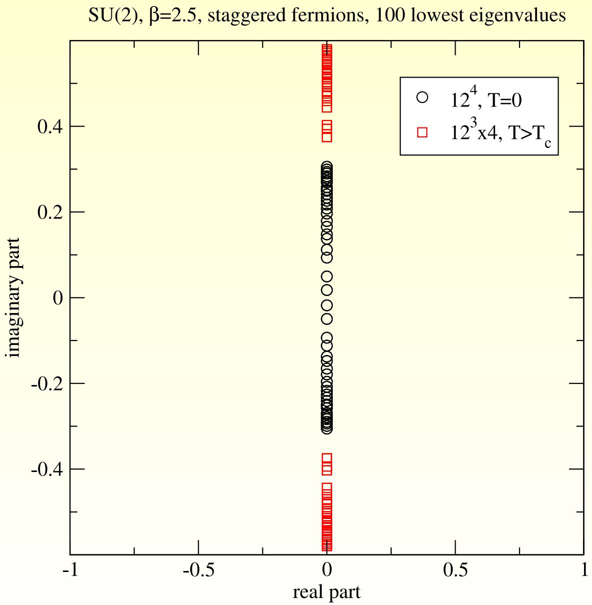
SU(2), β=2.5, staggered fermions, 100 lowest eigenvalues
12^4, T=0
12^3x4, T>T_c
0.4
0.2
0
-0.2
-0.4
imaginary part
-1
-0.5
0
0.5
1
real part

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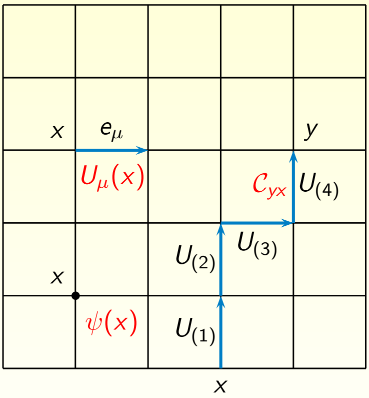
Lattice Formulation



matter fields on sites $x \to \psi(x)$

gauge fields on links (x, e_μ)

$(x, e_\mu) \to U_\mu(x) \in G$ (unitary)

$$U_{-\mu}(x+e_\mu)U_\mu(x) = \mathbb{1}$$

$$U_\mu(x) \approx \exp\left(iaA_\mu(x)\right)$$

parallel transport along $\mathcal{C}_{yx}$ with

$$\mathcal{W}_{\mathcal{C}_{yx}} = U_{(4)}U_{(3)}U_{(2)}U_{(1)}$$

gauge transformation

$$\psi(x) \to V_x\psi(x),\ V_x \in G$$

$$U_\mu(x) \to V_{x+e_\mu}U_\mu(x)V_x^{-1}$$

$$\mathcal{W}_{\mathcal{C}_{yx}} \to V_y\mathcal{W}_{\mathcal{C}_{yx}}V_x^{-1}$$

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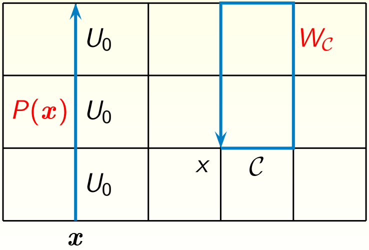
Wilson and Polyakov loops

- finite temperature:
  asymmetric $N_\tau \times N_s^3$-lattice, $N_\tau \ll N_s,\; V = N_\tau \cdot N_s^3$
- parallel transport along loop $\mathcal{C}_{xx} \Rightarrow$
  gauge invariant Wilson loop

$$W_{\mathcal{C}} = \mathrm{Tr}\,\mathcal{W}_{C_{xx}} \Longrightarrow \mathrm{Tr}\,\left(V_x \mathcal{W}_{C_{xx}} V_x^{-1}\right) = W_{\mathcal{C}}$$

- Polyakov loops



loops winding around periodic time direction

$$P_{\boldsymbol{x}} = \mathrm{Tr}\,\mathcal{P}_{\boldsymbol{x}}$$

$$\mathcal{P}_{\boldsymbol{x}} = \prod_{\tau=1}^{N_\tau} U_0(\tau, \boldsymbol{x})$$

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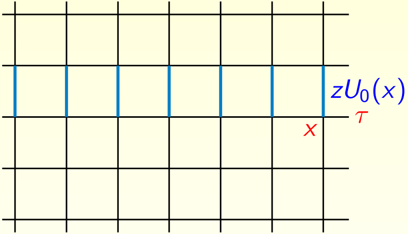
Center transformations

multiply all $U_0(\tau, \boldsymbol{x})$ in one time-slice with center elements z

$$\text{configuration } \{U_\mu(x)\} \longrightarrow \{{}^zU_\mu(x)\} \text{ twisted configuration}$$



transformation of Wilson loops (C_{xx} contractable)

$$\mathcal{W}_{\mathcal{C}_{xx}} \longrightarrow \mathcal{W}_{\mathcal{C}_{xx}} \Rightarrow S_{\rm w} \text{ invariant}$$

transformation of Polyakov loops

$$\mathcal{P}_{\boldsymbol{x}} \longrightarrow z\mathcal{P}_{\boldsymbol{x}}$$

$\mathcal{C}_{xx}$ winds n-times around periodic time direction:

$$\mathcal{W}_{\mathcal{C}_{xx}} \longrightarrow z^n \mathcal{W}_{\mathcal{C}_{xx}}$$

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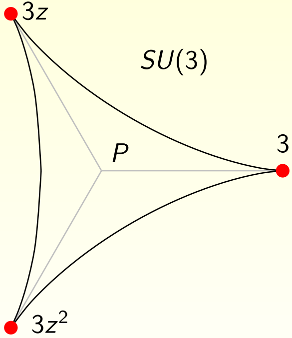
probability distribution for order parameter $P_{\boldsymbol{x}}$ proportional to

$$e^{-S_{\rm eff}[P]} = \int \mathcal{D}U\,\delta\big(P_{\boldsymbol{x}}, {\rm Tr}\,\mathcal{P}_{\boldsymbol{x}}\big)\,e^{-S_{\rm w}[U]}, \quad \mathcal{P}_{\boldsymbol{x}} = \prod_\tau U_0(\tau,\boldsymbol{x})$$

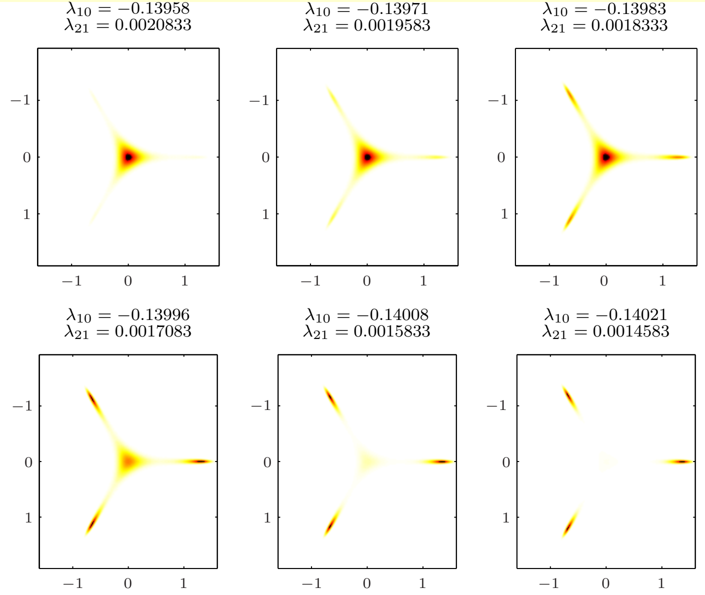


center symmetry of $S_{\rm w} \Rightarrow$

$$S_{\rm eff}[P] = S_{\rm eff}[z\cdot P]$$

high temperatures: $\langle P\rangle \neq 0 \Rightarrow$
breaking of center symmetry
low temperature: $\langle P\rangle = 0 \Rightarrow$
center symmetric phase

relation to Potts models:
Phys. Rev. D 74 114501 (2006)



Histogram of P

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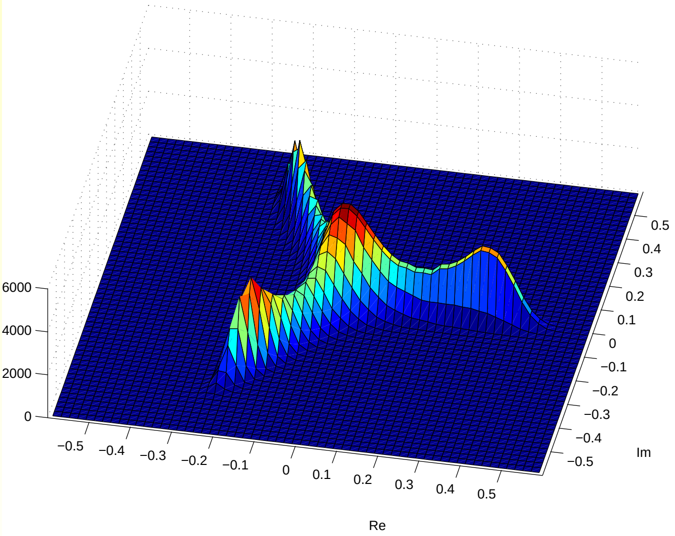

histogram of P

Dirac Operator

- forward/backward covariant derivatives: $D^{\rm f}_\mu = -(D^{\rm b}_\mu)^\dagger$:

$$\begin{aligned}(D^{\rm b}_\mu\psi)(x) &= \psi(x) - U_\mu(x-e_\mu)\psi(x-e_\mu)\\ (D^{\rm f}_\mu\psi)(x) &= U_{-\mu}(x+e_\mu)\psi(x+e_\mu) - \psi(x)\end{aligned}$$

- anti-hermitean naive Dirac operator with doublers

$$\not{D} = \tfrac{1}{2}\gamma^\mu\big(D^{\rm f}_\mu + D^{\rm b}_\mu\big), \quad \gamma_\mu = \gamma^\dagger_\mu$$

- doublers $\Rightarrow$ add hermitean covariant Laplacian

$$D^2 = \sum\nolimits_\mu D^{\rm b}_\mu D^{\rm f}_\mu = \sum\nolimits_\mu (D^{\rm f}_\mu - D^{\rm b}_\mu)$$

- Wilson operator (no doublers)

$$\mathcal{D} = -\not{D} + m - \tfrac{1}{2}D^2, \quad \gamma_5\mathcal{D}\gamma_5 = \mathcal{D}^\dagger$$

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- number of eigenvalues $= V \times 2^{[d/2]} \times N_c$
- characterstic polynom $P(\lambda) = \det(\lambda \mathbb{1} - \mathcal{D})$:

$$P(\lambda) = \det \gamma_5(\lambda \mathbb{1} - \mathcal{D})\gamma_5 = \det(\lambda \mathbb{1} - \mathcal{D}^\dagger) = P^*(\lambda^*)$$

  γ_5-hermiticity $\Rightarrow$ non-real eigenvalues come in pairs λ_n, λ_n^*
  same for staggered, overlap, ... fermions

- $\mathcal{D}$ with only nearest neighbour interaction

$$\mathcal{D}_{xy} = (m+d)\delta_{xy} \\ -\frac{1}{2}\sum_\mu \Big((1+\gamma^\mu)U_{-\mu}(y)\delta_{x,y-e_\mu} + (1-\gamma^\mu)U_\mu(y)\delta_{x,y+e_\mu}\Big)$$

- hop from site x to $x \pm e_\mu$: factor $-\frac{1}{2}(1 \mp \gamma^\mu)U_\mu(x)$
  staying at x: factor $(m+d)$
  $\mathcal{D}^\ell$: chains of ℓ or less hops on lattice

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Spectral Sums

- $\langle x|\mathcal{D}^\ell|x\rangle$ described by Wilson loops $\mathcal{W}_{\mathcal{C}_{xx}}$ with $|\mathcal{C}_{xx}| \le \ell$
- twisting:

$$U \to {}^zU, \quad \mathcal{D} \to {}^z\mathcal{D}, \quad \lambda_n \to {}^z\lambda_n$$

$$\ell < N_\tau \ll N_s \Rightarrow \text{all loops contractable} \Longrightarrow {}^z\mathcal{W}_\mathcal{C} = \mathcal{W}_\mathcal{C} \Longrightarrow$$

$$\langle x|\,{}^z\mathcal{D}^\ell|x\rangle = \langle x|\mathcal{D}^\ell|x\rangle \Longrightarrow \sum_{\text{center}} \bar{z}_k \big\langle x\big|\,{}^{z_k}\mathcal{D}^\ell\big|x\big\rangle = 0, \quad \ell < N_\tau$$

assumed: non-trival center $\Rightarrow \sum z_k = 0$.
trace $\Rightarrow$ constraints on spectral sums

$$\sum_{\text{center}} \bar{z}_k \Big(\sum\nolimits_n {}^{z_k}\lambda_n^\ell \Big) = 0, \quad \ell < N_\tau$$

- $\ell = N_\tau$: $\langle x|\mathcal{D}^{N_\tau}|x\rangle$ contains Polyakov loops, use $\sum \bar{z}_k z_k = |\mathcal{Z}| \Rightarrow$
  $\Rightarrow$ Gattringer formula

$$\sum_k \bar{z}_k \langle x|^{z_k}\mathcal{D}^{N_\tau}|x\rangle = \kappa' \mathcal{P}(\boldsymbol{x}), \quad \kappa' \propto V|\mathcal{Z}|$$

$$\sum_k \bar{z}_k \,\mathrm{Tr}\left({}^{z_k}\mathcal{D}\right)^{N_\tau} = \kappa\, L, \quad L = \frac{1}{V_s}\sum_{\boldsymbol{x}} P(\boldsymbol{x})$$

- first relation Polyakov loop $\leftrightarrow$ spectral data of $\mathcal{D}$
- valid for all covariant lattice operators with NN interaction
- Problem: in continuum limit $N_\tau \to \infty$,

$$L = \frac{1}{\kappa}\sum_k \bar{z}_k \Big(\sum_n {}^{z_k}\lambda^{N_\tau}\Big), \quad \kappa = (-1)^{N_\tau} 2^{[d/2]-1} V|\mathcal{Z}|$$

ill-defined $\Rightarrow$ need generalized spectral sums

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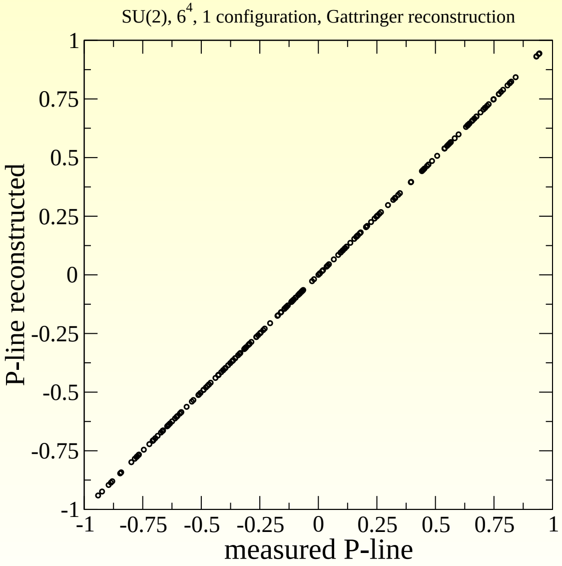
SU(2), 6^4, 1 configuration, Gattringer reconstruction
P-line reconstructed
measured P-line
1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-1

average change of λ_n by twisting:

$$\Delta_n \equiv \frac{1}{3}\big(|\lambda_n - {}^z\lambda_n| + |{}^z\lambda_n - {}^{\bar z}\lambda_n| + |{}^{\bar z}\lambda_n - \lambda_n|\big)$$

observations:

- Δ_n big above T_c, small below T_c
  Δ_n maximal for low-lying eigenvalues
- $(\lambda_n)^{N_\tau}$: main contribution to L from large eigenvalues: consider

$$\Sigma_n = \frac{1}{\kappa}\sum_k \bar z_k \sum_{p=1}^{n} {}^{z_k}\lambda_p^{N_\tau}$$

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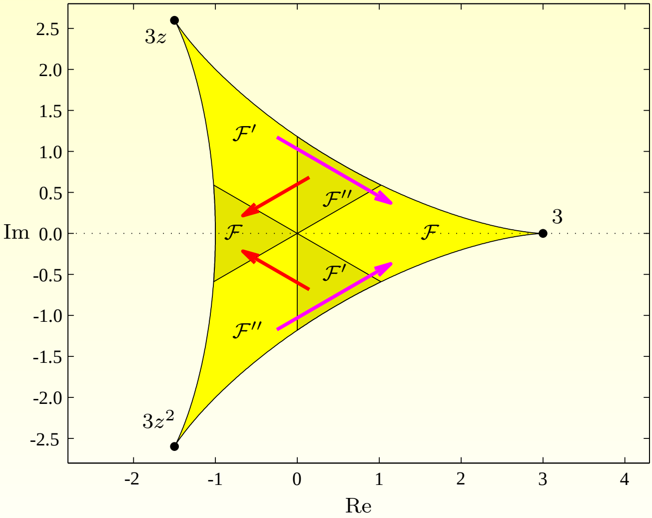
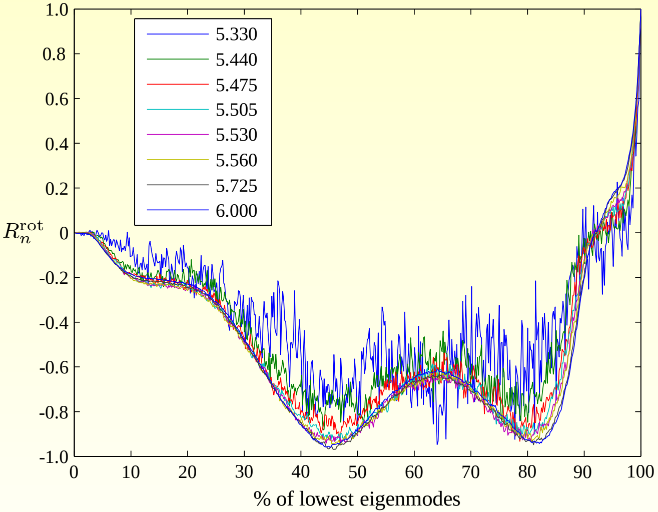


fundamental domain for P, definition of L^{rot}, Σ^{rot}



$R_n^{\rm rot} = \frac{\langle \Sigma_n^{\rm rot} \rangle}{\langle L^{\rm rot} \rangle}$, % of lowest eigenvalues ($\beta_c = 5.49$)

universal behaviour for $\langle L \rangle > 0.6$, $\beta > 5.5$

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$$\mathcal{S}_n(f) = \frac{1}{\kappa} \sum_k \bar{z}_k \sum_p f\left({}^{z_k}\lambda_p\right)$$

- sum over all eigenvalues $\Rightarrow$ traces

$$\mathcal{S}(f) = \frac{1}{\kappa} \sum_k \bar{z}_k \sum_{\text{all } p} f\left({}^{z_k}\lambda_p\right) = \frac{1}{\kappa} \sum_k \bar{z}_k \operatorname{Tr} f\left({}^{z_k}\mathcal{D}\right)$$

- $S(f)$ is order parameter for center symmetry

$$S(f) \xrightarrow{z} \frac{1}{\kappa} \sum_k \bar{z}_k \operatorname{Tr} f\left({}^{z_k z}\mathcal{D}\right) = \frac{z}{\kappa} \sum_k (\bar{z}_k \bar{z}) \operatorname{Tr} f\left({}^{z_k z}\mathcal{D}\right) = z S(f)$$

$\langle S(f) \rangle \neq 0 \Rightarrow$ center symmetry broken

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- choose f with support in infrared

$$\Sigma^{(-1)} = \frac{1}{\kappa}\sum_k \mathrm{Tr}\left(\frac{\bar{z}_k}{z_k\mathcal{D}}\right) \quad \text{and} \quad \Sigma^{(-2)} = \frac{1}{\kappa}\sum_k \mathrm{Tr}\left(\frac{\bar{z}_k}{z_k\mathcal{D}^2}\right)$$

- Σ^{-1} enters Banks-Casher relation
- hopping parameter expansion
  $\mathcal{D} = (m+d)\mathbb{1} - V$: expansion in powers of V,

$$\mathcal{D}^{-1} = \frac{1}{m+d}\sum_k \frac{1}{(m+d)^k}\left[(m+d)\mathbb{1} - \mathcal{D}\right]^k \Longrightarrow$$

$$\Sigma^{(-1)} = \frac{(-1)^{N_\tau}}{(m+d)^{N_\tau+1}}\Sigma^{(N_\tau)} + \dots \approx \frac{(-1)^{N_\tau}}{(m+d)^{N_\tau+1}}L$$

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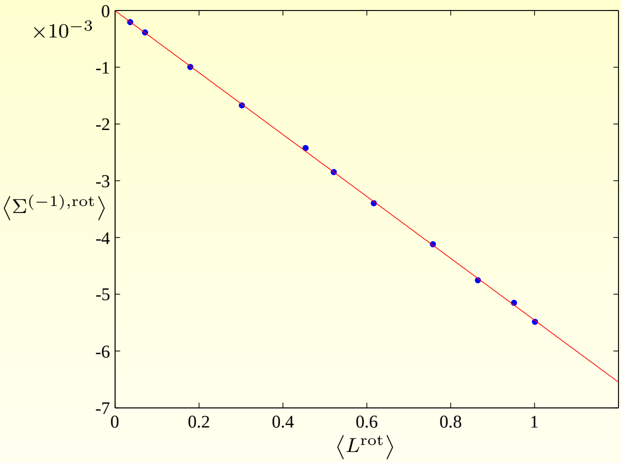

The expectation value of $\Sigma^{(-1),\mathrm{rot}}$ as function of L^{rot}

$$\langle \Sigma^{(-1),\mathrm{rot}} \rangle = -0.00545 \cdot \langle L^{\mathrm{rot}} \rangle - 4.379 \cdot 10^{-6} \; (\mathrm{rmse} = 2.978 \cdot 10^{-5})$$

expansion on $4^3 \times 3 : \langle \Sigma^{(-1)} \rangle = -0.004 \cdot \langle L \rangle$

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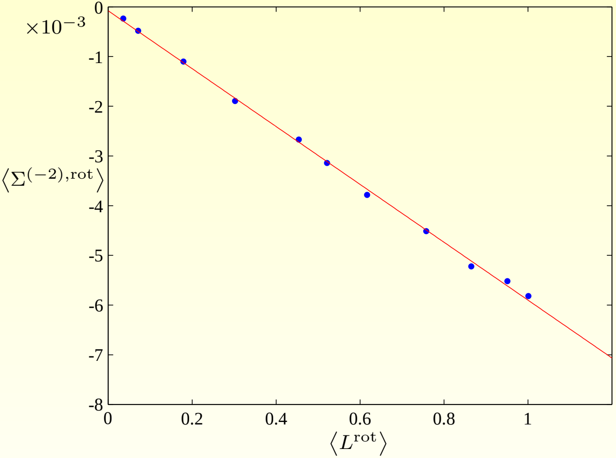

The expectation value of $\Sigma^{(-2),\mathrm{rot}}$ as function of L^{rot}

$$\langle \Sigma^{(-2),\mathrm{rot}} \rangle = -0.00582 \cdot \langle L^{\mathrm{rot}} \rangle - 8.035 \cdot 10^{-5}$$

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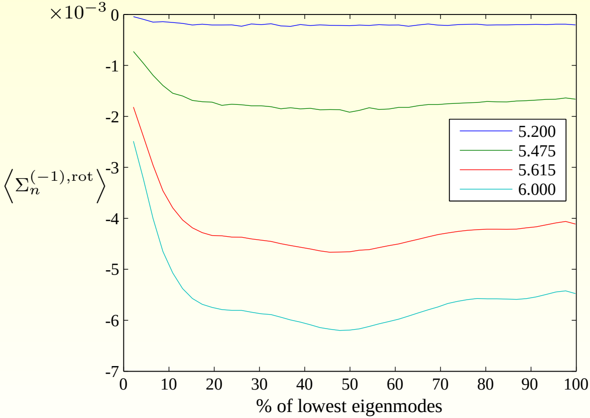
ultraviolet vs. infrared contributions: partial sums

$$\Sigma_n^{(-1)} = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{p=1}^{n} \frac{1}{z_k \lambda_p} \Rightarrow \text{lowest } 10\%$$



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- exponential spectral sums

$$\mathcal{E} = \frac{1}{\kappa}\sum_k \bar{z}_k \,\mathrm{Tr}\,\exp\left(-^{z_k}\mathcal{D}\right)$$

- Gaussian spectral sums

$$\mathcal{G} = \frac{1}{\kappa}\sum_k \bar{z}_k \,\mathrm{Tr}\,\exp\left(-^{z_k}\mathcal{D}^{\dagger\;z_k}\mathcal{D}\right)$$

- used in proper-time and zeta-function regularisation
  $\kappa\mathcal{G}$ well-behaved continuum limit
- expansion $\Rightarrow$

$$\mathcal{E} \approx \frac{(-1)^{N_\tau}}{N_\tau!}\, e^{-(m+d)}L$$

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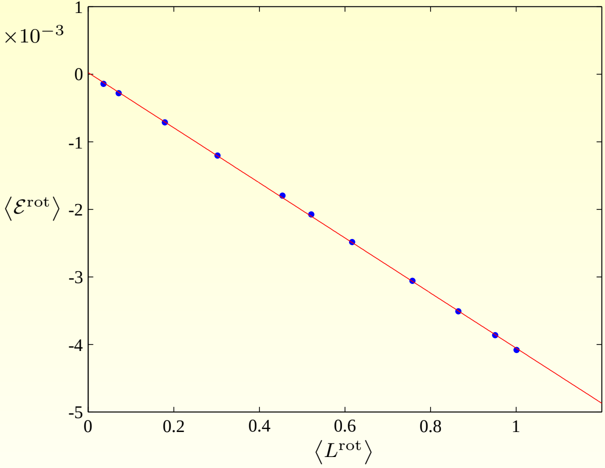

The expectation value of $\mathcal{E}^{\rm rot}$ as function of $L^{\rm rot}$

$$\langle\mathcal{E}^{\rm rot}\rangle = -0.00408\cdot\langle L^{\rm rot}\rangle + 2.346\cdot 10^{-5}\ (\mathrm{rmse} = 1.82\cdot 10^{-5})$$

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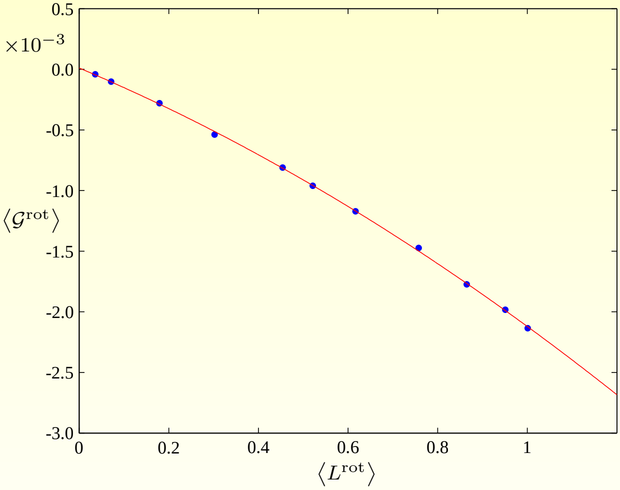

The expectation value of $\mathcal{G}^{\mathrm{rot}}$ as function of L^{rot}

$$\langle \mathcal{G}^{\mathrm{rot}} \rangle = -0.000571 \cdot \langle L^{\mathrm{rot}} \rangle^2 - 0.00156 \cdot \langle L^{\mathrm{rot}} \rangle + 1.061 \cdot 10^{-5}$$

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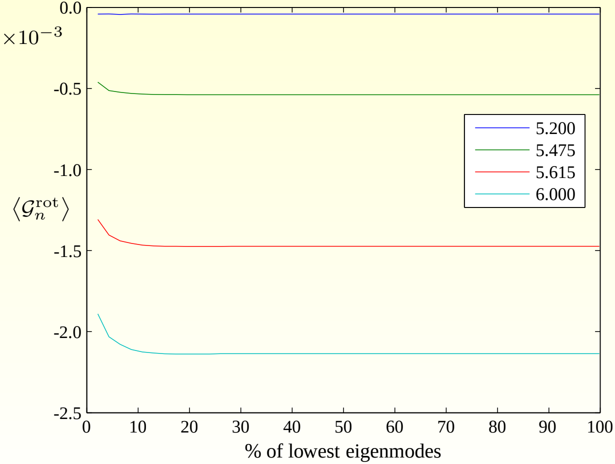
$$\mathcal{G}_n = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{p=1}^{n} e^{-|z_k\lambda_p|^2} \Rightarrow \text{lowest } 3\%$$



Flat connections and instantons

curvature free gauge field configurations

- $SU(N)$-field configuration

$$U_i(x) = \mathbb{1} \quad \text{and} \quad U_0(x) = U_0(\tau), \quad x = (\tau, \boldsymbol{x})$$

- gauge transformation

$$V(\tau) = \mathcal{P}^{-1}(\tau), \quad \mathcal{P}(\tau) = U_0(\tau - 1) \cdots U_0(2) U_0(1)$$

trivial $U_\mu(x) = \mathbb{1}$ and free Dirac operator, but

$$\psi(\tau + N_\tau, \boldsymbol{x}) = \mathcal{P}^{-1} \psi(\tau, \boldsymbol{x})$$

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- diagonalize Polyakov loop $\mathcal{P} \sim \left(e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_N}\right)$
- eigenfunctions: plane waves with $(p_0, \boldsymbol{p})$, non-periodic in x^0

  eigenvalues of $\mathcal{D}$

$$\lambda_p^{\pm} = m \pm i|\sin p_\mu| + 2r\sin^2\left(\frac{p_\mu}{2}\right)$$

twist $\equiv$ changing bc

$$z_k = e^{2\pi ik/N}\mathbb{1} \Longrightarrow \varphi_j \longrightarrow \varphi_j - \frac{k}{N}$$

example

$$\mathcal{P}(\theta) = \begin{pmatrix} e^{2\pi i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi i\theta} \end{pmatrix} \Longrightarrow L = 1 + 2\cos(2\pi\theta)$$

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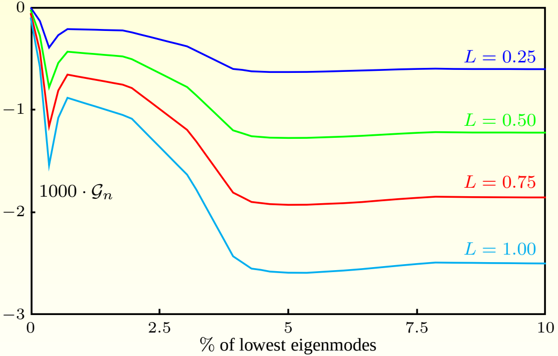
momenta in time direction

$$p_0(z_k) \in \left\{ \frac{2\pi}{N_\tau} (n_0 - \varphi_j - k/N) \right\}, \quad 1 \le j, k \le N$$

$\Rightarrow$ eigenvalues, spectral sums



The partial Gaussian sums $\mathcal{G}_n$ for flat connections with different L

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Beyond order Parameter

- correlations functions of Polyakov loops
  exact relation

$$\mathcal{P}(\boldsymbol{x})\mathcal{P}(\boldsymbol{y}) = \eta^2 \sum_{k,\ell} \bar{z}_k \bar{z}_\ell \langle x|^{z_k}\mathcal{D}^{N_\tau}|x\rangle\langle y|^{z_\ell}\mathcal{D}^{N_\tau}|y\rangle$$

- dominated by UV-modes; enhance IR-modes $\longrightarrow$

$$\mathcal{P}(\boldsymbol{x})\mathcal{P}(\boldsymbol{y}) \approx \eta'^2 \sum_{k,\ell} \bar{z}_k \bar{z}_\ell \langle x|f\left({}^{z_k}\mathcal{D}\right)|x\rangle\langle y|f\left({}^{z_\ell}\mathcal{D}\right)|y\rangle$$

- stringtension

$$\langle P(\boldsymbol{x})P(\boldsymbol{y})\rangle \propto e^{-\beta F_{q\bar{q}}(\boldsymbol{x}-\boldsymbol{y})}, \quad F_{q\bar{q}}(\boldsymbol{\xi}) \to \begin{cases} \sigma|\boldsymbol{\xi}| & |\boldsymbol{\xi}| \gg 1/\Lambda \\ c/|\boldsymbol{\xi}| & |\boldsymbol{\xi}| \ll 1/\Lambda \end{cases}$$

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- partial sums (propagator or exponential)
  only lowest modes $\Rightarrow$ $q\bar{q}$ not resolved $\Rightarrow$ only linear potential
  including all eigenvalues (traces) $\Rightarrow$ short range Coulomb visible
- continuum gauge theories on $\mathbb{T}^4$:
  flat connection no good $\Rightarrow$ Abelian instantons ('t Hooft)
  eigenvalues known; eigenfunction = elliptic functions
  work in progress (Langfeld, Synatschke, AW)
- first numerical results (large lattices)

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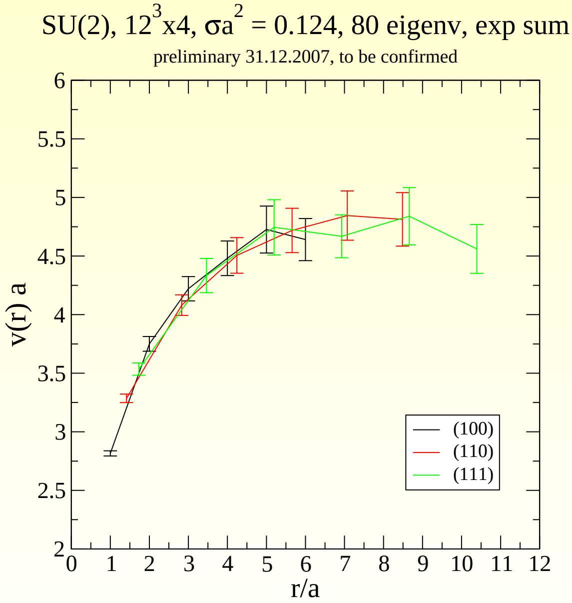
SU(2), 12³x4, σa² = 0.124, 80 eigenv, exp sum
preliminary 31.12.2007, to be confirmed
v(r) a
r/a
6
5.5
5
4.5
4
3.5
3
2.5
2
0
1
2
3
4
5
6
7
8
9
10
11
12
(100)
(110)
(111)

Conclusions, remarks

- direct relation order parameter $\mathcal{P}(x) \leftrightarrow \sigma(\mathcal{D})$
- if one allows $z \in U(1)$ and not only $z \in \mathcal{Z} \Rightarrow$ then

$$\int_{S^1} d\bar{z}\, \mathrm{Tr}\, f({}^zD) = \text{const} \cdot \tilde{L}$$

dressed Polyakov loop $\tilde{L}$: sum of loops with one winding

- choose $f(\mathcal{D}) = 1/\mathcal{D}$, average over gauge fields; $m \to 0 \Longrightarrow$

Banks-Casher traces $\leftrightarrow$ order parameter $\langle \tilde{L} \rangle$

(Bruckmann, Gattringer, unpublished)

- continuum limit

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