Pole structure of the Hamiltonian ζ-function for a singular potential

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(December 11, 2001)

We study the pole structure of the ζ-function associated to the Hamiltonian H of a quantum mechanical particle living in the half-line $\mathbf{R}^+$, subject to the singular potential $gx^{-2} + x^2$. We show that H admits nontrivial self-adjoint extensions (SAE) in a given range of values of the parameter g. The ζ-functions of these operators present poles which depend on g and, in general, do not coincide with half an integer (they can even be irrational). The corresponding residues depend on the SAE considered.

I. INTRODUCTION

In Quantum Field Theory under external conditions, quantities like effective actions and vacuum energies, which describe the influence of external fields or boundaries on the physical system, are generically divergent and require a renormalization to get a physical meaning.

In this context, a powerful and elegant regularization scheme is based on the use of spectral functions, such as the associated ζ-function [1,2] and heat-kernel (for recent reviews see, for example, [3–6]).

It is well known [7,8] that for an elliptic boundary value problem in a ν-dimensional compact manifold with boundary, described by a differential operator A of order ω, with smooth coefficients and defined on a domain of functions subject to local boundary conditions, the ζ-function

$$\zeta_A(s) \equiv Tr\{A^{-s}\} \qquad (1)$$

has a meromorphic extension to the complex s-plane presenting isolated simple poles at $s = (\nu - j)/\omega$, with $j = 0, 1, 2, \ldots$

In the case of positive definite operators, the ζ-function is related, via Mellin transform, to the trace of the heat-kernel of the problem, $Tr\{e^{-tA}\}$. The pole structure of $\zeta_A(s)$ determines the small-t asymptotic expansion of this trace [8],

$$Tr\{e^{-tA}\} \sim \sum_{j=0}^{\infty} a_j(A)\, t^{(j-\nu)/\omega}, \qquad (2)$$

where the coefficients are related to the residues by

$$a_j(A) = \left. \text{Res} \right|_{s=(\nu-j)/\omega} \Gamma(s)\, \zeta_A(s). \qquad (3)$$

However, for the case of a differential operator with coefficients presenting singularities, less is known about the structure of the ζ-function or the heat-kernel trace asymptotic expansion.

Callias [9–11] has argued that, when the coefficient in the zero-th order term in an elliptic, (essentially) self-adjoint, second order differential operator presents a singularity like $1/x^2$, the heat-kernel trace asymptotic expansion in terms of powers $t^{(j-\nu)/2}$ (as in (2)) is ill-defined, and an expansion including $\log t$ and perhaps more general powers of t (t^α with $\alpha \neq n/2$) would be in order. In particular, considering Hamiltonians H with these characteristics, it has been given in [9,10] a small-t asymptotic expansion for the diagonal element $e^{-tH}(x,x)$ which also presents $t^{(j-\nu)/2} \log t$ terms, and where some of the coefficients are distributions with support concentrated at the singularities.

It is the aim of the present article to analyze the pole structure of the ζ-function of a Hamiltonian H describing a quantum Schrœdinger particle living in the half-line $\mathbf{R}^+$, subject to a singular potential given by $V(x) = gx^{-2} + x^2$, with $g \in \mathbf{R}$.

For certain range of values of g, this Hamiltonian (a second order differential operator) admits nontrivial self-adjoint extensions in $\mathbf{L_2}(\mathbf{R}^+)$ (each one describing a different physical system). We will show that the associated ζ-function presents isolated simple poles which depend on g, which (in general) do not lie at $s = (1 - j)/2$ for $j = 0, 1, \ldots$, and can even be irrational numbers. Moreover, we will find that the residues at these simple poles depend on the self-adjoint extension of H considered.

This pole structure for the ζ-function implies a small-t asymptotic expansion for the heat kernel trace of the problem in terms of powers which (in general) are not half an integer. Moreover, the coefficients in this expansion depend on the selected self-adjoint extension.

The structure of the paper is the following: In Section II we specify the adjoint of the Hamiltonian operator and in Section III we determine its deficiency subspaces. The Hamiltonian self-adjoint extensions are characterized in Section IV, and in Section V is described the corresponding spectrum. In Section VI we give an integral representation for the ζ-function of each SAE of the Hamiltonian and in Section VII we discuss the structure of its singularities. In Section VIII we analyze some particular cases, and we establish our conclusions in Section IX. Appendix A is devoted to the construction of the clo-

sure of H, and in Appendix B we outline the necessary asymptotic expansions.

II. THE HAMILTONIAN AND ITS ADJOINT

Let us consider the operator

$$H = -\frac{d^2}{dx^2} + V(x), \qquad (4)$$

with

$$V(x) = \frac{g}{x^2} + x^2, \qquad (5)$$

densely defined on the domain $\mathcal{D}(H) = \mathcal{C}_0^\infty(\mathbf{R}^+)$, the linear space of functions $\varphi(x)$ with continuous derivatives of all order and compact support non containing the origin. It is easily seen that H is a symmetric operator.

In order to construct the SAE [12] of H we must get its adjoint, $H^\dagger$, and determine the deficiency subspaces.

The operator $H^\dagger$ is defined on the subspace of square-integrable functions $\psi(x)$ for which $(\psi, H\varphi)$ is a continuous linear functional of $\varphi \in \mathcal{D}(H)$. This requires the existence of $\chi(x) \in \mathbf{L_2}(\mathbf{R}^+)$ such that $(\psi, H\varphi) = (\chi, \varphi), \forall \varphi \in \mathcal{D}(H)$. If this is the case, then $\chi(x)$ is uniquely defined, since $\mathcal{D}(H)$ is dense in $\mathbf{L_2}(\mathbf{R}^+)$ and, by definition, $H^\dagger \psi = \chi$.

For $\psi \in \mathcal{D}(H^\dagger)$ and $\forall \varphi \in \mathcal{D}(H)$ we have

$$(\psi, H\varphi) = \int_0^\infty \psi(x)^* (-\varphi''(x) + V(x)\varphi(x)) =$$
$$= ((-\psi'' + V(x)\psi), \varphi) = (\chi, \varphi), \qquad (6)$$

where the derivatives of ψ are taken in the sense of distributions.

Equation (6) implies that $\psi''(x) = V(x)\psi(x) - \chi(x)$, a locally integrable function. Then, its primitive $\psi'(x)$ is absolutely continuous for $x > 0$.

Therefore, the domain of $H^\dagger$ is the subspace of square integrable functions having an absolutely continuous first derivative and such that

$$H^\dagger \psi(x) = -\psi''(x) + V(x)\psi(x) \in \mathbf{L_2}(\mathbf{R}^+) \qquad (7)$$

(without requiring any boundary condition at $x = 0$).

In the next Section we will determine the deficiency subspaces of H, $\mathcal{K}_\pm = \text{Ker}(H^\dagger \mp i)$.

III. DEFICIENCY SUBSPACES OF H

To compute the deficiency indices [12] of H, $n_\pm = \dim \mathcal{K}_\pm$, we must solve the eigenvalue problem

$$H^\dagger \phi_\lambda = -\phi_\lambda''(x) + V(x)\phi_\lambda(x) = \lambda\phi_\lambda, \qquad (8)$$

for $\phi_\lambda \in \mathcal{D}(H^\dagger)$ and $\lambda \in \mathbf{C}$, with $\Im(\lambda) \neq 0$.

By means of the following Ansatz (suggested by the expected behavior of the solutions of (8) for $x \to 0^+$ and $x \to \infty$),

$$\phi = x^\alpha e^{-\frac{x^2}{2}} F(x^2), \qquad (9)$$

with

$$\alpha = 1/2 + \sqrt{g + 1/4}, \qquad (10)$$

we get from (8) the Kummer's equation for F(z):

$$zF''(z) + (b - z)F'(z) - aF(z) = 0, \qquad (11)$$

where $a = (2\alpha + 1 - \lambda)/4$ and $b = \alpha + 1/2$.

For real α, we have $g \geq -1/4$ and $\alpha \geq 1/2$. In this case it can be seen [13] that the only solution of eq. (11) leading to a square-integrable at infinity solution of eq. (8) is given by the Kummer function $F(z) = U(a; b; z)$. Then, the eigenfunctions of $H^\dagger$ are proportional to

$$\phi_\lambda(x) = x^\alpha\, e^{-\frac{x^2}{2}}\, U\left(\frac{2\alpha + 1 - \lambda}{4}; \alpha + \frac{1}{2}; x^2\right). \qquad (12)$$

We must now study the behavior of ϕ_λ near the origin, where $U(a; b; z)$ behaves as $z^{-a}(1 + O(1/z))$ [13]. We must consider two different regions for the parameter α.

For $\alpha \geq 3/2$, $\phi_\lambda \in \mathbf{L_2}(\mathbf{R}^+) \Leftrightarrow a = (2\alpha + 1 - \lambda)/4 = -n$, with $n \in \mathbf{N}$. As a consequence, if $\lambda \notin \mathbf{R}$, $\phi_\lambda \notin L_2(\mathbf{R}^+)$, and the deficiency subspaces are trivial.

This means that, for $\alpha \geq 3/2$, H is essentially self-adjoint, and its discrete spectrum is given by the condition $-a \in \mathbf{N}$, i. e.

$$\lambda_n = 4n + 2\alpha + 1, \qquad (13)$$

with $n = 0, 1, 2, \ldots$ The corresponding eigenfunctions are

$$\phi_n = x^\alpha e^{-\frac{x^2}{2}} U\left(-n; \alpha + \frac{1}{2}; x^2\right). \qquad (14)$$

On the other hand, for $1/2 \leq \alpha < 3/2$, one can see [13] that $\phi_\lambda \in L_2(\mathbf{R}^+), \forall \lambda \in \mathbf{C}$. Then, the deficiency subspaces $\mathcal{K}_\pm$ are one-dimensional, and the deficiency indices $n_\pm = 1$[1]. In this region, H admits different self-adjoint extensions.

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[1] This is in accordance to Weyl's criterion [12] according to which, for continuous $V(x)$, H is essentially self-adjoint if and only if it is in the limit point case, both at infinity and at the origin.

In addition, if $V(x) \geq M > 0$, for x large enough, then H is in the limit point case at infinity. In consequence, in the present case H is essentially self-adjoint if and only if it is in the limit point case at zero.

In particular, for positive $V(x)$ $(g \geq 0)$, if $V(x) \geq 3/4\ x^{-2}$ for x sufficiently close to zero then H is in the limit point case at the origin. On the contrary, if $V(x) \leq (3/4 - \varepsilon)\ x^{-2}$, for some $\varepsilon > 0$, then H is in the limit circle case at zero.

This confirms our results concerning the self-adjointness of H in the different regions of the parameter g.

IV. SELF-ADJOINT EXTENSIONS OF H

Since $n_+ = 1 = n_-$ for $1/2 \leq \alpha < 3/2$, there exists a one-parameter family of self-adjoint extensions of H, which are in a one-to-one relationship with the isometries from $\mathcal{K}_+$ onto $\mathcal{K}_-$ [12].

The deficiency subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ are generated by $\phi_+ \equiv \phi_{\lambda=i}$ and $\phi_- \equiv \phi_{\lambda=-i} = \phi_+^*$, respectively. Then, each isometry $\mathcal{U}_\gamma : \mathcal{K}_+ \to \mathcal{K}_-$ can be identified with the parameter $\gamma \in [0, \pi)$ defined by

$$\mathcal{U}_\gamma \phi_+ = e^{-2i\gamma} \phi_-. \tag{15}$$

The corresponding self-adjoint operator, H_γ, is defined on a dense subspace [12]

$$\mathcal{D}(H_\gamma) \subset \mathcal{D}(H^\dagger) = \mathcal{D}(\overline{H}) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{16}$$

where $\overline{H}$ is the closure of H. Functions $\phi \in \mathcal{D}(H_\gamma)$ can be written as

$$\phi = \phi_0 + A\left(\phi_+ + e^{-2i\gamma}\phi_-\right), \tag{17}$$

with $\phi_0 \in \mathcal{D}(\overline{H})$ and A a constant. Since H_γ is a restriction of $H^\dagger$, we have

$$H_\gamma \phi = H^\dagger \phi = \overline{H}\phi_0 + iA\left(\phi_+ - e^{-2i\gamma}\phi_-\right). \tag{18}$$

In the following we take $g \geq 0 \Rightarrow 1 \leq \alpha < 3/2$. As we will see, condition (17) determines the behavior of $\phi \in \mathcal{D}(H_\gamma)$ near the origin. Taking the logarithmic derivative of ϕ we get,

$$\frac{\phi'}{\phi} = \frac{e^{i\gamma}\phi_0' + 2A\,\Re\left(e^{i\gamma}\phi_+'\right)}{e^{i\gamma}\phi_0 + 2A\,\Re\left(e^{i\gamma}\phi_+\right)}. \tag{19}$$

In this expression, the terms coming from ϕ_+ give the leading contributions for small x. In fact, in Appendix A we show that $\phi_0(x) = o(x^\alpha)$ and $\phi_0'(x) = o(x^{\alpha-1})$. Then, for the right hand side of eq. (19) we get [13] (see eq. (12)),

$$\frac{\phi'(x)}{\phi(x)} = \frac{1-\alpha}{x} + (2\alpha - 1)\frac{\Gamma(\frac{1}{2}-\alpha)}{\Gamma(\alpha - \frac{1}{2})} \times$$
$$\frac{\cos(\gamma - \gamma_1)}{\cos(\gamma - \gamma_2)} \cdot x^{2\alpha-2} + o(x^{2\alpha-2}), \tag{20}$$

where we have called $\gamma_1 = \arg\{\Gamma[(-2\alpha + 3 - i)/4]\}$ and $\gamma_2 = \arg\{\Gamma[(2\alpha + 1 - i)/4]\}$.

Thus, the limit of eq. (19) for $x \to 0^+$ gives the appropriate boundary condition for the functions in the domain of the particular SAE . As we will see, this boundary condition will finally determine a discrete spectrum for H_γ.

V. THE SPECTRUM

The boundary condition specified in eq. (20) characterizes the domain of a particular SAE of the operator H, H_γ. In order to determine its spectrum, we must find the solutions of (8), ϕ_λ as given in (12) with $\lambda \in \mathbf{R}$, which satisfy this boundary condition. Their behavior near the origin is given by (see eq. (12) and [13]),

$$\frac{\phi_\lambda'(x)}{\phi_\lambda(x)} = \frac{1-\alpha}{x} + (2\alpha - 1)\frac{\Gamma(\frac{1}{2}-\alpha)}{\Gamma(\alpha - \frac{1}{2})} \times$$
$$\frac{\Gamma\left[\frac{2\alpha+1-\lambda}{4}\right]}{\Gamma\left[\frac{-2\alpha+3-\lambda}{4}\right]} \cdot x^{2\alpha-2} + o(x^{2\alpha-2}). \tag{21}$$

Comparison of eqs. (20) and (21) immediately leads us to

$$\frac{\Gamma\left(\kappa - \frac{\lambda}{4}\right)}{\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)} = \beta(\gamma, \kappa), \tag{22}$$

where we have defined the parameters

$$\kappa = \frac{2\alpha + 1}{4} = \frac{1}{4}\left(2 + \sqrt{1 + 4g}\right) \in [3/4, 1)$$
$$\beta(\gamma, \kappa) = \cos(\gamma - \gamma_1)/\cos(\gamma - \gamma_2). \tag{23}$$

Eq. (22) determines a discrete spectrum for each SAE. In Figure 1. we plot both sides of eq. (22) as a function of λ, for $\kappa = 4/5$ and $\beta = 1$. The abscissæ of the intersections of this two functions give the corresponding spectrum.

Notice that each SAE can equivalently be characterized by $\beta \in \mathbf{R} \cup \{-\infty\}$. Then, we will also use the notation $H_{(\beta)}$ to design this SAE.
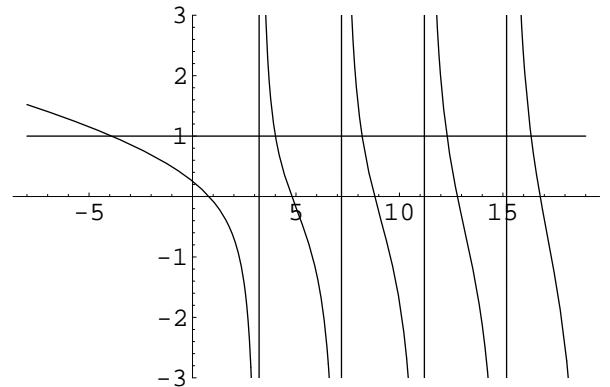

FIG. 1. $F(\lambda) \equiv \frac{\Gamma\left(\kappa - \frac{\lambda}{4}\right)}{\Gamma\left(\frac{1-\kappa-\lambda}{4}\right)}$ as a function of λ, for $\kappa = 4/5$. The solutions of $F(\lambda) = \beta$ give the spectrum of the SAE identified by β.

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The spectrum of $H_{(\beta)}$ is bounded from below, and presents a negative eigenvalue for those SAE characterized by $\beta > \Gamma(\kappa)/\Gamma(1-\kappa)$ (even though the potential $V(x) \geq 2\sqrt{g} \geq 0$). Moreover, there is no common lower bound; instead, any negative real is in the spectrum of some SAE.

For any value of g, there are two particular SAE for which the spectrum can be easily worked out (see eq. (22)):

- For $\beta = 0$ the spectrum is given by

$$\lambda_n = 4(n+1-\kappa), \tag{24}$$

with $n = 0, 1, 2, \ldots$

- For $\beta = -\infty$ the spectrum is given by

$$\lambda_n = 4(n+\kappa), \tag{25}$$

with $n = 0, 1, 2, \ldots$

For other values of β, the eigenvalues grow linearly with n,

$$4(n-1+\kappa) < \lambda_n < 4(n+\kappa). \tag{26}$$

The case with $g = 0$

It is instructive to consider the particularly simple case of the harmonic oscillator in the half-line, for which there is no singularity in the potential. Indeed, for $g = 0$ ($\alpha = 1$ or $\kappa = 3/4$), the boundary condition (eq. (20)) reads,

$$\frac{\phi'(x)}{\phi(x)} = -2\beta + O(x) \tag{27}$$

or, equivalently,

$$\lim_{x \to 0^+} \left\{ \phi'(x) + 2\beta\, \phi(x) \right\} = 0, \tag{28}$$

which corresponds to Robin boundary conditions at the origin. Dirichlet and Neumann boundary conditions are obtained for $\beta = -\infty$ and $\beta = 0$, respectively.

Let's now study the eigenfunctions and eigenvalues of the self-adjoint extensions of H corresponding to different values of β.

Dirichlet boundary conditions ($\beta = -\infty$)

Since $\kappa = 3/4$, the eigenvalues (see eq. (25)) are given by

$$\lambda_n = 4n + 3, \tag{29}$$

where $n = 0, 1, 2, \ldots$

Since the Hamiltonian (eq. (4)) corresponds in this case to a particle with mass $m = 1/2$ and frequency $\omega = 2$, the eigenvalues of this SAE can be written as $\lambda_n = \omega[(2n+1) +$

1/2], which coincides with the spectrum of the odd parity eigenvectors of the harmonic oscillator on the complete real line.

In fact, the eigenfunctions are given by (see eq. (12) and [13]),

$$\phi_n = 2^{-2n-1} e^{-\frac{x^2}{2}} H_{2n+1}(x). \tag{30}$$

Neumann boundary conditions ($\beta = 0$)

In this case, the eigenvalues (see eq. (24)) are given by

$$\lambda_n = 4n + 1, \tag{31}$$

where $n = 0, 1, 2, \ldots$. This eigenvalues can be written as $\lambda_n = \omega(2n + 1/2)$, which coincides with the even parity sector of the harmonic oscillator spectrum on the complete real line.

The eigenfunctions are now given by (see eq. (12)),

$$\phi_n = 2^{-2n} e^{-\frac{x^2}{2}} H_{2n}(x). \tag{32}$$

Robin boundary conditions ($\beta \neq 0, -\infty$)

For finite $\beta \neq 0$, the eigenfunctions are given by (eq. (12)),

$$\phi_\lambda = x e^{-\frac{x^2}{2}} U\left(\frac{3-\lambda}{4}; \frac{3}{2}; x^2\right), \tag{33}$$

and the corresponding eigenvalues are determined by the trascendental equation

$$\frac{\Gamma\left(\frac{3-\lambda}{4}\right)}{\Gamma\left(\frac{1-\lambda}{4}\right)} = \beta(\gamma, 3/4). \tag{34}$$

Notice that, for general Robin boundary condition, the ground state is negative (less than the minimum of the potential) for $\beta > \Gamma[\frac{3}{4}]/\Gamma[\frac{1}{4}]$.

VI. THE INTEGRAL REPRESENTATION FOR THE ζ-FUNCTION

The spectrum of each SAE of the operator H in (4) is determined by eq. (22), for any given $\beta \in [-\infty, \infty)$. In this section, we will study the pole structure of the associated ζ-function, defined as

$$\zeta_\beta(s) \equiv Tr\left\{H_{(\beta)}^{-s}\right\} = \sum_n \lambda_{\beta,n}^{-s}. \tag{35}$$

Notice that, since the eigenvalues grow linearly with n (see eq. (26)), $\zeta_\beta(s)$ is analytic in the half-plane $\Re(s) > 1$.

For finite β, let us define the holomorphic function,

$$f(\lambda) = \frac{1}{\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)} - \frac{\beta}{\Gamma\left(\kappa - \frac{\lambda}{4}\right)}, \qquad (36)$$

with $\frac{3}{4} \leq \kappa < 1$. The eigenvalues of the self-adjoint operator $H_{(\beta)}$ correspond to the zeroes of $f(\lambda)$ which, consequently, are all real. They are also positive, with the only possible exception of the first one, according to the discussion in the previous Section.

Moreover, the zeroes of $f(\lambda)$ are simple. To prove this, let's assume the converse is true, i. e. there is a $\lambda \in \mathbf{R}$ such that $f(\lambda) = f'(\lambda) = 0$. Taking into account that

$$f'(\lambda) = \frac{\psi\left(1 - \kappa - \frac{\lambda}{4}\right)}{4\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)} - \beta \frac{\psi\left(\kappa - \frac{\lambda}{4}\right)}{4\Gamma\left(\kappa - \frac{\lambda}{4}\right)} =$$
$$= \frac{1}{4} \left\{ \frac{\left[\psi\left(1 - \kappa - \frac{\lambda}{4}\right) - \psi\left(\kappa - \frac{\lambda}{4}\right)\right]}{\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)} + \psi\left(\kappa - \lambda/4\right) f(\lambda) \right\}, \qquad (37)$$

we see that our assumption requires that

$$\psi\left(1 - \kappa - \lambda/4\right) = \psi\left(\kappa - \lambda/4\right), \qquad (38)$$

which is not the case for any $\lambda \in \mathbf{R}$, if $\frac{3}{4} \leq \kappa < 1$.

Therefore, the ζ-function can be represented as the integral on the complex plane

$$\zeta_\beta(s) = \frac{1}{2\pi i} \oint_C \lambda^{-s} \frac{f'(\lambda)}{f(\lambda)} + \Theta(-\lambda_{0,\beta})\lambda_{0,\beta}^{-s}, \qquad (39)$$

where C is a curve which encircles the positive zeroes of $f(\lambda)$ counterclockwise. In eq. (39), $\Theta(y) = 1$ for $y > 0$ and $\Theta(y) = 0$ for $y \leq 0$.

Let us consider the dominant asymptotic behavior of the quotient

$$\frac{f'(\lambda)}{f(\lambda)} =$$
$$= \frac{\left[\psi\left(1 - \kappa - \frac{\lambda}{4}\right) - \psi\left(\kappa - \frac{\lambda}{4}\right)\right]}{4\left(1 - \beta\frac{\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)}{\Gamma\left(\kappa - \frac{\lambda}{4}\right)}\right)} + \frac{1}{4} \psi\left(\kappa - \lambda/4\right). \qquad (40)$$

For $|\arg(-\lambda)| < \pi$ and $|\lambda| \to \infty$, it is sufficient to write

$$\psi\left(\kappa - \lambda/4\right) = \log\left(-\lambda\right) + O\left(1\right), \qquad (41)$$

$$\psi\left(1 - \kappa - \lambda/4\right) - \psi\left(\kappa - \lambda/4\right) = O(\lambda^{-1}), \qquad (42)$$

$$\frac{\Gamma\left(1 - \kappa - \frac{\lambda}{4}\right)}{\Gamma\left(\kappa - \frac{\lambda}{4}\right)} = O(\lambda^{1-2\kappa}). \qquad (43)$$

Consequently, for $\Re(s) > 1$ the path of integration in (39) can be deformed to a vertical line, to get

$$\zeta_\beta(s) = \frac{-1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \lambda^{-s} \frac{f'(\lambda)}{f(\lambda)} \, d\lambda + h(s), \qquad (44)$$

where $h(s)$ (the contribution of the negative eigenvalue, if any) is a holomorphic function.

VII. POLE STRUCTURE OF THE ζ-FUNCTION

The integral in eq. (44) defines $\zeta_\beta(s)$ as an analytic function in the half-plane $\Re(s) > 1$, which can be meromorphically extended to the whole complex s-plane. It can be written as

$$\zeta(s) = -\frac{1}{2\pi i} \int_i^{i\infty} \frac{f'(\lambda)}{f(\lambda)} \lambda^{-s} \, d\lambda -$$
$$-\frac{1}{2\pi i} \int_{-i\infty}^{-i} \frac{f'(\lambda)}{f(\lambda)} \lambda^{-s} \, d\lambda + h_1(s) =$$
$$= -\frac{e^{-is\pi/2}}{2\pi} \int_1^\infty \frac{f'(i\mu)}{f(i\mu)} \mu^{-s} \, d\mu -$$
$$-\frac{e^{is\pi/2}}{2\pi} \int_1^\infty \frac{f'(-i\mu)}{f(-i\mu)} \mu^{-s} \, d\mu + h_1(s), \qquad (45)$$

where $h_1(s)$ is a holomorphic function.

In Appendix B we work out the asymptotic expansion of $f'(\lambda)/f(\lambda)$, which is given by

$$\frac{f'(\lambda)}{f(\lambda)} \sim \frac{1}{4} \log\left(-\lambda\right) + \frac{1}{4} \sum_{k=0}^\infty c_k\left(\kappa\right) \left(-\lambda\right)^{-k} +$$
$$+ \sum_{N=1}^\infty \sum_{n=0}^\infty C_{N,n}(\kappa, \beta) \left(-\lambda\right)^{-N(2\kappa-1)-2n-1}, \qquad (46)$$

where the coefficients $c_k\left(\kappa\right)$ are polynomials in κ whose explicit form is not needed for our purposes, and

$$C_{N,n}(\kappa, \beta) =$$
$$= -\left(4^{2\kappa-1}\beta\right)^N \left(2\kappa - 1 + \frac{2n}{N}\right) b_n(\kappa, N), \qquad (47)$$

with $b_n(\kappa, N)$ defined in eq. (B7) (see also eq. (B8)).

As can be seen from eq. (46), the asymptotic expansion of $f'(\lambda)/f(\lambda)$ contains the logarithmic term $\frac{1}{4} \log(-\lambda)$, and a series of non positive integer powers of λ, both coming from the ψ-function in the last term in the right hand side of (40). There is also a series of decreasing κ-dependent powers of λ, which comes from the first term in the right hand side of (40).

For the dominant logarithmic term we get from (45)

$$-\frac{1}{8\pi} \int_1^\infty \left[e^{-i\frac{\pi s}{2}} \log\left(e^{-i\frac{\pi}{2}}\mu\right) + e^{i\frac{\pi s}{2}} \log\left(e^{i\frac{\pi}{2}}\mu\right) \right] \mu^{-s} \, d\mu$$

$$= \frac{\sin\left(\frac{\pi s}{2}\right)}{8\left(s-1\right)} - \frac{\cos\left(\frac{\pi s}{2}\right)}{4\pi\left(s-1\right)^2} = \frac{1}{4}\frac{1}{\left(s-1\right)} + h_2(s), \qquad (48)$$

where $h_2(s)$ is holomorphic. The analytic extension of this term presents a unique simple pole at $s = 1$, with a residue equal to $1/4$.

The remaining terms in the asymptotic expansion of $f'(\lambda)/f(\lambda)$ are of the form $A_j(-\lambda)^{-j}$, for some $j \geq 0$ (see eq. (46)). Replacing this in eq. (45) we get

$$-\frac{A_j}{2\pi} \int_1^\infty \left[e^{-i\frac{\pi}{2}(s-j)} + e^{i\frac{\pi}{2}(s-j)} \right] \mu^{-s-j} \, d\mu =$$

$$= -\frac{A_j}{\pi} \cos\left(\frac{\pi}{2}(s-j)\right) \frac{1}{s-(1-j)} = \quad (49)$$

$$= -\frac{A_j \, \sin(\pi j)}{\pi} \frac{1}{s-(1-j)} + h_3(s),$$

where $h_3(s)$ is holomorphic.

So, from each power dependent term in the asymptotic expansion of $f'(\lambda)/f(\lambda)$, proportional to $(-\lambda)^{-j}$, we get a unique simple pole at $s = 1 - j$, with a residue given by $-(A_j/\pi) \sin(\pi j)$.

Notice that this residue vanishes for integer values of j. In particular, this is the case for all the contribution coming from the asymptotic expansion of $\psi(\kappa - \lambda/4)$ in the last term in the right hand side of eq. (40), except for the first one, the logarithmic term leading to eq. (48). In fact, this is the only singularity present in the $\beta = -\infty$ and $\beta = 0$ cases (see eq. (40)).

But in general, for $\frac{3}{4} \leq \kappa < 1$, there are also poles at non integer values of s, as follows from (46).

In conclusion, besides the pole at $s = 1$ with residue $1/4$, for each pair of integers

$$(N, n), \text{ with } N = 1, 2, 3, \ldots, \text{ and } n = 0, 1, 2, \ldots, \quad (50)$$

the ζ-function of the SAE of H characterized by the parameter β, $\zeta_\beta(s)$, has a contribution with a simple pole at the negative value

$$s = -N(2\kappa - 1) - 2n \in \left(-N - 2n, -\frac{N}{2} - 2n\right], \quad (51)$$

with a β-dependent residue given by

$$\operatorname{Res}(\cdot)|_{s=-N(2\kappa-1)-2n} =$$
$$= \frac{(-1)^N}{\pi} C_{N,n}(\kappa, \beta) \sin(2\pi N\kappa). \quad (52)$$

This is our main result, establishing the existence of κ-dependent poles of the ζ-function which, in general, are not located at a half an integer value of s. Moreover, the residues depend on the SAE considered.

Finally, notice that when κ is a rational number, there can be several (but a finite number of) pairs (N, n) contributing to the same pole. They must satisfy

$$\frac{n - n'}{N - N'} = \frac{1}{2} - \kappa = -\frac{p}{q} \in (-1/2, -1/4], \quad (53)$$

where $p, q \in \mathbf{N}$.

On the contrary, when κ is irrational the poles coming from different pairs (N, n), also irrational, are not coincident.

A. Poles and residues of $\zeta_\beta(s)$

Let us recall that the logarithmic term in the expansion (46) leads to a pole at $s = 1$ (see eq. (48)) with a residue given by

$$\operatorname{Res}(\cdot)|_{s=1} = \frac{1}{4}, \quad (54)$$

independently of the SAE considered.

The other poles can be organized in sequences characterized by the integer $N = 1, 2, \ldots$ In each sequence, successive poles differ by -2.

For example, the poles corresponding to the pairs $(N = 1, n)$, with $n = 0, 1, 2, \ldots$, are located at (see eq. (51))

$$-1 - 2n < s = 1 - 2\kappa - 2n \leq -\frac{1}{2} - 2n, \quad (55)$$

and have residues given by

$$\operatorname{Res}(\cdot)|_{s=1-2\kappa-2n} = -\frac{C_{1,n}(\kappa, \beta)}{\pi} \sin(2\pi\kappa). \quad (56)$$

Similarly, the poles arising from the $(N = 2, n \geq 0)$ terms in the asymptotic expansion (46) are located at

$$-2 - 2n < s = 2 - 4\kappa - 2n < -1 - 2n, \quad (57)$$

and have residues given by

$$\operatorname{Res}(\cdot)|_{s=2-4\kappa-2n} = \frac{C_{2,n}(\kappa, \beta)}{\pi} \sin(4\pi\kappa). \quad (58)$$

Notice that the poles in the N-th sequence have residues proportional to β^N (see eq. (47)).

Finally, let us stress that a pole of $\zeta_\beta(s)$ at a non integer $s = -N(2\kappa-1) - 2n$, as in (51), implies the presence of a term in the small-t asymptotic expansion of $\operatorname{Tr}\{e^{-t H_{(\beta)}}\}$ of the form

$$A_{[N(2\kappa-1)+2n]} \, t^{N(2\kappa-1)+2n}, \quad (59)$$

with a coefficient related to the residue by

$$A_{[N(2\kappa-1)+2n]} =$$
$$= \Gamma(-N(2\kappa - 1) - 2n) \operatorname{Res} \zeta_\beta(s)|_{s=-N(2\kappa-1)-2n}. \quad (60)$$

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B. ζ-function singularities from the asymptotic expansion of the eigenvalues

The singular behavior found for $\zeta_\beta(s)$ can be confirmed (at least for the first few poles) by determining from (22) the asymptotic expansion of the eigenvalues $\lambda_{\beta,n}$ for $n \gg 1$. Indeed, one can make the Ansatz

$$\frac{\lambda_{\beta,n}}{4} = 1 - \kappa + n + \varepsilon, \tag{61}$$

and self-consistently determine ε through successive corrections. For the first terms we get

$$\frac{\lambda_{\beta,n}}{4} \sim 1 - \kappa + n + \frac{\beta}{\pi} \sin(2\pi\kappa) n^{1-2\kappa} +$$
$$+ \frac{\beta}{\pi} \left(1 - 3\kappa + 2\kappa^2\right) \sin(2\pi\kappa) n^{-2\kappa} - \tag{62}$$
$$- \frac{\beta^2}{2\pi} \sin(4\pi\kappa) n^{2-4\kappa} + \dots,$$

where we have retained only powers of n greater than -2. This leads, for the ζ-function in eq. (35), to

$$\zeta_\beta(s) \sim 4^{-s} \zeta(s) + s\, 4^{-s} (\kappa - 1) \zeta(s+1) +$$
$$+ s\,(s+1)\, 4^{-s} \frac{(\kappa-1)^2}{2} \zeta(s+2) -$$
$$- s\, 4^{-s} \frac{\beta}{\pi} \sin(2\kappa\pi)\, \zeta(s + 2\kappa) -$$
$$- s\,(s+2\kappa)\, 4^{-s} \frac{\beta}{\pi} (\kappa-1) \sin(2\pi\kappa) \zeta(1 + s + 2\kappa) +$$
$$+ s\, 4^{-s} \frac{\beta^2}{2\pi} \sin(4\pi\kappa)\, \zeta(s - 1 + 4\kappa) + \dots, \tag{63}$$

where $\zeta(z)$ is the Riemann ζ-function, which presents a unique simple pole at $z = 1$, with a residue equal to 1.

This result shows a pole structure in agreement with the one previously described.

VIII. PARTICULAR CASES

In this Section we will show how our results reduce to the usual ones for $g = 0$ (when there is no singularity in the potential). We will also show that, for $\beta = 0$ and $\beta = -\infty$, the ζ-function presents a unique simple pole.

A. The $\beta = 0$ and $\beta = -\infty$ SAE

The ζ-function for the SAE characterized by $\beta = 0$ and $\beta = -\infty$ can be exactly evaluated, since in these cases

the spectrum was explicitly computed in eqs. (24) and (25) respectively. We get

$$\zeta_0(s) = 4^{-s} \sum_{n=0}^{\infty} (n + 1 - \kappa)^{-s} = 4^{-s} \zeta(s, 1 - \kappa),$$
$$\tag{64}$$
$$\zeta_{-\infty}(s) = 4^{-s} \sum_{n=0}^{\infty} (n + \kappa)^{-s} = 4^{-s} \zeta(s, \kappa),$$

where $\zeta(s, q)$ is the Hurwitz ζ-function, whose analytic extension presents only a simple pole at $s = 1$, with a residue $\mathrm{Res}\, \zeta(s, q)|_{s=1} = 1$. This leads, in both cases, to a unique simple pole for the ζ-function at $s = 1$, with a residue equal to $1/4$, in agreement with eq. (48).

In fact, from eqs. (52) and (47) it is evident that all the residues corresponding to negative poles vanish for $\beta = 0$. On the other hand, for $\beta = -\infty$, $f'(\lambda)/f(\lambda)$ reduces to $\frac{1}{4}\psi(\kappa - \lambda/4)$ (see eq. (40)), and the only term leading to a singularity is the logarithm in the asymptotic expansion (46), as already discussed (see eq. (48)).

B. The harmonic oscillator in the half-line

For the harmonic oscillator in the half-line ($g = 0$ or $\kappa = 3/4$) we still find a simple pole at $s = 1$, with residue $1/4$ (the only singularity for Dirchlet or Neumann boundary conditions, as previously discussed).

For finite β, the remaining singularities are located at (see eq. (51)),

$$s = -\frac{N}{2} - 2n, \; N = 1, 2, 3, \dots, \; n = 0, 1, 2, \dots, \tag{65}$$

with residues given by (see eq. (52)),

$$\mathrm{Res}\,(\cdot)|_{s = -\frac{N}{2} - 2n} =$$
$$= \frac{(-1)^N}{\pi} C_{N,n}\left(\kappa = 3/4, \beta\right) \sin\left(\frac{3\pi}{2} N\right), \tag{66}$$

which vanish for even N.

Then, each pole (except for the first one, at $s = 1$) corresponds to a negative half-integer,

$$s = -k - 1/2, \; k = 0, 1, 2, \dots. \tag{67}$$

Moreover, it is clear that for finitely many pairs (N, n) satisfying $N + 4n = 2k + 1$, the corresponding poles lie at the same point.

Therefore, the residue of $\zeta_\beta(s)$ at $s = -k - 1/2$ must be computed by adding all these contributions, characterized by $N = 2(k - 2n) + 1$, with $n = 0, 1, 2, \dots, [k/2]$. We get

$$\mathrm{Res}\,(\zeta_\beta(s))|_{s = -k - \frac{1}{2}} =$$
$$= \frac{(-1)^{k+1}}{\pi} \sum_{n=0}^{[k/2]} C_{[2(k-2n)+1], n}\left(\kappa = 3/4, \beta\right). \tag{68}$$

For example, for $k = 0$, the residue is

$$\text{Res}\,(\zeta_\beta(s))|_{s=-\frac{1}{2}} = -\frac{1}{\pi}\,C_{1,0}\,(\kappa = 3/4, \beta) = \frac{\beta}{\pi} \quad (69)$$

and, for $k = 1$,

$$\text{Res}\,(\zeta_\beta(s))|_{s=-\frac{3}{2}} = \frac{1}{\pi}\,C_{3,0}\,(\kappa = 3/4, \beta) = -\frac{4}{\pi}\,\beta^3. \quad (70)$$

IX. CONCLUSIONS

In this article we have analyzed the pole structure of the ζ-function of the Hamiltonian describing a quantum Schrœdinger particle living in the half-line $\mathbf{R}^+$, subject to the singular potential $V(x) = gx^{-2} + x^2$.

We have specified the domain of the adjoint of the Hamiltonian, $H^\dagger$, and determined the deficiency subspaces of H, initially defined on $\mathcal{C}_0^\infty(\mathbf{R}^+)$. We have shown that, for $-1/4 \leq g < 3/4$, H admits nontrivial self-adjoint extensions which depend on a continuous real parameter β.

For computational convenience, we have limited our analysis to the range $0 \leq g < 3/4$.

Once determined the closure of H (studied in Appendix A), we were able to characterize each SAE $H_{(\beta)}$ by the behavior (singular, in general) near the origin of the functions in the corresponding domain of definition. This relation also allowed for the identification of the spectrum of $H_{(\beta)}$ with the zeroes of an analytic function $f(\lambda)$.

The asymptotic expansion of $f(\lambda)$ (outlined in Appendix B) led to the determination of the poles and residues of the ζ-function associated with $H_{(\beta)}$, $\zeta_\beta(s)$.

We have shown that the poles of $\zeta_\beta(s)$ can be organized in sequences characterized by an integer $N = 1, 2, 3, \ldots$, and are located at $s = -N(2\kappa - 1) - 2n$, with $n = 0, 1, 2, \ldots$ and $\kappa = (1 + \sqrt{g + 1/4})/2 \in [3/4, 1)$. Notice that these values of s are not, in general, half an integer (which are the expected positions of the poles for a second order differential operator with smooth coefficients on a compact segment), and they are irrational numbers for irrational values of κ.

We have also found that the residues depend on the parameter β characterizing the SAE $H_{(\beta)}$.

We have confirmed this ζ-function pole structure (for the first poles) through the comparison with the results obtained from the asymptotic behavior of the eigenvalues.

These results also imply that the small-t asymptotic expansion of the heat kernel of $H_{(\beta)}$ contains powers of t which (in general) are not half an integer, and that the corresponding coefficients depend on the SAE.

Finally, several particular cases were analyzed, finding that our results are consistent with the known ones. In particular, for the harmonic oscillator ($g = 0$) in the half

line, subject to any local boundary condition at $x = 0$, the poles lie at half-integer values of s.

A final remark is in order: Notice that the unusual pole structure previously described is a consequence of having a potential with a *moderate* singular behavior near the origin. In fact, for $g \geq 3/4$, where the Hamiltonian H is essentially self-adjoint due to a stronger singular behavior of $V(x)$, the ζ-function simply reduces to $4^{-s}\,\zeta(s, \kappa)$ (see eq. (13)), which presents a unique pole at $s = 1$ with residue $1/4$.

A similar pole structure is obtained for the Hamiltonian ζ-function of charged Dirac particles living in $(2+1)$-dimensions, in the presence of both a uniform magnetic field and a singular magnetic tube with a non-integer flux. This problem was considered in [14], where it was shown that the Hamiltonian restricted to a *critical* angular momentum subspace admits nontrivial SAE, whose spectra are determined by a trascendental equation similar to (22). These results will be reported elsewhere.

ACKNOWLEDGEMENTS

APPENDIX A: CLOSURE OF H

In this Section we will justify to disregard the contributions from the functions in the domain of the closure $\overline{H}$ to the boundary condition, eq. (20). Indeed, we will show that if $\phi \in \mathcal{D}(\overline{H})$ then

$$\phi(x) = o(x^\alpha) \quad \text{and} \quad \phi'(x) = o(x^{\alpha-1}) \quad (A1)$$

near the origin, for any $\alpha < 3/2$.

In order to determine the closure of the Hamiltonian's graph we must consider those Cauchy sequences in $\mathcal{D}(H) = \mathcal{C}_0^\infty(\mathbf{R}^+)$, $\{\varphi_n\}_{n\in\mathbf{N}}$, such that $\{H\varphi_n\}_{n\in\mathbf{N}}$ are also Cauchy sequences. Notice that, since the coefficients in H are real (see eq. (4)), we can consider real functions.

Let us call $\varphi = \varphi_n - \varphi_m$, with $n, m \in \mathbf{N}$. Then $\varphi \to 0$ and $H\varphi \to 0$ as $n, m \to \infty$.

Consider first the scalar product

$$(\varphi, H\varphi) = \int_0^\infty \varphi\left(-\varphi'' + \frac{g}{x^2}\,\varphi + x^2\varphi\right)\,dx =$$

$$= \int_0^\infty \left(\varphi'^2 + \frac{g}{x^2}\,\varphi^2 + x^2\varphi^2\right)\,dx \leq \|\varphi\|\,\|H\varphi\| \to 0$$

$$(A2)$$

for $n, m \to \infty$. Therefore, for $g > 0$, we conclude that

$$\{\varphi'_n(x)\}_{n \in \mathbf{N}}, \quad \left\{\frac{\varphi_n(x)}{x}\right\}_{n \in \mathbf{N}} \quad \text{and} \quad \{x \, \varphi_n(x)\}_{n \in \mathbf{N}} \quad \text{(A3)}$$

are also Cauchy sequences.

We will now prove the following
Lemma: Let $\{\varphi_n\}_{n \in \mathbf{N}}$ be a Cauchy sequence in $\mathcal{D}(H) = \mathcal{C}_0^\infty(\mathbf{R}^+)$ such that, for $g > 0$, $1 \le a < 2$ and $g \ne (a^2 - 1)/4$,

$$\{H\varphi_n\}_{n \in \mathbf{N}}, \quad \left\{\frac{\varphi_n(x)}{x^a}\right\}_{n \in \mathbf{N}}, \quad \text{and} \quad \left\{\frac{\varphi'_n(x)}{x^{a-1}}\right\}_{n \in \mathbf{N}} \quad \text{(A4)}$$

are also Cauchy sequences. Then,

$$\left\{\frac{\varphi_n(x)}{x^{1+a/2}}\right\}_{n \in \mathbf{N}} \quad \text{and} \quad \left\{\frac{\varphi'_n(x)}{x^{a/2}}\right\}_{n \in \mathbf{N}} \quad \text{(A5)}$$

are Cauchy sequences too.
Proof: As before, let $\varphi = \varphi_n - \varphi_m$. First notice that, for $1 \le a < 2$,

$$\int_0^\infty \left(x^{1-a/2} \, \varphi(x)\right)^2 dx \le \int_0^1 (\varphi(x))^2 \, dx +$$
$$+ \int_1^\infty (x \, \varphi(x))^2 \, dx \le ||\varphi(x)||^2 + ||x \, \varphi(x)||^2. \quad \text{(A6)}$$

Then, from (A3), we see that $\left\{x^{1-a/2} \, \varphi_n(x)\right\}_{n \in \mathbf{N}}$ is also a Cauchy sequence.

A straightforward calculation shows that

$$\left(\frac{\varphi(x)}{x^a}, H\varphi(x)\right) = \int_0^\infty \left\{\left(\frac{\varphi'(x)}{x^{a/2}}\right)^2 + \right.$$
$$\left. + \left[g - \frac{a(a+1)}{2}\right]\left(\frac{\varphi(x)}{x^{1+a/2}}\right)^2 + \left(x^{1-a/2} \, \varphi(x)\right)^2\right\} dx. \quad \text{(A7)}$$

Similarly,

$$\left(\frac{\varphi'(x)}{x^{a-1}}, H\varphi(x)\right) = \int_0^\infty \left\{-\left(\frac{a-1}{2}\right)\left(\frac{\varphi'(x)}{x^{a/2}}\right)^2 + \right.$$
$$\left. g\left(\frac{a+1}{2}\right)\left(\frac{\varphi(x)}{x^{1+a/2}}\right)^2 + \left(\frac{a-3}{2}\right)\left(x^{1-a/2} \, \varphi(x)\right)^2\right\} dx. \quad \text{(A8)}$$

Now, taking into account that the sum of fundamental sequences is also a Cauchy sequence, we see that

$$\left(A \, \frac{\varphi(x)}{x^a} + B \, \frac{\varphi'(x)}{x^{a-1}}, H\varphi(x)\right) \to 0 \quad \text{(A9)}$$

when $n, m \to \infty$, for any pair of real numbers A and B.

The coefficients of $\left(\frac{\varphi'(x)}{x^{a/2}}\right)^2$ and $\left(\frac{\varphi(x)}{x^{1+a/2}}\right)^2$ in the expression of the scalar product in (A9) are

$$A - B\left(\frac{a-1}{2}\right) \quad \text{and}$$
$$A\left(g - \frac{a(a+1)}{2}\right) + B \, g\left(\frac{a+1}{2}\right) \quad \text{(A10)}$$

respectively. One can see that, by choosing one of them as zero, the other is non vanishing (except for $g = (a^2 - 1)/4$).

It is easily seen that these results prove the Lemma.

For $a = 1$, from (A3) and the Lemma, we conclude that

$$\left\{\frac{\varphi_n(x)}{x^{3/2}}\right\}_{n \in \mathbf{N}} \quad \text{and} \quad \left\{\frac{\varphi'_n(x)}{x^{1/2}}\right\}_{n \in \mathbf{N}} \quad \text{(A11)}$$

are Cauchy sequences.

Let us first suppose that g is an irrational number. Then, applying iteratively the Lemma from (A11) one can show that, for any positive integer k,

$$\left\{\frac{\varphi_n(x)}{x^{2\left[1-(1/2)^k\right]}}\right\}_{n \in \mathbf{N}} \quad \text{and} \quad \left\{\frac{\varphi'_n(x)}{x^{2\left[1-(1/2)^k\right]-1}}\right\}_{n \in \mathbf{N}} \quad \text{(A12)}$$

are Cauchy sequences.

Finally, for any given $\varepsilon > 0$ there are integers k_1 and k_2 such that $(1/2)^{k_1} \le \varepsilon \le (1/2)^{k_2}$. Taking into account that

$$\frac{1}{x^{2-\varepsilon}} \le \frac{1}{x^{2\left[1-(1/2)^{k_1}\right]}}, \quad \text{for } 0 < x \le 1,$$
$$\frac{1}{x^{2-\varepsilon}} \le \frac{1}{x^{2\left[1-(1/2)^{k_2}\right]}}, \quad \text{for } x \ge 1, \quad \text{(A13)}$$

one immediately concludes that $\left\{\frac{\varphi_n(x)}{x^{2-\varepsilon}}\right\}_{n \in \mathbf{N}}$ is a Cauchy sequence.

A similar conclusion is easily obtained for $\left\{\frac{\varphi'_n(x)}{x^{1-\varepsilon}}\right\}_{n \in \mathbf{N}}$.

Let us now suppose that g is a rational number. Then, from (A3) and (A11) it is seen that we can choose an irrational $a \in (1, 3/2)$ from which the Lemma can also be applied iteratively to arrive to the same conclusions.

In the following we will consider the behavior of the functions near the origin.
For any $\varepsilon > 0$, we can write

$$x^{-\alpha}\,\varphi(x) = \int_0^x \left(y^{-\alpha}\,\varphi(y)\right)'\,dy =$$

$$= \int_0^x y^{-\alpha+1-\varepsilon}\left\{-\alpha\,\frac{\varphi(y)}{y^{2-\varepsilon}} + \frac{\varphi'(y)}{y^{1-\varepsilon}}\right\}\,dy. \tag{A14}$$

So, for $x \leq 1$, $\alpha < 3/2$ and ε small enough, we have

$$\left|x^{-\alpha}\,\varphi(x)\right| \leq \left(\int_0^1 y^{2(-\alpha+1-\varepsilon)}dy\right)^{1/2}\left\{|\alpha|\left\|\frac{\varphi(y)}{y^{2-\varepsilon}}\right\| + \right.$$
$$\left. + \left\|\frac{\varphi'(y)}{y^{1-\varepsilon}}\right\|\right\} \to_{n,m\to\infty} 0. \tag{A15}$$

Therefore, the sequence $\{x^{-\alpha}\,\varphi_n(x)\}_{n\in\mathbf{N}}$, with $\alpha < 3/2$, is uniformly convergent in $[0,1]$, and its limit is a continuous function vanishing at the origin,

$$\lim_{n\to\infty}\left(x^{-\alpha}\,\varphi_n(x)\right) = x^{-\alpha}\,\phi(x), \tag{A16}$$

$$\lim_{x\to 0^+}\left(x^{-\alpha}\,\phi(x)\right) = 0. \tag{A17}$$

In particular, for $\alpha = 0$ we have the uniform limit

$$\lim_{n\to\infty}\varphi_n(x) = \phi(x), \tag{A18}$$

which coincides with the limit of this sequence in $\mathbf{L_2}(\mathbf{R}^+)$.

On the other hand, we can also write

$$\int_0^x y^{-\alpha+1}\,H\varphi(y)\,dy = -x^{-\alpha+1}\,\varphi'(x) +$$
$$+ \int_0^x y^{-\alpha+1-\varepsilon}\left\{(-\alpha+1)\,\frac{\varphi'(y)}{y^{1-\varepsilon}} + g\,\frac{\varphi(y)}{y^{2-\varepsilon}}\right\}\,dy + \tag{A19}$$
$$+ \int_0^x y^{-\alpha+2}\,y\,\varphi(y)\,dy.$$

Therefore, for $x \leq 1$, $\alpha < 3/2$ and ε sufficiently small, we have

$$\left|x^{-\alpha+1}\,\varphi'(x)\right| \leq \left(\int_0^1 y^{2(-\alpha+1)}\,dy\right)^{1/2}\|H\varphi(y)\| +$$
$$\left(\int_0^1 y^{2(-\alpha+1-\varepsilon)}\,dy\right)^{1/2}\left\{|\alpha-1|\left\|\frac{\varphi'(y)}{y^{1-\varepsilon}}\right\| + g\left\|\frac{\varphi(y)}{x^{2-\varepsilon}}\right\|\right\} +$$
$$+ \left(\int_0^1 y^{2(-\alpha+2)}\,dy\right)^{1/2}\|y\,\varphi(y)\| \to_{n,m\to\infty} 0. \tag{A20}$$

Consequently, the sequence $\{x^{-\alpha+1}\,\varphi'_n(x)\}_{n\in\mathbf{N}}$, with $\alpha < 3/2$, is uniformly convergent in $[0,1]$, and its limit is a continuous function vanishing at the origin,

$$\lim_{n\to\infty}\left(x^{-\alpha+1}\,\varphi'_n(x)\right) = x^{-\alpha+1}\,\chi(x), \tag{A21}$$

$$\lim_{x\to 0^+}\left(x^{-\alpha+1}\,\chi(x)\right) = 0. \tag{A22}$$

In particular, for $\alpha = 1$ we have the uniform limit

$$\lim_{n\to\infty}\varphi'_n(x) = \chi(x), \tag{A23}$$

which coincides with the limit of this sequence in $\mathbf{L_2}(\mathbf{R}^+)$ (see (A3)).

Let us now show that $\chi(x) = \phi'(x)$. Indeed, for $x \leq 1$, we have

$$\left|\phi(x) - \int_0^x \chi(y)\,dy\right| \leq$$

$$\leq |\phi(x) - \varphi_n(x)| + \left|\int_0^x \left(\chi(y) - \varphi'_n(y)\right)\,dy\right| \leq \tag{A24}$$

$$\leq |\phi(x) - \varphi_n(x)| + \|\chi - \varphi'_n\| \to_{n\to\infty} 0.$$

So, $\phi(x)$ is a differentiable function whose first derivative is $\chi(x)$.

Equations (A17) and (A22) imply that, given $\varepsilon_1 > 0$ and $\alpha < 3/2$,

$$|\phi(x)| < \varepsilon_1\,x^{\alpha} \quad \text{and} \quad |\phi'(x)| < \varepsilon_1\,x^{\alpha-1} \tag{A25}$$

if $x < \delta$, for some $\delta > 0$ small enough. This proves our assertion.

APPENDIX B: ASYMPTOTIC EXPANSIONS

In this appendix we will compute the asymptotic expansion for $f'(\lambda)/f(\lambda)$ as given in eq. (40).

The asymptotic expansion for the polygamma function appearing in the right hand side of eq. (40) can be easily obtained from Stirling's formula [13],

$$\psi\left(\kappa - \lambda/4\right) \sim \log\left(-\lambda\right) + \sum_{i=0}^{\infty} c_i(\kappa)\,(-\lambda)^{-k}, \tag{B1}$$

where the coefficients $c_i(\kappa)$ are polynomials in κ which we will not need to explicitly know for our purposes.

On the other hand, taking into account (43), we can write asymptotically for the first term in the right hand side of eq. (40)

$$\frac{\left[\psi\left(1-\kappa-\frac{\lambda}{4}\right)-\psi\left(\kappa-\frac{\lambda}{4}\right)\right]}{1-\beta\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}}\sim$$

$$\sum_{N=0}^{\infty}\beta^{N}\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]^{N}\left[\psi\left(1-\kappa-\frac{\lambda}{4}\right)-\psi\left(\kappa-\frac{\lambda}{4}\right)\right]=$$

$$=\sum_{N=0}^{\infty}\beta^{N}\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]^{N}4\frac{d}{d(-\lambda)}\log\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]=$$

$$=\sum_{N=0}^{\infty}\frac{\beta^{N}}{N}4\frac{d}{d(-\lambda)}\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]^{N}.$$

$$(B2)$$

From the Stirling's formula [13] we get

$$\log\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]\sim(1-2\kappa)\log(-\frac{\lambda}{4})+$$

$$+\left\{\sum_{m=1}^{\infty}a_{m}(\kappa)(-\lambda)^{-2m}\right\},$$

$$(B3)$$

where the coefficients in the series are given by

$$a_{m}(\kappa)=\frac{2^{4m-1}}{2m+1}\left\{\left[(1-\kappa)^{2m}-\kappa^{2m}\right]+\left(\frac{\kappa-1/2}{m}\right)\times\right.$$

$$\times\left[(1-\kappa)^{2m}+\kappa^{2m}\right]+(2m+1)\sum_{p=1}^{m}\frac{B_{2p}}{p(2p-1)}\times$$

$$\times\left(\begin{array}{c}2m-1\\2p-2\end{array}\right)\left[\kappa^{2(m-p)+1}-(1-\kappa)^{2(m-p)+1}\right]\right\}.$$

$$(B4)$$

Then,

$$\left[\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]^{N}\sim$$

$$(B5)$$

$$\sim\left(-\frac{\lambda}{4}\right)^{-N(2\kappa-1)}\sum_{n=0}^{\infty}b_{n}(\kappa,N)(-\lambda)^{-2n},$$

where

$$\sum_{n=0}^{\infty}b_{n}(\kappa,N)z^{-2n}\sim e^{N\sum_{m=1}^{\infty}a_{m}(\kappa)z^{-2m}}.\qquad(B6)$$

The coefficients $b_{n}(\kappa,N)$ are polynomials in κ and N given by

$$b_{n}(\kappa,N)=\sum_{r_{1}+2r_{2}+\ldots+nr_{n}=n}N^{r_{1}+r_{2}+\ldots+r_{n}}\times$$

$$(B7)$$

$$\times\frac{a_{1}(\kappa)^{r_{1}}a_{2}(\kappa)^{r_{2}}\ldots a_{n}(\kappa)^{r_{n}}}{r_{1}!\,r_{2}!\,\ldots\,r_{n}!},$$

where the sum extends over all sets of non negative integers $r_{1},r_{2},\ldots,r_{n}$ such that $r_{1}+2\,r_{2}+\ldots+n\,r_{n}=n$. For the first five coefficients we get

$$b_{0}(\kappa,N)=1,$$

$$b_{1}(\kappa,N)=\frac{8}{3}N\kappa\left(1-3\kappa+2\kappa^{2}\right),$$

$$b_{2}(\kappa,N)=\frac{32}{45}N\kappa\left(5N\kappa\left(1-3\kappa+2\kappa^{2}\right)^{2}+\right.$$

$$+6\left(-1+10\kappa^{2}-15\kappa^{3}+6\kappa^{4}\right))$$

$$b_{3}(\kappa,N)=\frac{256}{2835}N\kappa\left(1-3\kappa+2\kappa^{2}\right)\times$$

$$\times\left(360-18\left(-60+7N\right)\kappa+35N^{2}\kappa^{2}-\right.$$

$$-30\left(72-42N+7N^{2}\right)\kappa^{3}+$$

$$+5\left(216-378N+91N^{2}\right)\kappa^{4}-$$

$$-84N\left(-9+5N\right)\kappa^{5}+140N^{2}\kappa^{6}),$$

$$b_{4}(\kappa,N)=\frac{512}{42525}N\kappa\left(1-3\kappa+2\kappa^{2}\right)\times$$

$$\times\left(-45360+36\left(-3780+221N\right)\kappa-\right.$$

$$-252\left(60-9N+5N^{2}\right)\kappa^{2}+$$

$$+7\left(32400-8604N+540N^{2}+25N^{3}\right)\kappa^{3}-$$

$$-315\left(-240+36N-32N^{2}+5N^{3}\right)\kappa^{4}+$$

$$+21\left(-10800+9684N-2700N^{2}+275N^{3}\right)\kappa^{5}-$$

$$-63\left(-1200+3156N-1420N^{2}+175N^{3}\right)\kappa^{6}+$$

$$+6N\left(9468-10080N+1925N^{2}\right)\kappa^{7}-$$

$$-1260N^{2}\left(-12+5N\right)\kappa^{8}+1400N^{3}\kappa^{9}).$$

$$(B8)$$

Now, replacing eq. (B5) in eq. (B2) we get

$$\frac{\left[\psi\left(1-\kappa-\frac{\lambda}{4}\right)-\psi\left(\kappa-\frac{\lambda}{4}\right)\right]}{4\left[1-\beta\frac{\Gamma\left(1-\kappa-\frac{\lambda}{4}\right)}{\Gamma\left(\kappa-\frac{\lambda}{4}\right)}\right]}\sim-\sum_{N=1}^{\infty}\sum_{n=0}^{\infty}4^{N(2\kappa-1)}\times$$

$$\times\beta^{N}\left(2\kappa-1+\frac{2n}{N}\right)b_{n}(\kappa,N)(-\lambda)^{-N(2\kappa-1)-2n-1}\equiv$$

$$\equiv\sum_{N=1}^{\infty}\sum_{n=0}^{\infty}C_{N,n}(\kappa,\beta)(-\lambda)^{-N(2\kappa-1)-2n-1}.$$

$$(B9)$$

Finally, eqs. (B1) and (B9) lead to the asymptotic expansion for $f'(\lambda)/f(\lambda)$ in eq. (46).

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