Asymptotic Safety of O(N) Lattice Models

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Seminar Plymouth

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Asymptotic Safety

- perturbatively non-renormalizable theories can be predictable
- assumes: nontrivial renormalization UV-fixed point
  $\Rightarrow$ dimensionless couplings tend to finite values at fixed point
- at fixed point:
  small number of IR-relevant (UV-attractive) directions
  $\Rightarrow$ small number couplings need to be fine-tuned
- UV-completion for scalar field theory — K. Wilson and J. Kogut, 1974
- Quantum Gravity could be asymptotically safe — S. Weinberg, 1976
- accumulating evidence in past years — M. Reuter, F. Saueressig, R. Percacci, ...

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Conjecture: Theory has UV-attractive fixed point in $d + \epsilon$ dimensions if it is renormalizable and asymptotically free in d dimensions. S. Weinberg

- large-N Gross-Neveu model in 3 dimensions — Gawedzki and A. Kupiainen, 1985
- general 4-Fermi models in 3 dimensions (FRG) — Janssen, Gies
- O(N) models in 3 dimensions (FRG) — R. Percacci et al.
  - testbed for background field method
    bi-field method fully implemented
  - recent covariant high-order calculations — R. Flore et al 2013
    stability of UV-fixed point?
- 2d gravity: perturbatively renormalizable
  gravity in $2 + \epsilon$ dimensions show non-trivial FP — S. Weinberg, 1979
- gauge theories in 5 dimensions?

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- 4d Gravity should have nontrivial UV-fixed point — M. Reuter, 1996
  - based on functional methods
  - initially Einstein-Hilbert truncation
  - enlarged space of operators
  - so far: $f(R)$ and C^2 truncations
  - inclusion of matter fields

UV-fixed point in 4d quantum gravity 'stabilizes'

- issue of background field method

  FP really stable?

  $\Rightarrow$ increase theory-subspace further
- independent studies with different method
- e.g. lattice regularization

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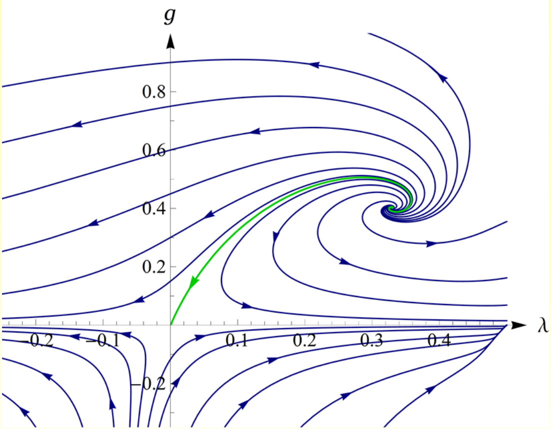
FRG-flow in 4d-quantum gravity



M. Reuter and F. Saueressing (2002)

Nonlinear O(N)-Models

- target-space = sphere: $\boldsymbol{n} \in \mathbb{R}^N$ and $\boldsymbol{n} \cdot \boldsymbol{n} = 1$
- classical action

$$S = \int \mathrm{d}^d x \, \partial_\mu \boldsymbol{n}(x) \cdot \partial^\mu \boldsymbol{n}(x)$$

- $N = 4$: effective model for chiral phase transition
- $N = 3$: Heisenberg model for ferromagnetism
- $N = 2$: shows Kosterlitz-Thouless PT in $d = 2$
- $N = 1$: ubiquitous Ising model
- $d = 2$: perturbatively renormalizable
  O(3)-model shares properties with *QCD*: asymptotic freedom, instantons, dynamical mass-generation, ...

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FRG for nonlinear O(N) models

- nonlinear curved target space
  - $\Rightarrow$ background field techniques
  - $\Rightarrow$ testing ground for conceptional issues
- study of bi-field method (cp. gravity)
- manifestly covariant setting
- configuration space

$$\mathcal{M} \equiv \{\varphi : \mathbb{R}^d \to S^{N-1}\}$$

- covariant classical action

$$S[\varphi] = \frac{\zeta}{2} \int \mathrm{d}^d x \; h_{ab}(\varphi)\partial_\mu \varphi^a \partial^\mu \varphi^b$$

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Geometry of target space

- target space is sphere

$$h_{ab} \to \Gamma^a_{bc} \to R_{abcd} = h_{ac}h_{bd} - h_{ad}h_{bc}$$

- dependence on N:

$$R_{ab} = (N-2)h_{ab}, \quad R = (N-1)(N-2)$$

- pull back of covariant derivative

$$\nabla_\mu v^a \equiv \partial_\mu v^a + \partial_\mu \varphi^b \Gamma_b{}^a{}_c v^c$$

- Laplacian

$$\Delta = -\delta^{\mu\nu}\nabla_\mu \nabla_\nu$$

Manifest covariant formulation

- geodesic $\varphi(s)$ from background-field φ to field ϕ:
- $\varphi(s=0)=\varphi$ and $\varphi(s=1)=\phi$
- tangent vectors $\xi^a(s)=\mathrm{d}\varphi^a(s)/\mathrm{d}s$, set $\xi^a(0)=\xi^a$
- derivative along geodesic $\nabla_s=\xi^a(s)\nabla_a$
- functional of field

$$
\begin{aligned}
F[\phi] &= \sum_{n\geq 0}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}F[\varphi(s)]\Big|_{s=0} = \sum_{n\geq 0}\frac{1}{n!}\nabla_s^n F[\varphi(s)]\Big|_{s=0} \\
&= \sum_{n\geq 0} F^n_{(a_1,\dots,a_n)}[\varphi]\xi^{a_1}\cdots\xi^{a_n} = F[\varphi,\xi]
\end{aligned}
$$

- $\phi(\varphi,\xi)$ determined by background field and tangential vector

Bi-fields formulation

- general local action up to fourth order in derivatives

$$\Gamma_k^{\mathrm{s}}[\phi] = \frac{1}{2}\int d^d x \, \Big(\zeta_k h_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + \alpha_k h_{ab}\Delta\phi^a\Delta\phi^b + T_{abcd}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b\partial_\nu\phi^c\partial^\nu\phi^d\Big)\Gamma_k[\phi(\varphi,\xi)]$$

$$T_{abcd} = L_{1,k} h_{a(c}h_{d)b} + L_{2,k} h_{ab}h_{cd}$$

- in background-field method split φ and ξ
- with wave function renormalization

$$\Gamma_k[\varphi,\xi] = \Gamma_k^{\mathrm{s}}[\phi(\varphi, Z_k^{1/2}\xi)]$$

- bi-field cutoff action

$$\Delta S_k[\varphi,\xi] = \frac{1}{2}\int \mathrm{d}^d x \, \xi^a \mathcal{R}^k_{ab}[\varphi]\xi^b$$

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- cutoff operator $\mathcal{R}^k_{ab}[\varphi] = Z_k h_{ab} R_k[\Delta]$
- plug into flow equation

$$k\partial_k \Gamma_k[\varphi,\xi] = \frac{1}{2}\mathrm{Tr}\left(\frac{k\partial_k \mathcal{R}_k[\varphi]}{\Gamma_k^{(0,2)}[\varphi,\xi] + \mathcal{R}_k[\varphi])}\right) .$$

- technical problems:
- $\Gamma_k^{(0,2)}[\varphi, 0]$ is 4^{th} order elliptic differential operator
- adapted 4^{th} order cutoff-operator

$$R_k(\Delta) = \left[\zeta_k(k^2 - \Delta) + \alpha_k(k^2 - \Delta^4)\right] \theta(k^2 - \Delta)$$

$\Rightarrow$ off-diagonal heat kernel method

- heat kernel expansion, Laplace/Mellin-transform

$\Rightarrow$ flow of couplings, β-functions, anomalous dimension, ...

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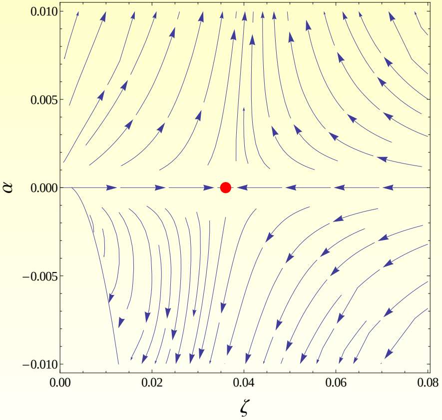
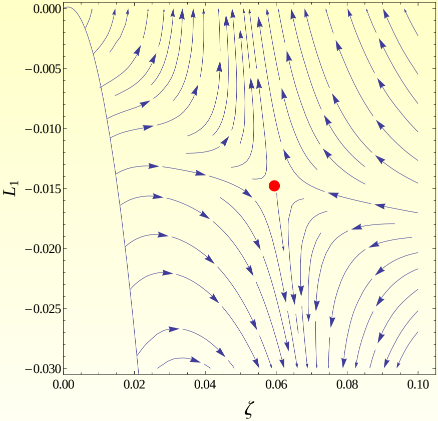
Some results

- two coupling O(3)-model has fixed point in (ζ_k, α_k)-plane
- arrows point to UV



- three coupling O(3)-model has fixed point in (ζ_k, α_k, L_1)-space
- slice $\alpha = 0$, one irrelevant direction



- critical exponent $\nu(N)$ can be estimated
- 4^{th} operator $\propto L_2$: fixed points goes away
  truncation artifacts
  reappears with higher order truncation (phase space flow)
- same with exponential regulator
- inclusion of 6^{th} order operators tedious
  $\Rightarrow$ compare with Lattice simulations

R. Flore, AW and O. Zanusso, Phys.Rev. D87 (2013) 065019

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Nonlinear O(N) Models on Lattice

- $\boldsymbol{n}^2 = 1 \Rightarrow$ derivative expansion for effective action

$$S[\boldsymbol{n}] = \sum_{\alpha=0}^{3} g_\alpha N S_\alpha[\boldsymbol{n}] + \mathcal{O}(\partial^6)$$

- all operators with not more than four derivatives

$$S_0 = -\int \mathrm{d}^d x \; \boldsymbol{n} \cdot \Delta \boldsymbol{n}$$

$$S_1 = \int \mathrm{d}^d x \; \boldsymbol{n} \cdot \Delta^2 \boldsymbol{n}$$

$$S_2 = \int \mathrm{d}^d x \; (\boldsymbol{n} \cdot \Delta \boldsymbol{n})^2$$

$$S_3 = \int \mathrm{d}^d x \; (\boldsymbol{n} \cdot \partial_\mu \partial^\nu \boldsymbol{n})(\boldsymbol{n} \cdot \partial^\mu \partial_\nu \boldsymbol{n})$$

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- mass-dimensions of couplings

$$[g_0] = d - 2\,, \quad [g_1] = [g_2] = [g_3] = d - 4$$

- discretization
- change to different operator basis
- RG transformation: field n on fine grid $(N, a) \Longrightarrow$ averaged field n' on coarser grid $(N' = N/b, a' = ba)$
- physical IR-cutoff is fixed, UV-cutoff lowered $\Lambda \to \Lambda' = \Lambda/b$
- $g_\alpha \to g'_\alpha$ due to quantum fluctuation with scales in $[\Lambda', \Lambda]$

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- here: combined blockspin-transformation and Demon method
- blockspin transformation: draw averaged field n' according to

$$\mathcal{P}(\boldsymbol{n}'_x) \propto \exp\left(C(g_\alpha)\,\boldsymbol{n}'_x \cdot \sum_{y\in\Box_x} \boldsymbol{n}_y\right)$$

- lattice perturbation theory $\to C(g_\alpha)$ A. Hasenfratz
- not good enough for larger couplings $\Rightarrow$
  $C(g_\alpha) > 0$ such that truncation errors in MCRG are minimized
- important for reliable global flow diagram
- previously: extensive studies in $2d$

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Monte Carlo Renormalization Group (MCRG)

- classical MCRG: define blocked observables
  - blockspin transformation with blocking kernel …
  - localize fixed point
  - linearize MCRG-transformation in vicinity of fixed point
    $\Rightarrow$ critical exponents
- powerful alternative implementation: Demon method
- direct with effective action or measure
- microcanonical: S fixed, no β — M. Creutz
- canonical: $\langle S \rangle$ fixed, β multiplier — Hasenbusch et al.

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Demon method

- action of system

$$S_{\rm Sys} = \sum \beta_\alpha S_\alpha$$

- action (energy) of auxiliary demon system

$$S_{\rm D} = \sum_\alpha \beta_\alpha E_\alpha, \quad E_\alpha \in [0, E_{\rm max}]$$

- partition function of joint systems

$$Z_{\rm total} = \int_0^{E_{\rm max}} \prod_\alpha \mathrm{d}E_\alpha \int \mathcal{D}\phi \, \mathrm{e}^{-S_{\rm Sys} - S_{\rm D}}$$

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- factorization $\Rightarrow$ averages of "Demon-energies"

$$\langle E_\alpha \rangle = -\frac{\mathrm{d}}{\mathrm{d}\beta_\alpha} \log \int_0^{E_{\max}} \mathrm{d}E \, \mathrm{e}^{-\beta_\alpha E}$$
$$= \frac{1}{\beta_\alpha} - \frac{E_{\max}}{\mathrm{e}^{\beta_\alpha E_{\max}} - 1} \approx \frac{1}{\beta_\alpha}$$

- $\langle E_\alpha \rangle$ from simulations $\Rightarrow \beta_\alpha$
- implementation of RG transformation
  - generate configuration in equilibrium on fine grid for some $\{\beta_\alpha\}$
  - blocking of configuration $\Rightarrow$ configuration on coarser grid
    assume: also distributed with $\mathrm{e}^{-S_{\mathrm{Sys}}}$ for some $\{\beta'_\alpha\}$

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Couplings on coarser grid?

- generate typical configuration on fine grid according to $\{\beta_\alpha\}$
- blocking of configuration
- microcanonical simulation of joint system on coarser grid
  - begin with blocked configuration and
    demon energies extracted from previous runs
  - $S_\alpha - E_\alpha$ fixed, unknown couplings do not enter
  - calculate $\langle E_\alpha \rangle \Longrightarrow \beta'_\alpha$
- generate configuration on fine grid accoring to $\{\beta'_\alpha\}$
- blocking of this configuration …

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Some Algorithmic Aspects

- assume linear dependence $C(g) = \sum c_\alpha g_\alpha$
- near nontrivial fixed point:

  $g_1, g_2, g_3 \ll g_0$

  $\Rightarrow$ only fine tuning of c_0 necessary
- near Gaussian fixed point:

  small c_1, c_2, c_3 improve quality of flow

  truncation: in general $\xi' \neq \xi/b$
- choose c_α such that $\xi' \approx \xi/b$

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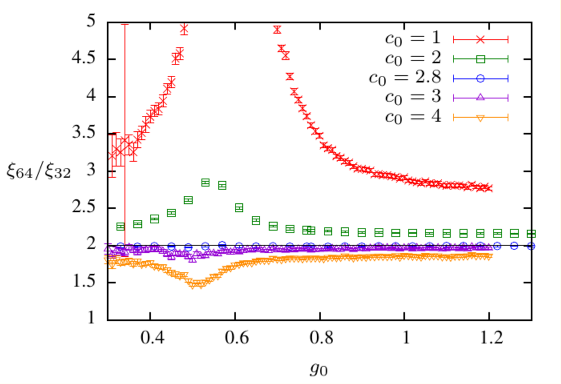
O(N) models in 2 dimensions

- well-studied ⇒ test of algorithm
- e.g. minimal one-coupling model $\xi/\xi' \approx 2 \Rightarrow c_0 \approx 2.8$



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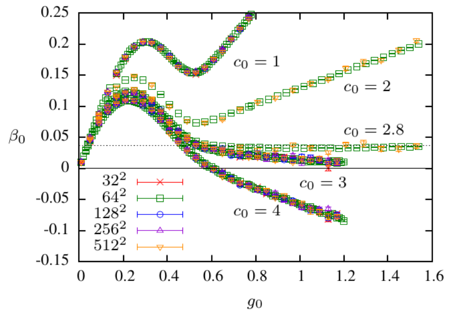
β-function in leading order of derivative expansion

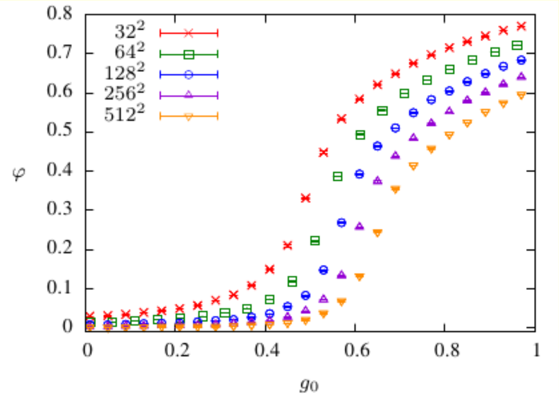
- optimal $c_0 = 2.8 \Rightarrow$
  for large coupling β-function tends to large-N result $\log(2)/(6\pi)$
- $c_0 < 2.8$: one fixed point at vanishing coupling
- $c_0 > 2.8$: additional fixed point at finite coupling: truncation artifact



- average field

$$\varphi = \left\langle \left| \frac{1}{V} \sum n_x \right| \right\rangle$$

- infinite volume: system $\forall g_0$ in symmetric phase
- finite volume: $\varphi > 0$ possible for strong coupling
- finite volume effects increase with g_0



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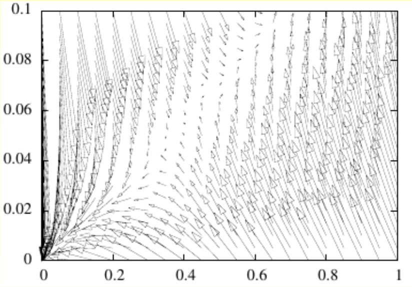
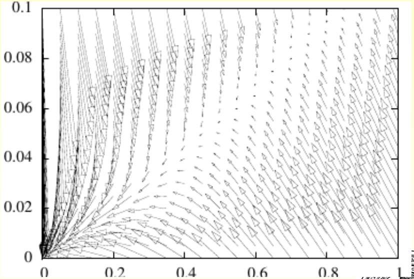
- 2-operator truncation
- flow volume-dependent: additional fixed point on small lattices
- larger lattices: Gaussian IR-fixed point at origin

  UV-attractive trivial fixed point at infinite couplings
- connected by renormalized trajectory



16^2 lattice



64^2 lattice

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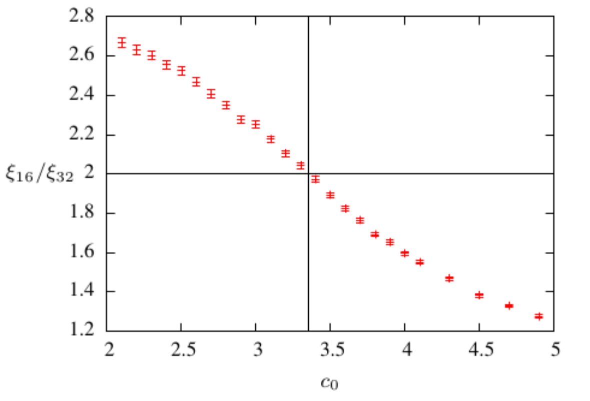
O(N) models in 3 dimensions

- simple truncation $S = (g_0 N) S_0$; blocking $32^3 \to 16^3$ sufficient
- on fine grid: $g_0^c = 0.22975(25)$
- thermodynamic limit: $g_0^c = 0.2287462(7)$ Campostrini et al. 2002)

$N = 3$:

$c_0^{\text{pert}} = 2.30$

$c_0^{\text{opt}} = 3.35$



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Nontrivial FP in simple truncation $Ng_0 S_0$

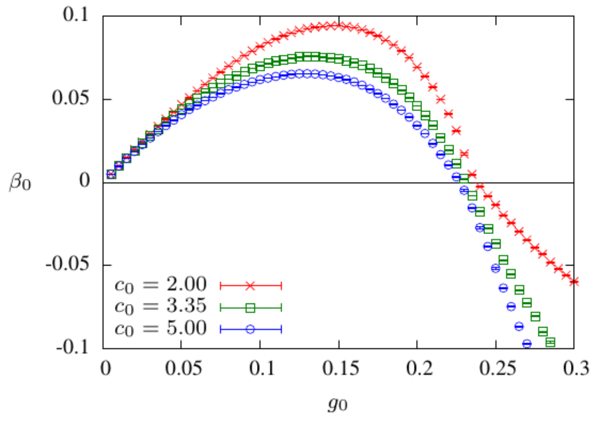
nontrivial fixed point "for all" c_0

$\beta(g_0^*) = 0$

$g_0^* = 0.2310(5)$

$g_0 < g_0^* \rightarrow$ disordered GFP

$g_0 > g_0^* \rightarrow$ ordered phase



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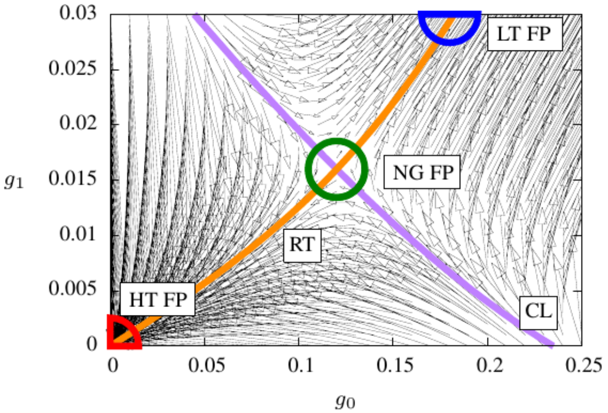
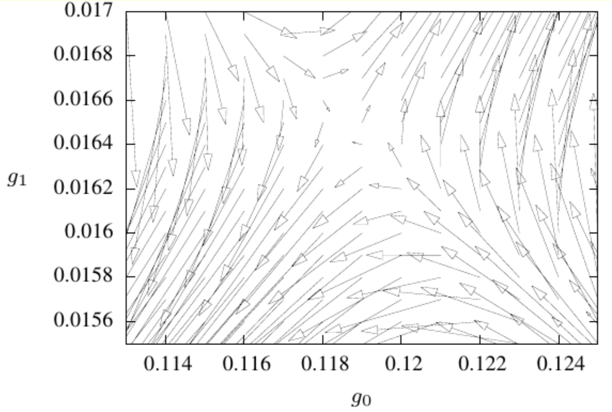
Include higher-order derivative terms

- fixed point stable?
- number of relevant directions?
- $2 \to 2$ truncation: $c_0 = 3.1$ and $c_1 = 2.5$



Vicinity of UV fixed point

- $2 \to 2$ truncation: $g_0 = 0.119(1)$ and $g_1 = 0.0164(2)$
- position of fixed points almost volume-independent



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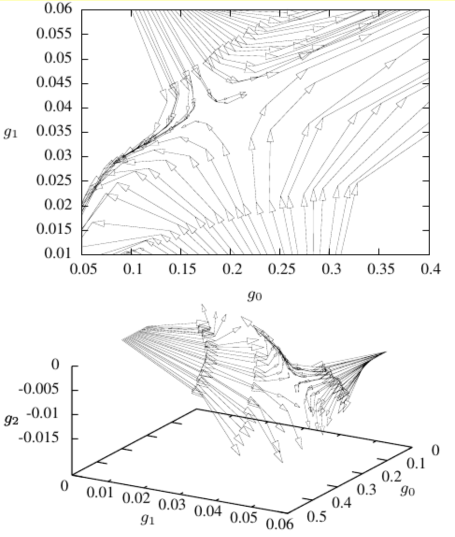
truncation with S_0, S_1, S_2

asymptotic safety:
only small number of UV-attractive (IR-relevant) directions



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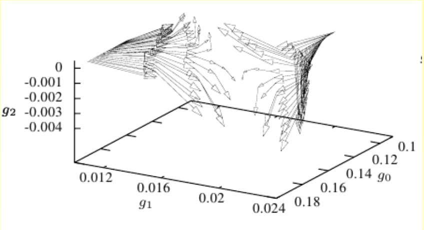
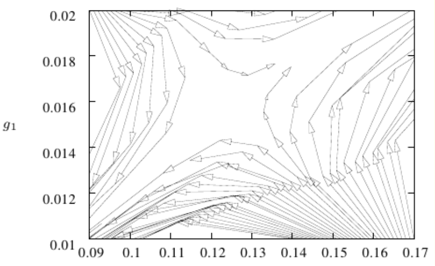
detailed picture of S_0, S_1, S_2 truncation

one UV-attractive direction

$$g_0 = 0.13(1)$$
$$g_1 = 0.016(1)(1)$$
$$g_2 = -0.0015(5)$$





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detailed picture of S_0, S_1, S_3 truncation

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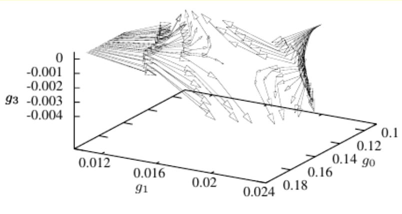

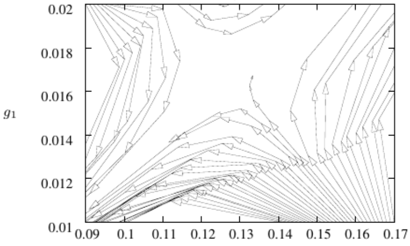

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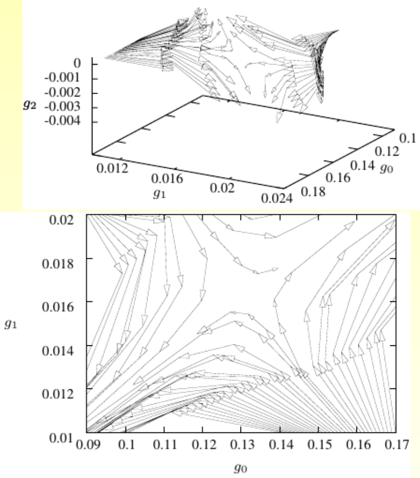

No instabilities as in FRG-approach seen!

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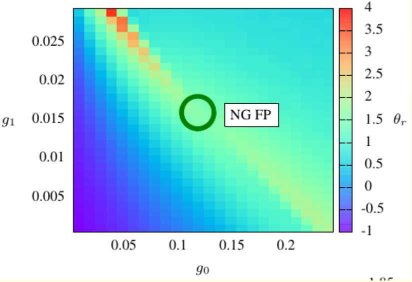
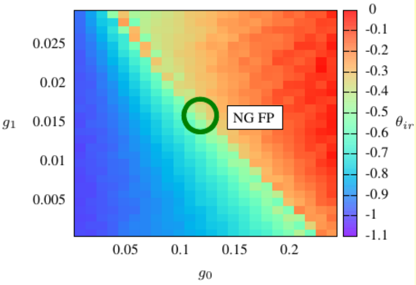
Critical exponents



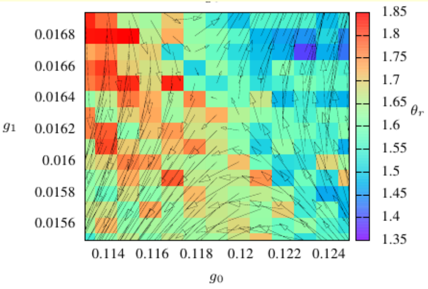


upper left $\theta_{\rm rel} = 1.61(4)$

$\nu = 0.62(3)$

$\nu_{\rm true} = 0.7112(5)$

upper right $\theta_{\rm irr} = -0.44$



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Results for critical exponent ν

Method	ν	ν/ν_{MCHT}
$1 \to 1$ trunc. ($c_0 = 3.35$)	0.51(1)	~ 0.72
$1 \to 2$ trunc. ($c_0 = 3.35$)	0.55(2)	~ 0.77
$2 \to 2$ trunc. ($c_0 = 3.1,\ c_1 = 2.5$)	0.62(3)	~ 0.87
$2 \to 2$ trunc. ($c_0 = 3.4,\ c_1 = 1.0$)	0.66(4)	~ 0.93
$3 \to 3$ trunc. ($c_0 = 3.1,\ c_1 = 2.5,\ c_2 = 0$)	0.64(3)	~ 0.90
FRG	0.704	~ 0.99
MCHT	0.7112(5)	1
MC	0.7116(10)	~ 1
RG	0.706	~ 0.99
HT	0.715(3)	~ 1

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Dependence on N

- optimal c_α in blocking kernel depends on N
- comparable global flow diagrams for $N = 3, 4, 5, \ldots$
- position of UV-FP varies with N
- critical exponents: comparison with large-N expansions
  comparison with RG-expansions

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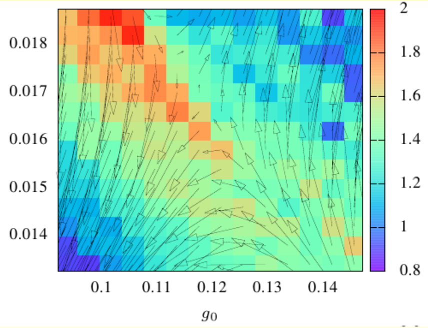
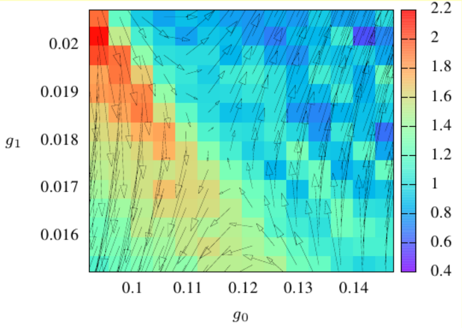
Flow diagrams and critical exponents for $N > 3$



O(4)-model



O(6)-model .

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Critical exponent ν

N	2	3	4	5	6
$1 \to 1$ truncation	0.42	0.51	0.57	0.63	0.65
$2 \to 2$ truncation	0.64(4)	0.66(4)	0.71(5)	0.78(6)	0.81(6)
FRG	-	0.704	0.833	-	0.895
HT exp.	0.677(3)	0.715(3)	0.750(3)	-	0.804(3)
RG exp.	0.607	0.706	0.738	0.766	0.790

N	7	8	9	10
$1 \to 1$ truncation	0.68	0.65	0.62	0.58
$2 \to 2$ truncation	0.86(7)	0.84(7)	0.89(8)	
FRG	-	0.912	-	0.920
HT exp.	-	0.840(3)	-	0.867(4)
RG exp.	0.811	0.830	0.845	0.859

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Conclusion

- FRG detects nontrivial UV fixed point for every N
- stability problem of FRG with one 4^{th}-order operator
  problem of covariant background-field method?
  $\Rightarrow$ further studies necessary
- in MCRG approach no stability problems (different truncation)
- FRG and MCRG in qualitative agreement (modulo instability)
- results sufficient for global picture on flow diagram
- for critical exponents: better methods are available

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