ETH-TH/91-15

Finite Temperature Schwinger Model

I. Sachs and A. Wipf

Institut für Theoretische Physik
Eidgenössische Technische Hochschule
Hönggerberg, Zürich CH-8093, Switzerland

12 August 1991; arXiv-ed 10 May 2010

Abstract

The temperature dependence of the order parameter of the Schwinger model is calculated in the euclidean functional integral approach. For that we solve the model on a finite torus and let the spatial extension tend to infinity at the end of the computations. The induced actions, fermionic zero-modes, relevant Green functions and Wilson loop correlators on the torus are derived. We find the analytic form of the chiral condensate for any temperature and in particular show that it behaves like $\langle\bar{\Psi}\Psi\rangle \sim -2T\exp(-\pi\sqrt{\pi}T/e)$ for temperatures large compared to the induced photon mass.

Keywords: Schwinger model; finite temperature; Euclidean path integral; fermionic zero modes; Wilson loop; condensate; effective action; field theory, torus; two-point function

Published as: Helvetica Physica Acta **65** (1992) 652-678

---

email (2010): ivo.sachs@physik.lmu.de, wipf@tpi.uni-jena.de

Contents

1 Introduction

The study of exactly soluble field theories has always received a good deal of attention in the hope that they might shed some light on more realistic theories. One such model is the Schwinger model or quantum electrodynamics for massless fermions in two space-time dimensions [1]. It serves as an important tool in illustrating various (related) field theoretical concepts such as mass generation, dynamical symmetry breaking, charge shielding or fermion trapping [2].

More recently, the Schwinger model and its extensions have been used as toy models for studying Baryon number violating processes [3]. Since the semiclassical expansion in the standard model breaks down at the interesting energies or temperatures, it is important to compare the realistic calculations to the corresponding ones for simpler models. As simple soluble models which, for the problem at hand, have features identical to the standard model, the non-linear sigma model [4] and the Schwinger model (coupled to a scalar field) [3] have been considered.

The influence of an external gravitational field on the ultra-violet and infra-red properties of the model has also been investigated [5]. This problem is not completely solved up to now but the partial results indicate that qualitatively all long-distance features of the flat-space-time solution persist in curved backgrounds.

The zero-temperature Schwinger model has been solved some time ago by using operator methods [6] and more recently in the path integral formulation [7]. Some properties of the model (e.g. the non-trivial vacuum structure) are more transparent in the operator approach and others (e.g. the role of the chiral anomaly) are better seen in the path integral approach.

When studying the finite temperature $O(3)$-sigma model coupled to fermions (as a relevant model for the B-number violating processes in the electro-weak theory [4]) we realized that no satisfactory solution of the simpler finite-temperature Schwinger model is available, at least in the path integral approach. Therefore we decided to investigate the Schwinger model at finite temperature and in particular determine the temperature dependence of the chiral condensate by using functional integral methods. To study the temperature effects we solve the Schwinger model on the two-dimensional torus to avoid the infra-red divergences, which sometimes plague the analysis of two-dimensional gauge theories.

Previous finite-volume solutions of QED_2 have clarified various aspects of the model. For example the role of the nonintegrable phase factor $\exp(ie\int A_\mu\, dx^\mu)$ has been understood by studying the model on a cylinder, that is by considering the zero-temperature model where the spatial dimension is given by a circle [8]. Recently the Schwinger model has been solved in the path integral approach on the 2-dimensional sphere and the role of the fermionic zero modes has been emphazise [9]. Also it has been considered on a 2-dimensional disk assuming the most general (local) self-adjoint boundary conditions for the fermions [10].

However, these geometries are not the proper ones to study finite temperature effects. For that we must assume that the fields are (anti)-periodic in the time direction with period $\beta = 1/T$. In order to justify the manipulations below (in particular the treatment of the Grassmannian integrals) we prefer to work on a finite space-time volume such that the relevant wave-operators possess discrete spectra. Thus we assume that the spatial extension of space-time is finite as well and consider the Schwinger model on a torus $[0,\beta]\times[0,L]$ with volume $V=\beta\cdot L$. We assume periodic boundary conditions in the spatial direction. Only at the end of our calculations do we let the spatial extension L tend to infinity.

The only other solution of the Schwinger model on the torus we are aware of, besides assuming periodic boundary conditions for Dirac-Kähler fermions instead of finite temperature ones, concentrated mainly on finding and discussing the fermionic zero-modes [11]. Also, our trivialization of the $U(1)$-bundle over the torus is different from the one in [11] but is very convenient for determining the explicit spectrum of the Dirac operator.

Our main results are the analytic form of the (zero-mode truncated) effective actions and zero-modes in the various topological sectors, the two-point Green function of $\Delta^2 - m_\gamma^2\Delta$ on the torus and as a consequence the explicit formulae (5.9), (5.10) and (5.12) for the temperature dependence of the chiral condensate $\langle\bar\Psi\Psi\rangle$. We find that $\langle\bar\Psi\Psi\rangle_T$ is equal to $-(m_\gamma/2\pi)\cdot e^\gamma$ for $T=0$, in agreement with the zero-temperature solutions

[7,9], and tends to $-2T\exp(-\pi T/m_\gamma)$ for temperature large compared to the induced photon mass $m_\gamma = e/\sqrt{\pi}$. Our high temperature result differs by a factor 2 from the only other (formal) finite temperature path integral solution we are aware of [13]. Also, the torus calculation nicely splits the chiral anomaly responsible for the non-vanishing condensate, into its high- and low energy parts. The first (coming from the gauge invariant ultraviolet regularisation) only affects the numerical value of $\langle\bar{\Psi}\Psi\rangle$, whereas the latter (coming from the chiral asymetry of the zero-energy sector) is responsible for the non-zero $\langle\bar{\Psi}\Psi\rangle$. This is the reason why ordinary perturbation theory, although leading to the correct UV-contribution via the wellknown anomaly graph, fails to predict a non-vanishing condensate. At the end we obtain analytic expressions for (thermal) Wilson loop correlators and in particular their temperature dependence as well as the gauge invariant fermionic two-point functions.

2 The Schwinger Model

In this section we recapitulate briefly the euklidean Schwinger model [7], using functional methods. We emphasize the less known features involved when considering the model on a torus instead on the plane. The Schwinger model is defined by the action

$$S[A,\bar{\Psi},\Psi] = \frac{1}{2}\int d^2x\, E^2 - \int d^2x\, \bar{\Psi} i \not{D}\, \Psi, \tag{1.1}$$

where

$$E = F_{01} = \partial_0 A_1 - \partial_1 A_0 \quad \text{and} \quad \not{D} = \gamma^\mu\big(\partial_\mu - ieA_\mu\big).$$

Note that in two space-time dimensions the electric charge has the dimension of an inverse length, contrary to the situation in four dimensions. The model is superrenormalizable and requires no infinite renormalization besides a trivial redefinition of the zero energy density. The generating functional for the Green functions is

$$Z[J,\eta,\bar{\eta}] = \mathcal{C}^{-1}\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\; e^{-S[A,\bar{\Psi},\Psi]-S_{gf}[A]+\int d^2x\big(A_\mu J^\mu+\bar{\eta}\Psi+\bar{\Psi}\eta\big)}, \tag{1.2}$$

where $S_{gf}[A]$ contains the terms due to gauge fixing and J^μ resp. $(\bar{\eta},\eta)$ are c-number and Grassman-valued currents respectively. The normalization constant $\mathcal{C}$ is chosen such that $Z[0,0,0]=1$. First we evaluate the fermionic path integral

$$Z[A;\eta,\bar{\eta}] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\; e^{\int \bar{\Psi} i\not{D}\,\Psi+\int d^2x\big(\bar{\eta}\Psi+\bar{\Psi}\eta\big)}, \tag{1.3}$$

and treat the 'photon'-field as external field in this first step.

On the torus the Dirac operator may possess normalizable zero-modes, $\not{D}\,\psi = 0$, and thus we must allow for such fermionic zero modes in the path integral (1.3). Let us assume that there are $|k|$ (our notation will become clear later on) normalizable zero-modes ψ_p. In addition there are an infinite number of excited modes ψ_q, $q > |k|$ with eigenvalues $\lambda_q \neq 0$. The number of zero-modes can be determined as follows:

First we note that since γ_5 anti-commutes with the Dirac operator the excited modes come in pairs

$$i\not{D}\,\psi_q = \lambda_q\psi_q \Longrightarrow i\not{D}\,(\gamma_5\psi_q) = -\lambda_q(\gamma_5\psi_q), \tag{1.4}$$

so that ψ_q and $\gamma_5\psi_q$ have eigenvalues λ_q and $-\lambda_q$ and are orthogonal to each other. It follows that the contribution of the excited states to the (supersymmetric) partition function

$$\mathrm{tr}\ \gamma_5\, e^{t\not{D}^2} = \sum_1^{|k|}(\psi_p,\gamma_5\psi_p) + \sum_{|k|+1}^{\infty} e^{-t\lambda_q^2}(\psi_q,\gamma_5\psi_q).$$

vanishes. Thus the partition function is just the index $n_+ - n_- \equiv k$, where n_+ is the number of right handed (chirality $\gamma_5 = 1$) and n_- the number of left handed ($\gamma_5 = -1$) zero modes. On the other hand, inserting the small- t expansion (see e.g. [14])

$$\langle x|e^{t\not{D}^2}|x\rangle \sim \frac{1}{4\pi t}\big(1 + \gamma_5 E t + O(t^2)\big) \tag{1.5}$$

one immediately obtains the index theorem (see e.g. [15])

$$k = \frac{1}{2\pi}\int d^2x\, E \equiv \frac{1}{2\pi}\Phi \tag{1.6}$$

which relates the index k of the Dirac operator to the flux of the electric field. This formula already shows that there are zero-modes for non-vanishing fluxes Φ. Also note that on the torus the flux is quantized in integer muliples of 2π. This is really a consequence of the single valuedness of the fermionic wave function (cocycle condition).

The pairing property (1.4) does not hold for the zero-modes. Indeed, they can be chosen to have fixed chirality since γ_5 commutes with the Dirac operator on the subspace spanned by the zero modes. Also (1.4) would not be valid for the excited modes if we would impose chirality-violating boundary conditions as in [10]. But (anti) periodic boundary conditions are compatible with the transformation $\psi \to \gamma_5\psi$ and the identity (1.4) holds in the present situation.

To determine n_+ and n_- separately we decompose the gauge potential as

$$A_\mu = \tilde{A}_\mu - \epsilon_{\mu\nu}\partial_\nu\phi \quad \text{where} \quad \phi = \frac{1}{\Delta}\big(E - \frac{1}{V}\Phi\big) + \mathrm{c} \Longrightarrow E = \frac{\Phi}{V} + \Delta\phi, \tag{1.7}$$

that is into a global "instanton"-type potential $\tilde{A}$ with constant field strength, $\tilde{E} = \Phi/V$ and a local fluctuation $\delta A_\mu = -\epsilon_{\mu\nu}\partial_\nu\phi$ about the instanton. The subtraction of the constant term in (1.7) is necessary since on the torus the Laplacian has the constant zero-mode and is invertible only on functions which integrate to zero. Using $\gamma_\mu\gamma_5 = -i\epsilon_{\mu\nu}\gamma_\nu$ it is now easy to see that

$$\not{D} = e^{\gamma_5\phi}\tilde{\not{D}}e^{\gamma_5\phi} \tag{1.8}$$

which shows that the number of fermionic zero-modes is independent of ϕ and hence is the same for A and $\tilde{A}$. But for the instanton potentials $\tilde{A}$

$$\tilde{\not{D}}^2 = \tilde{D}^2 + \gamma_5\frac{\Phi}{V} \tag{1.9}$$

and since $-\tilde{\not{D}}^2$ is a non-negative operator, all zero modes are either right- or left-handed for non-vanishing fluxes. Only in the zero-instanton sector $\Phi = 0$ can it happen that there are both right- and left-handed zero modes. For example, for a vanishing gauge potential there is one right-handed and one left-handed zero mode.
What we have shown then is that for $\Phi \neq 0$ there are exactly $|k| = |\Phi|/2\pi$ zero modes. If the flux is positive they are all right-handed, else they are all left-handed.

After having counted the number of zero modes we proceed by expanding the 'electron'-field in an adapted orthonormal eigen-base as

$$\Psi(x) = \sum_1^{|k|} \alpha_p \psi_p(x) + \sum_{|k|+1}^{\infty} \beta_q \psi_q(x)$$

and similarly $\bar{\Psi}$, so that

$$(\bar{\eta}, \Psi) = \sum (\bar{\eta}, \psi_p)\alpha_p + \sum (\bar{\eta}, \psi_q)\beta_q \qquad (\bar{\Psi}, \eta) = \sum \bar{\alpha}_p(\bar{\psi}_p, \eta) + \sum \bar{\beta}_q(\bar{\psi}_q, \eta)$$

split into a zero-mode part and an excited part. Inserting this decomposition into (1.3) and using $\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = \mathcal{D}\bar{\alpha}\mathcal{D}\alpha\mathcal{D}\bar{\beta}\mathcal{D}\beta$ the Grassmannian integral over the α's can easily be done since the action does not depend on them. This way one finds for the zero-mode contribution to (1.3)

$$\int \mathcal{D}\bar{\alpha}\mathcal{D}\alpha \prod_1^{|k|} e^{(\bar{\eta}, \psi_p)\alpha_p} \, e^{\bar{\alpha}_p(\bar{\psi}_p, \eta)} = \prod_1^{|k|} (\bar{\eta}, \psi_p)(\bar{\psi}_p, \eta). \qquad (1.10a)$$

The remaining β-integration is performed by shifting the β's (and similarly the $\bar{\beta}$'s) according to

$$\beta_q \longrightarrow \beta_q - \frac{1}{\lambda_q}(\bar{\psi}_q, \eta).$$

After this shift the β integration yields

$$\int \mathcal{D}\bar{\beta}\mathcal{D}\beta \prod_{|k|+1}^{\infty} e^{\lambda_q \bar{\beta}_q \beta_q + (\bar{\eta}, \psi_q)\beta_q + \bar{\beta}_q(\bar{\psi}_q, \eta)} = e^{-\int \bar{\eta}(x) G_e(A;x,y)\eta(y)} \cdot \det{}'(i\not{D}), \qquad (1.10b)$$

where $\det'$ is the determinant with the zero-eigenvalues omitted and

$$G_e(A; x, y) = \sum_{|k|+1}^{\infty} \frac{\psi_q(x)\psi_q^\dagger(y)}{\lambda_q}. \qquad (1.11a)$$

is the Green function on the space orthogonal to the zero modes. Clearly this function obeys the differential equation

$$i\not{D} G_e(A; x, y) = \delta(x-y) - P(x, y), \qquad \text{where} \quad P(x, y) = \sum_1^{|k|} \psi_p(x)\psi_p^\dagger(y) \qquad (1.11b)$$

is the projection kernel on the zero-mode subspace. Due to the pairing property of the excited modes the 'excited' green function anti-commutes with γ_5:

$$\gamma_5 G_e(A;x,y)\gamma_5 = -G_e(A;x,y). \tag{1.11c}$$

Inserting now (1.10) into the path integral for the (fermionic) partition function (1.3) we end up with

$$Z[A;\bar\eta,\eta] = \prod_1^{|k|}(\bar\eta,\psi_p)(\bar\psi_p,\eta)\ \ e^{-\int\bar\eta(x)G_e(A;x,y)\eta(y)}\ \det{}'(i\not{D}). \tag{1.12}$$

Thus to determine $Z(A;\bar\eta,\eta)$ for a fixed potential we need to compute the $|k|$ zero-modes of the Dirac operator, the determinant of $i\not{D}$ with the zero-modes omitted and the 'excited' Green function. The (unnormalized) fermionic $2n$-point functions for a given A-field are now obtained by differentiation with respect to the external currents

$$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi\ \Psi_{\alpha_1}(x_1)\bar\Psi_{\beta_1}(y_1)...\Psi_{\alpha_n}(x_n)\bar\Psi_{\beta_n}(y_n)\, e^{\int\bar\Psi i\not{D}\Psi}$$

$$= \frac{\delta^{2n}}{\delta\eta^{\beta_n}(y_n)\delta\bar\eta^{\alpha_n}(x_n)...\delta\eta^{\beta_1}(y_1)\delta\bar\eta^{\alpha_1}(x_1)} Z[A;\bar\eta,\eta]|_{\bar\eta=\eta=0} \tag{1.13}$$

Using (1.12) in this formula we can immediately read off that

1. The fermionic partition function

   $$Z[A;0,0] = \det(i\not{D}) \tag{1.14}$$

   which enters in the normalization constant $\mathcal{C}$ in (1.2) is only non-zero if the Dirac operator possesses no zero modes that is for gauge field with vanishing flux.

2. The 2-point functions are non-zero only if A admits either no or one zero-mode, that is for $\Phi = 0$ or $\Phi = \pm 2\pi$ and then

   $$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi\ e^{\int\bar\Psi i\not{D}\Psi}\ \bar\Psi_\alpha(x)\Psi_\beta(y) = \begin{cases} -G_{\alpha\beta}(x,y)\det(i\not{D}) & \Phi = 0 \\ -\psi^\dagger_\alpha(x)\psi_\beta(y)\det{}'(i\not{D}) & |\Phi| = 2\pi. \end{cases} \tag{1.15}$$

   From (1.11c) it follows that the expectation values of $\bar\Psi P_\pm\Psi$, where $P_\pm = \frac{1}{2}(1\pm\gamma_5)$ are the chiral projections, are non-vanishing only for $\Phi = \pm 2\pi$ in which cases the Dirac operator has one zero mode ψ of chirality ± 1, and then

   $$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi\ e^{\int\bar\Psi i\not{D}\Psi}\ \bar\Psi(x)P_\pm\Psi(x) = -\,\mathrm{tr}\,\big(\psi^\dagger(x)P_\pm\psi(x)\big)\,\det{}'(i\not{D}). \tag{1.16}$$

   The higher $2n$-point functions are obtained similarly. For example, for the 4-point functions only gauge potentials with no, one or two zero modes contribute. In particular for the expectation value of the operator with (chiral) charge zero

   $$O(\Psi) = (\bar\Psi P_+\Psi)(\bar\Psi P_-\Psi), \qquad O(e^{\alpha\gamma_5}\Psi) = O(\Psi) \tag{1.17}$$

only the trivial sector with no zero-modes yields a non-vanishing result. This explains why earlier solutions of the Schwinger model on the plane (in which the non-trivial sectors have been neglected) yield the correct result for this expectation value and thus (via clustering) the correct result for the chiral condensate.

Note that the excited fermionic Green function does not appear in the expectation values (1.16), and this simplifies the computation of the chiral condensate considerably. More generally, the expectation values of the charge $\pm n$ operators $(\bar{\Psi}P_{\pm}\Psi)^n$ are non-zero only for potentials with fluxes $\Phi = \pm 2\pi n$, and then only the zero modes and effective actions, but not the excited Green functions occur in the expectation values.

3 Effective Actions - Local Part

In this section we compute the ϕ-dependence of the effective actions $\det' i\not{D}$ for the general gauge potentials (1.7) by integrating the chiral anomaly. Since there are some subtleties when zero-modes are present we derive the local ϕ-dependent part of the effective actions, although some of our arguments are not new [16].
To determine the ϕ-dependence of the effective actions we consider the one-parametric family of Dirac-operators

$$\not{D}_\alpha = e^{\gamma_5\phi\alpha}\tilde{\not{D}}e^{\gamma_5\phi\alpha} \Longrightarrow \dot{\not{D}}_\alpha = \{\gamma_5\phi, \not{D}_\alpha\} \qquad (\dot{} = \frac{d}{d\alpha}) \tag{2.1}$$

which interpolates between $\tilde{\not{D}}$ and $\not{D}$, and calculate the variation of the zeta-function regularized determinants [17]

$$\log\det{}'(i\not{D}_\alpha) = \frac{1}{2}\log\det{}'(-\not{D}_\alpha^2) = -\frac{1}{2}\frac{d}{ds}\zeta_{\not{D}^2}(s)|_{s=0}. \tag{2.2a}$$

The zeta-function is defined as as the analytic continuation of

$$\zeta_{\not{D}^2}(s) = \sum_{|k|+1}^{\infty}\mu_q^{-s}, \qquad \Re(s) > 1, \tag{2.2b}$$

where $\mu_q = \lambda_q^2$ are the α-dependent excited eigenvalues of the squared Dirac operator, to the whole complex s-plane. Using (2.1) in the Feynman-Hellman formula

$$\dot{\lambda}_q = (\psi_q, i\dot{\not{D}}\psi_q) = 2\lambda_q(\psi_q, \gamma_5\phi\psi_q)$$

the variation of the zeta-function can be written as

$$\frac{d}{d\alpha}\zeta_{\not{D}_\alpha^2}(s) = -s\sum_{|k|+1}^{\infty}\mu_q^{-s-1}\dot{\mu}_q = -4s\sum\mu_q^{-s}(\psi_q, \gamma_5\phi\psi_q).$$

This can be further rewritten as a Mellin transform

$$\frac{d}{d\alpha}\zeta_{\not{D}^2_\alpha}(s) = -\frac{4s}{\Gamma(s)}\int dt\, t^{s-1}\sum_{|k|+1}^{\infty} e^{-t\mu_q}(\psi_q,\gamma_5\phi\psi_q) = -\frac{4s}{\Gamma(s)}\int dt\, t^{s-1}\,\mathrm{tr}'\Big(e^{t\not{D}^2_\alpha}\gamma_5\phi\Big),$$

where the trace is only to be taken over the excited states. Inserting the asymptotic expansion (1.5) from which we must subtract the projection density $P_\alpha(x,x) = P_\alpha(x)$ (see (1.11b) onto the zero modes we find

$$\frac{d}{d\alpha}\frac{d}{ds}\zeta_{\not{D}^2}(s)|_{s=0} = -\frac{2\alpha}{\pi}\int d^2x\, E\phi + 4\int \mathrm{tr}\,\big(P_\alpha(x)\gamma_5\phi(x)\big). \tag{2.3}$$

To integrate over α we observe that the zero-modes of $\not{D}_\alpha$ and $\tilde{\not{D}}$ are related as

$$\psi_p^{(\alpha)} = e^{\mp\alpha\phi}\tilde{\psi}_p, \qquad p = 1,\dots,|k| \tag{2.4}$$

with the negative sign in the exponent for right-handed and the positive sign for left-handed zero-modes. These modes are in general not orthonormal, and the normmatrix

$$\mathcal{N}_{pr}(\alpha) = \big(\psi_p^{(\alpha)},\psi_r^{(\alpha)}\big) \tag{2.5}$$

is not the identity so that the projection density reads

$$P_\alpha(x) = \sum_{pr}\psi_p^{(\alpha)}(x)\mathcal{N}^{-1}_{pr}(\alpha)\,(\psi_r^{(\alpha)})^\dagger(x).$$

Now it follows from (2.4) and (2.5) that

$$\frac{d}{d\alpha}\log\det\big(\mathcal{N}(\alpha)\big) = -2\int d^2x\,\mathrm{tr}\,\big(P_\alpha\gamma_5\phi\big) \tag{2.6}$$

and this formula can be used to integrate the anomaly equation (2.3) from $\alpha = 0$ to $\alpha = 1$. Together with (2.2) we end up with

$$\det{}'(i\not{D}) = \det\frac{\mathcal{N}}{\tilde{\mathcal{N}}}\det{}'(i\tilde{\not{D}})\ \exp\big(\frac{e^2}{2\pi}\int E\phi\big) \tag{2.7}$$

which is the (almost) factorization of the determinants into a global, $\tilde{A}$-dependent, and local, ϕ-dependent, part. The factorization is not complete since $\det\mathcal{N} = \det\mathcal{N}(1)$ still contains a coupling between the instanton potential $\tilde{A}$ and the local fluctuation δA via (2.4). Since E in (1.7) depends on both Φ and ϕ, there is also an apparent coupling in the last factor in (2.7). However we shall see later that not all functions ϕ are permitted and that for the allowed ones this factor does not depend on Φ. Also note that the local part does not depend on the geometry of the torus. In particular the induced photon mass $m_\gamma = e/\sqrt{\pi}$ is the same as on the infinite plane.

4 Effective Actions - Global Part

With the factorizations (2.4) and (2.7) of the zero modes and effective actions into local and global factors the problem of finding the 2-point functions (1.16) (or $2n$-point functions of charge $\pm n$ operators) reduces to the problem of computing the zero modes and effective actions for the instanton potentials $\tilde{A}$. For that we need to know the explicit form of these potentials.

An instanton potential can always be decomposed as

$$\tilde{A}_0 = -\frac{\Phi}{V}x^1 + \frac{2\pi}{\beta}h_0 + \partial_0\lambda \quad \text{and} \quad \tilde{A}_1 = \frac{2\pi}{L}h_1 + \partial_1\lambda \tag{3.1}$$

with constant h_μ. The least trivial term proportional to Φ yields the constant field strength and corresponds to a particular trivialization of the $U(1)$-bundle over the torus. In other words, the gauge potentials at (x^0, x^1) and (x^0, x^1+L) are necessarily related by a non-trivial gauge transformation

$$A_\mu(x^0, x^1+L) - A_\mu(x^0, x^1) = \partial_\mu\alpha, \qquad \text{where} \quad \alpha = -\frac{\Phi}{\beta}x^0. \tag{3.2a}$$

Accordingly the fermionic wave functions transform as

$$\psi(x^0, x^1+L) = e^{i\alpha}\psi(x^0, x^1). \tag{3.2b}$$

These boundary conditions must be supplemented by the finite temperature boundary conditions in the x^0-direction

$$A_\mu(x^0+\beta, x^1) = A_\mu(x^0, x^1) \qquad \text{and} \quad \psi(x^0+\beta, x^1) = -\psi(x^0, x^1). \tag{3.2c}$$

The constant terms in (3.1) are the harmonic pieces of the gauge potential. Even in the trivial sector $\Phi = 0$ they yield the (dynamical) nonintegrable phase factors

$$e^{i\int A_\mu dx^\mu} = \exp\big[2\pi i(h_0 n_0 + h_1 n_1)\big] \tag{3.3}$$

for loops which wind n_0-times around the torus in the x^0 direction and n_1-times in the x^1-direction.

The last term in (3.1) is a pure gauge term and will drop out at the end of the calculations. Now we consider the sector with $\Phi = 0$ and those with $\Phi \neq 0$ in turn:

4.1 Trivial sector $\Phi = 0$

In this sector $-\tilde{\not{D}}^2 = (\nabla - h)^2 Id$ is simple and possesses the double degenerate eigenvalues (of course, the spectrum is not affected by the gauge term in (3.1))

$$\mu_m = \big(\frac{2\pi}{\beta}\big)^2(m_0 - a_0)^2 + \big(\frac{2\pi}{L}\big)^2(m_1 - a_1)^2 \qquad \text{where} \quad (a_0, a_1) = (\frac{1}{2} + h_0, h_1). \tag{3.4}$$

The corresponding zeta-functions $\zeta(s)=\sum\mu_m^{-s}$ are the generalized Epstein functions [18]. Applying the Poisson-resummation formula the derivative at $s=0$ can be expressed in terms of Jacobi theta-functions as [19]

$$\frac{d}{ds}\zeta(s)|_{s=0}=-2\log\Big|\frac{1}{\eta(i\tau)}\Theta\Big[\begin{matrix}\frac{1}{2}+a_0\\ \frac{1}{2}-a_1\end{matrix}\Big](0,i\tau)\Big|,\qquad \text{where}\quad \tau=L/\beta, \tag{3.5a}$$

is the ratio of the two circumferences of the torus and

$$\eta(i\tau)=q^{1/24}\prod_{n>0}(1-q^n);\qquad q=e^{-2\pi\tau} \tag{3.5b}$$

Dedekind's eta-function. We have adopted the conventions in [20] for the theta-functions:

$$\Theta\Big[\begin{matrix}a\\ b\end{matrix}\Big](z,i\tau)=\sum_{Z}e^{-\pi\tau(n+a)^2+2\pi i(n+a)(z+b)}. \tag{3.5c}$$

Taking the degeneracy of the eigenvalues into account, the zeta-function regularized determinants (2.2a) in the sector with no zero modes are

$$\det(i\tilde{\not{D}})=\Big|\frac{1}{\eta(i\tau)}\Theta\Big[\begin{matrix}\frac{1}{2}+a_0\\ \frac{1}{2}-a_1\end{matrix}\Big](0,i\tau)\Big|^2. \tag{3.6}$$

4.2 The non-trivial sectors

We have seen that the excited eigenmodes of the Dirac operator come in pairs with opposite eigenvalues. Since γ_5 commutes with the squared Dirac operator the chiral projections $P_\pm\psi_q$ of these modes are eigenmodes of $-\not{D}^2$. Thus the excited eigenmodes of the squared Dirac operator come in pairs as well and two partners have the same energies $\mu_q=\lambda_q^2$ but opposite chiralities. We have also shown that there are exactly $|k|=|\Phi|/2\pi$ chiral zero modes of the Dirac operator and hence of $-\not{D}^2$. Since $\tilde{\not{D}}^2$ in (1.19) differs only by the constant $2\Phi/V$ in the two chiral sectors, the $|k|$ zero modes with chirality, say $+1$, are at the same time excited modes with energies $2\Phi/V$ and chirality -1. Due to pairing there are also $|k|$ excited modes with chirality $+1$, and so on. Thus $-\tilde{\not{D}}^2$ possesses the following spectrum

$$\mu_n=\begin{cases}0 & \text{degeneracy}=|\Phi|/2\pi\\ 2n|\Phi|/V & \text{degeneracy}=|\Phi|/\pi\end{cases} \tag{3.7}$$

Note that, contrary to the situation in the zero-instanton sector, the spectrum does not depend on the harmonic part h of the gauge potential $\tilde{A}$.
The zeta-function is now proportional to the ordinary Riemann zeta-function and one obtains

$$\det{}'(i\tilde{\not{D}})=\Big(\frac{\pi V}{|\Phi|}\Big)^{|\Phi|/4\pi} \tag{3.8}$$

for the determinant of the non-zero eigenvalues.

Inserting (3.6) and (3.8) into (2.7) we end up with the following formulae for the determinants in the sectors with $|k|$ zero modes:

$$\det(i\not{D}\,) = \Big|\frac{1}{\eta(i\tau)}\Theta\Big|^2 \ \exp\Big(\frac{m_\gamma^2}{2}\int\phi\Delta\phi\Big) \qquad\qquad k=0$$

$$\det{}'(i\not{D}\,) = \big(\frac{V}{2|k|}\big)^{|k|/2}\det\frac{\mathcal{N}}{\tilde{\mathcal{N}}}\ \exp\Big(\frac{m_\gamma^2}{2}\int\phi\Delta\phi+\frac{ek}{V}\int\phi\Big) \qquad k\neq 0,$$

where Θ is the theta-function in (3.6) and we have expressed E in terms of ϕ and Φ (see (1.7)) and made the dependence on the dimensionful electric charge explicit (recall that $m_\gamma = e/\sqrt{\pi}$).

5 Computing the Zero-Modes

According to (2.4) we only need to evaluate the zero-modes of $-\tilde{\not{D}}^2$. Also we shall only consider expectation values of gauge invariant operators and thus may set the gauge part in (3.1) to zero. Since the gauge potentials (3.1) do not depend on time the eigenmodes should be proportional to $\exp(ip_0x^0)$. The finite temperature boundary conditions (3.2c) then require that the momentum is quantized as $p_0 = (2p-1)\pi/\beta$ with integer p. If we further eliminate the h_1-part of the instanton potential we are lead to the following ansatz

$$\tilde{\chi}_p = e^{i(2p-1)\pi x^0/\beta}\, e^{i2\pi h_1 x^1/L}\ \xi_p(x^1)$$

for the zero modes. Inserting this ansatz into the zero mode equation $\tilde{\not{D}}^2\tilde{\chi}_p = 0$ yields

$$\big(\frac{d^2}{dy^2}-\frac{\Phi^2}{V^2}y^2+\frac{|\Phi|}{V}\big)\xi_p = 0, \qquad \text{where}\quad y = x^1+\frac{L}{k}(p-a_0)$$

which shows that ξ_p is the ground state wave function of a harmonic oscillator and thus

$$\xi_p = \exp\Big[-\frac{|\Phi|}{2V}\big\{x^1+\frac{L}{k}(p-a_0)\big\}^2\Big], \tag{4.1}$$

where we have used the index theorem (1.6). These functions do not obey the boundary condition (3.2b) but the correct eigenmodes can be constructed as superpositions of them. For that one notes that

$$\tilde{\chi}_p(x^0,x^1+L) = e^{-i\Phi x^0/\beta}\, e^{2i\pi h_1}\ \tilde{\chi}_{p+k}(x^0,x^1) \tag{4.2}$$

so that the sums

$$\tilde{\psi}_p = \big(\frac{2|k|}{\beta^2 V}\big)^{1/4}\sum_Z e^{2i\pi n h_1}\tilde{\chi}_{p+nk}, \qquad p=1,\dots,|k| \tag{4.3}$$

obey the boundary conditions and thus are the $|k|$ required zero-modes of the Dirac operator. The overall factor normalizes these functions to one (the norm can be computed by using 4.2)). Also, modes with different p are orthogonal to each other, so that the system (4.3) forms an orthonormal basis of the zero-mode subspace. These functions can be written in terms of Jacobi theta functions as (we drop a constant phase)

$$\tilde{\psi}_p = \Big(\frac{2|k|}{\beta^2 V}\Big)^{1/4} e^{2\pi i[h_0 x^0/\beta - kx^0 x^1/V]} \;\; \Theta\Big[\begin{matrix} x^1/L + (p - a_0)/k \\ kx^0/\beta + a_1 \end{matrix}\Big](0, i|k|\tau), \tag{4.4}$$

where τ has been introduced in (3.5a). Since these modes are orthonormal the determinant of the norm matrix $\tilde{\mathcal{N}}$ in (3.9) is just one. The unnormalized zero modes which enter the normmatrix $\mathcal{N}$ can now be obtained from the zero modes (4.4) by the transformation (2.4) with $\alpha = 1$. These completes our discussion of the zero-mode sector. We have computed all $|k|$ zero modes of $\not{D}$ explicitly in terms of Jacobi-theta functions.
For the two-point functions of interest (1.16) only the sector with one zero mode contributes, and since then the normalized zero mode is just $\psi/\sqrt{\mathcal{N}}$, the normalization constant cancels with $\mathcal{N}$ in (3.9) so that

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \; e^{\int \bar{\Psi} i \not{D} \Psi} \; \bar{\Psi}(x) P_\pm \Psi(x) = -\frac{1}{\beta} \, |\Theta_\pm|^2 \, e^{\mp 2\phi} \; \exp\Big(\frac{m_\gamma^2}{2}\int \phi\Delta\phi + \frac{ek}{V}\int \phi\Big), \tag{4.5a}$$

where $\Theta_\pm$ are the theta-functions in (4.4) with $p=0$ and $k = \pm 1$:

$$\Theta_\pm = \Theta\Big[\begin{matrix} x^1/L \mp a_0) \\ x^0/\beta \pm a_1 \end{matrix}\Big](0, i\tau) \tag{4.5b}$$

(we used that the modulus of this function is unchanged if the sign of the "lower parameter" is changed) and the $\pm$ sign refers to the chirality of the zero modes.
Our trivialization of the $U(1)$ bundle which leads to the form (3.1) of the instanton potential differs from the one chosen in [11] and so do our zero modes. But one can show [12] that the modes in [11] are obtained from our modes (4.4) by the corresponding change of trivialization, as required.

5.1 Calculating the bosonic path integral

After having solved the fermionic integration we are now left with the functional integrals over the A-field. In particular we shall evaluate the two point functions

$$\langle \bar{\Psi} P_\pm \Psi \rangle = \frac{\int \mathcal{D}A \, e^{-\frac{1}{2}\int E^2} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \; e^{\int \bar{\Psi} i \not{D} \Psi} \; \bar{\Psi}(x) P_\pm \Psi(x)}{\int \mathcal{D}A \, e^{-\frac{1}{2}\int E^2} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \; e^{\int \bar{\Psi} i \not{D} \Psi}}, \tag{5.1}$$

where E is given in terms of ϕ and Φ in (1.7). Clearly, since the integrands are expressed in terms of $\{\phi, h_\mu, \Phi\}$ it is natural to change the integration variables from A to these variables.

First one notes that there is a one to one correspondence between E and $\{\Delta\phi, \Phi\}$. This becomes a one to one correspondence to $\{\phi, \Phi\}$ if we demand that ϕ is orthogonal to the kernel of Δ, that is if it integrates to zero. Second, for given $\{\phi, \Phi\}$ the nonintegrable phase factors (3.3) are in one to one correspondence to the phases $\exp(2i\pi h_\mu)$, and thus to the h_μ modulo 1. Furthermore, from $\partial A = \Delta\lambda$ we see that there is also a one to one correspondence between the divergence of A and functions λ which integrate to zero. To summarize: we have shown that the transformation

$$A_\mu \longrightarrow \{\phi, \lambda, h_\mu, \Phi\} \qquad \text{where} \quad \int \phi = \int \lambda = 0 \qquad \text{and} \quad 0 \le h_\mu < 1, \tag{5.2}$$

defined in (1.7) and (3.1), is one to one.

Next we need to calculate the Jacobian of this transformation. This is conveniently done by expanding all fields in eigenmodes of $-\Delta$, e.g.

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{m\neq 0} \phi_m \, e^{2i\pi(m_0 x^0/\beta + m_1 x^1/L)}, \tag{5.3}$$

where $m = (m_0, m_1) = 0$ is excluded because of the constraint on ϕ in (5.2). In terms of the coefficients the transformations (1.7) and (3.1) read for $\Phi = 0$

$$\begin{pmatrix} A_{0,m} \\ A_{1,m} \end{pmatrix} = \frac{2i\pi}{L} \begin{pmatrix} -m_1 & \tau m_0 \\ \tau m_0 & m_1 \end{pmatrix} \begin{pmatrix} \phi_m \\ \lambda_m \end{pmatrix} \qquad (m \neq 0)$$

$$\begin{pmatrix} A_{0,0} \\ A_{1,0} \end{pmatrix} = 2\pi \begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & 1/\sqrt{\tau} \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix},$$

and the Jacobian of this transformation is just $J = (2\pi)^2 \det'(-\Delta)$ and thus independent of the dynamical fields. In the nontrivial sectors labelled by k we can write $A_\mu = -\Phi/V \cdot x^1 \delta_{\mu 0} + \delta A_\mu$ and the above transformation applies then to δA, so that

$$\int \mathcal{D}A_\mu = \sum_k \mathcal{D}\delta A_\mu = J \sum_k \int_0^1 dh_0 dh_1 \int \prod_{m\neq 0} d\phi_m d\lambda_m. \tag{5.4}$$

Since in expectation values the same Jacobian appears in the numerator and denominator we may neglect J in what follows. Also, gauge invariant operators (like in (5.1)) are independent of the gauge function λ and the integration over the λ_m drops in expectation values as well (in our decomposition $\lambda = 0$ corresponds to the ghost free Lorentz gauge). Inserting (3.9) and (4.5a) into (5.1) we find

$$\langle \bar\Psi P_\pm \Psi \rangle = \frac{-1}{\beta} e^{-2\pi^2/e^2 V} \, \frac{\int d^2h |\Theta_\pm|^2 \int \mathcal{D}\phi \; e^{-\Gamma[\phi] \mp 2e\phi(x)}}{\int d^2h |\frac{1}{\eta}\Theta|^2 \int \mathcal{D}\phi \, e^{-\Gamma[\phi]}} \tag{5.5a}$$

where

$$\Gamma[\phi] = \frac{1}{2} \int \phi(\Delta^2 - m_\gamma^2 \Delta)\phi \tag{5.5b}$$

and one integrates over ϕ which integrate to zero. $\Theta_\pm$ and Θ are the theta functions in (4.5b) and (3.6), repectively. Note that the last term in the exponent in (4.5a) is absent for the allowed ϕ's.
Finally, using

$$\int d^2h|\Theta_\pm|^2 = \int d^2h|\Theta|^2 = \sqrt{\frac{1}{2\tau}} \tag{5.5c}$$

and performing the Gaussian functional integral yields

$$\langle\bar{\Psi}P_\pm\Psi\rangle = -\frac{|\eta(i\tau)|^2}{\beta}\, e^{-2\pi^2/e^2V}\, e^{2e^2K(x,x)}, \quad \text{where} \quad K(x,x) = \langle x|\frac{1}{\Delta^2 - m_\gamma^2\Delta}|x\rangle \tag{5.6}$$

is the Green function on the space of functions which integrate to zero. Using the eigenmodes (5.3) of $-\Delta$ and observing that the excited modes span the space of functions ϕ permitted in the path integral (5.5a), we obtain

$$K(x,y) = \sum_{m\neq 0}\frac{\phi_m^\dagger(x)\phi_m(y)}{\mu_m^2 + m_\gamma^2\mu_m} \Longrightarrow K(x,x) = \frac{1}{m_\gamma^2V}\sum_{m\neq 0}\Big(\frac{1}{\mu_m} - \frac{1}{\mu_m + m_\gamma^2}\Big),$$

where the eigenvalues μ_m of $-\Delta$ are the ones in (3.4) with a_0 and a_1 set to zero. Using the identity

$$\sum_{m_1=-\infty}^{\infty}\frac{e^{2\pi i m_1\Phi}}{a^2 + m_1^2} = \frac{\pi}{a}\frac{\cosh(\pi a[1-2\Phi])}{\sinh(\pi a)} \qquad (\Phi \geq 0) \tag{5.7}$$

for $\Phi = 0$ the summation over m_1 can be carried out and one obtains ($n{=}m_1$)

$$m_\gamma^2 K(x,x) = \frac{1}{m_\gamma^2 V} - \frac{\coth(Lm_\gamma/2)}{2\beta m_\gamma} + \frac{\tau}{12} + \frac{1}{2\pi}\sum_{n>0}\Big[\frac{\coth(n\pi\tau)}{n} - \frac{\coth(\pi\tau\xi(n))}{\xi(n)}\Big] \tag{5.8}$$

where $\xi^2(n) = n^2 + (\beta m_\gamma/2\pi)^2$. The first sum on the right hand side can be expressed in terms of Dedekinds function as

$$\sum_{n>0}\frac{1}{n}\big(\coth(\pi\tau n) - 1\big) = 2\sum_{n,r>0}\frac{e^{-2\pi\tau nr}}{n} = -2\sum_r \log\big(1 - e^{-2\pi\tau r}\big) = -2\big(\frac{\pi\tau}{12} + \log\eta(i\tau)\big)$$

and thus cancels against the third term on the right hand side in (5.8) and the $|\eta^2|$ in (5.6). Also note the V-dependent exponential factor in (5.6) cancels against the first term on the right hand side in (5.8) so that finally

$$\langle\bar{\Psi}P_\pm\Psi\rangle = -\frac{1}{\beta}\exp\Big[-\frac{\pi}{\beta m_\gamma}\coth(\frac{1}{2}Lm_\gamma)\Big]\, e^{F(\beta m_\gamma)}\, e^{-2H(\beta m_\gamma,\tau)} \tag{5.9a}$$

where

$$F(x) = \sum_{n>0}\Big(\frac{1}{n} - \frac{1}{\sqrt{n^2 + (x/2\pi)^2}}\Big)$$
$$H(x,\tau) = \sum_{n>0}\frac{1}{\sqrt{n^2+(x/2\pi)^2}}\cdot\frac{1}{e^{\tau\sqrt{(2\pi n)^2 + x^2}} - 1}. \tag{5.9b}$$

The formula (5.9) is the exact form of the two point functions we have been aiming at. This formula simplifies considerably if we let $L \to \infty$, in which case $H(.,.) \to 0$ and $\coth(.) \to 1$ so that

$$\langle\bar{\Psi}\Psi\rangle = -\frac{2}{\beta}\exp\Big[-\frac{\pi}{\beta m_\gamma}\Big]\, e^{F(\beta m_\gamma)}, \tag{5.10}$$

where we have used that $P_+ + P_- = Id$.
The formula (5.9a) for the finite temperature chiral condensate on a finite intervall and the corresponding formula (5.10) after the limit $L \to \infty$ has been taken, are the main results of this paper. Let us now derive the low - and high temperature limits in turn:
To study the low temperature limit of the chiral condensate we regularize the indiviudal sums in (5.9b) as [21]

$$\sum \frac{1}{n^{1+s}} \sim \frac{1}{s} + \gamma + O(s)$$

$$\sum \frac{1}{(n^2+(x/2\pi)^2)^{\frac{1}{2}+s/2}} \sim \frac{1}{s} - \frac{\pi}{x} - \log\frac{x}{4\pi} + 2\sum_{l=1}^{\infty} K_0(lx) + O(s) \tag{5.11}$$

where $\gamma = 0.57721...$ is Eulers constant. The difference is finite as s tends to zero and inserting it into (5.9) and (5.10) yields

$$\langle\bar{\Psi}\Psi\rangle = -\frac{m_\gamma}{2\pi} e^{\gamma}\, e^{2I(\beta m_\gamma)} \qquad \text{where} \quad I(x) = \int_0^\infty \frac{1}{1-e^{x\cosh(t)}}dt, \tag{5.12}$$

where the last integral represents (up to a sign) just the last sum in (5.11). This altenative representation of the chiral condensate for $L = \infty$ has been derived in the bosonised Schwinger model on the cylinder in [8]. Since the integral I tends to zero for low temperatures we find

$$\langle\bar{\Psi}\Psi\rangle \longrightarrow -\frac{m_\gamma}{2\pi}\, e^{\gamma} \qquad \text{for} \quad T \to 0, \tag{5.13}$$

and this is the well known result for the zero temperature Schwinger model [7].
To study the high temperature limit we observe that $F(x)$ tends to zero as $x \ll 1$ or if the temperature is big compared to the induced photon mass. In this region

$$\langle\bar{\Psi}\Psi\rangle \longrightarrow -2T\, e^{-\pi T/m_\gamma} \qquad \text{for} \quad T \to \infty. \tag{5.14}$$

Thus the chiral condensate vanishes exponentially fast for $T \gg m_\gamma$.
For the intermediate temperatures $0 < T \sim m_\gamma$ we must retreat to numerical methods to evaluate the infinite sum which defines the function F appearing in (5.10). The result of the numerical calculations are depicted in Figure 1.
Note how the function $\langle\bar{\Psi}\Psi\rangle(T)$ resembles the behaviour of an order parameter in a system which suffers a second order phase transition. However, the chiral condensate does not really vanish at any finite temperature, it tends to zero 'only' exponentially, contrary to the phase transition case. Thus in a strict sense the chiral symmetry remains broken at all finite temperature.

6 Further Correlation Functions

In this section we determine the correlation functions of Wilson loops, products of the field strength, thermal Wilson loops and gauge invariant fermionic two-point functions. Clearly, for the purely bosonic operators only the zero-instanton sector yields non-vanishing expectation values. Since in this sector A =harmonics+$\epsilon\partial\phi$ one has

$$\langle O[A]\rangle = \sqrt{2\tau}\frac{\int \mathcal{D}\phi\, e^{-\Gamma[\phi]}\int d^2h\,|\Theta|^2\, O[\phi,h]}{\int \mathcal{D}\phi e^{-\Gamma[\phi]}}, \tag{6.1}$$

for an operator $O[A]$. Here we have used the second formula in (5.5c) and Θ is the theta-function in (3.6). Hence, up to the h-integration we remain with Gaussian integrals with respect to the effective action (5.5b).

6.1 Correlation functions of the field strength

The n-point functions of the field strength $E=\Delta\phi$ are particularly simple, since E does not depend on the harmonic pieces in the gauge potential. Because of the symmetry $\phi\to-\phi$ of Γ all expectation values of odd powers of E vanish. The $2n$-point functions can be calculated from

$$\langle E(x_1)\cdots E(x_{2n})\rangle = \frac{\int \mathcal{D}\phi\, e^{-\Gamma[\phi]}E(x_1)\cdots E(x_{2n})}{\int \mathcal{D}\phi\, e^{-\Gamma[\phi]}} = \Delta_1\cdots\Delta_{2n}\langle\phi(x_1)\cdots\phi_{2n})\rangle.$$

and applying Wick's theorem to the last expectation value allows one to express them in terms of the 2-point function

$$\langle E(x)E(y)\rangle = \Delta_x\Delta_y\langle\phi(x)\phi(y)\rangle = \Delta_x\Delta_y K(x{-}y,m_\gamma),$$

where the Green function K corresponding to Γ has been introduced above (5.7). Since $-\Delta\phi_m=\mu_m\phi_m$ we easily find that

$$\langle\ldots\rangle = \sum_{m\neq 0}\phi_m^\dagger(x)\phi_m(y)\big[1-\frac{m_\gamma^2}{\mu_m+m_\gamma^2}\big] = -\Delta_x G(x-y,m_\gamma). \tag{6.2}$$

Now we use the (almost) completeness of the excited eigenmodes ϕ_n and obtain

$$\langle E(x)E(y)\rangle = \delta(x-y)-\frac{1}{V}-m_\gamma^2 G(x{-}y;m_\gamma) \tag{6.3}$$

where G denotes the massive Klein-Gordon propagator. This shows that E is (up to contact terms) just a free massive pseudo-scalar field. Using (5.7) the propagator can be written as

$$G(\xi;m_\gamma) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{\cosh\big[\pi\tau\xi(n)(1-2\xi^1/L)\big]\,e^{2\pi in\xi^0/\beta}}{\xi(n)\sinh\big[\pi\tau\xi(n)\big]}-\frac{1}{m_\gamma^2V}, \tag{6.4a}$$

where $\xi(n)$ has been defined below (5.8). In the limit where L tends to infinity this simplifies to

$$G(\xi;m_\gamma)=\frac{1}{4\pi}\sum_n\frac{e^{-2\pi[\xi(n)\xi^1-in\xi^0]/\beta}}{\xi(n)}. \tag{6.5}$$

This shows explicitely that the 2-point function of the field strength falls off exponentially.

6.2 Wilson loops

Since loop integrals of local Wilson loops (that is loops without windings) also do not depend on the harmonic pieces one finds

$$\langle e^{ie\oint A}\rangle=\frac{\int\mathcal{D}\phi\, e^{-\Gamma[\phi]}\,\exp\Big(ie\int_{\mathcal{D}}\Delta\phi\Big)}{\int\mathcal{D}\phi\, e^{-\Gamma[\phi]}},$$

where the loop encloses the region $\mathcal{D}$ and we have used that the loop integral is equal to the flux of $E=\Delta\phi$ through $\mathcal{D}$. Using (6.3) the resulting Gaussian integral yields

$$\begin{aligned}\langle e^{ie\oint A}\rangle&=\exp\Big(-\frac{e^2}{2}\int_{\mathcal{D}\times\mathcal{D}}\langle E(x)E(y)\rangle d^2xd^2y\Big)\\&=\exp\Big(-\frac{e^2}{2}\Big[A_{\mathcal{D}}-\frac{A_{\mathcal{D}}^2}{V}-m_\gamma^2\int_{\mathcal{D}\times\mathcal{D}}G(x-y,m_\gamma)\Big]\Big),\end{aligned} \tag{6.6a}$$

where $A_{\mathcal{D}}$ is the area enclosed by the loop. For example, for a rectangle $\mathcal{D}$ one finds for $L=\infty$

$$\langle e^{ie\oint A}\rangle=\exp\Big(-\frac{e^2A_{\mathcal{D}}}{2}\big[1-m_\gamma^2\int_{2\mathcal{D}}G(\xi;m_\gamma)d^2\xi\big]\Big). \tag{6.6b}$$

For low temperatures (compared to the induced photon mass) and loops with edges long compared to the inverse photon mass we may use the zero-temperature propagator $G(\xi,m_\gamma)=K_0(m_\gamma|\xi|)/2\pi$ in (6.6b) for which the integral in the exponent is, up to an exponentially small term, just m_γ^{-2}. It follows then that at low temperature the interaction between two charges is exponentially small

6.3 Thermal Wilson loops

The expectation values of a string of thermal Wilson loops

$$\mathcal{P}(u)=\exp\Big(ie\int_0^\beta A_0(x^0,u)dx^0\Big) \tag{6.7}$$

depend on the harmonics because a thermal Wilson loop has winding number one. Since $\int A_0dt=2\pi h_0-\partial_1\int\phi dx^0$ the h-integral for a string of q loops becomes

$$\int d^2h|\theta|^2\,e^{2\pi iqh_0}=\sqrt{\frac{1}{2\tau}}e^{-\pi q^2/2\tau}.$$

The remaining ϕ-integration yields

$$\langle\prod_{i=1}^{q}\mathcal{P}(u_i)\rangle = e^{-\pi q^2/2\tau}\exp\Big[\frac{e^2}{2}\Big(\int d^2x d^2y\sum_{i=1}^{q}\delta(x^1-u_i)\ \partial_{x^1}^2 K(x,y)\sum_{i=1}^{q}\delta(y^1-u_i)\Big)\Big].$$

Inserting the explicit form of K (see above (5.7)) one finds the following one- and two-point functions

$$\langle\mathcal{P}(0)\rangle = \exp\Big(-\frac{\pi\beta m_\gamma}{4}\coth(\frac{1}{2}Lm_\gamma)\Big)$$

$$\langle\mathcal{P}(u)\mathcal{P}(0)\rangle = \langle\mathcal{P}(0)\rangle^2\exp\Big(-\frac{1}{2}\pi\beta m_\gamma S(u)\Big) \qquad (6.8a)$$

where S denotes the function

$$S(u) = \frac{\cosh\frac{1}{2}m_\gamma(L-2|u|)}{\sinh(\frac{1}{2}m_\gamma L)}. \qquad (6.8b)$$

The normalized higher n-point functions

$$\mathcal{P}(u_1,\dots,u_q) \equiv \frac{\langle\mathcal{P}(u_1)\cdots\mathcal{P}(u_q)\rangle}{\langle\mathcal{P}(0)\rangle^q} \qquad (6.9)$$

are then just products of the normalized two-point function

$$\mathcal{P}(u_1,\dots,u_q) = \prod_{i<j}\mathcal{P}(u_i,u_j). \qquad (6.10)$$

Since $-T\log\mathcal{P}(u_1,\dots,u_q)$ is to be intepreted as the (zero energy subtracted) free energy of q static charges at positions $u_1,\dots,u_q$, (6.10) shows that the free energy of q charges is just the sum of the free energies of the individual pairs.
For $L=\infty$ the one and two-point functions (and thus the q-point functions) simplify to

$$\langle P(u)\rangle = \exp\big(-\pi\beta m_\gamma/4\big) \qquad \text{and} \quad \mathcal{P}(u,0) = \exp\Big(-\frac{1}{2}\pi\beta m_\gamma\, e^{-m_\gamma|u|}\Big),$$

in complete agreement with the analytically continued one and two-point functions obtained via bosonization in [8]. Note in particular that the free energy of a single charge is finite

$$\Delta F = -T\log\langle\mathcal{P}(0)\rangle = \frac{\pi m_\gamma}{4} \qquad (6.11)$$

and that the (zero energy subtracted) free energy of two static charges falls off exponentially leading to a Yukawa force

$$F = \frac{e^2}{2}\, e^{-m_\gamma|u|} \qquad (6.12)$$

between them. We see that the classical Coulomb force between external charges is shielded.

6.4 Gauge invariant fermionic two-point functions

The gauge invariant chiral two-point functions

$$S_{\pm}(x,y) = \langle \bar{\Psi}(x)\; e^{ie\int_y^x A_\mu d\xi^\mu} P_{\pm}\Psi(y)\rangle \tag{6.13a}$$

may lead, via the LSZ reduction procedure, to chirality violating scattering amplitudes. They can be calculated in the same way as the chiral condensates (5.5a) which they must reproduce for coinciding points. One obtains

$$S_{+}(x,y) = \frac{-1}{\beta} e^{-2\pi^2/e^2 V}\; \frac{\int d^2h\, \tilde{\psi}_0^\dagger(x)\tilde{\psi}_0(y)\; \int \mathcal{D}\phi\; e^{-\Gamma[\phi]-e(\phi(x)+\phi(y))+ie\int_y^x A}}{\int d^2h |\frac{1}{\eta}\Theta|^2 \int \mathcal{D}\phi\, e^{-\Gamma[\phi]}}, \tag{6.13b}$$

where the zero mode $\tilde{\psi}_0(x)$ has been computed in (4.4). The integration over the harmonics yields

$$\int d^2h\, \tilde{\psi}_0^\dagger(x)\tilde{\psi}_0(y)\; e^{ie\int_y^x A} = \frac{C}{\sqrt{2\tau}}\; e^{-\frac{\pi}{2V}(x-y)^2}\; e^{-i\int_y^x \epsilon_{\mu\nu}\partial_\nu\phi d\xi^\mu}, \tag{6.14}$$

where C is some normalisation constant. The ϕ-integration is again Gaussian but due to the phase factor in (6.14) involves derivatives of the bosonic Green function:

$$\int \mathcal{D}\phi\; e^{-\Gamma[\phi]-e(\phi(x)+\phi(y))-ie\int \epsilon_{\mu\nu}\partial_\nu\phi d\xi^\mu} = \tilde{C}\exp\big[K(0,0)+K(x,y)\big]\times$$

$$\exp\Big[-\frac{1}{2}\int_x^y d\xi^\mu d\eta^\alpha \epsilon^\nu_\mu \epsilon^\beta_\alpha \partial_{\xi^\nu}\partial_{\eta^\beta} K(\xi,\eta)\Big], \tag{6.15}$$

where $\tilde{C}$ is again some constant. For simplicty we shall first consider the zero temperature and infinite volume limit only. There K depends only on the distance of its two arguments and the integral on the r.h.s of (6.15) vanishes identically. Hence we are then left with the computation of the Green function $K(x,y)$. This Green function, which is the difference of the masseless- and the massive Klein-Gordon propagator (see 5.7) simplifies to

$$K(x,y) = \frac{1}{m_\gamma^2}\big[G(r;m_\gamma=0) - G(r;m_\gamma)\big] = \frac{-1}{2\pi m_\gamma^2}\big[\log(\mu r) + K_0(m_\gamma r)\big], \tag{6.16}$$

where $r = |x-y|$ and μ is an infrared regularisation which would be undefined on the Euklidean space. Here it is fixed by demanding that $S_{\pm}(x,x)$ reproduces the chiral condensate (5.13). Finally substituting (6.14 -16) into (6.13) yields

$$\langle \bar{\Psi}(x)\; e^{ie\int A} P_+\Psi(y)\rangle = -\sqrt{\frac{m_\gamma e^\gamma}{2}} \frac{\exp[-\frac{1}{2}K_0(m_\gamma|x-y|)]}{2\pi\sqrt{|x-y|}}. \tag{6.17a}$$

For large separations $K_0(m_\gamma r)$ vanishes exponentially so that the chiral two-point function is long range:

$$\langle ...\rangle \to -\sqrt{\frac{m_\gamma e^\gamma}{2}} \frac{1}{2\pi\sqrt{|x-y|}}, \tag{6.17b}$$

for $|x-y|$ approaching infinity. Note that contrary to the bosonic correlators which fall off exponentially due to the mass gap in the spectrum, the chiral fermionic 2-point function falls of like $1/\sqrt{r}$. This is due to the zero mode fermionic states in the Hilbertspace.

In a next step we will increase the complications by considering the case of again infinite volume but finite temperature. This time however we must assume the difference of the two arguments of the two-point function to be strictly spacelike (eg: $x^0 - y^0 = 0$) for then the integral on the r.h.s. of (6.15) vanishes again. This becomes clear if we notice that now only "timelike" derivatives of the bosonic Green function appear and these are zero for $x^0 - y^0 = 0$ as can be seen using the sum representation of $K(x,y)$ (above (5.6)). Hence we have again reduced the problem to the computation of $K(x,y)$. However, as in the case of coinciding arguments (chiral condensate) we couldn't find simple expressions representing the infinite sums for finite temperature. Therefore we again retreat to numerical methodes to evaluate (6.15). The results are despicted in Figure 2. We will not reproduce here the lengthy formulas but emphasize that for non-zero temperature and for $|x^1 - y^1| \to \infty$, $S_+(x,y)$ falls of like $\exp[-2\pi T|x^1 - y^1|]$. This is the expected behaviour at large temperature.

7 Discussion and Conclusions

Returning to (5.6) and noting that

$$\frac{|\eta(i\tau)|^2}{\beta} = \frac{\eta(i\tau)\eta(i/\tau)}{\sqrt{V}} \qquad (\tau = L/\beta) \tag{7.1}$$

is left invariant under an exchange of β with L, and that the same is true for $K(x,x)$, we may view (5.10) as dependence of the zero-temperature chiral condensate on the spatial extension L. Thus Figure 1 can be intepreted as the change of $\langle\bar{\Psi}\Psi\rangle(L)$ due to finite (spatial) size effects. In particular, if the compactified spatial size shrinks to zero, the chiral condensate behaves as

$$\langle\bar{\Psi}\Psi\rangle(L) \longrightarrow -\frac{2}{L}\, e^{-\pi/Lm_\gamma} \qquad \text{for} \quad L \to 0 \tag{7.2}$$

and vanishes exponentially (and non-analytically).

In the decomposition (5.4) of the bosonic path integral we have integrated over different $U(1)$-principal bundles labelled by the index k. We have assumed that the relative normalization between functional integrals with different k is one. To recover the θ-vacuum structure [6] we must allow for a relative phase $\exp(i\theta)$ between the sectors k and $k+1$ (similarly as in the path integral for a particle on a punctured plane). Then the expectation values $\langle\bar{\Psi}P_\pm\Psi\rangle$ would pick up the phases $\exp(\pm i\theta)$ and thus the right hand sides of (5.10) and (5.12-14) would be multiplied by $\cos(\theta)$. The same happens if one adds the explicit θ-dependent term $i\theta\Phi/2\pi$ to the classical action (1.1).

In the conventional path integral solution of the Schwinger model on the plane the zero-temperature result (5.13) is derived indirectly, via clustering of the 4-point function (1.17). This seems to be necessary since a (naive) direct calculation of the chiral condensate yields the wrong result $\langle\bar{\Psi}\Psi\rangle = 0$. This may not come as a surprise since one integrates only over potentials with vanishing fluxes and on the other hand we have seen that on the torus only configurations with fluxes $\pm 2\pi$ contribute to the expectation value (5.13). Integrating over all gauge potentials should yield the correct result (5.13). More precisely, on the plane a gauge potential can be decomposed as

$$A_\mu = \tilde{A}_\mu - \epsilon_{\mu\nu}\partial_\nu\phi, \tag{7.3a}$$

where $\tilde{A}$ is a vortex field

$$\tilde{A}_\mu = -\frac{1}{2\pi}\frac{\Phi}{a^2+r^2}\epsilon_{\mu\nu}x^\nu, \qquad \Phi = \int E \tag{7.3b}$$

and $\delta A_\mu = -\epsilon_{\mu\nu}\partial_\nu\phi$ a local fluctuation with vanishing flux about the vortex potential. One can show that $\tilde{\not{D}}$ possesses exactly $|k|$ zero modes, where $|k|$ is the integer part of $|\Phi|/2\pi$ [22] (besides these zero modes there is also a resonance state). Since only potentials with $|k| = 1$ contribute to the chiral condensate one obtains

$$\langle\bar{\Psi}P_+\Psi\rangle = \frac{\int\limits_{1<\Phi\leq 2} d\Phi\,\rho(\Phi)\int\mathcal{D}\phi\, e^{-\Gamma[\phi]}\,\mathrm{tr}\left(\psi^\dagger P_+\psi\right)}{\int\limits_{-1\leq\Phi\leq 1} d\Phi\,\rho(\Phi)\int\mathcal{D}\phi\, e^{-\Gamma[\phi]}}, \tag{7.4}$$

where we have allowed for a Jacobian ρ of the transformation (5.2) (without harmonics h). Again the effective action Γ contains two pieces, namely a local part coming from the chiral anomaly and a global vortex determinant. Since on the plane there aren't different topological sectors it would be interesting to see how the θ-parameter emerges in such a calculation.

Acknowledgments: The authors wish to thank G. Felder and C. Schmid for helpful discussions. In addition A. Wipf thanks J. Balog, M.P. Frey, K. Kirsten and E. Seiler for useful discussions.

References

[1] J. Schwinger, *Phys. Rev.* **128** (1962) 2425.

[2] S. Coleman, R. Jackiw and L. Susskind, *Ann. Phys.* **93** (1975) 267; L.S. Brown, *Nuovo Cimento* **29** (1963) 617.

[3] A.I. Bocharev and M.E. Shaposhnikov, *Mod. Phys. Lett.* **A2** (1987) 991; J. Kripfganz and A. Ringwald, *Mod. Phys. Lett.* **A5** (1990) 675.

[4] E. Mottola and A. Wipf, *Phys. Rev.* D 39 (1989) 588

[5] R. Gass, *Phys. Rev.* **D 27** (1983) 2893; J. Barcelos-Neto and A. Das, *Phys. Rev.* D 33 (1986) 2262

[6] J.H. Loewenstein and J.A. Swieca, *Ann. Phys.* **68** (1971) 172; A.K. Raina and G. Wanders, *Ann. Phys.* **132** (1981) 404; A.Z. Capri and R. Ferrari, *Nuovo Cimento* **A62** (1981) 273. P. Becher, *Ann. Phys.* **146** (1983) 223

[7] N.K. Nielsen and B. Schroer, *Nucl. Phys.* **B 120** (1977) 62; K.D. Rothe and J.A. Swieca, *Ann. Phys.* **117** (1979) 382; M. Hortascu, K.D. Rothe and B. Schroer, *Phys. Rev.* **D 20** (1979) 3203; N.V. Krasnikov et.al, *Phys. Lett Phys. Lett.* **B97** (1980) 103; R. Roskies and F. Schaposnik, *Phys. Rev.* **D 23** (1981) 558.

[8] N. Manton, *Ann. Phys.* **159** (1985) 220; J.E. Hetrick and Y. Hosotani, *Phys. Rev.* **38** (1988) 2621.

[9] C. Jayewardena, *Helv. Phys. Acta* **61** (1988) 636.

[10] J. Balog and P. Hrasko, *Nucl. Phys.* **245** (1984) 118.

[11] H. Joos, *Nucl. Phys.* **B 17** (Proc. Suppl.) (1990) 704; *Helv. Phys. Acta* **63** (1990) 670.

[12] I. Sachs, *diplom thesis* ETH-Zuerich

[13] Y.C. Kao, *preprint* UCB-PTH 83/14.

[14] H.P. McKean and I.M. Singer, *J. Diff. Geom* **1** (1967) 43.

[15] P.B. Gilkey, *The Index Theorem and the Heat Equation*, Publish or Perish, Boston, 1974

[16] S. Blau, M. Visser and A. Wipf, *Int. J. Mod. Phys.* **6** (1989) 1467.

[17] For a lucid discussion of various regularizations see E. Seiler, in: Gauge theories: fundamental interactions and rigorous results, eds. P. Dita, V. Georgescu and R. Purice. (Birkhäuser, Boston, 1982) p.263.

[18] P. Epstein, *Math. Ann.* **63** (1907) 205.

[19] S. Blau, M. Visser and A. Wipf, *preprint* MPI-PAE/PTh 57/88

[20] D. Mumford, *Tata Lectures on Theta*, Birkhäuser, Boston, 1983

[21] K. Kirsten and V.F. Müller, *Z. Phys.* **C45** (1989) 159

[22] R. Musto, L. O'Raifeartaigh and A. Wipf, *Phys. Lett.* **175 B** (1986) 433