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SCATTERING THEORY, U(1) ANOMALY AND INDEX THEOREMS FOR COMPACT AND NON-COMPACT MANIFOLDS

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The L_2 index theorem on even dimensional non-compact manifolds is related to the corresponding APS result for compact manifolds with boundaries. We show that generally the two index theorems are slightly different. For the two-dimensional Dirac operator on the disk we formulate (modified) nonlocal boundary conditions such that the two index theorems coincide. Exploiting the supersymmetric structure of $\not{D}^2$ we explicitly evaluate the supersymmetric partition function in this case.

1. Introduction

The aim of this paper is to relate the Atiyah-Patodi-Singer (APS) index theorem [1] valid for compact manifolds with boundaries to the corresponding index theorem for special non-compact manifolds (asymptotically R^{2n}) [2–4], based on scattering theory. In fact we show on various simple examples that generally the two index theorems are slightly different (assuming of course that we can take a meaningful limit in the APS index theorem by taking the boundary to infinity) but a subtle modification of the APS boundary conditions is enough to recover the scattering theory results. We also point out the connection of the results with the anomaly calculations.

Though we work out explicitly only simple two-dimensional examples it is clear how to generalize our results to higher (even) dimensions. In particular, since in four dimensions there exist finite energy self-dual solutions to the Yang-Mills equations

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which have non-integer topological charge [5] (sometimes called hyperbolic monopoles) our results are not purely of academic interest.

Our paper is organized as follows: in the second section we review the chiral anomalies, then after recalling the APS index theorem we consider the case of "cylinder manifolds" in sect. 3. In sect. 4 we analyze thoroughly the two-dimensional disk example and we show how to modify the APS boundary conditions to recover the well-known (supersymmetric) Bohm-Aharanov result on R^2 when the radius of the boundary tends to infinity.

2. Chiral U(1) anomaly

Ever since the discovery that in a gauge theory the classical chiral symmetry is broken at the quantum level [6] these axial anomalies have been playing an increasingly important role in the developments of quantum field theories. It is generally believed that anomalies are a true aspect of the quantum theory, not just consequences of technical problems in perturbation theory.

Let us recall the basic feature of the chiral anomaly: The effective action, Γ, for a euclidean fermionic theory interacting with background gauge fields is formally written

$$\Gamma = \log \int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi \exp\left(\int \bar{\psi} i(\not{D} + m)\psi \right) = \log\det i(\not{D} + m). \tag{2.1}$$

We assume that we work in a $d = 2n$-dimensional euclidean manifold which is not necessarily compact (e.g. R^2, R^4). Denoting the generalization of γ_5 by $\tilde{\gamma} = i^n \gamma_1 \cdots \gamma_{2n}$, under an abelian (local) chiral transformation

$$\psi = e^{\alpha\tilde{\gamma}}\psi', \qquad \bar{\psi} = \bar{\psi}' e^{\alpha\tilde{\gamma}}, \tag{2.2}$$

eq. (2.1) becomes

$$\Gamma = \log J(\alpha) + \log\det e^{\alpha\tilde{\gamma}} i(\not{D} + m) e^{\alpha\tilde{\gamma}}, \tag{2.3}$$

where $J(\alpha)$ is the jacobian associated with the transformation (2.2), $\mathrm{D}\bar{\psi}\,\mathrm{D}\psi = J(\alpha)\mathrm{D}\bar{\psi}'\,\mathrm{D}\psi'$, which is nontrivial as pointed out by Fujikawa [7]. For infinitesimal chiral variations we get formally

$$2\,\mathrm{tr}\,\tilde{\gamma}\alpha = 2im\,\mathrm{tr}\frac{1}{i\not{D} + im}\tilde{\gamma}\alpha + i\,\mathrm{tr}\frac{1}{i\not{D} + im}\gamma_\mu\tilde{\gamma}\alpha_{,\mu}, \tag{2.4}$$

where the right-hand side is the chiral variation of Γ, $\delta\Gamma$, and the left-hand side is the anomaly. Of course (2.4) is purely formal as $\log\det i(\not{D} + m)$ is UV-divergent, and also the measure in (2.1) should be properly defined. A natural way of defining the measure $\mathrm{D}\bar{\psi}\,\mathrm{D}\psi$ is à la Berezin [8]: introducing an orthonormal basis $\{\phi_n(x)\}$

and expanding $\bar{\psi}$ and ψ as

$$\bar{\psi}(x) = \sum \bar{a}_n \phi_n^\dagger(x), \qquad \psi(x) = \sum a_n \phi_n(x), \tag{2.5}$$

where $\{a_n\}$ are grassmannian coefficients, the measure in (2.1) is

$$\mathrm{D}\bar{\psi}\,\mathrm{D}\psi = \prod_n \mathrm{d}\bar{a}_n \mathrm{d}a_n. \tag{2.6}$$

Assuming that $i\not{D}$ is self-adjoint we can work in the $i\not{D}$ basis:

$$i\not{D}\psi_\lambda = \lambda\psi_\lambda. \tag{2.7}$$

In (2.7) it is assumed that the spectrum of $i\not{D}$ is discrete, which is most often achieved by compactifying the base space (e.g. $\mathrm{R}^4 \to \mathrm{S}^4$). This compactification procedure is applicable only if the background gauge fields satisfy strong regularity constraints (e.g., the Dirac operator on R^2 with a smooth gauge field with noninteger flux $\Phi = (1/4\pi)\int \varepsilon_{\mu\nu} F_{\mu\nu} \mathrm{d}^2x$, cannot be defined on the compactified $\mathrm{R}^2 \sim \mathrm{S}^2$ consistently.) Another way of achieving the discreteness of the spectrum of $i\not{D}$ is by introducing a finite volume, that is we modify the base space to a space X with boundary ∂X. However by introducing boundaries – which can be viewed as an infrared regulator – one must impose boundary conditions on the Dirac operator such that it is self-adjoint. Also since eventually one would like to take the infinite volume limit it is natural to choose boundary conditions which preserve the original (L_2) properties of the Dirac operator as much as possible. Let us first choose a representation of the γ-matrices such that

$$i\not{D} = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix}, \tag{2.8}$$

$$\tilde{\gamma} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \tag{2.9}$$

and f and g in $\psi = (f, g)$ are the right- and left-handed components of the Dirac spinor ψ. One sees at once that the popular Dirichlet boundary conditions lead to an over-determination of the Dirac problem and hence are inconsistent [9]. Other local boundary conditions which ensure that L and $L^\dagger$ are adjoints of each other were widely used in various physical applications (bags etc.) [10]. However it is not hard to see that these destroy the $\tilde{\gamma}$-invariance and/or the charge conjugation invariance. More importantly, the zero modes of $i\not{D}$ are not in this domain anymore, since the zero mode equations

$$L^\dagger f = 0, \qquad Lg = 0 \tag{2.10}$$

are first order, homogeneous equations. Therefore the "regulated" Dirac operator has no zero modes [11] and hence its (analytic) index is zero:

$$\text{index}\, i\not{D} = \dim\ker L^{\dagger} - \dim\ker L = 0\,. \tag{2.11}$$

Since the zero modes of the Dirac operator play a crucial role, the above regulator is rather unsatisfactory. However by enlarging one's point of view and if one allows for non-local boundary conditions it becomes possible to introduce boundaries and nevertheless keeping the index intact. The global boundary conditions consistent with a non-zero index were set up by Atiyah, Patodi and Singer (APS) [1], and we shall present them in some detail in sect. 3.

Let us return at this point to the anomaly eqs. (2.4). To give some meaning to it we must regularize both sides, since tr $\tilde{\gamma}$ does not exist. Conventionally one takes the heat kernel regularization for the left-hand side of (2.4):

$$Z(\beta) = \operatorname{tr}\tilde{\gamma}\,\mathrm{e}^{-\beta\not{D}^2} = \int \mathrm{d}x\, \operatorname{Tr}\langle x|\mathrm{e}^{-\beta LL^{\dagger}} - \mathrm{e}^{-\beta L^{\dagger}L}|x\rangle\,. \tag{2.12}$$

(Tr runs over the Dirac and internal symmetry indices.) The two terms at the right-hand side of (2.4) have a natural interpretation as the global ($\sim \alpha$), low energy part and the local ($\sim \alpha_{\mu}$), high energy part of the anomaly [3]. The global part is UV-finite so the second term on the right-hand side is UV-divergent and the trace must be regularized (e.g. by ζ-function techniques). For a constant α the global part of the anomaly is conveniently written as

$$m^2 \operatorname{tr}\left(\frac{1}{LL^{\dagger}+m^2} - \frac{1}{L^{\dagger}L+m^2}\right) = A(m^2)\,. \tag{2.13}$$

This fundamental object has been used in Callias paper [2] which first really addressed index theorems in infinite spaces (for fermions coupled to a external Yang-Mills-Higgs field). Since

$$\int_0^{\infty} \mathrm{d}\beta\, \mathrm{e}^{-\beta m^2} Z(\beta) = G(m^2)\,, \tag{2.14}$$

where

$$G(m^2) = \operatorname{tr}\left(\frac{1}{LL^{\dagger}+m^2} - \frac{1}{L^{\dagger}L+m^2}\right), \tag{2.15}$$

we see that the global part of the anomaly is simply related to the Laplace back transform of the (supersymmetric) partition function $Z(\beta)$. In particular, after a partial integration in (2.14)

$$A(m^2) = \lim_{\beta\downarrow 0} Z(\beta) + \int_0^{\infty} \mathrm{e}^{-m^2\beta} Z'(\beta)$$

and one sees at once that the large energy limit of $A(m^2)$ coincides with the high temperature limit of $Z(\beta)$, i.e. that

$$\lim_{m^2\to\infty} A(m^2) = \lim_{\beta\downarrow 0} Z(\beta). \tag{2.16}$$

Also, by writing $Z(\beta) = Z(\infty) + \tilde{Z}(\beta)$, where of course $\tilde{Z}(\beta)$ tends to zero when β approaches infinity, one can establish that the difference $|A(m^2) - Z(\infty)|$ is smaller than

$$m^2\int_0^a |\tilde{Z}(\beta)| + \max_{\beta\geqslant\alpha} |\tilde{Z}(\beta)|$$

for any positive a. Therefore the difference must tend to zero when $m^2 \to 0$ (take $a \sim 1/m$). In this way we see that

$$\lim_{m^2\downarrow 0} A(m^2) = \lim_{\beta\to\infty} Z(\beta), \tag{2.17}$$

i.e. the global part of the anomaly is just $Z(\infty)$.

When the chiral variation α is localized (such that boundary terms vanish) we can recast (2.4) into the standard form

$$2\alpha_0(x) = 2mJ_5(x) - \partial_\mu J_\mu^5(x), \tag{2.18}$$

where J_μ^5 is the axial current

$$J_\mu^5(x) = i\int \mathrm{d}\lambda\, \frac{\psi_\lambda^\dagger \gamma_\mu \tilde{\gamma} \psi_\lambda}{\lambda + im}, \tag{2.19}$$

$J_5(x)$ is given as

$$J_5(x) = i\int \mathrm{d}\lambda\, \frac{\psi_\lambda^\dagger \tilde{\gamma} \psi_\lambda}{\lambda + im}, \tag{2.20}$$

and where $\alpha_0(x)$ denotes the anomaly density. If the space is compact and has no boundaries then the spectrum of $i\not{D}$ is purely discrete and the last term of (2.18), integrated over the space, is obviously zero. Now using the heat kernel techniques to compute the anomaly density [12] and the fact that $m\int J_5(x)\,\mathrm{d}x = A(m^2)$ is equal to the index of $i\not{D}$ in this case [3], one recovers the celebrated form

$$n_+ - n_- = \int \alpha_0(x)\,\mathrm{d}x = \kappa\int \Phi(x)\,\mathrm{d}x, \tag{2.21}$$

where $\Phi(x)$ is the Chern-Pontryagin density (generalized flux density). However if one works on an open space or on a manifold with boundaries the anomaly equation (2.21) must change since in this situation the right-hand side is not necessarily an integer. Actually it was Callias [2] who first addressed index theorems in infinite spaces. However, in his case (fermions coupled to external Yang-Mills and asymptotically constant Higgs fields) the continuum if $i\not{D}$ is separated from zero (the energy gap is determined by the asymptotic value of the Higgs field) and (2.21) still holds [3]. However, in more general background fields (see Niemi and Semenoff [2]) this is not the case anymore. Then the global part of the anomaly is no longer equal to $n_+ - n_-$, since the spectrum of $i\not{D}$ has a continuous part which stretches down to zero. In this case both the zero modes and the continuum contribute to the anomaly. In fact, the continuum is responsible for the fractional part of the anomaly [3,4]. For the anomaly "tr $\tilde{\gamma}$" the heat kernel method still gives the right-hand side of (2.21) [12]. In fact as shown in [3,4] the anomaly equation yields

$$\kappa\int_X \Phi(x) = n_+ - n_- + \frac{1}{\pi}\sum_k (\delta_k^+ - \delta_k^-), \tag{2.22}$$

where $\delta_k^\pm$ denotes the left- (right-) handed phase shifts of the operator $-\not{D}^2$. This is a form of the index theorem for non-compact manifolds. In (2.22) it is assumed of course that the generalized flux integral at the left-hand side exists, and that L_2 (scattering) boundary conditions are imposed on the Dirac operator.

On the other hand by introducing boundaries and applying the APS global boundary conditions the anomaly is equal to the index of $i\not{D}$ but there are two contributions to it. One part is again the generalized flux and the other part a non-trivial boundary contribution so that (2.21) is replaced by

$$n_+ - n_- = \kappa\int_X \Phi(x) - f(\partial X), \tag{2.23}$$

where $f(\partial X)$ is the boundary contribution, which is roughly the fractional part of the generalized flux. This shows how careful one must be when calculating the anomaly on non-compact spaces.

In this paper we shall derive and rederive eqs. (2.22), (2.23) for various simple examples in detail and compare the corresponding index theorems for the two different ways of calculating the anomaly for non-compact spaces, namely to work on the non-compact space and deal with the continuous spectrum and L_2-boundary conditions or introduce boundaries and apply the APS method. Clearly for a meaningful comparison we shall assume that the generalized flux integral is convergent over the non-compact manifold. We shall relate the APS η-invariant on ∂X to the phase shifts on the non-compact space. Since we deal with two different operators – the Dirac operator on the open space with a L_2-dense domain and the Dirac operator on a bounded subspace with APS domain – only in the limit when the boundary tends to infinity can they coincide. This relation between infinite space index and η-invariant was actually first developed by Niemi and Semenoff [2]. What is really interesting is that despite the above problem one can make a meaningful comparison between the two approaches and the resulting index theorems are the same if one introduces a slight modification in the original APS boundary conditions.

3. η-invariant and phase shifts. Non-compact manifolds and manifolds with boundaries

Let us start by recalling the Atiyah-Patodi-Singer (APS) index theorem [1,13] for a Dirac operator, $i\not{D}$, defined on a $2n$-dimensional, compact, euclidean spin manifold X with a $2n-1$ dimensional boundary ∂X:

$$\mathrm{index}(i\not{D}) = \int_{\mathrm{X}} \alpha_0(x) - \tfrac{1}{2}(\eta(0)+h), \tag{3.1}$$

where $\alpha_0(x)$ is the "anomaly" density defined by the left-hand side of (2.4) (e.g. on a two-dimensional flat manifold $\alpha_0(x)=(1/2\pi)F_{12}(x)$, where $F_{12}(x)$ is the field-strength tensor); h is the number of zero modes of the restriction of $i\not{D}$ to the boundary ∂X denoted by B; $\eta(0)$ is the famous η invariant of APS which measures the spectral asymmetry of B.

Parametrizing X near the boundary as $(u, y) \in I \times \partial$X where the normal coordinate u vanishes on ∂X, we assume that the operators L and $L^\dagger$ can be written as

$$L = -\partial_u + B, \tag{3.2a}$$

$$L^\dagger = \partial_u + B, \tag{3.2b}$$

where B is a self-adjoint (first order and elliptic) operator. It has been shown by APS [1] that if $\{\omega_n\}$ denotes the set of eigenvalues of B, the spectral function, $\eta(s)$ of B may be written as

$$\eta(s) = \sum_{\substack{n \\ \omega_n \neq 0}} \frac{\mathrm{sign}(\omega_n)}{|\omega_n|^s}. \tag{3.3}$$

They have also shown that $\eta(0)$ is finite for all self-adjoint elliptic operators on odd-dimensional manifolds. The anomaly density $\alpha_0(x)$ is the constant term in the high temperature expansion ($\beta \to 0$) of the diagonal of the heat kernel

$$Z(\beta, x, x) = \sum_\lambda \mathrm{e}^{-\beta\lambda}\left(\langle f_\lambda(x), f_\lambda(x)\rangle - \langle g_\lambda(x), g_\lambda(x)\rangle\right),$$

where (f_λ, λ) and (g_λ, λ) denote the normalized eigenfunctions and the eigenvalues of $LL^\dagger$ and $L^\dagger L$ respectively on the double of X. Since the double of X is a compact manifold without boundary all the usual heat kernel expansions can be applied directly [12].

A crucial point for the validity of the APS index theorem is to impose suitable boundary conditions at ∂X for the Dirac operator, which guarantees self-adjointness and are compatible with the non-trivial topology of the gauge fields (and/or the manifold). These non-local boundary conditions require that the right- (left-) handed spinor restricted to the boundary ∂X should lie in the subspace spanned by the eigenfunctions $\{e_n\}$ of B with the corresponding eigenvalues, ω_n being negative (positive). Let us expand f and g near the boundary in terms of an orthonormal basis of eigenfunctions of B

$$f(u,y) = \sum f_n(u) e_n(y), \qquad g(u,y) = \sum g_n(u) e_n(y).$$

Then the APS boundary conditions read

$$f_n(0) = 0 \qquad \text{for } \omega_n \geqslant 0, \tag{3.4a}$$

$$g_n(0) = 0 \qquad \text{for } \omega_n < 0, \tag{3.4b}$$

and guarantee that L and $L^\dagger$ (more precisely their closures) are the adjoints of each other.

The conditions (3.4a, b) imply the following boundary conditions for the second order operators, $H^+ = LL^\dagger$ and $H^- = L^\dagger L$ (for simplicity we assume that B has no zero modes):

$$\begin{aligned} f_n(0) &= 0 \qquad \text{for } \omega_n > 0, \\ (L^\dagger f)_n(0) &= 0 \qquad \text{for } \omega_n < 0, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} g_n(0) &= 0 \qquad \text{for } \omega_n < 0, \\ (Lg)_n(0) &= 0 \qquad \text{for } \omega_n > 0. \end{aligned} \tag{3.5b}$$

For a simple interpretation of these non-local boundary conditions, see refs. [1, 13].

It has been shown in ref. [1] that from the point of view of the operator ∂_u the *η-invariant* may be expressed as

$$\eta(0) = -2a_0, \tag{3.6}$$

where a_0 is the temperature independent term in the high temperature (small β) expansion of

$$Z(\beta) = \mathrm{tr}(\mathrm{e}^{-\beta H^+} - \mathrm{e}^{-\beta H^-}) \sim a_0 + a_1\sqrt{\beta} + \cdots \tag{3.7}$$

over the cylinder $\partial X \times [0, \infty) = \{(y, u)\}$. Here the boundary operator B on ∂X is trivially (u-independently) extended to the cylinder (and so are L and $L^\dagger$ in (3.2) and $H^{\pm}$) therefore we can diagonalize B over this cylinder by using the basis $\{e_n\}$. Writing

$$Z(\beta) = \sum_n Z_n(\beta), \tag{3.8}$$

where $Z_n(\beta)$ is the trace of the heat kernel in a given ω_n sector, $Z_n(\beta)$ has the following integral representation [1]:

$$Z_n(\beta) = -\tfrac{1}{2}\operatorname{sign}(\omega_n)\operatorname{erfc}\left(\sqrt{\beta\omega_n^2}\right), \tag{3.9}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \tag{3.10}$$

The spectral function, $\eta(s)$, is the Mellin transform of $Z(\beta)$ (we assume that B has no zero modes)

$$\eta(s) = -\frac{s\sqrt{\pi}}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_0^\infty \beta^{s/2-1} Z(\beta)\, d\beta. \tag{3.11}$$

For details of the original derivation of (3.11) we refer to ref. [1].

We find it instructive to derive (3.9) by exploiting the supersymmetry structure of the Dirac operator. In our case the supersymmetry boils down to the following: if f_λ is an eigenfunction of $LL^\dagger$

$$LL^\dagger f_\lambda = \lambda^2 f_\lambda \tag{3.12}$$

satisfying (3.5a) then $L^\dagger f_\lambda$ is an eigenfunction of $L^\dagger L$ satisfying the non-local boundary conditions (3.5b). In other words the boundary conditions are compatible with supersymmetry. Let us now introduce the Green functions G^+ and G^- as

$$\left(H_u^+ + m^2\right) G^+(u, u') = \delta(u - u'), \tag{3.13a}$$

$$\left(H_u^- + m^2\right) G^-(u, u') = \delta(u - u'), \tag{3.13b}$$

where $G^{\pm}$ are symmetric in (u, u') and fulfill (3.5a, b) at the boundary $u = 0$ (or $u' = 0$) and the usual L_2 condition at $u = \infty$ (or $u' = \infty$). First, using the spectral representation for $G^{\pm}$

$$G^{+}(u,u') = \int \mathrm{d}\lambda\, \frac{f_\lambda(u) f_\lambda^\dagger(u')}{\lambda^2+m^2}\,, \tag{3.14a}$$

$$G^{-}(u,u') = \int \mathrm{d}\lambda\, \frac{g_\lambda(u) g_\lambda^\dagger(u')}{\lambda^2+m^2}\,, \tag{3.14b}$$

together with the equations

$$L^\dagger f_\lambda = \lambda g_\lambda\,, \tag{3.15a}$$

$$L g_\lambda = \lambda f_\lambda\,, \tag{3.15b}$$

we derive the following supersymmetry relations between $G^+(u,u')$ and $G^-(u,u')$:

$$L_u^\dagger G^+(u,u') = L_{u'} G^-(u,u')\,, \tag{3.16a}$$

$$L_{u'}^\dagger G^+(u,u') = L_u G^-(u,u')\,, \tag{3.16b}$$

which will turn out to be extremely useful. Using (3.13) and (3.16) one obtains

$$L_u^\dagger L_{u'}^\dagger G^+(u,u') = \delta(u-u') - m^2 G^-(u,u')\,. \tag{3.17}$$

Since we diagonalized B over the "half cylinder", to actually compute the Green functions $G^{\pm} = \Sigma G_n^{\pm}$ we have to solve

$$(-\partial_u^2 + m^2 + \omega_n^2) G_n^{\pm}(u,u') = \delta(u-u') \tag{3.18}$$

subject to the boundary conditions (3.5a, b) at $u=0$ and $u'=0$ and L_2 boundary conditions at $u=\infty$ and $u'=\infty$. Let us first assume that $\omega_n > 0$. Then we can immediately write down the solution of (3.18) for the right-handed ($\tilde{\gamma} = +$) Green function:

$$G_n^+(u,u') = \frac{1}{2\sigma_n}\left(\mathrm{e}^{-\sigma_n|u-u'|} - \mathrm{e}^{-\sigma_n(u+u')}\right), \tag{3.19}$$

where $\sigma_n = \sqrt{m^2+\omega_n^2}$. From eq. (3.17) we can now trivially derive the left-handed ($\tilde{\gamma} = -$) Green function:

$$G_n^-(u,u') = \frac{1}{2\sigma_n}\left(\mathrm{e}^{-\sigma_n|u-u'|} + \frac{\sigma_n-\omega_n}{\sigma_n+\omega_n}\mathrm{e}^{-\sigma_n(u+u')}\right). \tag{3.20}$$

Since we wish to calculate

$$G_n(m^2) = \int_0^\infty (G_n^+(u,u) - G_n^-(u,u))\,\mathrm{d}u \tag{3.21}$$

it is useful to observe that G_n can be converted into surface terms by using the following relation between G_n^+ and G_n^- on their diagonal:

$$G_n^+(u,u) - G_n^-(u,u) = -\frac{1}{2\omega_n}\partial_u(G_n^+(u,u) + G_n^-(u,u)). \tag{3.22}$$

Eq. (3.22) follows from (3.16a) and (3.16b) by adding these two equations and then restrict the sum to the diagonal $u = u'$. For $\omega_n < 0$ the calculation of $G_n(m^2)$ is very similar and finally we obtain for (3.21)

$$G_n(m^2) = -\frac{1}{2}\frac{\mathrm{sign}(\omega_n)}{\sigma_n(\sigma_n + |\omega_n|)}. \tag{3.23}$$

As one can easily check [14] the Laplace back-transform of eq. (3.23) yields eq. (3.9).

An alternative way of calculating $Z(\beta)$ is by using "phase shifts" which gives the above result significantly faster than the previous "Green function method". The key formula – which holds for both the L_2 and the APS boundary conditions – for applying the phase shifts is:

$$\mathrm{tr}(F(H^+) - F(H^-)) = \frac{1}{\pi}\sum_n \int_0^\infty \mathrm{d}E\,F(E)\frac{\mathrm{d}\delta_n(E)}{\mathrm{d}E}, \tag{3.24}$$

where the energy, $E = \lambda^2$; $\delta_n(E) = \delta_n^+(E) - \delta_n^-(E)$ denotes the difference between the right- and left-handed phase shifts; the function $F(x)$ should tend to zero at least as $1/x$ as $x \to \infty$. We refer to appendix A for the derivation of (3.24) and the general definition of the phase shifts.

Let us choose $F(x) = \exp(-\beta x)$. For $\omega_n > 0$ the left- and right-handed "scattering" states are (up to an overall normalization)

$$f_n = \sin(ku),$$
$$g_n = \frac{1}{k}L^\dagger f_n = \sin(ku + \delta), \tag{3.25}$$

where $k = \sqrt{E - \omega_n^2}$ and $\delta = \arctan(k/\omega_n)$.

Using appendix A it is easy to see that $\delta(E)$ in (3.25) is indeed the difference $\delta^- - \delta^+$ between the left- and right-handed phase shifts. From (3.24) we therefore

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$$Z(\beta) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}k\, \mathrm{e}^{-\beta(\omega^2+k^2)}\frac{\omega}{\omega^2+k^2} = -\tfrac{1}{2}\mathrm{erfc}\left(\sqrt{\beta\omega^2}\right), \tag{3.26}$$

ɪst eq. (3.9).

ɪportant to note that the heat-kernel on the cylinder is temperature :, although the boundary conditions (3.5a, b) preserve the supersymmetry. :trum of $i\not{D}$ would be purely discrete this could not happen. However, it is nting out that the existence of a continuous spectrum is not sufficient to the temperature dependence of $Z(\beta)$. Since β has dimension [length]^2 ı is dimensionless, clearly another dimensionful parameter is required to n-trivial $Z(\beta)$. The point is that in the present case (on the cylinder) the :s of B have dimension [length]$^{-1}$ so $Z(\beta)$ could be any function of the ılless quantity $(\omega^2\beta)$.

e would like to connect the above results to our anomaly considerations. 3) and (3.23) we observe that for $m \to 0$

$$A(m^2) \sim -\tfrac{1}{4}m^2\sum_n \frac{\mathrm{sign}(\omega_n)}{\omega_n^2} = -\tfrac{1}{4}m^2\eta(2), \tag{3.27}$$

ˀ energy part of the anomaly vanishes. On the other hand, the high energy (m^2), $m \to \infty$ is equal to

$$A = c\Phi + b, \tag{3.28}$$

nd b are constants and $\Phi = \int_X \alpha_0(x)\,\mathrm{d}x$ is the generalized flux. In deriving assumed that the eigenvalues of B are linear in Φ. To establish (3.28) it is to compute $\partial^2 A/\partial\Phi^2$ which can be seen to vanish for $m \to \infty$ as m^{-2}. (3.28) fails for those exceptional values of Φ where new bound states

ite illuminating to calculate the anomaly when the original manifold, X, is ·:

$$\mathrm{X} = [a, b] \times \mathrm{S}^1$$

ve have two boundaries, $\partial \mathrm{X}^1 = \mathrm{S}^1_{(a)}$ and $\partial \mathrm{X}^2 = \mathrm{S}^1_{(b)}$. We denote the es by $x^0 = u$, $x^1 = \theta$. Let us choose the following representation for the γ

$$\gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

the gauge choice $A_0 = 0$ and also assume for simplicity that $A_1 = \Phi(u)$, so the Dirac operator $\not{D} = \not{\partial} + i\not{A}$ becomes

$$i\not{D} = \begin{pmatrix} 0 & L \\ L^{\dagger} & 0 \end{pmatrix}, \tag{3.29}$$

where

$$L = -\partial_u + \left(\frac{1}{i}\partial_\theta + \Phi\right), \tag{3.30a}$$

$$L^{\dagger} = \partial_u + \left(\frac{1}{i}\partial_\theta + \Phi\right). \tag{3.30b}$$

Clearly the operator at the right should be identified with the boundary operator

$$B = \frac{1}{i}\partial_\theta + \Phi, \tag{3.31}$$

with eigenvalues $\omega_n = n + \Phi_a$ and $\omega_n = -(\omega_n + \Phi_b)$.

In the present case when $m \to \infty$ we can compute both constants c and b in (3.28) explicitly. It is not difficult to see that

$$\lim_{m \to \infty} \partial_\Phi A(m^2) = c$$

can be represented by the integral

$$-\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{3/2}} = -1.$$

Furthermore, by observing that

$$A(m^2, \Phi) = A(m^2, \Phi + 1)$$

and that A is an odd function of Φ together with the fact that $A(\Phi)$ vanishes at $\Phi = \pm\frac{1}{2}$ (for these values of Φ the spectrum of B is symmetric) we finally obtain

$$A = \tfrac{1}{2} - \langle \Phi_b \rangle, \tag{3.32}$$

where $0 \leqslant \langle \Phi \rangle < 1$ denotes the fractional part of the flux, Φ (see fig. 1).

It is not too difficult to work out the η-invariant as well. At the right boundary we obtain

$$\eta_b(s) = \zeta(s, -\langle \Phi_b \rangle) - \zeta(s, 1 + \langle \Phi_b \rangle), \tag{3.33}$$

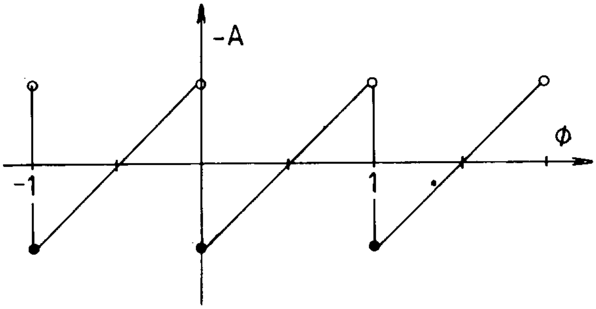

Fig. 1. The high energy limit $A(\infty)$ of $A(m^2)$ as function of the flux.

where $\zeta(s, q)$ is the modified ζ-function

$$\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}.$$

Since [15]

$$\zeta(0, q) = -q + \tfrac{1}{2}, \tag{3.34}$$

the contribution to the η-invariant from the right boundary is

$$\eta_b(0) = 2\langle \Phi_b \rangle - 1 = -2A. \tag{3.35}$$

The contribution to the η-invariant and the anomaly from the left boundary ($u = a$) is calculated in the same way and we find for the total boundary term (taking into account the change of orientation at $u = a$ with respect to $u = b$)

$$\eta_b + \eta_a = 2(\langle \Phi_b \rangle - \langle \Phi_a \rangle). \tag{3.36}$$

For the sake of completeness let us now count the number of zero modes. Assuming for example, $\Phi_b > \Phi_a > 0$, there are no left-handed zero modes satisfying the non-local boundary conditions. The right-handed ones are given by

$$f_n(u) = \exp\left\{ -\int^u (n + \Phi(u'))\, \mathrm{d}u' \right\}, \tag{3.37}$$

where the boundary conditions imply the following restriction for the allowed range of n:

$$-\Phi_b < n < -\Phi_a,$$

which in turn fixes the index to be

$$\mathrm{ind}(i\not{D}) = [\Phi_b] - [\Phi_a]. \tag{3.38}$$

Since the total flux is

$$\frac{1}{2\pi}\int_X F_{01}\,\mathrm{d}x = \Phi_b - \Phi_a$$

using (3.36) and (3.38) we verified the APS index theorem for this simple example.

Finally we calculate the anomaly of the Dirac operator on the *infinite* cylinder, $\mathrm{R}\times\mathrm{S}^1$. We shall assume that

$$\Phi(u) \to \Phi_{\pm} \qquad \text{for } u \to \pm\infty$$

(with finite $\Phi_{\pm}$) thus guaranteeing that the total flux is finite. In fact it is given by

$$\frac{1}{2\pi}\int_X F_{01}\,\mathrm{d}x = \Phi_+ - \Phi_- .$$

In this "non-compact" case the natural boundary conditions are dictated as being L_2. For computing the anomaly we shall employ the heat-kernel regularization as in (2.12). We shall exploit again the key formula (3.24) which gives the result in a very quick way. All we need to calculate now are the usual scattering phase shifts $\delta_n = \delta_n^+ - \delta_n^-$. The right-handed spinor solution has the asymptotic form

$$f \to \begin{cases} \mathrm{e}^{-ik_- u} & \text{for } u \to -\infty \\ a_{\mathrm{R}}(k)\mathrm{e}^{-ik_+ u} + b_{\mathrm{R}}(k)\mathrm{e}^{ik_+ u} & \text{for } u \to \infty , \end{cases}$$

where the dispersion relation is

$$E = k_{\pm}^2 + \left(\omega_n^{\pm}\right)^2 \qquad \text{with } \omega_n^{\pm} = n + \Phi_{\pm}$$

and we took the normalization convention from appendix A. Using again supersymmetry and taking the same normalization the left-handed spinor solution is given by

$$g = \frac{1}{-ik_- + w_-} L^{\dagger} f ,$$

so that

$$a_{\mathrm{L}}(k) = \frac{ik_+ - \omega_+}{ik_- - \omega_-} a_{\mathrm{R}}(k) . \tag{3.39}$$

Since the phase shift is given as

$$\mathrm{e}^{2i(\delta_{\mathrm{R}} - \delta_{\mathrm{L}})} = \frac{a_{\mathrm{R}}(-k)}{a_{\mathrm{R}}(k)} \frac{a_{\mathrm{L}}(k)}{a_{\mathrm{L}}(-k)} \tag{3.40}$$

(3.39) – the unknown function $a_{\mathrm{R}}(k)$ drops out from the phase shifts. (3.40) the result

$$\delta = \delta_{\mathrm{R}} - \delta_{\mathrm{L}} = -\arctan\frac{k_+}{\omega_+} + \arctan\frac{k_-}{\omega_-}. \tag{3.41}$$

h δ_{R} and δ_{L} are the same function of k as $\delta(k)$ in eq. (3.25), when plugging into (3.24) we can make use of (3.26) to write down $Z_{\mathrm{c}}(\beta)$, where $Z_{\mathrm{c}}(\beta)$ s that part of $Z(\beta)$ which comes purely from the scattering states. Taking count the jump of the phase shift at a bound state we end up with

$$Z_n(\beta) = n_{\mathrm{R}} - n_{\mathrm{L}} - \tfrac{1}{2}\left(\operatorname{erfc}\left(\sqrt{\omega_+^2\beta}\right) - \operatorname{erfc}\left(\sqrt{\omega_-^2\beta}\right)\right). \tag{3.42}$$

ain the anomaly we carry out the summation over n and take the limit $\beta \to 0$ ially obtain

$$A = \operatorname{ind}(i\not{D}) + (\langle\Phi_+\rangle - \langle\Phi_-\rangle). \tag{3.43}$$

43) is in complete agreement with the APS index theorem in the limit when boundaries ($u = a, b$) tend to $\mp$infinity.

emark that since the L_2 and APS boundary conditions appear to be different t a priori obvious that one should get the same results. What seems to be hat surprising is that the scattering phase shifts are actually the same for both case and in the outer region of the finite cylinder with the APS boundary ons.

hermore we think it is interesting that the fractional part of the anomaly is always zero on compact manifolds without boundaries) can be viewed as a ry effect, associated with the presence of a continuous spectrum on the one nd the APS η-invariant on the other hand. Finally we would like to mention is cylinder example has also been thoroughly analysed in a very clear paper e [16], from a somewhat different point of view but our results are of course plete agreement with his.

ial anomaly for the 2-dimensional disk and modified APS boundary conditions

his section we shall mostly concentrate on the simple but illuminating e of a U(1) gauge field defined on a disk in R^2, with radius R. As it will be e disk is somewhat more complicated than the cylinder case. The Dirac r rewritten in polar coordinates is given as

$$i\not{D} = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix}, \tag{4.1}$$

where

$$L = e^{i\theta}\left(-\partial_r + \frac{1}{r}\left(\frac{1}{i}\partial_\theta + \Phi\right)\right), \tag{4.2a}$$

$$L^\dagger = e^{-i\theta}\left(\partial_r + \frac{1}{r}\left(\frac{1}{i}\partial_\theta + \Phi\right)\right), \tag{4.2b}$$

and we assume for simplicity a rotationally symmetric gauge field

$$A_\mu = -\varepsilon_{\mu\alpha}\frac{x^\alpha}{r^2}\Phi(r). \tag{4.3}$$

Clearly from (4.2) the boundary operator is not self-adjoint (using the induced scalar product) because of the $\exp(\pm i\theta)$ factors which then prevent us from applying the APS method directly. We remark here that it is a generic feature in R^{2n} when one employs radial-type coordinates, already for the free Dirac operator that near the boundary it is not of the "standard" APS form.

However one can easily get rid of the troublesome $\exp(\pm i\theta)$ factors by applying the unitary transformation

$$U = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{4.4}$$

on $i\not{D}$:

$$Ui\not{D}U^{-1} = \begin{pmatrix} 0 & -\partial_r - \frac{1}{2r} + \left(\frac{1}{i}\partial_\theta + \Phi\right)\Big/r \\ \partial_r + \frac{1}{2r} + \left(\frac{1}{i}\partial_\theta + \Phi\right)\Big/r & 0 \end{pmatrix}. \tag{4.5}$$

Note, that the corresponding transformation on the wave function

$$\psi \to U\psi$$

maps single valued spinors to double valued ones. Since the transformation (4.4) is not well defined at the origin, we are forced to cut a hole (with radius δ, say) there. This is in agreement with the fact that on the disk only single valued spinors are allowed. The punctured disk is topologically equivalent to a circle, S^1, with R as its universal covering space where multivalued functions

$$\psi(2\pi) = e^{i2\pi\alpha}\psi(0) \tag{4.6}$$

are admissible.

If we start with polar coordinates, that is when the line element

$$ds^2 = dr^2 + r^2 d\theta^2$$

we get directly (4.5) for the Dirac operator. The terms $\pm 1/2r$ come from the spin connection

$$\omega_\mu = \tfrac{1}{4} i \left[\gamma^\rho, \partial_\rho \gamma_\mu \right],$$

which is present because of our use of curvilinear coordinates. Now, already the coordinate system itself is singular at $r = 0$ (and the spin connection as well) so a "hole" is induced by our choice of coordinates.

Having derived (4.5) in one or the other way it is easy to bring L and $L^\dagger$ to the standard form by transforming away the spin connection. Indeed defining

$$\psi = \frac{1}{\sqrt{r}} \tilde{\psi}, \tag{4.7}$$

thus changing the measure from $r\,dr$ to dr, we get

$$i \not{D} \psi = \frac{1}{\sqrt{r}} \begin{pmatrix} 0 & \tilde{L} \\ \tilde{L}^\dagger & 0 \end{pmatrix} \tilde{\psi}, \tag{4.8a}$$

where

$$\tilde{L} = -\partial_r + \left(\frac{1}{i} \partial_\theta + \Phi \right) \Big/ r,$$

$$\tilde{L}^\dagger = \partial_r + \left(\frac{1}{i} \partial_\theta + \Phi \right) \Big/ r. \tag{4.8b}$$

Note that the adjoint of $-\partial_r$ is ∂_r with respect to the induced scalar product. There is no problem now in reading off the boundary operator, B, from (4.8) – its eigenvalues at the "outer" boundary $r = R$ are

$$\omega_n = -\frac{n + \alpha + \Phi(R)}{R}, \tag{4.9}$$

where we imposed the following condition on the eigenfunctions of B:

$$e_n(2\pi) = e^{i 2\pi\alpha} e_n(0), \qquad 0 \leqslant \alpha < 1,$$

in agreement with (4.6). Now we still have to cope with the induced boundary at $r = \delta$. Let us assume that $\Phi(r) \to 0$ as $r \to 0$, so that the field strength is regular at the hole. At $r=\delta$ the corresponding eigenvalues of B are

$$\omega_n = \frac{n+\alpha+\Phi(\delta)}{\delta}, \tag{4.10}$$

where the relative minus sign between (4.9) and (4.10) is due to the change of orientation at $r=\delta$ with respect to $r=R$. The calculation of $\eta_R(0)$ and $\eta_\delta(0)$ now is essentially the same as for the cylinder and we obtain

$$\eta_R(0) = 2\langle \alpha+\Phi(R)\rangle - 1, \tag{4.11a}$$

$$\eta_\delta(0) = 1 - 2\langle \alpha+\Phi(\delta)\rangle. \tag{4.11b}$$

So according to the APS index theorem

$$\operatorname{ind}(i\not{D}) = [\Phi(R)+\alpha] - [\Phi(\delta)+\alpha], \tag{4.12}$$

which agrees of course with the counting of the explicitly known zero modes. For $\alpha=\frac{1}{2}$ we recovered Ma's result [17]. Since $\Phi(\delta)\to 0$ as $\delta\to 0$ in this limit we get for the disk without the hole

$$\operatorname{ind}(i\not{D}) = [\Phi(R)+\alpha]. \tag{4.13}$$

However as it stands this $\delta\to 0$ limit is somewhat questionable since we seem to end up with multi-valued spinors over a disk. Furthermore, though there is no "physical hole" (the fields are well behaved) we still get a non-vanishing contribution to $\eta_\delta(0)$ for $\alpha\neq\frac{1}{2}$. This argument suggests already that to make connection with the original problem (non-singular fields on the disk) $\alpha=\frac{1}{2}$ is the only allowed value. Indeed, remembering that the back transformation of $i\not{D}$ from polar to cartesian coordinates involves

$$U = \operatorname{diag}(e^{i\theta/2}, e^{-i\theta/2})$$

clearly it is only with $\alpha=\frac{1}{2}$ when we get single valued spinors and in this case the B operator has half-integer eigenvalues. For $\alpha=\frac{1}{2}$ the APS index theorem can be illustrated as in fig. 2.

Note that the η-invariant is periodic with periodicity one, which is a consequence of course, of the periodicity of the spectrum of B. (If $\alpha\neq\frac{1}{2}$ then in the two figures on the left-hand side there is a shift by $\alpha-\frac{1}{2}$.)

In what follows it is essential to observe that starting with $i\not{D}$ as in (4.1) and (4.2) on the disk, the angular momentum operator

$$J = \frac{1}{i}\partial_\theta + \Phi - \tfrac{1}{2}\gamma_5$$

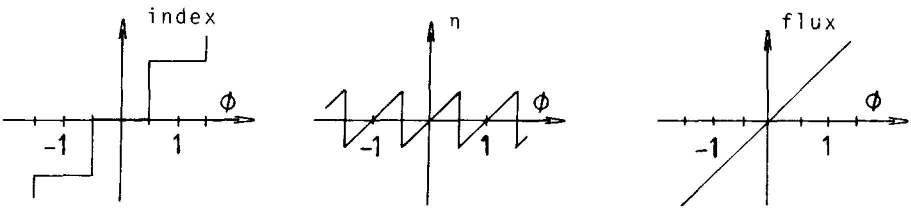

Fig. 2. The flux-dependence of the index and the η-invariant on the two-dimensional cylinder with APS boundary conditions.

commutes with $i\not{D}$. Therefore we can work in a given J sector and we immediately recover $i\not{D}$ as given in (4.5) and $-i\partial_\theta - \frac{1}{2}\gamma_5$ having half-integer eigenvalues. This argument seems to indicate that the boundary operator can be identified with $J + \Phi$.

To compactify the disk to S^2 it is necessary to have an integer flux (assuming that the transition function is single valued on S^2 as required by the bundle picture). In this case the index (on S^2) is equal to the flux. Therefore one might expect that integer values of the flux play a somewhat distinguished role. Since the boundary $r = R$ can be thought of as an infrared regularization of the anomaly – (note that there is a long range gauge field) it would be natural to demand that by removing the regulator (i.e. $R \to \infty$) we recover the well known L_2 result which can be depicted as [3, 4] in fig. 3.

One sees that the fractional part – in contradistinction to the previous case – is not periodic and the index jumps at integer and not at half-integer values of the flux. For example, when $\Phi > 0$, the explicit right-handed L_2 zero modes are

$$f = \exp\left\{ -\int^{r} \frac{n + \Phi(r')}{r'} \, \mathrm{d}r' \right\}, \tag{4.14}$$

where n denotes the (integer) eigenvalues of the orbital angular momentum operator. One can easily establish that the condition for normalizability at $r \to \infty$ is $n - 1 + \Phi(\infty) > 0$ which differs from the APS condition $j + \Phi > 0$. Since the

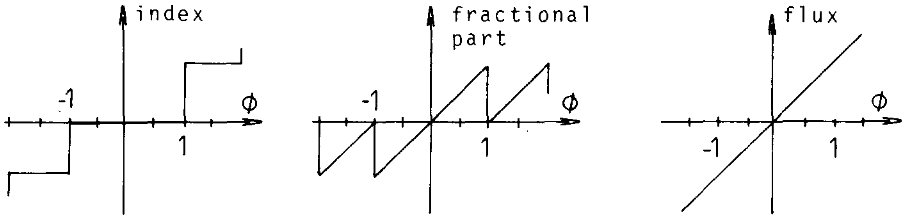

Fig. 3. The index and the fractional part of the anomaly on R^2 with L_2 boundary conditions.

physical bound states should be L_2 normalizable – using the APS boundary conditions with the boundary operator as J we would allow for zero modes which would correspond to unphysical ones in the limit $R \to \infty$. All these arguments indicate that the boundary operator is not $J + \Phi$, as one would have naively thought. In addition, there is another seemingly technical reason against identifying B with $J + \Phi$. When computing the originally well defined anomaly, $A(m^2)$ as in (2.13) – in the outside region $r > R$ using the APS boundary conditions by first diagonalizing J and then attempt to calculate the "trace" (summing over j) – we find that the resulting sum is ill defined (see appendix B). On the other hand if we insist on $B = (1/i)\partial_\theta + \Phi$ with the usual (integer valued) spectrum of $(1/i)\partial_\theta$ we then obtain a convergent sum (over n) for $A(m^2)$. We hope that this point will become clear from the explicit calculations below.

Motivated by the above arguments (both physical and mathematical) in the following we take $B = (1/i)\partial_\theta + \Phi$ as our boundary operator in (4.2). Using (4.1) as the Dirac operator poses some difficulties, since L and $L^\dagger$ are not the adjoints of each other due to the $\exp(\pm i\theta)$ factors, with the usual APS boundary conditions. It is however not so difficult to find the modification of the APS conditions to ensure the adjointness of L and $L^\dagger$ and thus the self-adjointness of the Dirac operator. For the technical details we refer to appendix C. The result is the following: expanding f and g near the boundary in the B eigenbasis

$$Be_n = \omega_n e_n \qquad \text{where } \omega_n = n + \Phi ,$$

we obtain the following boundary conditions for

$$f = \sum f_n e_n \qquad \text{and} \qquad g = \sum g_n e_n$$

at $r = R$: for $\Phi > 0$

$$\begin{aligned} f_n(R) &= 0 \qquad \text{for } \omega_n > 1 , \\ g_n(R) &= 0 \qquad \text{for } \omega_n \leqslant 0 , \end{aligned} \tag{4.15a}$$

and for $\Phi < 0$

$$\begin{aligned} f_n(R) &= 0 \qquad \text{for } \omega_n \geqslant 0 , \\ g_n(R) &= 0 \qquad \text{for } \omega_n < -1 . \end{aligned} \tag{4.15b}$$

The adjointness conditions do not fix the boundary conditions completely – in deriving (4.15a, b) we required that $Z(\beta)$ should vanish in the outside region when $\Phi(r) = 0$.

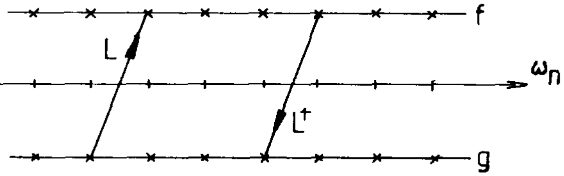

Fig. 4. The slantwise action of the supersymmetric operators L and $L^{\dagger}$.

The corresponding boundary conditions for the second order operators $LL^{\dagger}$ and $L^{\dagger}L$ following from (4.15) are

$$f_n(R)=0,\ \omega_n>1;\qquad \left(\partial_r+\frac{1}{r}\omega_n\right)f_n|_R=0,\qquad \omega_n\leqslant 1, \tag{4.16a}$$

$$g_n(R)=0,\ \omega_n\leqslant 0;\qquad \left(-\partial_r+\frac{1}{r}\omega_n\right)g_n|_R=0,\qquad \omega_n>0. \tag{4.16b}$$

for $\Phi>0$ and similarly for $\Phi<0$.

Our calculations will be drastically simplified by the fact that the boundary conditions (4.16) preserve supersymmetry just like the APS boundary conditions. In fact supersymmetric boundary conditions are equivalent to demanding that L and $L^{\dagger}$ be the adjoints of each other.

It is important to note that the supersymmetry operators L and $L^{\dagger}$ (in 4.2) change the orbital angular momentum and hence the eigenvalues of B by $+1$ (resp. -1). This "slantwise" action of the operator is illustrated in fig. 4.

We now proceed to calculate the anomaly using again the phase shift method and the key formula (3.24), exploiting heavily the supersymmetry. First we have to find the phase shifts of the second order operators. Therefore we have to solve the equations

$$LL^{\dagger}f_n=\left(-\partial_r^2-\frac{1}{r}\partial_r+\frac{\omega_n^2}{r^2}\right)f_n=\lambda^2 f_n, \tag{4.17a}$$

$$L^{\dagger}Lg_n=\left(-\partial_r^2-\frac{1}{r}\partial_r+\frac{\omega_n^2}{r^2}\right)g_n=\lambda^2 g_n \tag{4.17b}$$

in the outer region $r>R$ subject to the boundary conditions (4.16). As it is well known (4.17) is just the Bessel equation whose solution is given as:

$$f_n=\alpha J_{\omega_n}(\lambda r)+\beta J_{-\omega_n}(\lambda r), \tag{4.18a}$$

$$g_{n-1}=\frac{1}{\lambda}L^{\dagger}f_n=\alpha J_{\omega_n-1}(\lambda r)-\beta J_{1-\omega_n}(\lambda r), \tag{4.18b}$$

where $L^\dagger$ is of course taken in the B eigenbasis: $L^\dagger = \partial_r + \omega_n/r$. Using the asymptotic expansion for the Bessel function ($r \to \infty$) and the phase shift definition as in appendix A one can establish that the phase shift for the right-handed function, f, is

$$\delta_{\rm R}^{(n)} = \tfrac{1}{4}\pi + \arctan\left(\frac{1-\beta/\alpha}{1+\beta/\alpha}\tan\frac{\pi}{2}\omega_n\right). \tag{4.19}$$

The phase shift of the supersymmetric partner $L^\dagger f$ is given by

$$\delta_{\rm L}^{(n-1)} = \tfrac{1}{4}\pi + \arctan\left(\frac{1+\beta/\alpha}{1-\beta/\alpha}\tan\frac{\pi}{2}(\omega_n - 1)\right), \tag{4.20}$$

which is obtained from (4.19) simply by the substitution $\alpha \to \alpha$, $\beta \to -\beta$, $\omega_n \to \omega_n - 1$. From (4.20) one can easily check that

$$\delta_{\rm R}^{(n)} \pm \tfrac{1}{2}\pi = \delta_{\rm L}^{(n-1)}, \tag{4.21}$$

where the $(+)$ sign is for $\omega_n > 1$ and the $(-)$ sign is for $\omega_n \leqslant 1$. To apply the key formula (3.24) we evaluate the limit of

$$\delta^{(N)} = \sum_{-N}^{N}\left(\delta_{\rm R}^{(n)} - \delta_{\rm L}^{(n)}\right) \tag{4.22}$$

when $N \to \infty$.

We can make use of (4.21) by rearranging the finite sum as follows:

$$\delta^{(N)} = \sum_{-(N+1)}^{N}\left(\delta_{\rm R}^{(n+1)} - \delta_{\rm L}^{(n)}\right) - \left(\delta_{\rm R}^{(N+1)} - \delta_{\rm L}^{-(N+1)}\right). \tag{4.23}$$

In (4.23) the sum is actually zero because of (4.21) (there is an equal number of positive and negative ω terms in the sum). The above rearrangement of (4.22) is illustrated below in fig. 5.

So we find that the sum (4.22) telescopes to

$$\delta^{(N-1)} = \delta_{\rm L}^{(-N)} - \delta_{\rm R}^{(N)}. \tag{4.24}$$

From this telescoping property one can already anticipate that the only contribution to the anomaly comes from the very large angular momenta. To calculate $\delta_{\rm L}^{(-N)} - \delta_{\rm R}^{(N)}$ for large values of N we observe that $\omega_N > 0$ and $\omega_{-N} < 0$, hence the boundary conditions (4.16a, b) imply

$$\beta/\alpha = -\frac{J_{\omega_N}(\lambda R)}{J_{-\omega_N}(\lambda R)} \tag{4.25a}$$

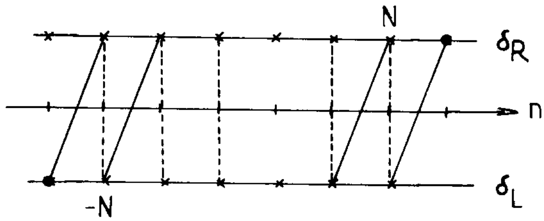

Fig. 5. The remaining "boundary terms" after the rearrangement of the finite sum (4.22).

in the right-handed sector, whereas in the left-handed sector:

$$\beta/\alpha = -\frac{J_{\omega_{-N}}(\lambda R)}{J_{-\omega_{-N}}(\lambda R)}. \tag{4.25b}$$

From the formulae for the asymptotic expansion for large orders of the Bessel functions it follows at once that β/α in (4.25a) tends to zero exponentially for $\omega_N \to \infty$ and β/α in (4.25b) tends to infinity (also exponentially fast) as $\omega_{-N} \to -\infty$.

Then the phase shift formulae (4.19) and (4.20) yield

$$\delta^{(N-1)} = \delta_{\mathrm{L}}^{(-N)} - \delta_{\mathrm{R}}^{(N)} \to -\pi\langle\Phi\rangle\theta(E), \tag{4.26}$$

where $E = \lambda^2$. In eq. (4.26) the fractional part of the flux, $\langle\Phi\rangle$, comes from the correct choice of the branch of the arctan function in (4.19) and (4.20) which is fixed by the requirement that $0 \leqslant \delta < \pi$. Also in (4.26) the phase shifts are zero for $E < 0$ because both $LL^\dagger$ and $L^\dagger L$ are non-negative.

Since

$$\frac{\mathrm{d}}{\mathrm{d}E}\delta^{(N)}(E) = -\pi\langle\Phi\rangle\delta(E), \tag{4.27}$$

where $\delta(E)$ denotes the usual delta function, the key formula (3.24) yields:

$$\mathrm{tr}(F(H^+) - F(H^-)) = -\langle\Phi\rangle F(0). \tag{4.28}$$

From (4.28) we immediately get for $Z(\beta)$ (in the outside region with the boundary conditions (3.15a, b)):

$$Z(\beta) = -\langle\Phi\rangle, \tag{4.29}$$

which is clearly temperature independent contrary to the case of the cylinder.

This temperature independence seems surprising at first since the spectra of the operators L, $L^\dagger$ are continuous, but it can be understood from the following simple dimensional analysis. In the present case the eigenvalues of the boundary operator,

B, are dimensionless contrary to the case of the cylinder where the B eigenvalues had dimension [length]$^{-1}$. The only intrinsic dimensionful parameter now is R coming from the boundary conditions, so the (dimensionless) $Z(\beta)$ can only be a function of β/R^2. Since we assume that $\Phi(R) \to \Phi(\infty)$ for R large enough, the eigenvalues of B are independent of R. Also the index of $i\not{D}$ becomes independent of R when $R \to \infty$. Therefore $Z(\beta)$ should be independent of R in this limit.

It is also true that the anomaly $A(m^2)$ does not depend on the energy scale, so this is a way to understand why the anomaly can be viewed as either coming from the ultraviolet or from the infrared region.

In appendix B we compute the anomaly using the Green function method, where the calculations are more involved (and are also more explicit). The results are of course in complete agreement with the phase shift calculations.

Finally we would like to point out that the zero modes as given by eq. (4.14) satisfy the boundary conditions (4.15) when $\omega_n = n + \Phi > 1$. To ensure normalizability at the origin $n < 1$, so we find that the index of $i\not{D}$ is given by $[\Phi]$ for $\Phi > 0$. (For $\Phi < 0$ the index of $i\not{D}$ is $-[-\Phi]$.) Using eq. (4.29) we can see again how the index theorem is satisfied in the present case. We remark that the index theorem derived this way is slightly different from the APS index theorem. This is of course a consequence of our modification of the original APS boundary conditions.

To conclude this section we would like to compare the above results over the disk with the L_2 result in R^2.

First we observe that the supersymmetry relation between f and g is still valid, that is if f is a scattering state regular at the origin, then $L^{\dagger}f$ is also a regular scattering state. For large r the solutions of (4.17a, b) are again given by (4.18a, b). Since in the derivation of (4.24) we used only the supersymmetry property it remains true that $\delta^{(N)}$ is determined by the very large angular momenta. Furthermore we can argue that β/α tends to zero in the f sector for $\omega_N \to \infty$ and β/α tends to (minus) infinity for $\omega_{-N} \to -\infty$. Indeed, for large values of ω the solution f can be written as

$$f \sim \frac{\alpha}{\sqrt{2\pi\omega}}\left(\frac{e\lambda r}{2\omega}\right)^{\omega} + \beta \sin\omega\pi \cdot \sqrt{\frac{2}{\pi|\omega|}}\left(\frac{e\lambda r}{2|\omega|}\right)^{-|\omega|}. \tag{4.30}$$

When $\omega \to \infty$ the second term in (4.30) explodes exponentially and cannot be matched with a regular solution unless $\beta \to 0$ in this limit. This argument is essentially the same as the one employed for $\lambda \to 0$ (low energy) in ref. [3]. Taking into account (4.14) we find the expected L_2 result as shown in fig. 3.

In this case (in R^2) the temperature independence of $Z(\beta)$ is quite obvious since the only contribution to the anomaly comes from the very large angular momentum sectors so only the large distance behaviour of the gauge fields is relevant. In other words, although for dimensional reasons a regular gauge field (at $r = 0$) must depend at least on one dimensionful parameter (characterizing the short-range field strength) $Z(\beta)$ actually does not depend on this parameter. It only depends on the dimensionless coefficient of the asymptotic $1/r$-tail of the gauge field. Thus, as in the case of the APS boundary conditions, $Z(\beta)$ is independent of the dimensionful inverse temperature β.

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Appendix A

Here we establish the key formula (3.24). This formula allows one to calculate supersymmetric traces in terms of the phase shifts $\delta_n = \delta_n^+ - \delta_n^-$ [18]. It holds for both the L_2 and the APS boundary conditions.

Consider the second order equation (3.12) for the right-handed spinor

$$\left(-\partial_u^2 + V\right) f_\lambda = \lambda^2 f_\lambda \tag{A.1}$$

and assume $V \to \omega^2$ as $u \to \infty$. Applying $\mathrm{d}/\mathrm{d}\lambda$ (denoted by a dot) to eq. (A.1) we obtain

$$\left(-\partial_u^2 + V\right) \dot{f}_\lambda = \lambda^2 \dot{f}_\lambda + 2\lambda f_\lambda . \tag{A.2}$$

Using these two equations we can immediately write down the wronskian identity between $\dot{f}_\lambda$ and $f_\lambda^\dagger$:

$$W\left(\dot{f}_\lambda, f_\lambda^\dagger\right)\Big|_a^b = \left(\dot{f}_\lambda \partial_u f_\lambda^\dagger - \partial_u \dot{f}_\lambda f_\lambda^\dagger\right)\Big|_a^b = 2\lambda \int_a^b f_\lambda f_\lambda^\dagger \,\mathrm{d}u . \tag{A.3}$$

Subtracting the corresponding identity for the left-handed spinor yields

$$\left\{ W\left(\dot{f}_\lambda, f_\lambda^\dagger\right) - W\left(\dot{g}_\lambda, g_\lambda^\dagger\right)\right\}\Big|_a^b = 2\lambda \int_a^b \left\{ f_\lambda f_\lambda^\dagger - g_\lambda g_\lambda^\dagger \right\} \mathrm{d}u . \tag{A.4}$$

The point is that the term on the right-hand side is related to the spectral representation of the kernel of $F(H^+) - F(H^-)$ on the diagonal $u = u'$:

$$\left(F(H^+) - F(H^-)\right)(u, u) = \int F(\lambda^2)\left(f_\lambda f_\lambda^\dagger - g_\lambda g_\lambda^\dagger\right) \mathrm{d}\lambda^2 . \tag{A.5}$$

The scattering solutions in (A.5) are normalized by the usual completeness relations

$\int f_\lambda(u) f_\lambda^\dagger(u')\,d\lambda^2 = \delta(u-u')$ etc. From (A.4) we see that the trace

$$\operatorname{tr}(F(H^+) - F(H^-)) = \int du \int d\lambda^2\, F(\lambda^2)\left(f_\lambda f_\lambda^\dagger - g_\lambda g_\lambda^\dagger\right)$$

can be converted into surface terms,

$$\operatorname{tr}(F(H^+) - F(H^-)) = \lim_{\substack{a\to 0\\ b\to\infty}} \int d\lambda\, F(\lambda^2)\left\{W\left(\dot f_\lambda, f_\lambda^\dagger\right) - W\left(\dot g_\lambda, g_\lambda^\dagger\right)\right\}\Big|_a^b. \quad (A.6)$$

(On the cylinder $a\to 0$ must be replaced by $a\to -\infty$.) It remains to be shown that the right-hand side of (A.6) is expressible purely in terms of the phase shifts of f_λ and g_λ.

We apply (A.6) to the *half cylinders* $u \geqslant 0$ with the APS boundary conditions at $u=0$. Using (3.5) one sees at once that the wronskians vanish at this boundary. It remains to compute the contributions from the far zone ($kb \gg 1$). For $ku \gg 1$ the scattering solutions have the asymptotic form

$$f_\lambda \sim \frac{1}{\sqrt{\pi k}}\sin(ku+\delta^+), \qquad g_\lambda \sim \frac{1}{\sqrt{\pi k}}\sin(ku+\delta^-), \qquad k^2 = \lambda^2-\omega^2, \quad (A.7)$$

and we can now trivially derive the difference of the wronskians

$$W\left(\dot f_\lambda, f_\lambda^\dagger\right) - W\left(\dot g_\lambda, g_\lambda^\dagger\right) \sim \frac{1}{\pi}\left\{\dot\delta - \frac{\dot k}{k}\cos(2ku+\delta^+ +\delta^-)\sin\delta\right\}. \quad (A.8)$$

Inserting (A.8) into (A.6) and applying the Riemann-Lebesgue lemma to the rapidly oscillating term $\sim\cos(2kb+\delta^+ +\delta^-)$ we immediately find the key formula (3.24).

For the other two cases, i.e. the cylinder and the disk (both with L_2 boundary conditions) one may employ the same technique. In this or some other way (e.g. by using analycity arguments [19]) one derives the following result in the *cylinder* case: Normalizing the scattering solutions on the real line at the far left ($u\to -\infty$) as

$$f_\lambda \sim e^{-ik_- u}; \qquad k_-^2 = \lambda^2 - V(-\infty), \quad (A.9)$$

we can expand them at the far right ($u\to\infty$) as

$$f_\lambda \sim a(k)e^{-ik_+ u} + b(k)e^{ik_+ u}; \qquad k_+^2 = \lambda^2 - V(\infty), \quad (A.10)$$

where $1/a$ and b/a are the transmission- and reflection coefficients of a left-moving wave. With this normalization one can show that the right-handed scattering phase shifts defined as

$$e^{2i\delta^+} = -a(-k)/a(k), \quad (A.11)$$

together with the left-handed ones (calculated in the same way) appear in the key formula.

On the *disk* the phase shifts are defined similarly. Far from the origin ($r \to \infty$) the regular solutions have the form

$$f_\lambda \sim \frac{1}{\sqrt{r}}\left(a(k)e^{-ikr} + b(k)e^{ikr}\right), \tag{A.12}$$

where as on the half cylinder $k^2 = \lambda^2 - V(r=\infty)$. The L_2 boundary conditions (at $r=0$) are independent of k and therefore $a(-k) = b(k)$. So the right-handed phase shifts are now given by

$$e^{2i\delta^+} = -a(-k)/a(k) = -b(k)/a(k). \tag{A.13}$$

Beside the explicit derivation of the key formula given above one may employ the S-matrix theory directly to derive the same result. Indeed, in ref. [3] the spectral representation of the S-matrix of $H^\pm$ has been used to show that

$$d\mu(\lambda) = \frac{1}{\pi}\dot{\delta}(\lambda), \tag{A.14}$$

where $\mu(\lambda)$ is the difference of the trace measures of H^+ and H^-, i.e. $\mathrm{tr}(F(H^+) - F(H^-)) = \int F(\lambda)\, d\mu(\lambda)$ for sufficiently fast decaying functions F. Of course, (A.14) is equivalent to the key formula (3.24).

Appendix B

In this appendix we present a detailed calculation of the anomaly $A(m^2)$ for the disk (in the outside region) using the Green function method.

The quantity we are going to calculate is

$$G(m^2) = \mathrm{tr}\left(\frac{1}{LL^\dagger + m^2} - \frac{1}{L^\dagger L + m^2}\right) = \mathrm{tr}\left(G^+(m^2) - G^-(m^2)\right) \tag{B.1}$$

in a given ω_n sector. The Green functions

$$G^+_\omega(r,r') = \langle r|\frac{1}{LL^\dagger + m^2}|r'\rangle, \qquad G^-_\omega(r,r') = \langle r|\frac{1}{L^\dagger L + m^2}|r'\rangle$$

satisfy

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(m^2 + \frac{\omega_n^2}{r^2}\right)\right)G^\pm_\omega = -\frac{1}{r}\delta(r-r'). \tag{B.2}$$

Of course, G_{ω}^{+} is subject to the boundary conditions (4.16a), e.g. for $\Phi > 0$

$$G_{\omega}^{+}(r=R, r'>R) = G_{\omega}^{+}(r>R,\ r'=R) = 0 \qquad \text{for } \omega_n > 1\,,$$

$$\left(L_r^{\dagger} G_{\omega}^{+}\right)(r=R,\ r'>R) = \left(L_r^{\dagger} G_{\omega}^{+}\right)(r>R,\ r'=R) = 0 \qquad \text{for } \omega_n \leqslant 1\,,$$

($L_r^{\dagger} = \partial_r + \omega_n/r$) and G^{-} has to satisfy in (r, r') the conditions as in (4.16b). Let us write the solution of (B.2) as

$$G_{\omega} = G_{\omega}^{\mathrm{I}} + G_{\omega}^{\mathrm{H}}\,, \tag{B.3}$$

where G_{ω}^{I} denotes the inhomogeneous part and G_{ω}^{H} denotes the homogeneous part of the Green function.

In fact G^{I} can be written in terms of the modified Bessel functions as

$$G^{\mathrm{I}} = I_{\omega}(mr_{<})K_{\omega}(mr_{>})\,, \tag{B.4}$$

where $r_{<} = \min(r, r')$ and $r_{>} = \max(r, r')$. An easy way to see that G^{I} does produce the δ function at $r = r'$ in (B.2) is the following: first we observe that in (B.2) the δ-function can come only from $(\sqrt{r}\,G^{\mathrm{I}})''/\sqrt{r}$ since the other terms in (B.2) are continuous as $r \to r'$. Therefore it is enough to show that the first derivative of $\sqrt{r}\,G^{\mathrm{I}}$ has a jump of $-1/\sqrt{r}$ at $r = r'$. Indeed

$$\lim_{r \to r'} \left\{ \left(\sqrt{r}\, I_{\omega}(mr_{<})K_{\omega}(mr_{>})\right)'_{r>r'} - \left(\sqrt{r}\, I_{\omega}(mr_{<})K_{\omega}(mr_{>})\right)'_{r<r'} \right\}$$

$$= \frac{x}{\sqrt{r}}\left(I_{\omega}(x)K_{\omega}'(x) - I_{\omega}'(x)K_{\omega}(x)\right) = -\frac{1}{\sqrt{r}}\,, \tag{B.5}$$

using the familiar identity

$$I_{\omega}(x)K_{\omega+1}(x) + I_{\omega+1}(x)K_{\omega}(x) = \frac{1}{x}\,, \tag{B.6}$$

where $x = mr$, together with the (supersymmetric) recursion relations

$$I_{\omega}(x) = I_{\omega+1}'(x) + \frac{\omega+1}{x} I_{\omega+1}(x)\,,$$

$$I_{\omega}(x) = I_{\omega-1}'(x) - \frac{\omega-1}{x} I_{\omega-1}(x)\,,$$

$$K_{\omega}(x) = -K_{\omega+1}'(x) - \frac{\omega+1}{x} K_{\omega+1}(x)\,,$$

$$K_{\omega}(x) = -K_{\omega-1}'(x) + \frac{\omega-1}{x} K_{\omega-1}(x)\,. \tag{B.7}$$

We note that the inhomogeneous part, G_{ω}^{I}, is the same for both the G^{+} and the G^{-}.

Let us assume for simplicity that the flux $0 < \Phi < 1$ so that $\omega_0 = \langle \Phi \rangle$. The Green functions G_ω^+ and G_ω^- are finally given as

$$\begin{aligned}
&\text{for } \omega_n > 1: && G_\omega^+ = G_\omega^{\mathrm{I}} - \alpha_\omega K_\omega(mr) K_\omega(mr'),\\
&\text{for } \omega_n < 1: && G_\omega^+ = G_{-\omega}^{\mathrm{I}} + \alpha_{1-\omega} K_\omega(mr) K_\omega(mr'),\\
&\text{for } \omega_n > 0: && G_\omega^- = G_\omega^{\mathrm{I}} + \alpha_{\omega+1} K_\omega(mr) K_\omega(mr'),\\
&\text{for } \omega_n < 0: && G_\omega^- = G_{-\omega}^{\mathrm{I}} - \alpha_{-\omega} K_\omega(mr) K_\omega(mr'),
\end{aligned} \tag{B.8}$$

where $\alpha_\omega = I_\omega(mR)/K_\omega(mR)$. It is easy to check that the Green function in (B.8) do indeed satisfy the modified APS boundary conditions at $r = R$ and at $r' = R$ and are regular for $r \to \infty$ (and $r' \to \infty$). The trace we have to calculate can be written as

$$\operatorname{tr}\left(G^+(m^2) - G^-(m^2)\right) = \sum_{\omega_n} \int_R^\infty \mathrm{d}r\, r\left(G_\omega^+(r,r) - G_\omega^-(r,r)\right). \tag{B.9}$$

We split the sum over the ω_n's into 3 terms:

$$\begin{aligned}
G(m^2) = &- \sum_{\omega_n > 1} \int \mathrm{d}r\, r(\alpha_\omega + \alpha_{\omega+1}) K_\omega^2(mr) + \sum_{\omega_n \leqslant 0} \int \mathrm{d}r\, r(\alpha_{1-\omega} + \alpha_{-\omega}) K_\omega^2(mr)\\
&+ (\alpha_{\omega_0 - 1} - \alpha_{\omega_0+1}) \int \mathrm{d}r\, r K_{\omega_0}^2(mr).
\end{aligned} \tag{B.10}$$

In writing down the third term we used the defining equation for K_ω in terms of I_ω and $I_{-\omega}$. First we point out that [20]

$$\int_R^\infty \mathrm{d}r\, r K_\omega^2(mr) = \tfrac{1}{2} R^2\left(K_{\omega-1}(mR) K_{\omega+1}(mR) - K_\omega^2(mR)\right). \tag{B.11}$$

So the first sum in (B.10) becomes

$$-\frac{R}{2m} \sum_{\omega_n > 1} \left(\frac{K_{\omega-1}}{K_\omega} - \frac{K_\omega}{K_{\omega+1}} \right) = -\frac{R}{2m} \frac{K_{\omega_0}}{K_{\omega_0+1}}, \tag{B.12}$$

in other words the infinite sum telescopes to a single term. The same telescoping property holds for the second sum in (B.10), which then becomes

$$\frac{R}{2m} \frac{K_{\omega_0}}{K_{1-\omega_0}}.$$

The third term can be recast as

$$\frac{\omega_0}{m^2}\left(1-\frac{K_{\omega_0}^2}{K_{\omega_0-1}K_{\omega_0+1}}\right).$$

Now, since

$$\frac{K_{\omega_0}}{K_{1-\omega_0}}-\frac{K_{\omega_0}}{K_{1+\omega_0}}=\frac{2\omega_0}{mR}K_{\omega_0}^2\frac{1}{K_{\omega_0+1}K_{\omega_0-1}}$$

we finally obtain

$$G(m^2)=\frac{\omega_0}{m^2}=\frac{\langle\Phi\rangle}{m^2}. \tag{B.13}$$

So the global part of the anomaly as in (2.13) is

$$A(m^2)=\langle\Phi\rangle.$$

We conclude this appendix by commenting on the problem with identifying the total angular momentum J as boundary operator. If we would make this identification then we should first diagonalize J, i.e. evaluate

$$G_j(m^2)=\int \mathrm{d}r\, r\left(G_\omega^+(r,r)-G_{\omega-1}^-(r,r)\right) \tag{B.14}$$

in a given J-sector ($n=j\pm\frac{1}{2}$ in the right- (resp. left-) handed sector), and then sum over j

$$G(m^2)=\sum_j G_j(m^2) \tag{B.15}$$

in order to calculate $G(m^2)$ in (2.15).

Using the identities

$$\begin{aligned}\partial_x(xI_\omega K_{\omega-1})&=x(I_{\omega-1}K_{\omega-1}-I_\omega K_\omega),\\ \partial_x(xK_\omega K_{\omega-1})&=-x\left(K_{\omega-1}^2+K_\omega^2\right),\end{aligned} \tag{B.16}$$

and the explicit form of the Green functions in (B.8) one easily checks the following identities (the analog to the supersymmetric relations (3.22)):

$$\begin{aligned}&\text{for } \omega>1: && x(G_\omega^+-G_{\omega-1}^-)=\partial_x(-xI_\omega K_{\omega-1}+x\alpha_\omega K_{\omega-1}K_\omega),\\ &\text{for } \omega\leqslant 1: && x(G_\omega^+-G_{\omega-1}^-)=\partial_x(xI_{1-\omega}K_\omega-x\alpha_{1-\omega}K_{1-\omega}K_\omega),\end{aligned} \tag{B.17}$$

which allows one to recast $G_j(m^2)$ into a surface integral at infinity (at $r=R$ the surface contribution vanishes). Employing the asymptotic expansion of the Bessel functions for large arguments we end up with

$$\text{for } \omega>1: \qquad G_j(m^2) = -\frac{1}{2m^2},$$

$$\text{for } \omega\leqslant 1: \qquad G_j(m^2) = \frac{1}{2m^2}, \tag{B.18}$$

which is of course in complete agreement with (4.21). Note that the sum (B.15) is ill defined. In this way one sees explicitly that one cannot evaluate $G(m^2)$ as in (B.15).

Appendix C

In sect. 4 we have calculated the anomaly on the disk with the modified APS boundary conditions (4.15). Here we wish to demonstrate that these boundary conditions follow naturally by requiring the self-adjointness and the charge conjugation invariance of the Dirac operator. Expanding the right- and left-handed spinor components as

$$f=\sum f_n(r)\mathrm{e}^{in\theta}, \qquad g=\sum g_m(r)\mathrm{e}^{im\theta}, \tag{C.1}$$

one sees at once that (with L and $L^\dagger$ as in (4.2))

$$\begin{aligned}(L^\dagger f,g)-(f,Lg) &= R\int_{r=R} f^\dagger \mathrm{e}^{i\theta} g\,\mathrm{d}\theta \\ &= 2\pi R\sum f^\dagger_{n-1}(R)g_n(R).\end{aligned} \tag{C.2}$$

Let us assume $\Phi>0$ and the APS-like boundary conditions (domain for $L^\dagger$)

$$g_m(R)=0 \qquad \text{for } \omega_m\leqslant\alpha \tag{C.3a}$$

for the left-handed spinor. For $L^\dagger$ being the adjoint of L the coefficients $f_n(R)$ must vanish for $\omega_n>\beta$ with $\beta\geqslant\alpha+1$. Actually, according to the definition of the domain of the Hilbert space adjoint we must allow for all right-handed spinors for which the boundary term in (C.2) vanishes. In this way we find the right-handed boundary conditions

$$f_n(R)=0 \qquad \text{for } \omega_n>\alpha+1. \tag{C.3b}$$

(Indeed, the conditions (C.2) guarantee that the defect indices of both $LL^\dagger$ and $L^\dagger L$ vanish.)

To have charge conjugation invariance we must ensure that with ψ also the C-conjugated spinor

$$\psi_C = \gamma_0(\psi^*)^{\mathrm{T}}, \qquad \gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{C.4}$$

is in the domain of the Dirac operator. Since C should transform solutions which couple to eA_μ into those which couple to $-eA_\mu$, we first write down the boundary conditions for $\Phi < 0$. According to our previous considerations we demand

$$\begin{aligned} f_n(R) &= 0 \qquad \text{for } \omega_n \geqslant \tilde{\alpha}, \\ g_m(R) &= 0 \qquad \text{for } \omega_m < \tilde{\alpha} - 1, \end{aligned} \tag{C.5}$$

in this case.

Assume now $\Phi > 0$ and that (f, g) fulfill the boundary conditions (C.3). One can easily check that the C-conjugated spinor $\Psi_C = (f_C, g_C)$, where

$$f_C = i \sum g_m^\dagger(r) \mathrm{e}^{-im\theta}, \qquad g_C = -i \sum f_n^\dagger(r) \mathrm{e}^{-in\theta} \tag{C.6}$$

obeys the boundary conditions (C.5) if $-\omega_n \geqslant \tilde{\alpha}$ and $-\omega_n < \tilde{\alpha} - 1$. These conditions are equivalent to (C.2) (and thus ensure C-invariance) only if $\alpha + \tilde{\alpha} = 0$. This constraint allows for two choices of $(\alpha, \tilde{\alpha})$ such that the "mass gaps" $[\alpha, \alpha+1]$ and $[\tilde{\alpha} - 1, \tilde{\alpha}]$ in the spectrum of the boundary operator are symmetrically situated. For $\alpha = -\frac{1}{2}$, $\tilde{\alpha} = \frac{1}{2}$ the gaps coincide and one recovers the APS boundary conditions with boundary operator J. As explained in sect. 4 (and also since $Z(\beta) \to \text{const} \neq 0$ when $\Phi \to 0$ in this case) these boundary conditions are undesirable and we take the only other choice $\alpha = \tilde{\alpha} = 0$ which guarantees self-adjointness and C-invariance. In this way one obtains the modified APS boundary conditions (4.15).

References

[1] M.F. Atiyah, V.K. Patodi and I.M. Singer, Math. Proc. Camb. Phil. Soc. 77 (1975) 43

[2] C. Callias, Comm. Math. Phys. 62 (1978) 213;
A.J. Niemi and G.W. Semenoff, Nucl. Phys. B269 (1986) 131; Phys. Reports 135 (1986) 99

[3] R. Musto, L. O'Raifeartaigh and A. Wipf, Phys. Lett. B175 (1986) 433

[4] J. Kiskis, Phys. Rev. D15 (1977) 2329;
N.H. Christ, Phys. Rev. D21 (1980) 1591;
R. Akhoury and A. Comtet, Nucl. Phys. B246 (1984) 253;
D. Boyanovsky and R. Blankenbeckler, Phys. Rev. D31 (1985) 3234;
A.J. Niemi and G.W. Semenoff, Phys. Rev. D32 (1985) 471;
T. Jaroszewicz, Phys. Rev. D34 (1986) 3128, 3141;
D. Bolle, P. Dupont and D. Roekaerts, preprint-KUL-TF-87/1

[5] P. Forgacs, Z. Horvath and L. Palla, Phys. Rev. Lett. 46 (1981) 392;
H. Boutaleb-Joutei, A. Chakrabarti and A. Comtet, Phys. Rev. D23 (1981) 1781, D24 (1981) 3146;
A. Chakrabarti, Journal Math. Phys. 27 (1986) 340;
M.F. Atiyah, Comm. Math. Phys. 93 (1984) 437

[6] S.L. Adler, Phys. Rev. 177 (1969) 2426;
J.S. Bell and R. Jackiw, Nuovo Cim. A60 (1969) 47;
W.A. Bardeen, Phys. Rev. 184 (1969) 1848
[7] K. Fujikawa, Phys. Rev. D21 (1980) 2848
[8] F. Berezin, The method of second quantization (Academic Press, 1966)
[9] M. Hortacsu, K.D. Rothe and B. Schroer, Nucl. Phys. B171 (1980) 530
[10] K. Johnson, Acta Phys. Pol. B6 (1975) 865
[11] P. Hrasko and J. Balog, Nucl. Phys. B245 (1984) 118
[12] P. Gilkey, The index theorem and the heat equation (Publish or Perish, Boston, 1974)
[13] M. Ninomiya and C.I. Tan, Nucl. Phys. B245 (1985) 199;
H. Roemer and P.B. Schroer, Phys. Lett. B21 (1977) 182
[14] Tables of integral transform I (Bateman Manuscript Project) ed. A. Erdelyi (McGraw-Hill, New York, 1954), p. 177, eq. 9
[15] I.S. Gradshteyn and I.M. Ryzhik, Tables of integrals, series and products (Academic Press, 1980), p. 931, eq. 252.4; p. 1073
[16] M. Stone, Ann. of Phys. (NY) 155 (1984) 56
[17] Z.Q. Ma, J. Phys. A19 (1986) L317
[18] K. Gottfried, Quantum mechanics I (W.A. Benjamin, 1966)
[19] A. Wipf, Helv. Phys. Acta 58 (1985) 531
[20] Y.L. Luke, Integrals of Bessel functions (McGraw Hill, 1962) p. 255, eq. 4