Susy Lattice Models

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earlier work: Annals Phys. **303**, **315**, **316**, arXiv:0705.2212

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Introduction

Lattice Susy

- susy breaking, phase diagrams, treatment of fermions
- how much susy we give up on lattice: off- and on-shell
- low lying masses, Ward identities, matching anomalies
- fine tuning: improvement program in doubt

- RELATED EARLIER WORK:
  Dondi, Nicolai: exact susy on lattice, 77
  Elitzur, Schwimmer: $\mathcal{N} = 2$ Wess-Zumino in d=2, 82
  Sakai, Sakamoto: lattice susy & Nicolai mapping, 83
  Beccaria, Curci, D'Ambrosio: simulation & Nicolai map, 98
- RELATED RECENT WORK:
  Catterall et.al: exact susy on lattice, 01 –
  Cohen, Kaplan, Katz, Unsal: susy with N charges, 02 –
  Feo, Bonini, et.al: simulations of WZ-models, 02 –
  Sugino: extended SYM and exact susy on lattice, 04 –
  Giedt, Poppitz: perturbation theory, det D, 04 –
- PARTICLE PHYSICS:
  Montvay et.al: $\mathcal{N} = 1$ SYM on lattice, 01 –

$d = \mathcal{N} = 2$ Supersymmetric Landau-Ginzburg Models

- Φ chiral superfield: complex ϕ, Dirac ψ, complex auxiliary F:

$$\mathcal{L}(x) \propto \int d^2\theta d^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left[\int d^2\,\theta W(\Phi) + \text{h.c.}\right]$$

- $K(z, \bar{z})$: potential on Kählerian target space $\mathcal{M}$,

$$g_{p\bar{q}} = \frac{\partial^2 K}{\partial z^p \partial \bar{z}^{\bar{q}}}, \quad 1 \le p, \bar{q} \le n.$$

  $W(z)$: holomorphic superpotential

- K and W enter scalar potential

$$V = g^{p\bar{q}} W_p \bar{W}_{\bar{q}}, \quad W_p = \partial W / \partial z_p, \quad \bar{W}_{\bar{q}} = (W_q)^*$$

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- W not renormalized in perturbation theory (R-symmetries)
- But $K \to K_{\rm eff}$ from matching microscopic and IR lattice data simulations $\Rightarrow$ nonperturbative renormalization $K \to K_{\rm eff}$
- first step: $\to \langle\phi\rangle,\ m_{\rm light},\dots$ constrains

$$V_\phi,\quad V_{\phi\phi},\quad V_{\phi\bar\phi}$$

$\mathcal{N}=2$ Wess-Zumino model: $\mathcal{M}=\mathbb{C},\ K=\bar z z$ and

$$\mathcal{L}_{\rm wz} \propto \int d^2\theta d^2\bar\theta\,\bar\Phi\Phi + \left[\int d^2\theta\, W(\Phi) + \text{h.c.}\right]$$

Supersymmetric $O(3)$ model: $\mathcal{M}=CP^1, W=0$ and

$$\mathcal{L}_\sigma \propto \int d^2\theta d^2\bar\theta \log\left(1+\bar\Phi\Phi\right)$$

or Calabi-Yau models

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Wess-Zumino Model

- Euclidean off-shell Lagrangian density for component fields

$$\mathcal{L}_{\mathrm{wz}} = 2\bar{\partial}\bar{\phi}\partial\phi + \bar{\psi}M\psi + \frac{1}{2}|W'(\phi)|^2 - \frac{1}{2}\left|F + \bar{W}'(\bar{\phi})\right|^2$$

  On-shell density bounded below:

$$\mathcal{L}_{\mathrm{wz}} = 2\bar{\partial}\bar{\phi}\partial\phi + \bar{\psi}M\psi + \frac{1}{2}|W'(\phi)|^2$$

- Dirac operator with scalar and pseudoscalar Yukawa coupling

$$M = \not{\partial} + W''(\phi)P_+ + \bar{W}''(\bar{\phi})P_-$$

- Here: superpotential

$$W(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$$

Symmetries

- $(2,2)$ supersymmetry:

$$\delta\phi \sim \bar{\epsilon}\psi, \quad \delta\psi \sim \bar{\epsilon}(\partial\phi + F), \quad \delta F \sim \bar{\epsilon}\partial\psi$$

  unbroken for continuum model → susy multiplets (Witten index)
  conflict with lattice regularization

- R-type symmetries:

$$\begin{aligned} U(1)_V \quad &; \quad \psi \to e^{i\lambda}\psi,\ \bar{\psi} \to \bar{\psi}e^{-i\lambda} \\ \mathbb{Z}_2 \quad &: \quad \phi \to -(\phi + m/g),\ \psi \to \gamma_*\psi,\ \bar{\psi} \to -\bar{\psi}\gamma_* \end{aligned}$$

  $\mathbb{Z}_2$ probably broken for all $g \neq 0$.

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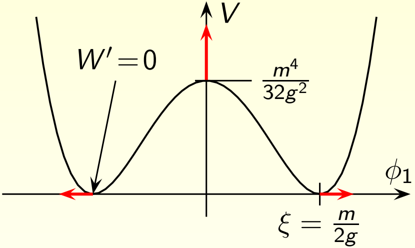
Potentials for WZ-Models

Classical potential for $\phi = (\phi_1 - \xi) + i\phi_2$ with shift $\xi = m/2g$:

$$V = \frac{1}{2}|W'|^2 = \frac{g^2}{2}\Big\{(\phi_1^2 + \phi_2^2)^2 + 2\xi^2\left(\phi_2^2 - \phi_1^2\right) + \xi^4\Big\}.$$



$$W'' = 2g(\phi_1 + i\phi_2)$$

$$S(\phi,\psi) = \text{kinetic} + \text{Yukawa} + \frac{1}{2}|W'(\phi)|^2$$

Finite one-loop on-shell result

$$U^{(1)}_{\rm on}(\varphi) = \frac{1}{2}|W'|^2 - \frac{|W''|^2}{8\pi}\Big(\log(1-X^2) + 2X\,\text{artanh}(X)\Big),$$

where

$$X = \frac{|W'''W'|}{|W''|^2} = \frac{|W'|}{2g|\varphi|^2}$$

extreme at classical minima $\pm\xi$, where $W' = 0$. Effective mass

$$m^2_{\rm eff} = m^2\Big(1 - \frac{1}{\pi}\frac{g^2}{m^2}\Big) \Rightarrow g_c = \sqrt{\pi}m$$

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$$S(\phi,\psi,F)=\text{kinetic}+\text{Yukawa}-\frac{1}{2}|F|^2+\frac{1}{2}FW'+\frac{1}{2}\bar{F}\bar{W}'$$

Finite one-loop off-shell result

$$U^{(1)}_{\text{off}}(\varphi)=\frac{1}{2}|W'|^2-\frac{|W''|^2}{8\pi}\Big(\log(1-X^2)+2X\,\text{artanh}(X)\Big),$$

where now

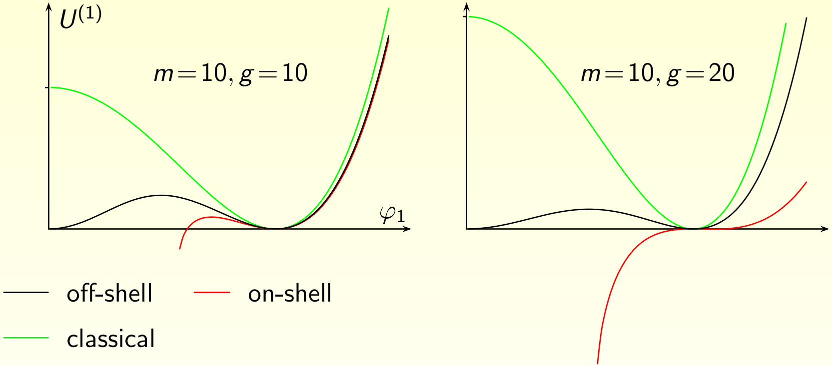
$$X=\frac{|FW'''|}{|W''|^2},\quad |F|+\frac{|W'''|}{4\pi}\text{artanh}\left(\frac{|FW'''|}{|W''|^2}\right)-|W'|=0$$

minimal at classical minima $\pm\xi$, where $W'=0$. Effective mass

$$m^2_{\text{eff}}=\frac{m^2}{1+\frac{g^2}{\pi m^2}}=m^2\left(1-\frac{1}{\pi}\frac{g^2}{m^2}+\dots\right)$$

supersymmetry unbroken

$U_{\rm on}^{(1)}$ non-positive, complex, unstable for strong couplings
$U_{\rm off}^{(1)}$ real, stable for all g



$U_{\rm on}^{(1)} \neq U_{\rm off}^{(1)}$ although $U_{\rm on} = U_{\rm off}$

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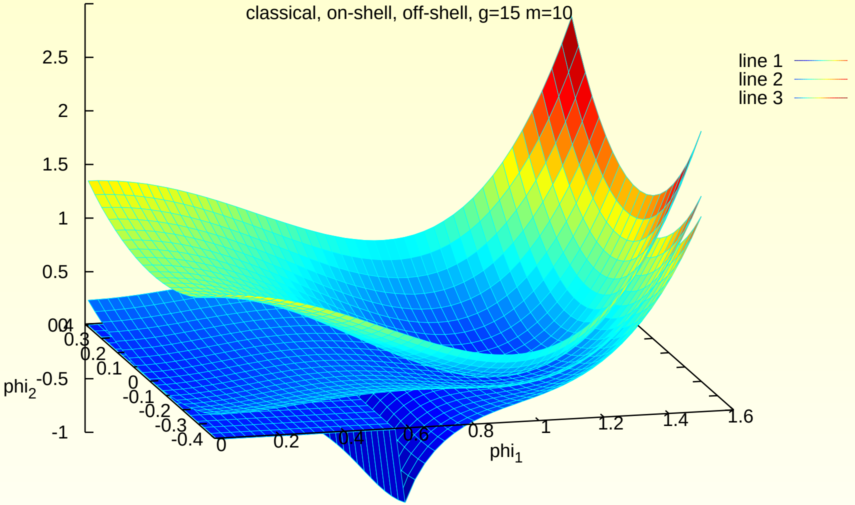

classical, on-shell, off-shell, g=15 m=10
line 1
line 2
line 3
2.5
2
1.5
1
0.5
0
-0.5
-1
phi_2
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
phi_1

Zero-momentum toy model for constant (ψ, ϕ, F)

- classical action

$$S = \bar{\psi} M \psi + \frac{1}{2}(FW' + \bar{F}\bar{W}' - \bar{F}F), \quad M = W'' P_+ + \bar{W}'' P_-$$

potentials via Legendre transform of $w(j)$ in

$$e^{w(j)} = \int d(\psi, \phi, F)\, e^{-S + j\phi_1}$$

- Saddle-point approximations before and after eliminating F:

$$\begin{aligned} U^{(1)} &= \frac{1}{2}|W'|^2 + \frac{1}{2}\log\left(1 - X^2\right), \qquad X = \frac{|W''' F|}{|W''|^2} \\ |F| &= |W'| \qquad \text{on-shell} \\ |F| &= \left(|F|^2 - |W''|^2\right)\left(|F| - |W'|\right) \qquad \text{off-shell} \end{aligned}$$

- exact constraint effective potential

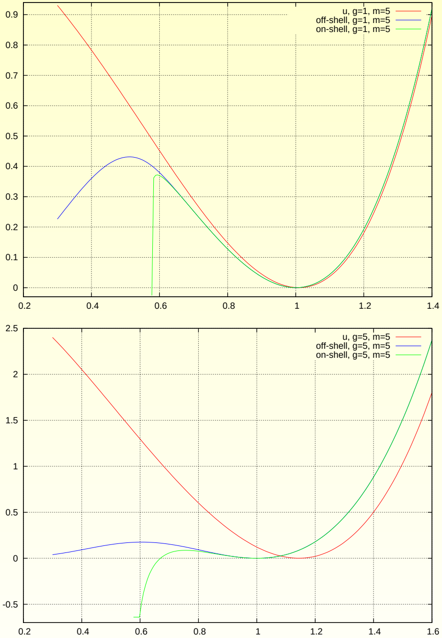
$$\tilde{u}^{(1)}(\varphi) = -\log \int e^{-S}\delta(\phi_1-\varphi) = V(\varphi,0) - \log\Delta(\varphi)$$

$$\Delta(\varphi) = 4\sqrt{gH}\, e^{H^2}\left\{g\varphi^2 K_{1/4}(H^2) + H\left(K_{3/4}(H^2) - K_{1/4}(H^2)\right)\right\}$$

- on- and off-shell 'masses' ($\alpha = 2g/m^2$)

$$\text{on-shell:}\qquad m_{\text{eff}}^2 = m^2\left(1-\alpha^2\right)$$

$$\text{off-shell:}\qquad m_{\text{eff}}^2 = \frac{m^2}{1+\alpha^2}$$



weak coupling

$U_{\rm on}^{(1)}$ complex, unstable
shifted minimas $\Rightarrow$
wave function $Z \neq 1$

strong coupling

Simulations

- no Leibniz $\rightarrow$ susy broken for naive discretization, $\mathcal{Z}$ not central
- fermion-doubling $\xrightarrow{\text{susy}}$ boson-doubling
- if $\partial \neq -\partial^T \Rightarrow$ large lattice artifacts in $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\}$
- Wilson fermions:
  no doubling, ultra-local, cheap, non-chiral, fine tuning
  with improvement: $S = S_{\mathrm{st}} + S_{\mathrm{impr}}$ with $\langle S_{\mathrm{st}}\rangle \approx \langle S_{\mathrm{impr}}\rangle \gg \langle S\rangle$
- Slac fermions $D_s = \gamma^\mu \partial_\mu^{\mathrm{slac}} + m$:
  no doubling, chiral, antisymmetric, 'exact', non-local
  one-loop renormalizable (Bergner)
  with improvement $\langle S_{\mathrm{st}} - S_{\mathrm{impr}}\rangle \ll \langle S\rangle \Rightarrow$ reweighting?
- HMC, Fourier accelerated; direct inverter in $1d$ and very small lattices in $2d$; else pseudo fermions; reweighting for $g \ll m$ (with Frommer) so far no PHMC or RHMC needed (checked for small EV)

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- Nicolai map of bosonic variables

$$\phi \longrightarrow \xi(\phi), \quad \mathbb{R}^n \to \mathbb{R}^n$$
$$S_{\rm susy} = \frac{1}{2}\sum_x \xi_x(\phi)^2 + \sum_{x,y} \bar{\psi}_x \frac{\partial \xi_x}{\partial \phi_y} \psi_y$$

exact susy: $\delta^{(1)}\phi_x = \bar{\epsilon}\psi_x, \;\; \delta^{(1)}\psi_x = 0, \;\; \delta^{(1)}\bar{\psi}_x = -\bar{\epsilon}\xi_x$

- choice: $\xi_x(\phi) = (\partial\phi)_x + W_x$

$$\begin{aligned} S_{\rm susy} &= \frac{1}{2}\sum_x ((\partial\phi)_x + W_x)^2 + \sum_{x,y} \bar{\psi}_x (\partial_{xy} + W_{xy})\psi_y \\ &= S_{\rm naive} + \sum_x W_x(\phi)(\partial\phi)_x \end{aligned}$$

Nicolai-improvement exists also for $d = \mathcal{N} = 2$-Wess-Zumino discretize supersymmetric quantum mechanics:

$$S_{\rm cont} = \int d\tau \Big(\frac{1}{2}\dot\phi^2 + \frac{1}{2}W'^2 + \bar\psi\dot\psi + \bar\psi W''\psi + \text{improvement} \Big)$$

model	$k_{\rm bos}$	$m_{\rm bos}(0)$	$k_{\rm ferm}$	$m_{\rm ferm}(0)$
$S_{\rm w}^{\rm naiv}$	139.52 ± 8.45	12.23 ± 0.08	-186.25 ± 4.98	18.40 ± 0.05
$S_{\rm w}$	-136.85 ± 5.22	16.68 ± 0.05	-146.10 ± 3.84	16.73 ± 0.04
$S_{\rm slac}$	-25.22 ± 6.24	16.92 ± 0.07	-33.64 ± 2.52	16.97 ± 0.03
$S_{\rm w}^{\rm susy}$	-135.11 ± 7.36	16.68 ± 0.07	-138.50 ± 2.85	16.64 ± 0.03
$S_{\rm slac}^{\rm susy}$	-17.97 ± 2.41	16.84 ± 0.03	-18.53 ± 0.91	16.81 ± 0.01

Table: linear interpolations for masses, $m_{\rm phys} = 16.865 \ \ (g/m^2 = 1)$

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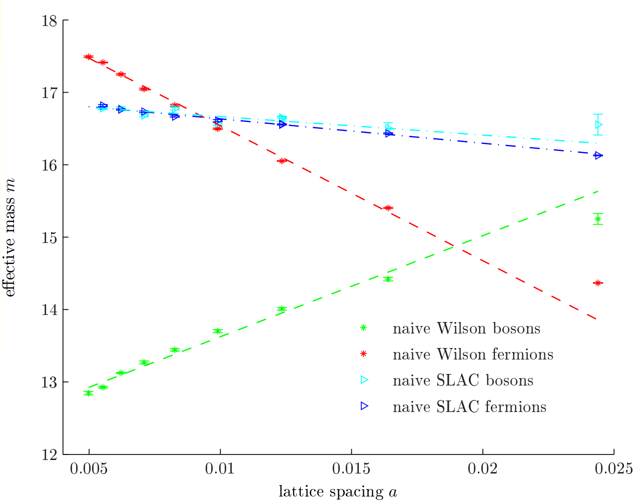

Figure: Boson and fermion masses for Wilson fermions without improvement. and non-supersymmetric SLAC fermions.

- continuum action

$$S_{\rm cont} = \int \left(2\bar\partial\bar\phi\partial\phi + \frac{1}{2}|W'|^2 + \bar\psi M\psi \right), \quad M = \partial\!\!\!/ + W''P_+ + \bar W''P_-$$

$\mathcal{N} = 2$ supersymmetry broken by naive discretization

- Nicolai variables

$$\xi_x = 2(\bar\partial\bar\phi)_x + W_x, \quad S_{\rm bos} = \frac{1}{2}\sum_x \bar\xi_x\xi_x$$

Nicolai improved action

$$S_{\rm bos} = \sum_x \left(2(\bar\partial\bar\phi)_x(\partial\phi)_x + \frac{1}{2}|W_x|^2 + W_x(\partial\phi)_x + \bar W'_x(\bar\partial\bar\phi)_x \right)$$

$$S_{\rm ferm} = \sum_{x,y} \bar\psi_x M_{xy}\psi_y, \quad M_{xy} = \gamma^\mu\partial_{\mu,xy} + W_{xy}P_+ + \bar W_{xy}P_-$$

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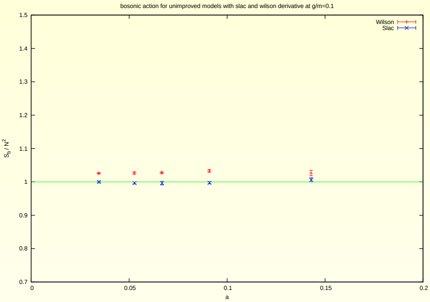
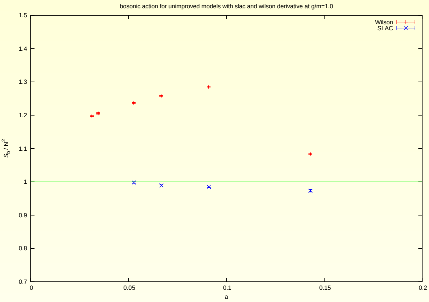
one (out of 4) susy remains ⇒ supersymmetric Ward identities , e.g.

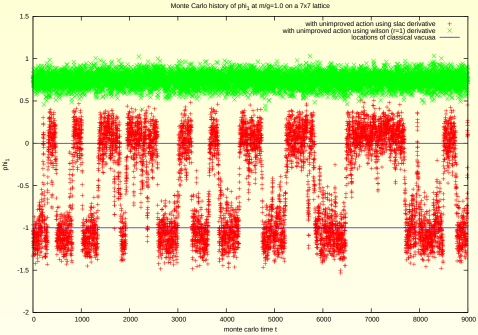
$$\langle S_B \rangle = N^2 \quad \text{for improved models (cp. Catterall)}$$





susy breaking for unimproved models increases with g/m (0.1 and 1); SLAC much better for strong coupling

Unimproved models: Monte Carlo history of ϕ_1 at $g/m = 1$ (7×7)



red: SLAC; green: Wilson; blue lines: classical vacua

Wilson term breaks $\mathbb{Z}_2$ r-symmetry, so do improvement terms

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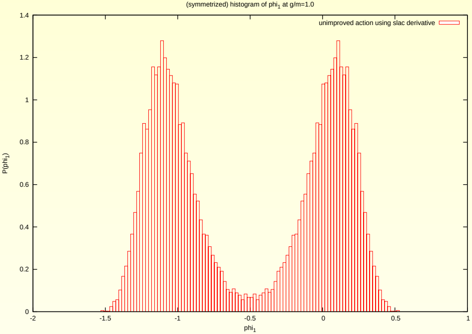
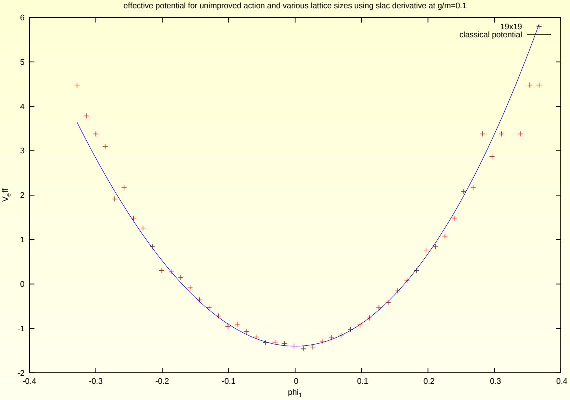
histogram for φ_1 shows $\mathbb{Z}_2$ symmetry for $SLAC, g/m = 1$



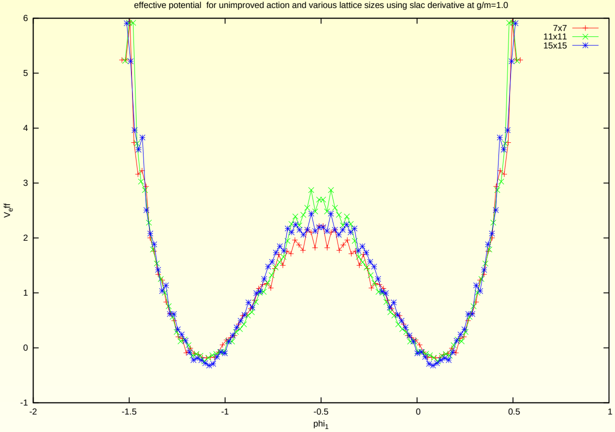
unimproved model with SLAC: $\mathbb{Z}_2$ (spontaneously) broken

constrained effective potential for weak coupling $g/m = 0.1$ and SLAC



well approximated by classical potential $\frac{1}{2}|W'|^2$, perturbative regime (lit)

cep for strong coupling $g/m = 1$, various lattice sizes and SLAC



minimas further out; mass extraction under way (statistics)

Nonlinear Sigma Models (with Wozar and Georg)

Here CP^1 model with $K = \log(1+\bar z z)$, $\Phi \sim (\psi, u, F)$, eliminate

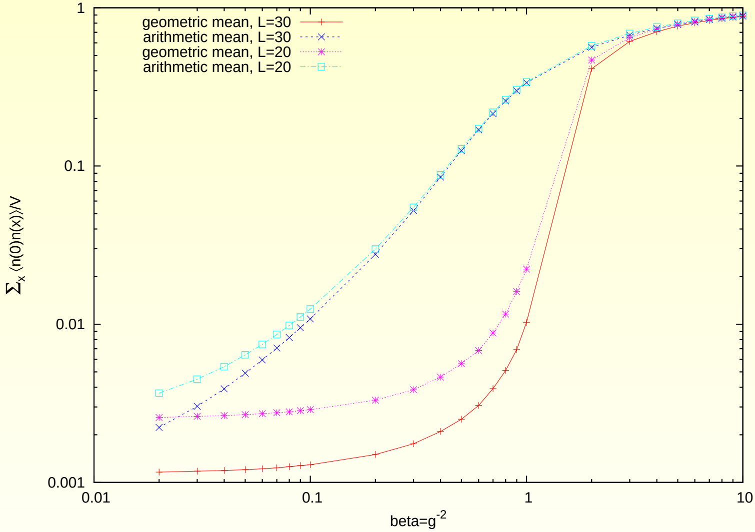
$$\mathcal{L}_\sigma = \frac{\rho}{2}\Big(\partial_\mu \bar u \partial^\mu u - \tfrac{i}{2}\bar\psi \partial\!\!\!/ \psi + \tfrac{i}{2}\partial_\mu \bar\psi \gamma^\mu \psi + \omega_\mu \bar\psi\gamma^\mu\psi + \rho\,(\bar\psi\psi)^2\Big)$$
$$\rho = (1+\bar u u)^{-2}, \quad \omega_\mu = i\sqrt{\rho}\,(\bar u \partial_\mu u - u\partial_\mu \bar u)$$

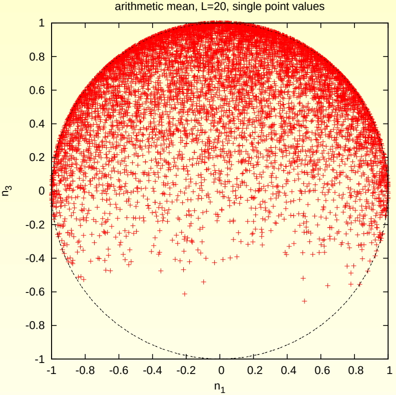
Simulation on curved target space? 4-fermi term: when does Hubbard-Stratonovich work? θ-term; purely bosonic sector:

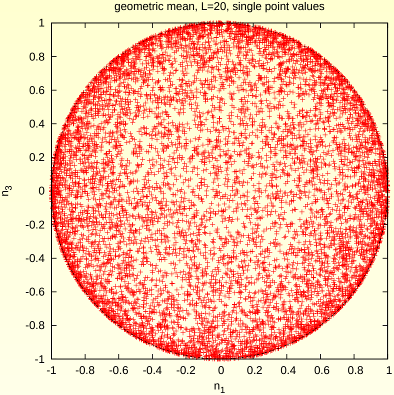
$$\mathcal{L}_{\rm B} \propto \frac{\partial_\mu \boldsymbol{u} \partial^\mu \boldsymbol{u}}{(1+\boldsymbol{u}^2)^2} \equiv \rho\, \partial_\mu \boldsymbol{u}\partial^\mu \boldsymbol{u}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

discretizations: well-kown ordering problem

$$S_a = 4\beta \sum_{\langle x,y\rangle} \frac{\rho_x+\rho_y}{2}(\boldsymbol{u}_x - \boldsymbol{u}_y)^2\,,\; S_g = 4\beta\sum_{\langle x,y\rangle}\sqrt{\rho_x\rho_y}(\boldsymbol{u}_x-\boldsymbol{u}_y)^2$$

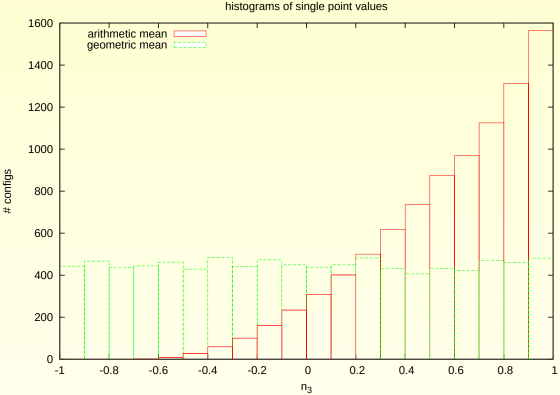

geometric mean, L=30
arithmetic mean, L=30
geometric mean, L=20
arithmetic mean, L=20
1
0.1
0.01
0.001
0.01
0.1
1
10
$\Sigma_x \langle n(0)n(x)\rangle/V$
beta=g^{-2}





$S^2 = SO(3)/SO(2)$: action S_a not $SO(3)$ invariant. Here

$$S_g \equiv \frac{\beta}{2}\sum_{x,\mu}(\boldsymbol{n}_{x+\mu} - \boldsymbol{n}_x)^2 = -\beta \sum_{\langle x,y\rangle} \boldsymbol{n}_x \boldsymbol{n}_y + \text{const}$$

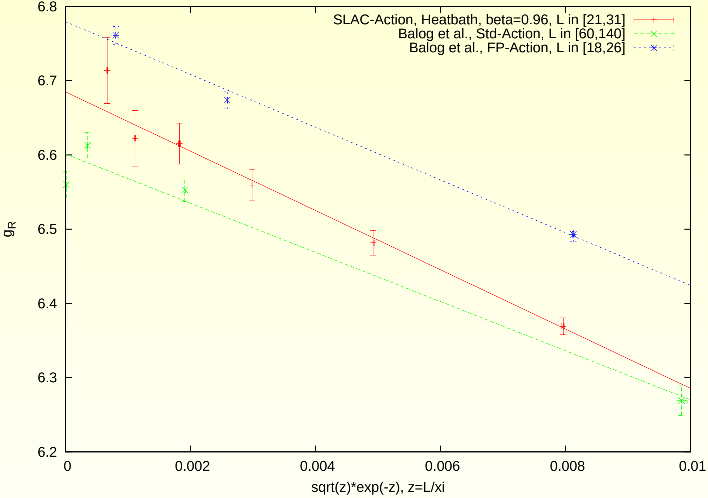


a must for coset models $\mathcal{M} = G/H$: choose G-invariant discretization

non-coset sigma-models (path integrals on curved $\mathcal{M}$)?

dependency on discretization (e.g. derivative) for n-formulation

standard action → SLAC derivative → fixpoint action (cp. Bergner)



Conclusions

- Fourier accelerated HMC-code works fine (JenLaTT, many tests!)
- Wilson and SLAC-fermions, with and without improvement
  Wilson: strong susy and r-symmetry breaking for $g/m \geq 1$
- numerics for WZ difficult: extremely flat potential for $g > m$
  related to pletoria of susy ground state of H_{impr}
- soon masses and precise U_{eff} for $g/m \approx 1$; discriminate

$$m_{\text{eff}}^2 = m^2\left(1 - \frac{1}{\pi}\frac{g^2}{m^2}\right) \longleftrightarrow m_{\text{eff}}^2 = m^2\left(1 + \frac{g^2}{\pi m^2}\right)^{-1}$$

- See no difficulties $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ (Pfaffian, sign,..)
  HMC-code for CP^n model: work in progress; CP^1 soon
  Nicolai maps for Sigma- or even Landau-Ginzburg models?
- Four dimensions: beyond local cluster $\rightarrow$ next year