From Susy-QM to Susy-LFT

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Outline:

- Generalities
- From Susy-QM to Susy-LT
- vacuum sector
- susy vs. lattice derivatives

Motivation, Problems

Why Susy?

- Only possible extension of Poincaré invariance to larger spacetime symmetry (HSL)
- Physics beyond Standard Model (hierarchy, gauge coupling unification, dark matter candidates)
- Superstring theory (susy needed for consistent QG?)
- Tool to obtain results in strongly coupled QFT.

Why Susy-QM?

- Most simple susy-FT
- IR-dynamics of susy-FT in finite V
- Matrix theory description of M theory
- Index theorems, isopectral deformations, integrable systems, susy inspired approximations ...

Why Susy-LT?

Nonperturbative dynamics (spectrum, CSB)

Confirm/extend existing results ($\mathcal{N}=2$ SYM)

Check conjectured results ($\mathcal{N}=1$ SYM)

How to deal with fermions

Gauge Theories

- Main focus on $\mathcal{N}=1$ SYM:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}\not{D}\psi$$

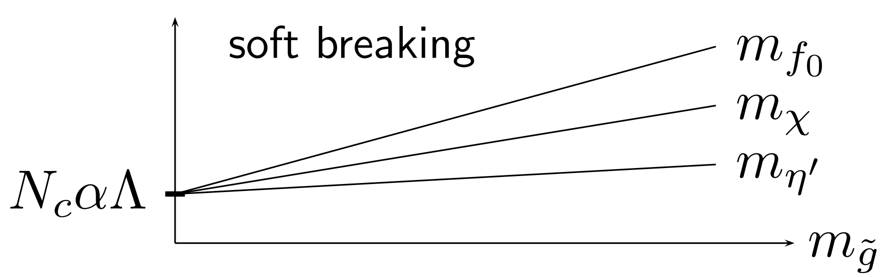
$$U_A(1) \stackrel{\text{anomaly}}{\longrightarrow} \mathbb{Z}_{2N_c} \stackrel{\text{condensate}}{\longrightarrow} \mathbb{Z}_2$$

N_c ground states $\langle\bar{\lambda}\lambda\rangle = c\Lambda^3 e^{2\pi ik/N_c},\ c=?$

($SU(2)$ and $SU(3)$: DESY-Münster)

hadron spectrum (VY action from WI)

mass splitting for $m_{\tilde{g}} \neq 0$ (Evans et.al)



beyond VY (Farrar et.al, Louis et.al)

glue/gluinoball mixing (Peetz et.al: small)

which type of fermions?

- Wilson: non-chiral, undoubled, ultra-local, cheap, hermitean
  susy broken by lattice + Wilson term
  susy and chiral limit at $m_{\tilde{g}} \to 0$
  condensate, smallest masses, Ward identities: Montvay et.al, Peetz et.al, . . .
- Ginsparg-Wilson (overlap, domain walls): chiral, undoubled, local, expensive, non-hermitean
  no fine-tuning for chiral limit
  condensate: Kogut et.al

- $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM:
  2 or 6 scalars $\rightarrow$ fine tuning gets worse
  euclidean S_B unbounded from below?
  new approaches (Kaplan et.al, Sugino, Itoh et.al)

Problems specific to Lattice regularization

- No discrete version of susy $\Rightarrow$ plethora of unwanted relevant operators, fine tuning $\Rightarrow$ models with subset of exact susy? (WZ; SYM with $\mathcal{N} > 1$)

- No Leibniz rule on lattice Λ:

$$f : \Lambda \to \mathbb{C}, \ (f, g) = \sum_x \bar{f}(x) g(x)$$

$$(Df)(x) = \sum_y D_{xy} f(y) \quad \text{linear}$$

$$D(fg) = (Df)g + f(Dg) \Longrightarrow D = 0$$

- ultralocal forward/backward derivatives (a=1):

$$\begin{aligned}
(\partial_\mu^f f)(x) &= f(x + \hat{\mu}) - f(x) \\
(\partial_\mu^b f)(x) &= f(x) - f(x - \hat{\mu}) \\
(f, \partial_\mu^f g) &= -(\partial_\mu^b f, g).
\end{aligned}$$

weighted sum: $r \in [-1, 1]$

$$\partial_\mu^r = \frac{1}{2}(1+r)\partial_\mu^f + \frac{1}{2}(1-r)\partial_\mu^b = \partial_\mu^a + \partial_\mu^s,$$

antisymmetric and symmetric parts

$$\partial_\mu^a = \frac{1}{2}(\partial_\mu^f + \partial_\mu^b) \quad , \quad \partial_\mu^s = \frac{r}{2}(\partial_\mu^f - \partial_\mu^b)$$

$r = 0$: antisymmetric and doubling;

$$\partial_\mu^s = \frac{r}{2}\underbrace{\partial_\mu^f \partial_\mu^b}_{\triangle} \Longrightarrow \sum_\mu \partial_\mu^s = \frac{r}{2} a \triangle$$

- Wilson fermions

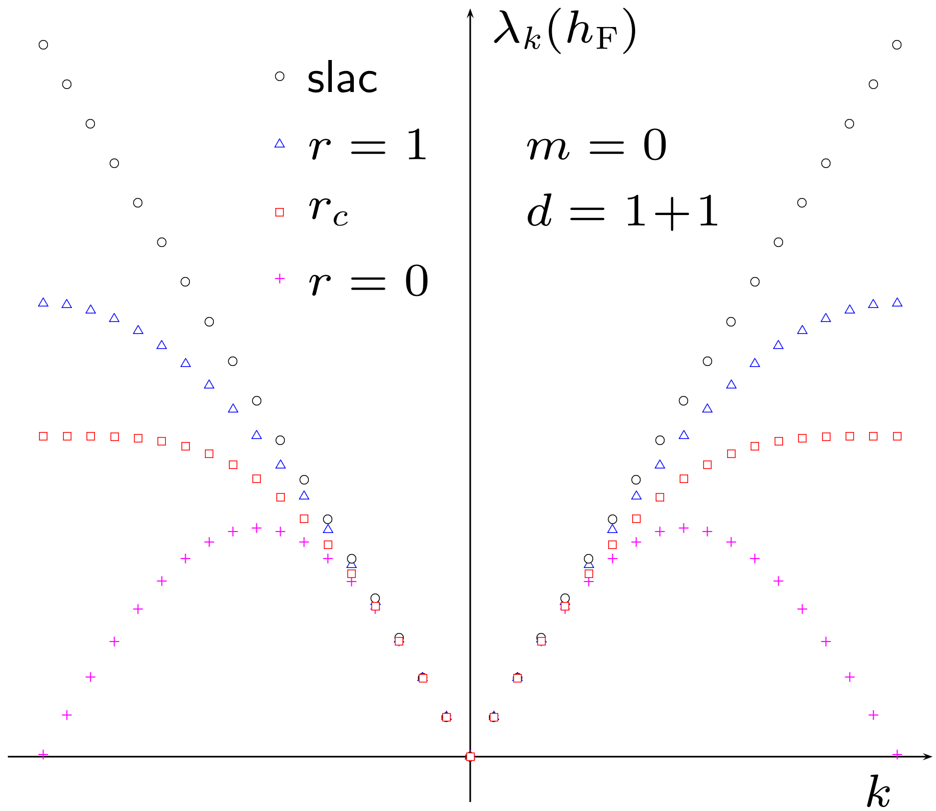
$$S_{\rm F} = (\bar{\psi}, D_w \psi), \quad D_w = \underbrace{i\gamma^\mu \partial_\mu^a - m}_{\rm doubling} + \underbrace{\tfrac{r}{2} a \triangle}_{\rm W-term}$$

- Hamiltonian form: continuous t, discrete space

$$h_{\rm F}^r = -i\alpha_M^i \partial_i^a + \beta\left(m - \tfrac{r}{2} a \triangle\right) = h_{\rm F}^{r\,\dagger}$$

- Slac fermions (Weinstein et.al): chiral, no doubling

$$h_{\rm F}=-i\alpha^i_M\partial^{\rm slac}_i+\beta m=h^\dagger_{\rm F},\quad \partial^{\rm slac}_i \text{ antisymmetric}$$



- $\partial^{\rm slac}_\mu$ non-local, 'identical' spectrum as $\partial^{\rm cont}_\mu$
- Ginsparg-Wilson fermions (e.g. Neuberger)

$$\gamma_5 D+D\gamma_5=aD\gamma_5 D,\quad D^\dagger=\gamma_5 D\gamma_5$$

Models, Methods and selected results

- Divergences depend on fermion type:
  $(d, \mathcal{N}) = (2,2)$ and $(4,1)$ Wess-Zumino: tadpoles linearly divergent for W, finite for WG (Fujikawa)

- No Leibniz rule:
  - superfield $\cdot$ superfield $\neq$ superfield
  - would be central charges not central:

$$\sum_x \left(\text{div} W(\phi)\right)(x) \neq \sum_{x,i} \frac{\partial W_i(\phi)}{\partial \phi(x)} \frac{\partial \phi(x)}{\partial x^i}$$

  - restoration of rule for $a \to 0$ (Fujikawa et.al)

- Exact (partial) susy on lattice:
  $\Rightarrow$ decrease number of fine tunings
  applied to $d = 2, 4$ WZ and extended SYM

- covariant quantization:
  typically non-local fermions;

sometimes non-local interaction (no Leibniz rule!)

- keep subset of extended susy: $4 \to 1$ (Catterall)
- keep Leibniz rule via noncommutativity (D'Adda)

(Kaplan, Sugino, Campostrini,. . .)

- Hamiltonian formalism:

  time continuous, keep spectral subalgebra (?) of

$$\{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_\beta\} = 2\left(\delta^{IJ} \not{P}_{\alpha\beta} + i\delta_{\alpha\beta}\mathcal{Z}^{IJ}_A + i\gamma_{*\alpha\beta}\mathcal{Z}^{IJ}_S\right)$$

$\mathcal{Z}$: central charges, $\mathcal{Q}^I_\alpha$ real supercharges

Example: WZ in 2 dimensions: $\mathcal{Z}^{IJ}_A = 0$,

$$\mathcal{N} = 1 : \quad \mathcal{Z}_S = \int dx\, \partial_x W$$

$$\mathcal{N} = 2 : \quad F(\phi^1 + i\phi^2) = W + iU$$

$$\left(\mathcal{Z}^{IJ}_S\right) = \sigma_3 \int dx\, \partial_x W - \sigma_1 \int dx\, \partial_x U,$$

Susy-QM

- Susy FT on finite space lattice = susy-QM
- lattice derivative: most simple case

$$\mathcal{Q} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad A = \partial + W, \;\; A^\dagger = \partial^\dagger + W,$$

$$H = \mathcal{Q}^2 = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}$$

discretize H:

$$\begin{aligned} (AA^\dagger)_L &= -\partial^2 + W^2 + W' \\ (A^\dagger A)_L &= -\partial^2 + W^2 - W' \end{aligned}$$

discretize Q: isospectral operators

$$\begin{aligned} A_L A_L^\dagger &= \partial\partial^\dagger + W^2 + \partial W + W\partial^\dagger \\ A_L^\dagger A_L &= \partial^\dagger\partial + W^2 + \partial^\dagger W + W\partial \end{aligned}$$

Difference: no Leibniz, $\partial^\dagger \neq -\partial$, central charge

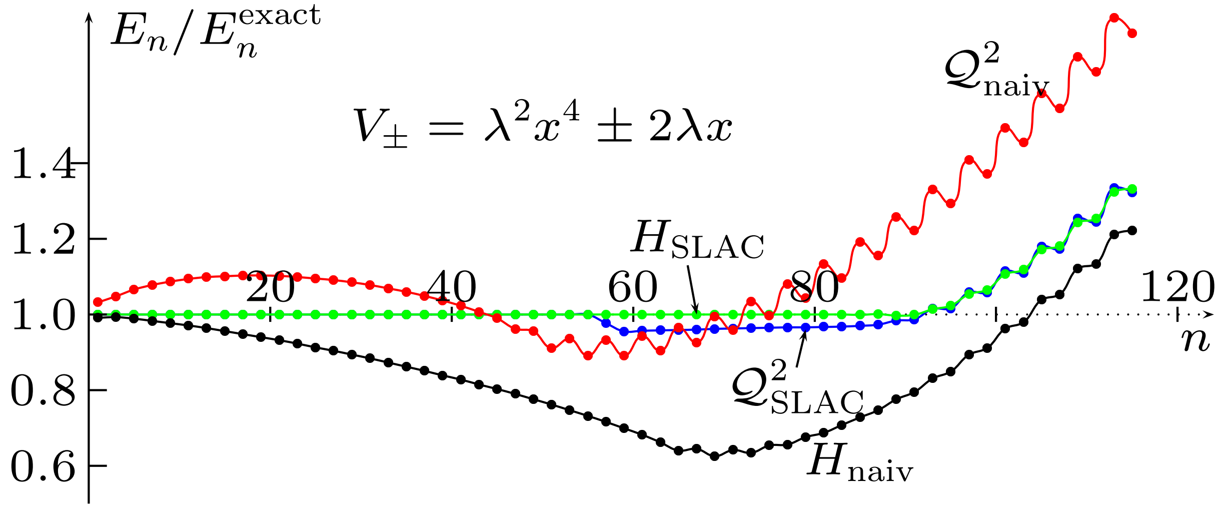


Figure 1: *EV of* $(\mathcal{Q}_L)^2$ *and* H_L *for SLAC derivative and forward-derivative for* $N = 180$*,* $L = 30$*,* $\lambda = 1$*.*

non-local SLAC-derivative:

- much more accurate for WZ-models (first studies)

- further improvement for gauge theories (Slavnov)

- generalization $\Rightarrow$ lattice FT (here $\mathcal{N} = 2$):
  $\{\psi, \psi^\dagger\} = \mathbb{1}$, nilpotent complex charge:

$$Q = \psi \otimes (\partial + W), \quad \psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow 2H = \{Q, Q^\dagger\}$$

$2N$ dimensions: $x \in \mathbb{R}^{2N},\ \psi^1, \ldots, \psi^{2N}$

$$W_n(x) = \partial_n \chi(x) \quad , \quad \{\psi^n, \psi^{m\dagger}\} = \delta^{mn}$$

$$Q = \sum_{n=1}^{2N} \psi^n \left(\partial_n + \partial_n \chi\right) = e^{-\chi} Q_0 e^{\chi}, \quad Q^2 = 0$$

Hamiltonian of $2N$-dimensional $\mathcal{N} = 2$ SQM:

$$H = -\frac{1}{2}\Delta + \frac{1}{2}(\nabla\chi, \nabla\chi) + \frac{1}{2}\Delta\chi - \psi^{n\dagger}\chi_{,nm}\psi^m$$

$$\mathcal{H} = \underbrace{\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}}_{N-\text{times}}, \qquad \mathfrak{h} = L_2(\mathbb{R}^2) \otimes \mathbb{C}^4$$

SQM $\longrightarrow$ Lattice-WZ$_{2,2}$ via identifications

$$\phi(n) = \begin{pmatrix} x^{2n-1} \\ x^{2n} \end{pmatrix}, \quad \psi(n) = \begin{pmatrix} \psi^{2n-1} \\ \psi^{2n} \end{pmatrix}$$

$$H = (\psi, h_F \psi) + \ldots \Longrightarrow \chi = \tfrac{1}{2}(\phi, h_F \phi) + \ldots$$

χ real: $\Rightarrow\ \gamma^0 = \sigma_3,\ \gamma^1 = i\sigma_1,\ \gamma_* = -\sigma_2 \Rightarrow$

$$h_F = \begin{pmatrix} m & \partial_x \\ \partial_x^\dagger & -m \end{pmatrix} = -i\gamma_* \partial_x^a + m\gamma^0 - i\gamma^1 \partial_x^s$$

unusual Wilson term $\sim \partial^s$ ($\mathcal{N} = 1$: usual one)

$$h_F^2 = (\partial\partial^\dagger + m^2)\mathbb{1}_2$$

Interacting fields

$$\chi = \tfrac{1}{2}(\phi, h_F \phi) + \sum f\big(\phi(n)\big), \quad \Delta f = 0.$$

Super-Hamiltonian: $H = P_0 + \mathcal{Z}$

$$P_0 = \frac{1}{2}(\pi, \pi) + \frac{1}{2}(\phi, \partial\partial^\dagger \phi) + \frac{1}{2}(\nabla_\phi f, \nabla_\phi f) + (\psi, h_F^0 \psi) - \big(\psi, \gamma^0 \Gamma(\phi)\psi\big)$$

$$\Gamma(\phi) = f_{,11}(\phi) - i\gamma_* f_{,12}(\phi), \quad \text{Yukawa}$$

f harmonic $\Rightarrow$

$$\frac{\partial f}{\partial \phi_1} = \frac{\partial g}{\partial \phi_2} \quad \text{and} \quad \frac{\partial f}{\partial \phi_2} = -\frac{\partial g}{\partial \phi_1}$$

Would be central charge

$$\mathcal{Z} = -\left(\nabla_\phi g, \partial_x^a \phi\right) + \left(\nabla_\phi g, \sigma_3 \partial_x^s \phi\right)$$

- weak coupling:

  free massive model for $f = \frac{1}{2}m(\phi_2^2 - \phi_1^2)$:

  exist one unique susy ground state (for all ∂)

- strong coupling: (Jaffe et.al)

  Neglect ∂ in controlled way $\Rightarrow H = \sum_n h_n$

  h_n single site operators on $L_2(\mathbb{R}^2) \times \mathbb{C}^4$

$$H = -\frac{1}{2}\triangle + \frac{1}{2}(\nabla f, \nabla f) - \psi^\dagger f'' \psi.$$

$$f(\phi) = \frac{\lambda}{p} \Re\left(\phi_1 + i\phi_2\right)^p$$

Elitzur, Schwimmer: $p-1$ susy ground states $\Rightarrow$

Lattice WZ has $(p-1)^N$ susy ground states

Can prove: Holds true for all $\lambda \neq 0$; e.g.

$$f(\phi) = \frac{\lambda}{p} \Re \left(\phi_1 + i\phi_2\right)^p + o(\phi^p) \Longrightarrow (p-1)^N$$

Highly degenerate vacuum sector!!

- From strong to weak coupling:

  Step 1: Perturbation $\mathcal{Q}(\epsilon) = \mathcal{Q}_0 + \epsilon \mathcal{Q}_1$

  hermitean $\mathcal{Q}_i$, commute with $\Gamma = \Gamma^\dagger$, $\Gamma^2 = \mathbb{1}$

$$\mathcal{Q}(\epsilon)\psi(\epsilon) = \lambda(\epsilon)\psi(\epsilon)$$

$$\mathcal{Q}_0\psi_0 = 0, \quad \Gamma\psi_0 = \psi_0$$

  $\Rightarrow \lambda(0) = 0$; formal power series

$$\psi(\epsilon) = \psi_0 + \sum_{k=1}^{\infty} \epsilon^k \psi_k, \quad \lambda(\epsilon) = \sum_{k=1}^{\infty} \epsilon^k \lambda_k$$

  Proposition: (Jaffe, Lesniewski and Lewenstein)

  as formal power series $\lambda(\epsilon) = 0$ and $\Gamma\psi(\epsilon) = \psi(\epsilon)$

Step 2: In Majorana representation

$$\{\mathcal{Q}_1^1, \mathcal{Q}_1^1\} = \{\mathcal{Q}_1^2, \mathcal{Q}_1^2\} = 2(P_0 + \mathcal{Z})$$

$$\{\mathcal{Q}_1^1, \mathcal{Q}_2^2\} = 0, \quad \mathcal{Z} = \mathcal{Z}_S^{11} = -\mathcal{Z}_S^{22}$$

$$\mathcal{Q}_1^1 = \underbrace{(\pi, \psi_1) + (\nabla W, \psi_2)}_{B_0 \text{ strong coupling}} + \underbrace{(\partial\phi^1, \psi_2^1) - (\partial^\dagger\phi^2, \psi_2^2)}_{B_1 \text{ perturbation}}$$

B_0 essentially s.a. on $\mathcal{D}(B_0) = C_c^\infty(\mathbb{R}^{2N}) \otimes \mathbb{C}^D$

use B_0 to define energy norm

- bounds on $a\|f\|^2 + \|B_0 f\|^2, \ f \in \mathcal{D}(B_0)$

$\Longrightarrow \mathcal{D}(\bar{B}_0)$ weighted Sobolov space

- For all $\lambda \in \mathbb{R}, \ \epsilon > 0$ exists $C_\epsilon > 0$ such that

$$\|\lambda B_1 f\| \leq \epsilon\|B_0 f\| + C_\epsilon\|f\|, \quad \forall f \in \mathcal{D}(\bar{B}_0)$$

Kato-Rellich: $Q_1^1(\lambda)$ s.a. on $\mathcal{D}(\bar{B}_0)$

λB_1 is B_0 bounded with arbitrary small bound

$\Rightarrow Q_1^1(\epsilon)$ analytic family (Kato)

- Spectrum of $Q_1^1(\epsilon)$ discrete $\Rightarrow$ $\lambda(\epsilon)$ analytic

KLW, Annals of Physics 316 (2004) 357

- applies to $\mathcal{N} = 1$ in $d = 2$:

  $\deg W = p$ even:

  susy unbroken in sc and pt

  $\deg W = p$ odd:

  susy broken in sc; may be unbroken in pt; example

$$W(\phi) = g_2\phi^3 + g_0\phi$$

  strong coupling: susy broken $\forall g_0$ (correct)

  weak coupling: susy for $g_0 < 0$

- applies to $\mathcal{N} = 2$ in $d = 2$:

  puzzle

  $(p-1)^N$ susy ground states on finite lattice

  $(p-1)$ susy ground states in continuum (Jaffe et.al)

  conjecture $H = P_0 + \mathcal{Z}, \ \ \mathcal{Z} \stackrel{a\to 0}{\longrightarrow} -c < 0$

- rigoros result independent on choice of ∂
- $\mathcal{N}=1$ WZ in 4 dimensions

  analog results for ground states of $\mathcal{Q}_1^2 = H - P_3$.
- Not applicable to potentials with flat directions!!
- For arbitrary ∂: 'would be central charges'

$$\begin{aligned}
\mathcal{Z}_S^{IJ} &= \sigma_3^{IJ}(W', \partial^a \phi) - \sigma_1^{IJ}(U', \partial^a \phi) \\
\mathcal{Z}_L^{IJ} &= -\sigma_3^{IJ}\gamma^0(W', \partial^s \phi) + \sigma_1^{IJ}\left(\gamma^0(U', \partial^s \phi)\right. \\
&\qquad \left. -i\,(\pi, I\partial^s \phi) - \tfrac{i}{2}(\psi, I\partial^s \psi)\right)
\end{aligned}$$

  cont. algebra, no doubling, chiral symmetry $\Rightarrow$

  Slac derivative favoured (gauge theories ?)
- checked index theorem on lattice of $D^{\mathrm{slac}}[U]$
- how to implement (detailed balance, ergodic,. . .)