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Conformal- and Thermodynamic Properties of a Family of

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Abstract: We investigate Thirring-like models containing fermionic and scalar fields propagating in 2-dimensional space time. The corresponding conformal algebra is studied and we disprove a conjecture relating the finite size effects to the central charge. Some new results concerning the fermionic determinant on the torus with chirally twisted boundary conditions and a chemical potential are presented. In particular we show how the thermodynamics of the Thirring model depends on the current-current interaction.

The dependence of expectation values on the temperature, particle density, space region, imposed boundary conditions or external fields is of importance in all the branches of physics [1]. In the present work we address these questions for the 2-dimensional model defined by the action

$$S=\int\sqrt{-g}\Big[i\bar\psi\gamma^\mu(\nabla_\mu-ig_1\partial_\mu\lambda+ig_2\eta_\mu{}^\nu\partial_\nu\phi)\psi + g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi+\partial_\mu\lambda\partial_\nu\lambda)-g_3\mathcal{R}\lambda\Big], \tag{1}$$

for the fermionic, scalar and pseudo scalar fields ψ,λ and ϕ respectively. For $g_3=0$ and $g_1^2=-g_2^2=g^2$ (1) belongs to the well known Thirring model [2,3]. For $g_1=g_2=0$ the theory decouples into free fields and non-minimally coupled scalars describing the minimal models in CFT. g_3 measures the deviation from minimal coupling to gravitation.

On flat space-time ($\mathcal{R}=0$) the action (1) defines a conformal field theory admitting a $U(1)$-Kac-Moody symmetry algebra. But the conformal algebra is deformed relative to that of the Thirring model. Part of this deformation can already be seen on the classical level. Indeed the energy momentum tensor

$$\begin{aligned} T^{\mu\nu} = & \frac{i}{2}\big[\bar\psi\gamma^{(\mu}D^{\nu)}\psi - (D^{(\mu}\bar\psi)\gamma^{\nu)}\psi\big] \\ & + 2\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi \quad + \quad (\phi \leftrightarrow \lambda) \\ & - 2g_3(g^{\mu\nu}\nabla^2 - \nabla^\mu\nabla^\nu)\lambda \\ & + \frac{1}{2}j^\mu\,(g_1\nabla^\nu\lambda - g_2\eta^{\nu\alpha}\nabla_\alpha\phi) \quad + \quad (\mu \leftrightarrow \nu) \\ & + g_2 g^{\mu\nu} j^\alpha \eta_{\alpha\beta}\nabla^\beta\phi - 2g_2 j^\alpha \eta_\alpha{}^{(\mu}\nabla^{\nu)}\phi\,, \end{aligned} \tag{2a}$$

has trace

$$T^\mu{}_\mu = g_3^2\mathcal{R}. \tag{2b}$$

For $g_3=0$ the trace vanishes, and the theory becomes Weyl-invariant. Hence it reduces to a conformal field theory in the flat spacetime limit [4]. However, (1) can be made Weyl invariant even for $g_3\neq 0$ by adding a nonlocal Wess-Zumino-type term to the action

$$S \to S' = S - \frac{g_3^2}{4}S_p \quad \text{where} \quad S_p = \int \sqrt{-g}\mathcal{R}\frac{1}{\triangle}\mathcal{R}. \tag{2c}$$

The field equations are not affected by the WZ term but the energy momentum tensor is modified in such a way that its trace vanishes and thus for $g_{\mu\nu}\to\eta_{\mu\nu}$ the Lagrangian corresponds to a conformal field theory in Minkowski spacetime.

The conformal weights of the fundamental fields are obtained computing their poisson brackets [5] with the generator T_f of the conformal symmetry transformations. In light cone coordinates $x^\pm = x^0 \pm x^1$, $T_f = \int dx^- f(x^-)T_{--}$ and

$$\begin{aligned} \{T_f,\phi\} &= f\partial_-\phi \\ \{T_f,\lambda\} &= f\partial_-\lambda - \frac{g_3}{2}\partial_- f \\ \{T_f,\psi_+\} &= f\partial_-\psi_+ + \frac{1}{2}(1-ig_1g_3)\psi_+\partial_- f \\ \{T_f,\psi_+^\dagger\} &= f\partial_-\psi_+^\dagger + \frac{1}{2}(1+ig_1g_3)\psi_+^\dagger\partial_- f, \end{aligned} \tag{3}$$

where $\psi_+ = \frac{1}{2}(1+\gamma_5)\psi$ denotes the right moving fermions. ϕ and ψ_+ are primary fields with conformal weights $h_\phi = 0$ and $h_{\psi_+} = \frac{1}{2}(1 - ig_1 g_3)$, respectively. The non-primary character of λ is linked with the g_3-dependent term in the transformation of the Dirac field. Since ψ is not a scalar under conformal transformation the term $\bar\psi(...)\psi$ in (1) is only conformally invariant because λ transforms inhomogenously like a spin connection. It may be surprising that the symmetry transformations depend on the coupling constant g_3 which is not present in the flat space time Lagrangean. However, the same happens in 4 dimensions if one couples a scalar field conformally to gravity. Although the Lagrangeans for the minimally and conformally coupled particles are the same on Minkowski spacetime, their energy momentum tensors are not. The same happens for the conformally invariant nonabelian Toda theories wich admit several energy momentum tensors and hence several conformal structures [6].
The current transforms as

$$\{T_f, j_-\} = f\partial_- j_- + j_-\partial_- f \tag{4a}$$

and hence is a primary field with weight 1. For the energy momentum tensor we find

$$\{T_f, T_{--}\} = f\partial_- T_{--} + 2T_{--}\partial_- f - g_3^2\partial_-^3 f \tag{4b}$$

and thus a classical central charge $c = 24\pi g_3^2$.

The *quantum analogues* of (3,4) are encoded in the short distance expansion of the fields with the stress tensor. Stress tensor insertions are gotten by differentiation w.r.t. the metric as

$$\begin{aligned} &Z(g)\langle O_1(x_1)...O_n(x_n)T_{y_1y_1}...T_{y_ny_n}\rangle = \\ &\qquad \frac{(-2)^n}{\sqrt{g(y_1)...g(y_n)}}\frac{\delta^n}{\delta g^{y_1y_1}...\delta g^{y_ny_n}}\langle O_1(x_1)...O_n(x_n)\rangle Z(g). \end{aligned} \tag{5}$$

For a primary field O with conformal weight h one then finds the transformation law

$$\frac{1}{i}\oint dz f(z)\langle O(x)\, T_{zz}\rangle = f(x)\partial_x O(x) + hO(x)\partial_x f(x). \tag{6}$$

Note that we have switched to the Euclidian region for the quantum considerations. The metric dependence of the effective action follows essentially from the trace anomaly. One finds

$$Z(g)=e^{(\frac{2}{24\pi}+g_3^2)S_L}Z(\hat{g}),\quad \text{where}\quad g=e^{2\sigma}\hat{g} \tag{7}$$

and $S_L=-\int\sqrt{g}\sigma\triangle\sigma$ is the Liouville action. Application of (5) and (6) then yields

$$\begin{aligned} c &= 3+24g_3^2\pi \qquad h_j=1 \\ h_{\psi_0} &= \frac{1}{2}+\frac{1}{16\pi}g_1^2-\frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi+g_2^2}-\frac{ig_1g_3}{2} \\ h_{\psi_1^\dagger} &= (h_{\psi_0})^\dagger \\ \bar{h}_{\psi_0} &= \frac{1}{16\pi}g_1^2-\frac{1}{16\pi}\frac{2\pi g_2^2}{2\pi+g_2^2}-\frac{ig_1g_3}{2}\,. \end{aligned} \tag{8}$$

The classical results for the λ- and ϕ fields are not modified. The central extension of the Kac-Moody algebra and the corresponding charges of the fermionic fields are the same as in the original Thirring model [4]. In the Thirring model limit $g_3=0$ and $g_1=g_2=g$, the different contributions in (8) add up to give the known anomalous dimension appearing in the Thirring model [4]. The last classical term is a peculiar feature of the solution. For the conformal weights to be real we must choose an imaginary g_3.

Let us now quantize the system on a sphere. The presence of the length scale breaks the conformal invariance and gives rise to *finite size effects.* An effective method to compute finite size effects has been developped in [7]. It is based on the following observation: Any conformal transformation $z\to w(z)$ is a composition of a diffeomorphism (defined by the same w) and a compensating Weyl transformation $g_{\mu\nu}\to e^{2\sigma}g_{\mu\nu}$ with

$$e^{2\sigma}=\frac{dw(z)}{dz}\frac{d\bar{w}(\bar{z})}{d\bar{z}},\qquad z=x^0+ix^1. \tag{9}$$

Therefore, chosing a diffeomorphism invariant regularization one has for the effective action Γ

$$0=\delta\Gamma_{Diff}=\delta\Gamma_{Conf}-\delta\Gamma_{Weyl}. \tag{10}$$

Integrating the conformal anomaly we end up with

$$\delta\Gamma=\frac{g_3^2}{4}\int\sqrt{g}\mathcal{R}\frac{1}{\Delta}(\mathcal{R}-\frac{8\pi}{V})-\frac{3}{24\pi}\int\sqrt{\hat{g}}\hat{\mathcal{R}}\sigma+\frac{3}{24\pi}\int\sqrt{\hat{g}}\sigma\hat{\Delta}\sigma. \tag{11}$$

Now we can see why the finite size conjecture generally fails to be true, although it holds for theories without background charge on domains with boundaries [7]. Take the simple case of a dilatation $w(z)=az$. Then, the conformal angle is a constant $\sigma=\log a$ and $(\mathcal{R}-8\pi/V)=0$. The first term in (11) vanishes and the finite size effect does not depend on g_3^2. It is given by

$$\delta\Gamma = -\frac{3}{24\pi}\log a \int \sqrt{\hat{g}}\hat{\mathcal{R}} = -\log a \tag{12}$$

and does not lead to the correct central charge c in (8) which depends on g_3. Thus we have disproved the conjecture. On other Riemannian surfaces one would find similar results.

To investigate the *thermodynamics* of (1) we quantize the model on a flat torus [8] with coordinates such that $x^\mu \in [0, L]$ and

$$g_{\mu\nu} \equiv \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix},$$

where $\tau = \tau_1 + i\tau_0$ is the Teichmueller parameter. Furthermore we introduce a chemical potential for the conserved $U(1)$-charge. Two new features appear which will have important consequences below:
1.) In the euclidean theory the chemical potential is equivalent to a constant imaginary gauge potential [9]. Therefore one has to give a sensible definition for fermionic determinants in complex gauge potentials.
2.) In order to recover the Thirring model on the torus one has to add constant (harmonic) contributions h_μ to the auxillary field. Hence we complete the action (1) by adding $g_0 \frac{2\pi}{L} h_\mu j^\mu + (\frac{2\pi}{L})^2 h_\mu h^\mu$, where $j^\mu = \psi^\dagger \gamma^\mu \psi$, to the Lagrangean in (1).
In order to compare our results with previous ones in the literature [10,11] we allow for twisted boundary conditions for the fermions

$$\begin{aligned} \psi(x^0 + L, x^1) &= -e^{2\pi i(\alpha_0 + \beta_0\gamma_5)}\psi(x^0, x^1) \\ \psi(x^0, x^1 + L) &= -e^{2\pi i(\alpha_1 + \beta_1\gamma_5)}\psi(x^0, x^1), \end{aligned} \tag{13}$$

where $\gamma_5 = \sigma_3$. α_i and β_i represent vectorial and chiral twists, respectively. In fact, the chiral twists are equivalent to chemical potentials. For the scalar field we impose periodic boundary conditions. As a first step in computing the *partition function* Z of our model, we determine the fermionic determinant. Due to the scaling property

$$\begin{aligned} \not{D} &= \gamma^\nu D_\nu = e^{ig_1\lambda + g_2\gamma_5\phi}\,\hat{\not{D}}\,e^{-ig_1\lambda + g_2\gamma_5\phi}, \quad \text{where} \\ \hat{\not{D}} &= \gamma^\mu\big(\partial_\mu - \frac{2\pi i}{L}[g_0 h_\mu + \mu_\mu]\big), \\ \mu_\mu &= -i\frac{\tau_0 L}{2\pi}\mu\,\delta_{\mu 0}, \end{aligned} \tag{14}$$

the dependence of $\det(i\not{D})$ on λ and ϕ can be found integrating the chiral anomaly [12] to be

$$\det(i\not{D}) = \det(i\hat{\not{D}}) \exp\Big(\frac{1}{2\pi}\int \sqrt{g}\phi\triangle\phi\Big). \tag{15}$$

Using standard grassmann integration rules [8,10] one further obtains

$$\det i\hat{\not{D}} = \prod_n \lambda_n^+ \lambda_n^- , \tag{16a}$$

where

$$\begin{aligned}
\lambda_n^+ =& \frac{2\pi}{\tau_0 L}[\bar\tau(\frac{1}{2}+a_1+\beta_1+n_1)-(\frac{1}{2}+a_0+\beta_0+n_0)]\\
\lambda_n^- =& \frac{2\pi}{\tau_0 L}[\tau(\frac{1}{2}+a_1-\beta_1+n_1)-(\frac{1}{2}+a_0-\beta_0+n_0)],\\
a_\mu =& \alpha_\mu - h_\mu - \mu_\mu .
\end{aligned} \tag{16b}$$

One may be tempted so identify

$$\det(D_+D_-) \sim \prod \lambda_n^+\lambda_n^- \quad \text{and} \quad \det D_+ \det D_- \sim \prod \lambda_n^+ \prod \lambda_m^- \tag{16c}$$

and thus conclude that the determinant is a product, $f(\tau)\bar f(\tau)$, that is factorizes into holomorphic and anti-holomorphic pieces. However, the infinite product (16a) must be regularized and the two expressions in (16c) may differ. To continue we recast the infinite product in the form

$$\prod^\infty \lambda_n^+\lambda_n^- = \prod_{n\in Z^2} \Big(\frac{2\pi}{L}\Big)^2 g^{\mu\nu}(\frac{1}{2}+c_\mu+n_\mu)(\frac{1}{2}+c_\nu+n_\nu) \tag{17a}$$

where

$$c_\mu = a_\mu + i\eta_\mu{}^\nu\beta_\nu, \quad ; \quad (\eta_\mu{}^\nu) = -\frac{1}{\tau_0}\begin{pmatrix}\tau_1 & -|\tau|^2\\ 1 & -\tau_1\end{pmatrix}. \tag{17b}$$

The point is that for real c_μ, that is for vanishing chiral twists β_μ and chemical potential the zeta function defined by

$$\zeta(s) = \sum_n \big(\lambda_n^+\lambda_n^-\big)^{-s} \tag{17c}$$

has a well defined analytic continuation to $s<1$ via the Poisson resummation. However, for complex c_μ the Poisson resummation is not applicable and $\zeta'(0)$ cannot be calculated by direct means. To circumvent these difficulties we note that the infinite product (17c) defining the ζ-function for $s>1$ is a meromorphic function in c. Thus we may first continue to $s<1$

for real c_μ and then continue the result to complex values. In this way we end up with

$$\det(i\hat{\not{D}}) = \frac{1}{|\eta(\tau)|^2}\Theta\begin{bmatrix}-c_1\\ c_0\end{bmatrix}(0,\tau)\bar{\Theta}\begin{bmatrix}-\bar{c}_1\\ \bar{c}_0\end{bmatrix}(0,\tau). \tag{18}$$

It can be shown that this determinant is *gauge invariant*, i.e. invariant under $\alpha_\mu \to \alpha_\mu+1$, but not invariant under chiral transformatins, $\beta_\mu \to \beta_\mu+1$, as expected. Furthermore it transforms covariantly under modular transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$. The result (18) also follows with operator methods and differs from previous results in the literature [10]. In particular there is no holomorphic factorization.

Having determined the fermionic determinant we are left with the integration over the auxillary fields ϕ, λ and h. These are Gaussian and one finds

$$\frac{Z}{\mathcal{N}_0} = \frac{1}{|\eta(\tau)|^2}\sqrt{\frac{2\pi+g_2^2}{2\pi+g_0^2}}\,\Theta\begin{bmatrix}u\\ w\end{bmatrix}(\Lambda)\,, \tag{19.a}$$

where [13]

$$\Theta\begin{bmatrix}u\\ w\end{bmatrix}(\Lambda) = \sum_{n\in Z^2} e^{i\pi(n+u)\Lambda(n+u)+2\pi i(n+u)w} \tag{19.b}$$

is the theta function with characteristics

$$u = -\begin{pmatrix}1\\ 1\end{pmatrix}(\alpha_1 + i\eta_1{}^\nu\beta_\nu) \quad \text{and} \quad w = \begin{pmatrix}1\\ -1\end{pmatrix}(\alpha_0 + i\eta_0{}^\nu\beta_\nu - \mu_0) \tag{19.c}$$

and covariance

$$\Lambda = \begin{pmatrix}\tau & 0\\ 0 & -\bar{\tau}\end{pmatrix} + i\frac{\pi g_0^2\tau_0}{2\pi+g_0^2}\begin{pmatrix}g_0^2 & -4\pi-g_0^2\\ -4\pi-g_0^2 & g_0^2\end{pmatrix}. \tag{19.d}$$

In (19.a) we have divided by the partition function $\mathcal{N}_0$ for non-interacting auxillary fields, so the number of degrees of freedom is the same as in the original Thirring model. In the Thirring model limit $g_2 = g_0$ and the square-root in (19a) disappears.

To investigate the thermodynamics of the model we must choose $\tau = i\beta/L$, where β is the inverse temperature. Then

$$\Omega = -\frac{1}{\beta}\log\frac{Z}{\mathcal{N}_0} \tag{20}$$

is the *grand canonical potential*. In the zero temperature limit the saddle point approximation to the theta function in (19.a) becomes exact. For *vanishing chiral twists and chemical potential* this yields the the ground state energy

$$E_0(L,\alpha_1,\beta_1{=}0) = -\frac{\pi}{6L} + \frac{2\pi}{L}\frac{2\pi}{2\pi+g_0^2}\big(\alpha_1 - [\frac{1}{2}+\alpha_1]\big)^2, \tag{21}$$

in agreement with [11]. Only for anti-periodic boundary conditions, that is for $\alpha_1{=}0$, does this Casimir energy coincide with the corresponding result for free fermions. For $g_0^2{\ge}4\pi$ the Casimir force is always attractive whereas for $g_0^2{<}4\pi$ it can be attractive or repulsive, depending on the value of α_1. For *small twists and chemical potential* the grand potential becomes

$$\Omega(\beta\to\infty) = -\frac{\pi}{6L} + \frac{2\pi}{L}\frac{2\pi}{2\pi+g_0^2}\alpha_1^2, \tag{22}$$

and hence, is independent of chemical potential and chiral twist. Here we disagree with [11]. The discrepancy is due to the breakdown of holomorphic factorization, a property which has been presupposed in [11]. In order to show that the result (22) is physically reasonable we note that for massless fermions the Fermi energy is just μ and at $T=0$ all electron states with energies less then μ and all positron states with energies less then $-\mu$ are filled. The other states are empty. Since $d\Omega/d\mu$ is the expectation value of the electric charge in the presence of μ we conclude that it must jump if μ crosses an eigenvalue of the first quantized Dirac Hamiltonian h. For vanishing twists the eigenvalues of h are just $E_n{=}(n-\frac{1}{2})\pi/L$. Indeed, from (22) one finds that the electric charge

$$\langle Q\rangle = \frac{d\Omega}{d\mu} = 2\big[\frac{1}{2}+\frac{\mu L}{2\pi}\big] = 2n \qquad \text{for} \quad E_n\le\mu<E_{n+1} \tag{23.a}$$

jumps at these values for μ. Further observe, that in the *thermodynamic limit* $L\to\infty$ the density

$$\frac{\Omega}{L} \to -\frac{2\pi}{2\pi+g_0^2}\frac{\mu^2}{2\pi}, \tag{23.b}$$

reduces for $g_0{=}0$ to the standard result for free electrons.

Let us now discuss the *equation of state*. Using the transformation properties of the theta functions under modular transformations [13] the pressure is given for $L\to\infty$ and *small twists* by

$$\beta p = \frac{\pi}{6\beta} - \frac{2\pi}{\beta}\frac{2\pi}{2\pi+g_0^2}\big(\alpha_0 + i\frac{\beta\mu}{2\pi}\big)^2. \tag{24.a}$$

In particular it becomes independent on the chiral twist β_0 in agreement with the earlier result that for small twists Ω is independent of β_1. For the thermal boundary conditions $\alpha_0=0$, we are lead to the following equation of state

$$p(\beta,\mu,\alpha_0=0)=\frac{\pi}{6\beta^2}+\frac{\mu^2}{2\pi}\frac{2\pi}{2\pi+g_0^2}, \tag{24.b}$$

which for small β_0 relates the pressure to the chemical potential and temperature. This result is consistent with the renormalization of the electric charge which is conjugate to the chemical potential. It shows in particular that the thermodynamic behaviour of the Thirring model is not just the one of free fermions as has been claimed in [14]. Indeed, the zero point pressure is multiplied by a factor $2\pi/(2\pi+g_0^2)$. This modification arises from the coupling of the current to the harmonic fields. It can not be seen if only the local part of the auxillary field is considered, which is the case if one quantizes the model on the infinite Euclidean space.

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