Lattice Supersymmetry

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work in progress

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1. Introduction

2. Questions in lattice-susy

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4. Susy Quantum Mechanics

5. 2d-Wess-Zumino Models

Introduction

Why susy on lattice?

- nonperturbative definition of QFT
- spectra, susy breaking, phase diagrams, ...
- careful treatment of fermions
- check conjectured results ($\mathcal{N} = 1$ SYM)
- confirm/extend existing results ($\mathcal{N} = 2$ SYM)
- integrable systems, susy inspired approximations ...

Questions in lattice-susy

fine tuning problem

- classification of local ct for correct continuum limit
- susy broken $\rightarrow$ additional symmetry-restoring ct

$\longrightarrow$ fine tuning

- no fine tuning if susy partly intact

Q-exact models: $S = QX, \quad Q^2 = 0$

$\Downarrow$

local Nicolai-map exists

either case: exact susy WI $\longrightarrow$ absence of ct

models without fine tuning

- supersymmetric quantum mechanics (testbed)
- 2 dimensions:
  N=2 Wess-Zumino
  SYM_2 needs no susy-restoring ct (Sugino)
  twisted nonlinear σ-models (Catterall)
- 4 dimensions:
  Wess Zumino with GW fermions (Feo & Bonini),
  SYM_4 with Ginsparg-Wilson fermions
  (chiral = supersymmetric limit)
- claim: super-QCD, extended SYM
  partial results by Kaplan et.al, Sugino, Itoh et.al

additional work required

stability problem

Euclidean S_B unbounded below for $N=2,4$ SYM_4:

$$\mathcal{L}_B = \tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}\,(D_\mu A)^2 + \tfrac{1}{2}\mathrm{Tr}\,(D_\mu B)^2$$
$$-\tfrac{1}{2}g^2\,\mathrm{Tr}\left([A,B]^2\right), \quad A=-A^\dagger,\ B=B^\dagger$$

various arguments:
Wick-rotation, dimensional reduction $\rightarrow$ non-compact R-symmetry

focus on $\mathcal{N}=1$ SYM (DESY-Münster-Roma)

$$U_A(1) \stackrel{\text{anomaly}}{\longrightarrow} \mathcal{Z}_{2N_c} \stackrel{\text{condensate}}{\longrightarrow} \mathcal{Z}_2 \quad \text{(VY, Shifman)}$$

N_c ground states $\langle\bar{\lambda}\lambda\rangle = c\Lambda^3 z^k,\ k=1,\dots,N_c$

particle spectrum?

- RELATED EARLIER WORK:
  Dondi, Nicolai: exact susy on lattice, 77
  Elitzur, Schwimmer: N=2 Wess-Zumino in d=2, 82
  Sakai, Sakamoto: lattice susy & Nicolai mapping, 83
  Beccaria, Curci, D'Ambrosio: simulation & Nicolai map, 98
- RELATED RECENT WORK:
  Catterall et.al: exact susy on lattice, 01 –
  Cohen, Kaplan, Katz, Unsal: susy with N charges, 02 –
  Feo, Bonini, et.al: simulations of WZ-models, 02 –
  Sugino: extended SYM and exact susy on lattice, 04 –
  Giedt Poppitz: perturbation theory, det D, 04 –
- PARTICLE PHYSICS:
  Montvay et.al: N=1 SYM on lattice, 01 –

Reconciling lattice and supersymmetry

- all susy broken for naive discretisation of S or H

  reason: no Leibniz rule on finite lattice:

$$(\partial f)(x) = \sum_y \partial_{xy} f(y) \quad \text{linear}$$

$$\partial(f \cdot g) = (\partial f) \cdot g + f \cdot (\partial g) \Longrightarrow \partial \equiv 0$$

$\Rightarrow$ central charges not central

superfield $\cdot$ superfield $\neq$ superfield

- $\partial \neq -\partial^T \Rightarrow$ large lattice artifacts in $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\}$
- fermion-doubling $\xrightarrow{\text{susy}}$ boson-doubling $\overset{?}{\Rightarrow}$ Ginsparg-Wilson fermions $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$

Lattice-derivatives & Dirac operator

ultralocal derivatives

- For all derivatives: $\gamma_* D\gamma_* = -D^\dagger \Longrightarrow \lambda, \lambda^*$
- forward/backward derivatives $\partial^f_\mu,\ \partial^b_\mu$

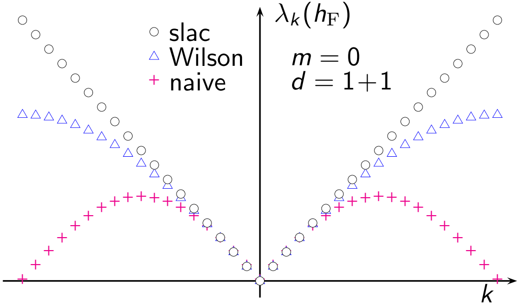
$$\partial^a_\mu \equiv \tfrac{1}{2}(\partial^f_\mu + \partial^b_\mu) = -(\partial^a_\mu)^T$$

- naive: $D_n = \gamma^\mu \partial^a_\mu + m$
  chiral, ultra-local, cheap, 2^d-doubling
  Wilson: $D_w = D_n - \frac{r}{2}\triangle$
  undoubled, ultra-local, cheap, non-chiral
- used for N=1 SYM_4, susy broken $\rightarrow$ fine tuning

Slac-fermions:

chiral, no doubling, antisymmetric, 'exact' spectrum, non-local

spectrum of h_F in $1+1$:



WZ with slac-derivative perturbatively renormalizable (Bergner)

∂_{slac} very accurate (see below)

Susy Quantum Mechanics

Hamiltonian formulation

- supercharge

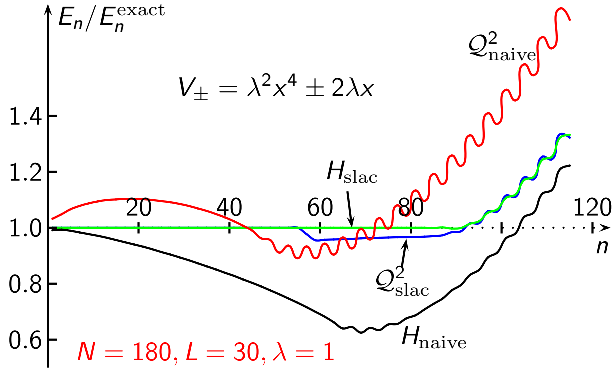
$$\mathcal{Q} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} = \mathcal{Q}^\dagger, \quad A = \partial + W$$

- super-Hamiltonian $H = \mathcal{Q}^2$

$$H = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = (-\partial^2 + W^2)\mathbb{1} \pm W'\sigma_3$$

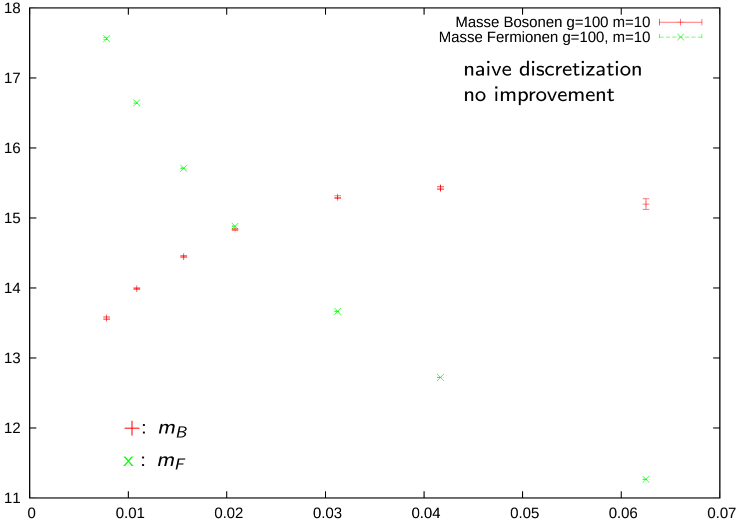
- space-lattice $\Rightarrow$

$$\begin{aligned} \text{discretize } \mathcal{Q} \quad &\Longrightarrow \quad \mathcal{Q}^2_{\text{naive}}, \mathcal{Q}^2_{\text{slac}} \\ \text{discretize } \mathcal{Q}^2 \quad &\Longrightarrow \quad H_{\text{naive}}, H_{\text{slac}} \end{aligned}$$



- $H_{\mathrm{slac}} - \mathcal{Q}^2_{\mathrm{slac}}$ from missing Leibniz-rule
- $H_{\mathrm{e}} - \mathcal{Q}^2_{\mathrm{naive}}$ from $\partial \neq -\partial^T$
- Slac: $\sim N/3$ lowest energies with 10^{-4} accuracy
- first MC-results for Wess-Zumino (HMC) with slac -:)

What can go wrong?



What can go wrong?

$$\partial_\tau + m + P(\tau), \quad \tau = [0,\beta], \quad \text{periodic BC}$$
$$\mathcal{P} = \int P(\tau), \quad \mathcal{W} = m\beta + \mathcal{P}$$

- continuum result:

$$\det{}_{\rm cont} \equiv \det\left(\frac{\partial + W}{\partial + m}\right) = \frac{\sinh\left(\frac{1}{2}\mathcal{W}\right)}{\sinh(\frac{1}{2}m\beta)}$$

- forward & slac derivatives ($N \to \infty$):

$$\det{}_{\rm forward} \to e^{-\mathcal{P}/2}\det{}_c \quad , \quad \det{}_{\rm slac} \to \det{}_c$$

- Gaussian model for $\xi = (\xi_1, \dots, \xi_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$

$$Z = \int \mathcal{D}\xi \mathcal{D}\bar{\alpha} \mathcal{D}\alpha \, e^{-S_0[\xi,\alpha,\bar{\alpha}]}, \quad S_0 = \tfrac{1}{2}(\xi,\xi) + (\bar{\alpha},\alpha)$$

symmetry of S_0

$$\begin{aligned} \delta\xi &= A\bar{\epsilon}\alpha \quad , \quad \delta\alpha = 0 \\ \delta\bar{\alpha} &= -\bar{\epsilon}A^T\xi + S\alpha, \quad S = S^T \end{aligned}$$

- Nicolai map: $x \to \xi(x)$ invertible $\mathbb{R}^n \to \mathbb{R}^n$.

$$\begin{aligned} Z &= \int \mathcal{D}x \det \xi' \, \mathcal{D}\bar{\alpha}\mathcal{D}\alpha \, e^{-S_0[\xi(x),\alpha,\bar{\alpha}]} \\ &= \int \mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S[x,\psi,\bar{\psi}]}, \quad \xi' = (\xi_{i,j}) \\ S &= S_0[\xi(x), \psi, \bar{\alpha}(\bar{\psi})] \quad \text{with} \quad \bar{\alpha} = \xi'^T(x)\bar{\psi} \end{aligned}$$

locality of symmetry in new variables?

$$\alpha = \psi, \quad \delta\xi = \xi'\delta x \stackrel{!}{=} A\bar{\epsilon}\alpha = A\bar{\epsilon}\psi$$

simple for δx if $A = \xi' \Rightarrow \delta x = \bar{\epsilon}\psi \Rightarrow$

$$\delta\bar{\alpha} = \delta\xi'^T\bar{\psi} + \xi'^T\delta\bar{\psi} \stackrel{!}{=} -\bar{\epsilon}A^T\xi + S\alpha = -\bar{\epsilon}\xi'^T\xi + S\psi$$

$$\text{solution:} \quad \delta\bar{\psi} = -\bar{\epsilon}\xi \quad \text{and} \quad S_{ij} = \frac{\partial^2\xi}{\partial x_i \partial x_j}\bar{\epsilon}\bar{\psi}_p$$

exact lattice supersymmetry

$$S = \frac{1}{2}(\xi(x), \xi(x)) + (\bar{\psi}, \xi'(x)\psi)$$

$$\delta x = \bar{\epsilon}\psi, \quad \delta\psi = 0, \quad \delta\bar{\psi} = -\bar{\epsilon}\xi(x)$$

- particular choice: $\xi(x) = \partial x + W'(x) \Rightarrow$ SQM

$$\begin{aligned} S &= \tfrac{1}{2}(\partial x, \partial x) + \tfrac{1}{2}(W', W') + (W', \partial x) \\ &+ (\bar{\psi}, (\partial + W'')\psi) \end{aligned}$$

- supersymmetric for all ∂

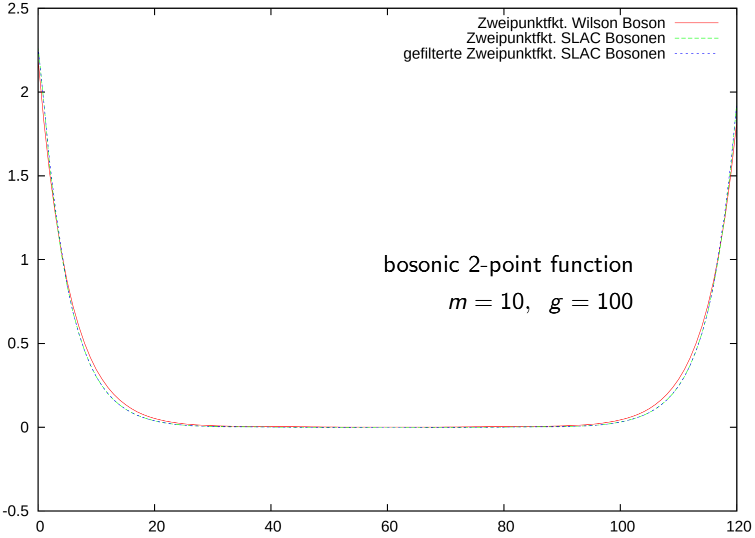
$$\delta x = \bar{\epsilon}\psi, \;\; \delta\psi = 0, \;\; \delta\bar{\psi} = -\bar{\epsilon}(\partial x + W'(x)), \quad \delta^2 = 0$$

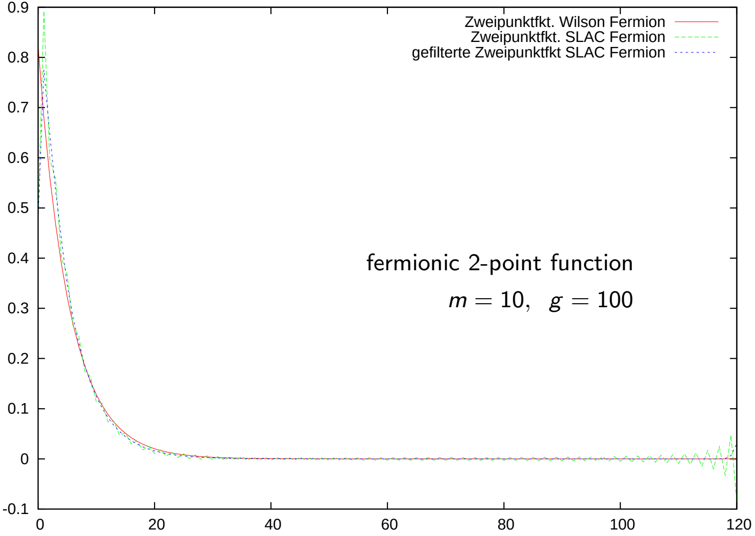
- no Leibniz rule $\Rightarrow (W'(x), \partial x) \neq 0$
  improvement essential for susy Ward identities
- second continuum susy for $(W', \partial x) \to -(W', \partial x)$

$$\delta' x = \bar{\psi}\epsilon, \;\; \delta'\bar{\psi} = 0, \;\; \delta'\psi = (\partial x - W'(x))\epsilon, \quad \delta'^2 = 0$$

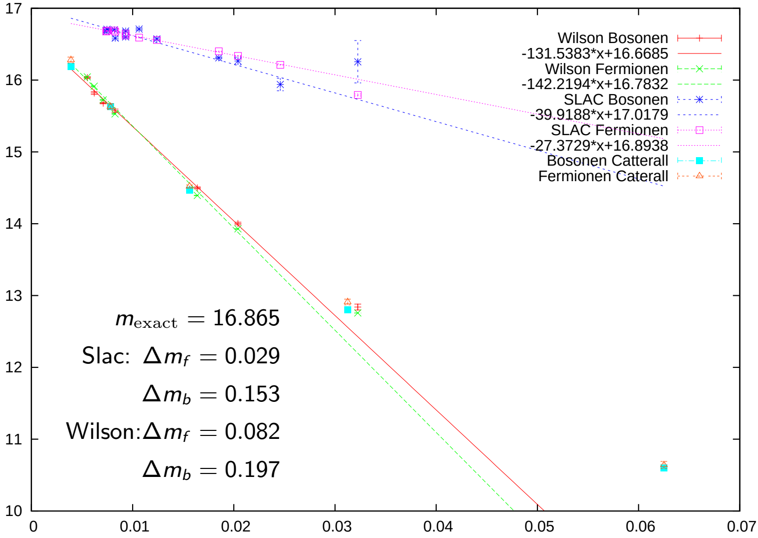
Results of Simulations

- Gibbs-phenomen for slac derivative
- data-filtering improves accuracy
- optimal filtering a la Tanner
- exponential fit to filtered 2-point function
- Slac on 'moderate' lattices superior
- comparable to fix-point actions

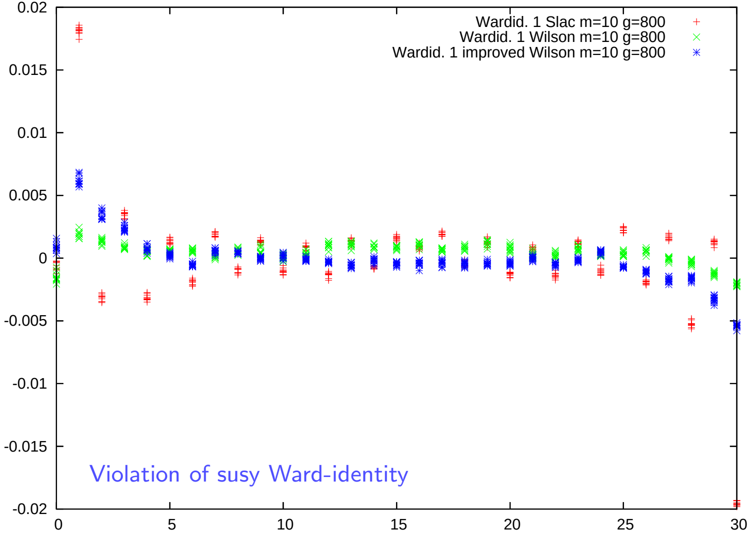

Zweipunktfkt. Wilson Boson
Zweipunktfkt. SLAC Bosonen
gefilterte Zweipunktfkt. SLAC Bosonen
bosonic 2-point function
$m = 10, \quad g = 100$
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100
120

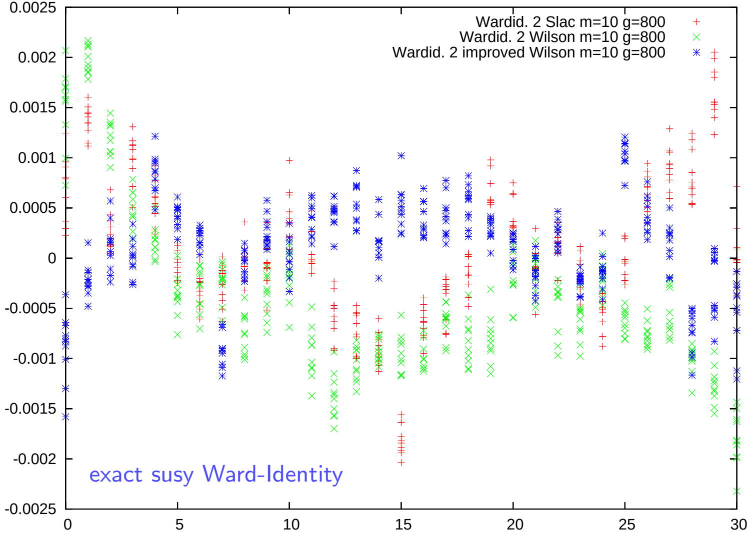


Zweipunktfkt. Wilson Fermion
Zweipunktfkt. SLAC Fermion
gefilterte Zweipunktfkt SLAC Fermion
fermionic 2-point function
$m = 10, \quad g = 100$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
20
40
60
80
100
120



Wilson Bosonen
-131.5383*x+16.6685
Wilson Fermionen
-142.2194*x+16.7832
SLAC Bosonen
-39.9188*x+17.0179
SLAC Fermionen
-27.3729*x+16.8938
Bosonen Catterall
Fermionen Caterall
$m_{\text{exact}} = 16.865$
Slac: $\Delta m_f = 0.029$
$\Delta m_b = 0.153$
Wilson: $\Delta m_f = 0.082$
$\Delta m_b = 0.197$


Wardid. 1 Slac m=10 g=800
Wardid. 1 Wilson m=10 g=800
Wardid. 1 improved Wilson m=10 g=800
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0
5
10
15
20
25
30
Violation of susy Ward-identity


Wardid. 2 Slac m=10 g=800
Wardid. 2 Wilson m=10 g=800
Wardid. 2 improved Wilson m=10 g=800
0.0025
0.002
0.0015
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
0
5
10
15
20
25
30
exact susy Ward-Identity

2d-Wess-Zumino Models

- 2 supersymmetries: complex scalar $\xi = \xi_1 + i\xi_2$
  $\xi_1, \xi_2 \in \mathbb{R}^n$ lattice fields; Gaussian integral

$$Z = \int \mathcal{D}\xi_1 \mathcal{D}\xi_2 \, e^{-S_0}, \quad S_0 = (\vec{\xi}, \vec{\xi}), \quad \vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

- Nicolai: holomorphic $U(\phi) + iV(\phi) = W(\phi_1 + i\phi_2)$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \not{\partial} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \partial_{\phi_1} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (\gamma^1 = \sigma_3, \; \gamma^2 = \sigma_1)$$

Cauchy-Riemann for U, V, any ∂_1, ∂_2

$$S_B[\phi] = S_0[\xi(\phi)] = \left(\nabla\vec{\phi}, \nabla\vec{\phi}\right) + \left(\nabla_\phi U, \nabla_\phi U\right) + 2\sum\nolimits_{i,j} \left(\partial_{\phi_j} W_i, \partial_i \phi_j\right)$$

no Leibniz $\Rightarrow$ last term $\neq 0$!

- Jacobian

$$\left(\frac{\partial \xi_i}{\partial \phi_j}\right) = \partial\!\!\!/ + U_{,11} + i\gamma_* U_{,12}\,, \quad i\gamma_* = \gamma^0\gamma^1 = i\sigma_2$$

- theory for ϕ, ψ with $S = S_B + S_F$,

$$S_F[\psi, \bar{\psi}] = \left(\bar{\psi}, \{\partial\!\!\!/ + U_{,11} + i\gamma_* U_{,12}\,\}\psi\right)$$

- exact nilpotent lattice-supersymmetry

$$\delta\vec{\phi} = \bar{\epsilon}\psi, \quad \delta\psi = 0, \quad \delta\bar{\psi} = -\bar{\epsilon}\left(\partial\!\!\!/\vec{\phi} + \partial_{\phi_1}\vec{W}\right)$$

generated by $\mathcal{Q}$ with $\mathcal{Q}^2 = 0$

- Theorem: (Kirchberg, Laenge, AW): On a spatial lattice with N lattice points and for superpotential $W = \lambda\phi^p + \ldots$ there are $(p-1)^N$ normalizable zero-modes of $H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$

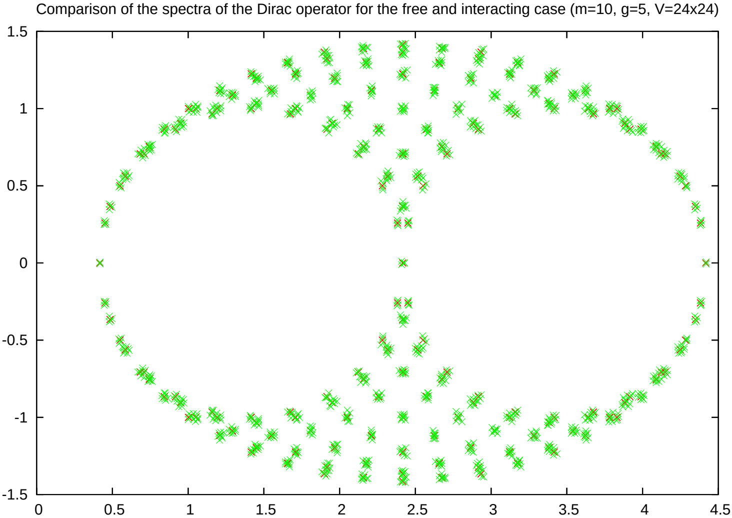
$\Rightarrow$ ϕ^4-model has 2^N susy ground states.

- should be detected in simulations?
- First simulations (jenLaTT & Linux-cluster)
  dynamical fermions via HMC with PF-fields

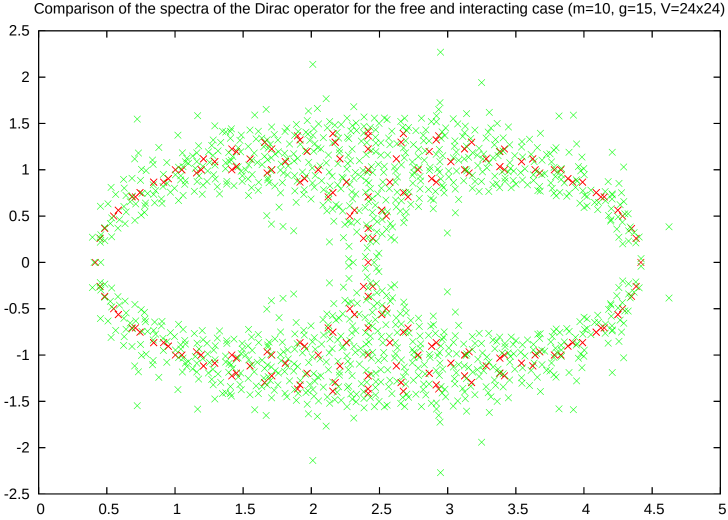
$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-(\bar{\psi},M\psi)} = \pm \int \mathcal{D}\vec{\chi}\, e^{-(\vec{\chi},A\vec{\chi})}, \quad A = (MM^t)^{-1}$$

- previously: ultralocal derivatives
  now: Slac, overlap and fix-point: inverter need MV-operations!

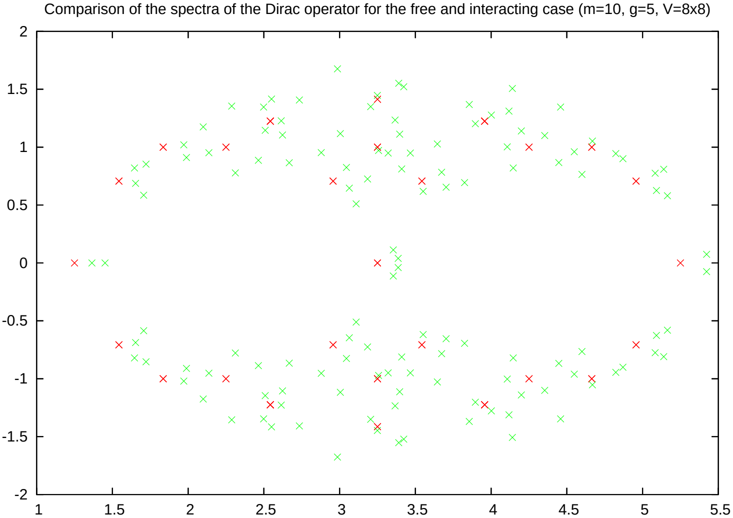
det $M \in \mathbb{R}$ changes sign?



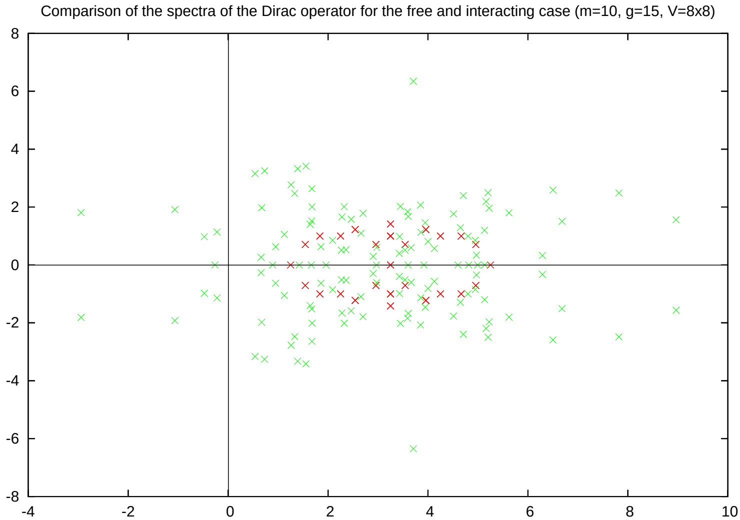
$m = 10,\ g = 5,\ V = 24 \times 24,$ × free, × interacting



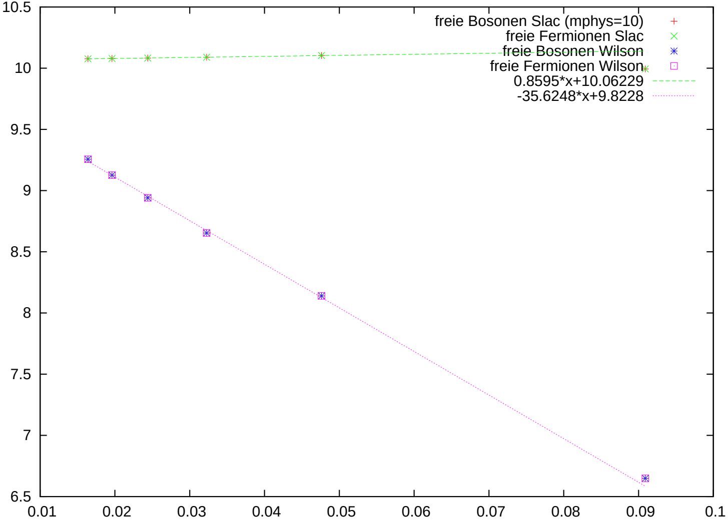
$m = 10,\ g = 15,\ V = 24 \times 24,$ × free, × interacting



$m = 10,\ g = 5,\ V = 8 \times 8,$ × free, × interacting



$m = 10,\ g = 15,\ V = 8 \times 8,$ × free, × interacting



$m = 10$, $+$: $Slac_B$, $\times$: $Slac_F$, $*$: Wil_B, $\square$: Wil_F

Summary

- simulations with $L = 8^2 ... 64^2$ on the way.
- mass-degeneracy of bosons and fermions
  Ward identities
- Nicolai-improvement for $N = 1$ models in $d = 2, 4$?
  must treat sign($Pf(M)$) numerically
- susy-breaking, phases of $N = 1$ models
- parallized code ⇒ 4-dimensional WZ-models
- supersymmetric Yang-Mills theories with dynamical fermions?