Interacting Fermions in two and three dimensions

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1. Relativistic four-Fermi theories

2. Inhomogeneous phases in 2*d* Gross-Neveu model at finite μ

3. Phases and phase transitions in 3*d* Thirring model

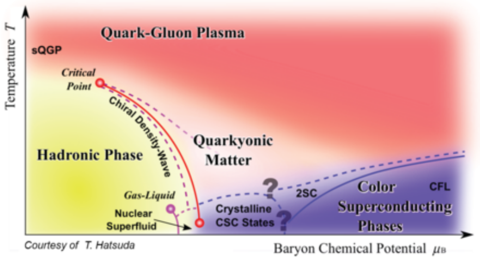
4. Critical flavour number with chiral lattice fermions

- Thirring model (Th), Gross-Neveu model (GN), ...

$$\mathcal{L}_{\text{Th}} \propto (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \quad \mathcal{L}_{\text{GN}} \propto (\bar{\psi}\psi)^2, \quad \dots$$

- effective models for CPT in QCD
- 2 spacetime dimensions:
  - massless Th: soluble — Thirring, Klaiber, Sachs+AW, ...
  - GN: asymptotically free, integrable — Gross-Neveu, Coleman, ...
- 3 spacetime dimensions:
  - not renormalizable in PT
  - renormalizable in large-N expansion — Gawedzki, Kupiainen; Park, Rosenstein, Warr
  - interacting UV fixed point $\rightarrow$ asymptotically safe — de Veiga, da Calen, Magnen, Seneor
  - parity breaking at low T?
- lattice theories:
  - generically: sign problem even for $\mu = 0$
  - partial solution — Schmidt, Wellegehausen, Lenz, AW

possible QCD-phase diagram



- crystalline LOFF phase (colour superconductive phase)
  $\mu \neq 0 \Rightarrow$ complex fermion determinant
  Taylor-expansion in μ: radius of convergence
- effective models: fluctuations of order parameter neglected?
- gauge theories without sign-problem (SU(2), G_2): large μ beyond reach
- simpler models which show crystal phase

- GN-Model in $d = 1 + 1$ at finite T and μ
- $\mathrm{N_f}$ flavours of Dirac-spinors $\Psi = (\psi_1, \dots, \psi_{\mathrm{N_f}})^T$
  ψ_a has two components
- massless Gross-Neveu model (GNM)

$$\mathcal{L}_{\mathrm{GN}} = \sum_a \bar\psi_a \mathrm{i}\partial\!\!\!/ \psi_a + \frac{g^2}{2\mathrm{N_f}}(\bar\Psi\Psi)^2, \quad \bar\Psi\Psi = \sum_{a=1}^{\mathrm{N_f}} \bar\psi_a\psi_a$$

- asymptotically free

$$\beta(g) = -\frac{\mathrm{N_f} - 1}{2\pi} g^3 + O(g^5)$$

- order parameter for discrete chiral symmetry $\langle\bar\Psi\Psi\rangle = \sum\langle\bar\psi_a\psi_a\rangle$
- realizations in condensed matter
  - conducting polymers (Trans- and Cis-polyacetylen) Su, Schrieffer, Heeger
  - quasi-one-dimensional inhomogeneous superconductor Mertsching, Fischbeck

- chemical potential μ for fermion number
- Hubbard-Stratonovich transformation

$$\mathcal{L}_\sigma = \sum_a \bar{\psi}_a \,\mathrm{i} D \psi_a + \frac{\mathrm{N_f}}{2g^2}\sigma^2, \quad D = \partial\!\!\!/ + \sigma + \mu\gamma^0 \neq D^\dagger$$

- expectation value in grand canonical ensemble

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \,\mathrm{e}^{-S_\psi}\,\mathcal{O} = \frac{1}{Z}\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\sigma\,\mathrm{e}^{-S_\sigma}\,\mathcal{O}$$

- chiral condensate: Ward-identity $\Rightarrow$

$$\langle \bar{\psi}(x)\psi(x)\rangle = \frac{\mathrm{i}\mathrm{N_f}}{g^2}\langle \sigma(x)\rangle \equiv \mathrm{i}\Sigma$$

- partition sum

$$Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\sigma \mathrm{e}^{-S_\sigma} = \int \mathcal{D}\sigma\, e^{-\mathrm{N_f} S_{\mathrm{eff}}[\sigma]} \stackrel{\mathrm{N_f}\to\infty}{\longrightarrow} \mathrm{e}^{-\mathrm{N_f}\min_\sigma S_{\mathrm{eff}}[\sigma]},$$

$$S_{\mathrm{eff}}[\sigma] = \frac{1}{2g^2}\int \sigma^2 - \log\det D$$

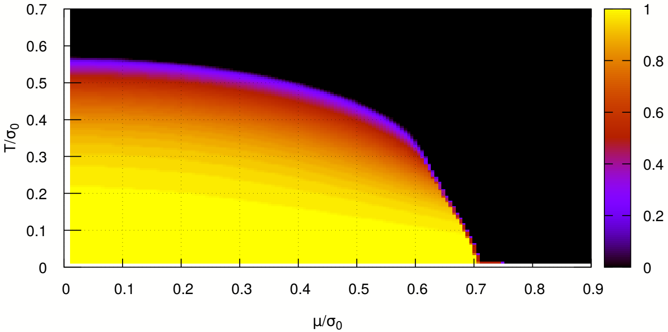
- for $\mathrm{N_f} \to \infty$: saddle point approximation exact,

$$Z = \mathrm{e}^{-\mathrm{N_f} \min_\sigma S_{\text{eff}}[\sigma]}$$

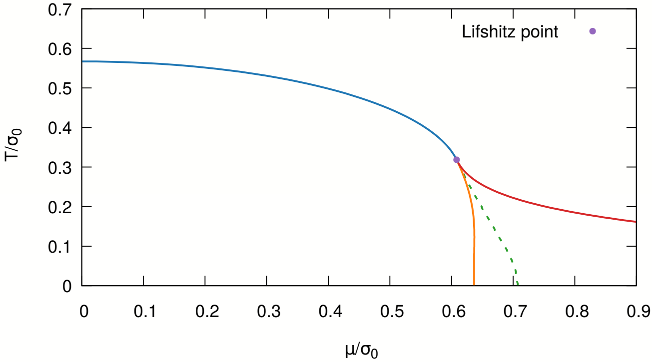
- translational invariance $\Rightarrow$ constant condensate $\Sigma =$ minimizing σ
- renormalized $S_{\text{eff}} = \beta L\, U_{\text{eff}}$

$$U_{\text{eff}} = \frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{\sigma_0} - 1\right) - \frac{1}{\pi}\int_0^\infty \mathrm{d}p\, \frac{p^2}{\varepsilon_p}\left(\frac{1}{1+\mathrm{e}^{\beta(\varepsilon_p+\mu)}} + \frac{1}{1+\mathrm{e}^{\beta(\varepsilon_p-\mu)}}\right)$$

- one-particle energies $\epsilon_p = \sqrt{p^2+\sigma^2}$
- renormalization at $\langle\sigma\rangle_{T=\mu=0} = \sigma_0 \Rightarrow$ energy scale σ_0



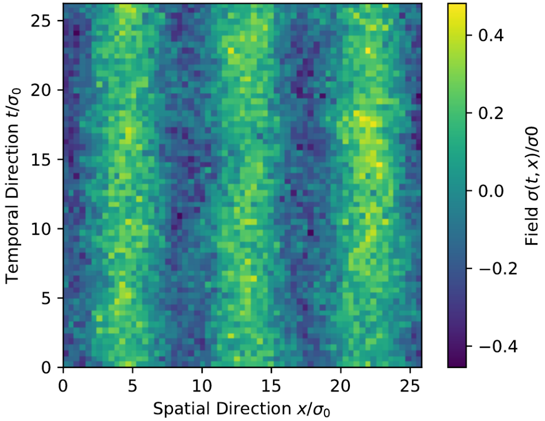
- symmetric (massless, gapless) phase for large T, μ
- homogeneous broken (massive, gapped) phase for small T, μ Wolff, Barducci
- special points: $(T_c, \mu) = (e^\gamma/\pi, 0)$ and $(T, \mu_c) = (0, 1/\sqrt{2})$
- Lifschitz-point at $(T, \mu_0) \approx (0.318, 0.608)$

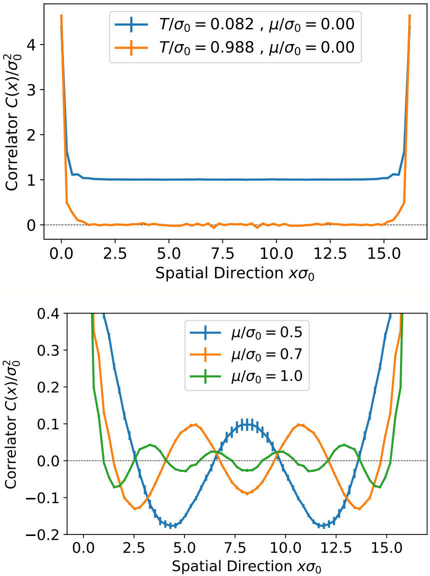


- inhomogeneous condensate for low T, large μ
- breaking of translational invariance for $\mathrm{N_f} \to \infty$
- all transitions are continuous
- baryons are with $0, 1, 2, \ldots, \mathrm{N_f}$ fundamental fermions

- homogeneous $\langle\bar\psi\psi\rangle$ breaks $\mathbb{Z}_2$-symmetry
  should not happen for $T > 0$
- inhomogeneous $\langle\bar\psi\psi\rangle$ breaks translation invariance $\to$
  massless Goldstone-excitations $\to$
  should not exist in $d = 1 + 1$
- No-go theorems not valid for $\mathrm{N_f} \to \infty$
- phase diagram $=$ artefact of $\mathrm{N_f} \to \infty$?
- is there a inhomogeneous condensate for $\mathrm{N_f} < \infty$?
- number of massless Goldstone excitations:
  n_k number of type k Goldstone modes
  type 1: $\omega \sim |\boldsymbol{k}|^{2n+1}$, e.g. relativistic dispersion relation
  type 2: $\omega \sim |\boldsymbol{k}|^{2n}$, e.g. non-relativistic dispersion relation
  inner symmetries $n_1 + 2n_2 =$number of broken directions
  spacetime symmetries $n_1 + 2n_2 \leq$ number of broken directions
- large μ: dispersion relation need not be relativistic

- simulations for $\mathrm{N_f} = 8$, $L = 64$, chiral fermions
- low temperature $T = 0.038\,\sigma_0$, medium density $\mu = 0.5\sigma_0$
- typical configuration





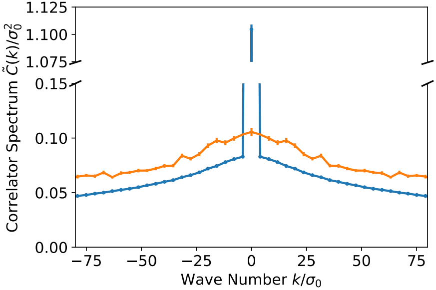
spatial correlations
of chiral condensate

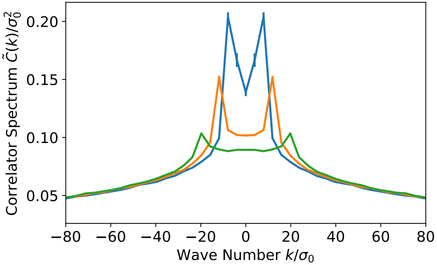
$$C(x) = \frac{1}{L} \sum_y \langle \sigma(y,t)\sigma(y+x,t) \rangle$$

- $\mathrm{N_f} = 8, L = 64$
  naive fermions
- above: homogeneous phases

  $\mu = 0$
  $T/\sigma_0 \in \{0.082, 0.988\}$
- below: inhomogeneous phase

  $T = 0.082\sigma_0$
  $\mu/\sigma_0 \in \{0.5, 0.7, 1.0\}$





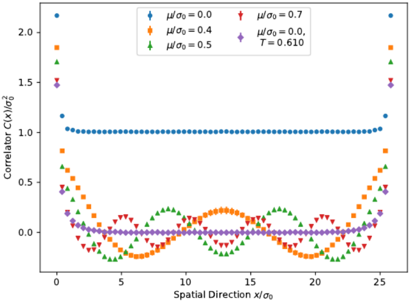
Fourier-transform of spatial correlation function

$$\tilde{C}(k) \propto \sum_x \mathrm{e}^{\mathrm{i}kx} C(x)$$

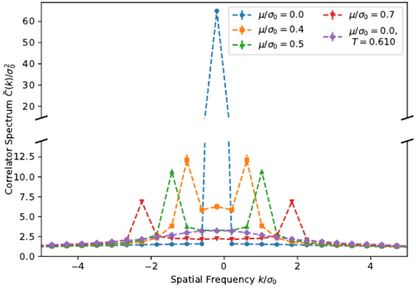
- $\mathrm{N_f} = 8$, $L = 64$
  naive fermions
- above: homogeneous phases

  $\mu = 0$
  $T/\sigma_0 \in \{0.082, 0.988\}$
- below: inhomogeneous phase

  $T = 0.082\,\sigma_0$
  $\mu/\sigma_0 \in \{0.5, 0.7, 1.0\}$



(a) Real space.

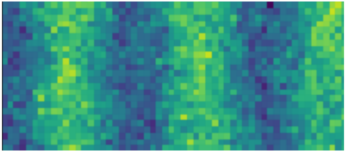


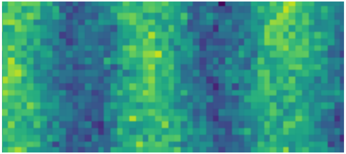
spatial correlation function and Fourier-transform

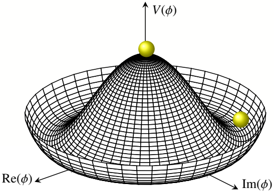
- $\mathrm{N_f} = 8$, $L = 64$
  SLAC-fermions
- low temperature $T = 0.038\sigma_0$
- different chemical potentials

  $\mu/\sigma_0 \in \{0, 0.4, 0.5, 0.7\}$
- violet:
  symmetric phase
  $\mu = 0,\ \ T = 0.61\,\sigma_0$

intepretation



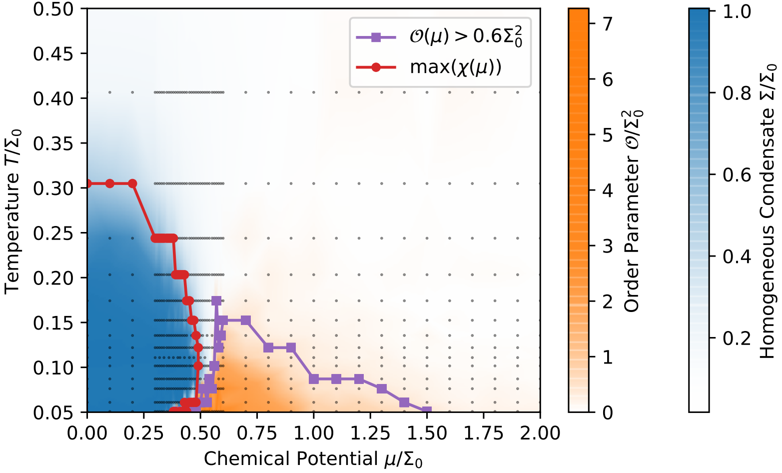




- relatively shifted configurations
- have same

$$C(x) \propto \int \mathrm{d}y \, \langle \sigma(y,t)\sigma(y+x,t) \rangle$$

- configurations related by translation symmetry $\Rightarrow$ same C
- position of maxima/minima does not move or very slowly for $\mathrm{N_f} = 12$
- compare with Mexican hat: ball does not move in valley
- but: all results very similar to $\mathrm{N_f} = \infty$

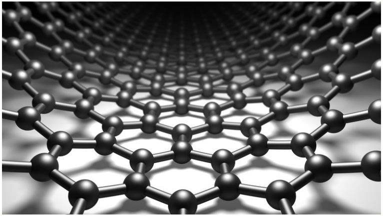


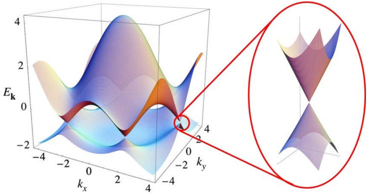
- phase diagram for $\mathrm{N_f} = 8$ with chiral naive/SLAC fermions
- qualitatively the same as for $\mathrm{N_f} = \infty$

- first numerical results for finite μ, $T, \mathrm{N_f}$
  (no analytic results are known)
  collaboration Frankfurt – Jena Lenz, Pannullo, Wagner, Wellegehausen, AW
- comparable results for $\mathrm{N_f} = 8$ and $\mathrm{N_f} = 16$
  for naive and SLAC fermions (both are chiral)
- no sign problem for even $\mathrm{N_f}$
- phase diagram similar as for $\mathrm{N_f} \to \infty$ Thies
- first simulations for $\mathrm{N_f} = 2$ on very large lattices ($L \lessapprox 1024$)
  $\to$ long-range correlations for $\mathrm{N_f} \to \infty$ Witten
- what happens to Goldstone-theorem?, ...
- are there baryons as for $\mathrm{N_f} = \infty$?
- NJL-model, higher dimensions, ...

low-energy description of interacting electrons

- low energy description of tight binding model with NN hopping
- graphen's honeycomb lattice (GN)
- increase coupling $\Rightarrow$ PT semi-metal $\rightarrow$ (Mott) insulator
- long-range order $\Rightarrow$ AF, CDW, QAHS





- applications to Dirac materials: two Dirac points
  $\Rightarrow$ reducible ψ with 4 components

$$\mathcal{L}_{\mathrm{Th}} = \bar{\Psi} \mathrm{i} \partial\!\!\!/ \Psi - \frac{g^2}{2\mathrm{N_f}} (\bar{\Psi} \Gamma^\mu \Psi)^2$$

reducible models: ψ four components

- euclidean 4×4 gamma-matrices $\Gamma_1, \Gamma_2, \Gamma_3$ in $\partial\!\!\!/ = \Gamma^\mu \partial_\mu$
- in addition $\Gamma_4, \Gamma_5 \Rightarrow$ two notions of "chirality"
- $\mathrm{N_f}$ reducible flavors, symmetry $\mathbb{Z}_2 \times U(2\mathrm{N_f})$
- $\langle \bar{\Psi} \Psi \rangle$ order parameter for $\mathbb{Z}_2$ parity only

irreducible models: ψ two components

- euclidean 2×2 gamma-matrices $\gamma_1, \gamma_2, \gamma_3$ in $\gamma^\mu \partial_\mu = \partial\!\!\!/$
- no notion of chirality in $d = 3$
- $\mathrm{N_f^{ir}}$ irreducible flavors, symmetry $\mathbb{Z}_2 \times U(\mathrm{N_f^{ir}})$
- $\langle \bar{\Psi} \Psi \rangle$ order parameter for $\mathbb{Z}_2$ parity only

- massless case: models equivalent $\mathrm{N}_\mathrm{f}^\mathrm{ir} = 2\mathrm{N}_\mathrm{f}$
- massive case: $\mathcal{L}_\mathrm{red}$ parity invariant
  $\mathcal{L}_\mathrm{irred}$ not parity invariant
- Hubbard-Stratonovich:

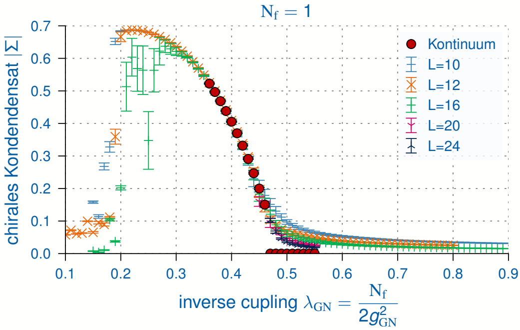
$$\mathcal{L}_\mathrm{Th} = \bar{\Psi}\left(\mathrm{i}\not{D} \otimes \mathbb{1}_{\mathrm{N}_\mathrm{f}}\right)\Psi + \frac{\mathrm{N}_\mathrm{f}}{2g^2} V_\mu V^\mu, \quad \not{D} = \not{\partial} - \mathrm{i}\not{V}$$

- fermionic integration

$$Z_\mathrm{Th} = \int \mathcal{D} V_\mu \left(\det \mathrm{i}\not{D}\right)^{\mathrm{N}_\mathrm{f}} \mathrm{e}^{-\mathrm{N}_\mathrm{f}/2g^2 \int \mathrm{d}^3 x\, V_\mu V^\mu}$$

- previous lattice studies for reducible models
  $\rightarrow$ Wilson/staggered fermions break parity invariance and $U(2\mathrm{N}_\mathrm{f})$
  $\rightarrow$ may end up in wrong universality class

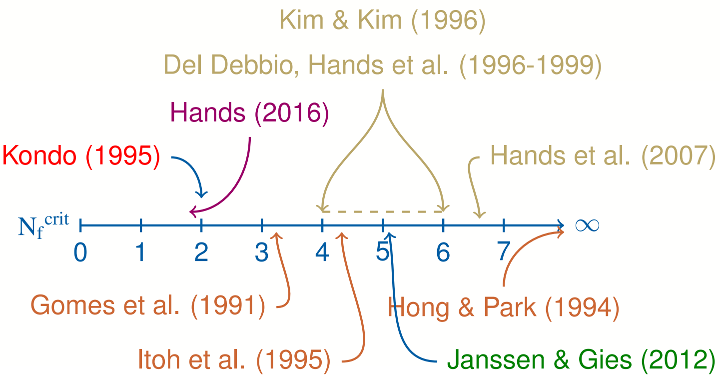
- Jena group: first simulations with chiral fermions!
- chiral SLAC fermions
  - ∂_μ real, antisymmetric
  - $\mathrm{i}\not{\partial}$ hermitean
  - no additional sign-problem due to discretization
  - internal symmetries in continuum $\equiv$ internal symmetries on lattice
  - no doublers, non-local
- successfully applied to
  - scalar field theories
  - non-linear sigma-models — Körner, Wozar, AW
  - supersymmetric Yukawa models — Wozar, Wellegehausen, Bergner, Kästner, AW



- $N_f^{ir} \geq 2$: PhD Thesis Daniel Schmidt, Jena
- $N_f^{ir} = 1$: Thesis Julian Lenz, Jena (dual formulation)
- $N_f^{ir} = 1$: $\langle\bar{\psi}\psi\rangle \neq 0$; $N_f \to \infty$: $\langle\bar{\psi}\psi\rangle_{m\to 0} = 0$

expect critical flavor number $N_f{}^{crit}$

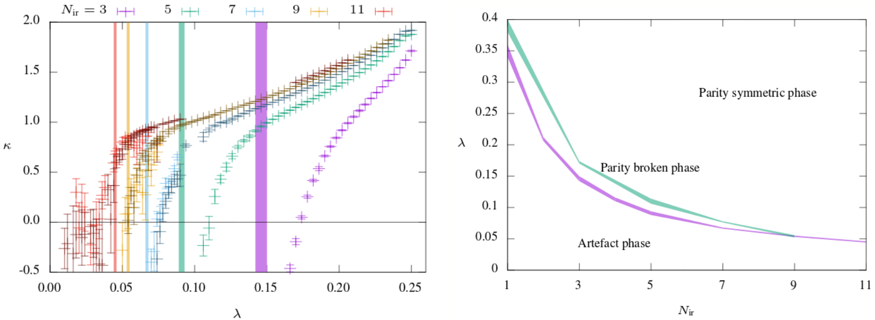
- Results from SD equations, $1/\mathrm{N_f}$-expansion, functional renormalization group, simulations with staggered and domain wall fermions



- spontaneous symmetry breaking signaled by

$$\kappa = V''_{\text{eff}}(x)\big|_{x=0}$$

- even $\mathrm{N_f^{ir}}$: $\kappa > 0 \Rightarrow$ no SSB for all reducible Thirring models



- curvatures for $3 \le \mathrm{N_f^{ir}} \le 11$ odd (bars: estimates for λ^*)

$N_{\mathrm{f,irr}}^{\mathrm{crit}} \approx 9$ for odd $\mathrm{N_f^{ir}}$ and $\mathrm{N_f^{crit}} \approx 0.8$

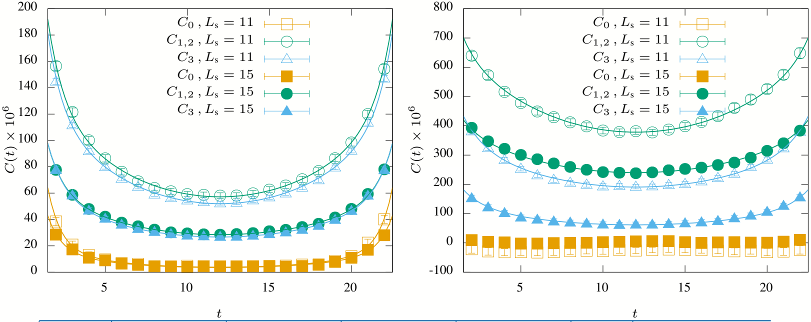
Phys. Rev. D96 (2017) 094504 and arXiv:1905.00137

- Simon Hands (2019): „domain wall“ fermions
  chiral condensat, spektrum of Goldstone-bosons
  no SSB for $N_f = 2, 3, \ldots$ reducible fermion species
  there could be SSB for $N_f = 1$ PRD 99, 2019
- Dabelow, Gies, Knorr (2019): FRG
  momentum-dependent coupling
  $N_f^{\rm crit}$ decreases as compared to more crude truncations
  new results could be compatible with $N_f < 1$ arXiv:1903.07388
- Lenz, Wellegehausen, Wipf (2019): chiral SLAC-fermions
  no SSB for $N_f > 0.80$ reducible fermion species
  $\to$ newest results improve and confirm results from 2017 arXiv:1905.00137

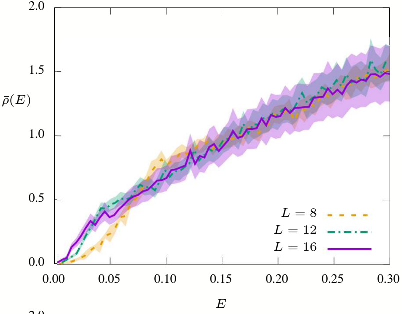
- transition with order parameter $\langle\bar{\psi}\psi\rangle$ for $\mathrm{N_f} \leq 0.8$
- transition without order parameter for $\mathrm{N_f} \geq 0.8$
- we calculated
  - chirale condensate $\Sigma_{L,\mathrm{N_f}}$
  - mean spectral density $\bar{\varrho}_{L,\mathrm{N_f}}(E)$
  - masses of light mesons $\mathrm{N_f} \in \{0.8, 1.0\}$
    symmetry of meson spectrum
  - susceptibilities
- simulations with „small“ fermion masses
- nuisance:

  lattice-artefact transition for all $\mathrm{N_f}$ and strong coupling

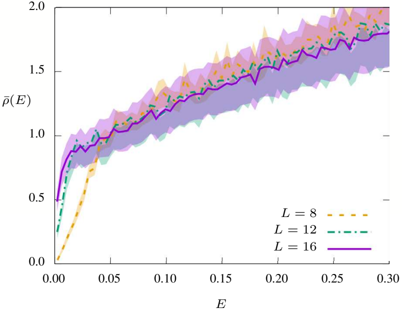
$C_a(t)$: symmetric ($\mathrm{N_f} = 1$) and broken ($\mathrm{N_f} = 0.8$) phases



C	$m(11)$	$m(15)$	$m^*(11)$	$m^*(15)$	$\mathrm{N_f}$	Symm.
C_0	0.21(2)	0.21(2)	1.27(6)	1.22(7)	1.0	
$C_{1,2}$	0.134(3)	0.128(2)	1.03(5)	1.02(3)	1.0	U(2)
C_3	0.138(2)	0.131(2)	1.08(4)	0.98(3)	1.0	
C_0	–	–	–	–	0.8	
$C_{1,2}$	0.103(2)	0.095(3)	1.04(12)	0.93(17)	0.8	U(1)× U(1)
C_3	0.109(4)	0.127(7)	0.81(7)	0.81(10)	0.8	

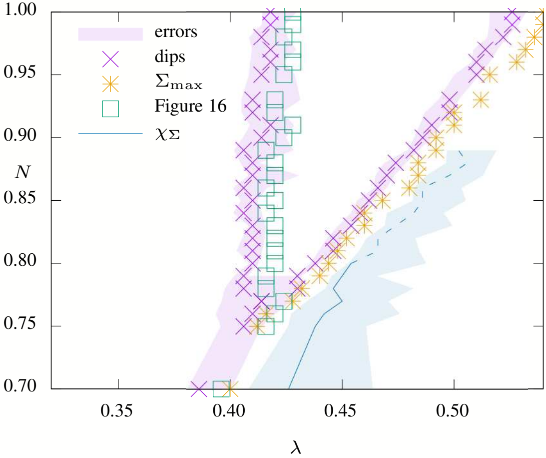


mean spectral density of $\not{D}$
symmetric phase
$N_f = 1.0$, $L = 8, 12, 16$
no condensation of low lying modes
$\Rightarrow$ symmetric phase



mean spectral density of $\not{D}$
broken phase
$N_f = 0.8$, $L = 8, 12, 16$
condensation of low lying modes
$\Rightarrow$ broken phase

Phase diagram in $(\lambda, \mathrm{N_f})$-plane



- do phase transition lines meet?
- probably second order transitions

- reliable simulations of 3*d* theories with chiral fermions
  - no sign-problem with reducible fermions
  - sign-problem with odd number of irreducible fermions, under control
- no SSB for all all reducible models with $\mathrm{N_f} = 1, 2, 3, \ldots$
  staggered fermions, early functional results ☹
  consistent with domain wall fermions, improved functional methods ☺
- odd $\mathrm{N_f^{ir}}$: parity breaking PT for $\mathrm{N_f^{ir}} \lessapprox 9$, corresponds to $\mathrm{N_f} \lessapprox 4.5$
- parity breaking PT with order parameter $\mathrm{N_f} \leq 0.8$
  new PT without order parameter $\mathrm{N_f} \geq 0.8$
- can construct continuum limit for all $\mathrm{N_f}$
- staggered fermions fail in theory with strongly interacting fixed point

cp. Anna Hasenfratz