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DETERMINANTS, DIRAC OPERATORS, AND ONE-LOOP PHYSICS

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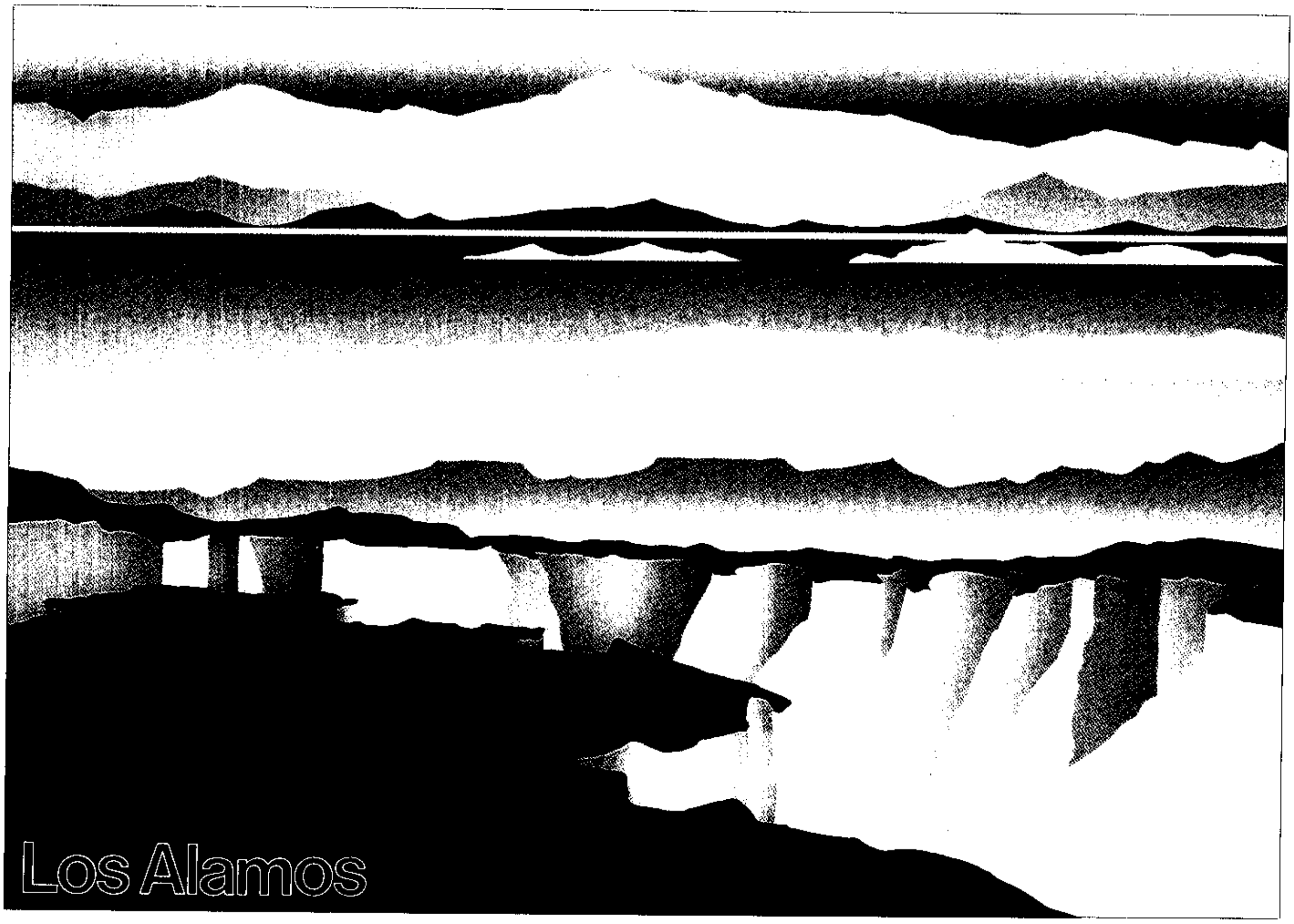

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Abstract

We consider the Dirac operator. Its determinant is examined and in two Euclidean dimensions is explicitly evaluated in terms of geometrical quantities. Our analysis is relevant to a number of interesting systems: Schwinger models and Thirring models on curved two-manifolds; string theories with world-sheet vectors; and as an exploration of possible directions in evaluating determinants in four dimensions.

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1. Introduction

The study of two dimensional physical systems has provided many important insights into four dimensional physics. In this paper we shall (among other things) investigate the behaviour of spinors coupled to gauge fields on arbitrary Euclidean two dimensional manifolds. In particular we shall be interested in the determinant of the Dirac operator. Indeed the determinant of any wave operator contains all the one-loop physics associated with that operator; it is just the exponential of the one-loop effective action. Our analysis is relevent to a number of physical models of considerable interest, including:

1. Schwinger models on curved two-manifolds.
2. Thirring models on curved two-manifolds * .
3. Superstrings with the addition of world-sheet vector fields [2].

We evaluate the determinant using the zeta function technique. In the interests of completeness a brief overview of the technique will be presented. We shall then prove a theorem that generates an ordinary differential equation for the determinant in terms of a Seeley-de Witt coefficient [3]. Integration of this equation shall be our main technique for evaluating determinants. This integration is most easily carried out in two dimensions, and we shall concentrate on this case. We also indicate how the techniques developed in this paper may be used in evaluating determinants of wave operators defined on four dimensional spacetime.

The organization of this paper is as follows. In section 2 we define the zeta function and determinant of a differential operator. In section 3 we derive a differential equation relating the determinants of a family of Dirac operators. In section 4 we consider the conformal dependence of the ungauged Dirac operator. In section 5 we add an Abelian gauge field. We find that on compact manifolds

* Note that Thirring models have been introduced into modern string theory by Bagger *et al.* [1].

with noncontractible loops there are gauge fields with vanishing field strength which nevertheless influence both the eigenvalues and the determinant. In section 6 we address the non-Abelian case. Finally we have a few words to say about Laplacians and also about four dimensions.

2. Defining the Determinant

Among the various definitions of determinant in use we find that for our purposes the most convenient is the zeta function regulated determinant [4,5,6]. (An alternative definition of determinant is discussed in appendix IV). For any two Hermitian matrices P,D we define

$$\zeta_{P,D}(s) = \mathrm{tr}'(PD^{-s}) = \sum{}' \langle n|P|n\rangle \; (\lambda_n)^{-s}. \tag{2.1}$$

Here λ_n and $|n\rangle$ are the eigenvalues and eigenvectors of D; and the primes on tr$'$ and $\sum'$ indicate that we should not include the zero eigenvalues of D. For finite matrices:

$$\det{}'(D) = \exp\left(- \frac{d}{ds}\zeta_{I,D}(s)\bigg|_{s=0}\right). \tag{2.2}$$

Here $\det'(D)$ denotes the product of the nonzero eigenvalues of D. The matrix D can be replaced by an arbitrary elliptic differential operator on a compact manifold; equation (2.2) still yields a finite quantity which one defines to be the zeta function regulated determinant. To see that this definition is sensible we proceed by rewriting the above equation by use of a Mellin transform

$$\begin{aligned}
\zeta_{P,D} &= \sum{}' \langle n|P|n\rangle (\lambda_n)^{-s} \\
&= \sum{}' \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1} \exp(-\lambda_n t) \langle n|P|n\rangle \\
&= \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1} \mathrm{tr}'(Pe^{-Dt}) \\
&= \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1} \left\{ \mathrm{tr}(Pe^{-Dt}) - \mathrm{tr}(PK) \right\}.
\end{aligned} \tag{2.3}$$

Here K is the projector onto the kernel of D, *i.e.*, $K = \sum_{i=1}^{\#} |i\rangle\langle i|$, where # is the number of zero–modes of D. Now suppose D is an elliptic second order differential operator on a compact d-dimensional manifold. A classical result due to Weyl [7] shows that asymptotically

$$\lambda_n \sim C_d \cdot \left(\frac{n}{\text{Volume}}\right)^{2/d}. \tag{2.4}$$

This is enough to show that the series converges absolutely for $Re(s) > d/2$; and thus that $\varsigma(s)$ is analytic in this region. To analytically extend $\varsigma(s)$ to other values of s we note that the heat kernel $e^{-tD}(\xi, \xi')$ on a compact Riemann manifold has an asymptotic (small t) expansion {The Minakshisundaram-Pleijel asymptotic expansion [8] } :

$$[e^{-tD}](\xi, \xi') = (4\pi t)^{-d/2} \cdot \exp[-d(\xi, \xi')^2/4t] \cdot \left\{ \sum_0^N a_n(\xi, \xi') t^n + O(t^{N+1}) \right\}. \tag{2.5}$$

Here $d(\xi, \xi')$ is the geodesic distance between ξ and ξ'. The case of main interest for our purposes is when the operator P is diagonal in position space, *i.e.*, simply a function $P(x)$. In this case it is sufficient to investigate only the diagonal part of the heat kernel

$$[e^{-tD}](\xi, \xi) = (4\pi t)^{-d/2} \left\{ \sum_0^N a_n(\xi) t^n + O(t^{N+1}) \right\}. \tag{2.6}$$

Inserting this asymptotic expansion into the Mellin transform one verifies [5, 9] that $\varsigma_{P,D}(s)$ is a meromorphic function of s posessing only simple poles. If d is even $\varsigma(s)$ has a finite number of simple poles at $s = d/2, d/2-1, \ldots, 2, 1$; with residues $(4\pi)^{-d/2} \int P(\xi)\, a_0(\xi) \sqrt{g}\, d^d\xi, \ldots, (4\pi)^{-d/2} \int P(\xi)\, a_{d/2-1}(\xi) \sqrt{g}\, d^d\xi$. Further the <u>value</u> of $\varsigma(0)$ in this case is $\int P(\xi)\{(4\pi)^{-d/2} a_{d/2}(\xi) - K(\xi)\}\sqrt{g}\, d^d\xi$. If d is odd there are an infinite number of simple poles at $s = d/2, d/2-1, \ldots, d/2-n, \ldots$, with residues $(4\pi)^{-d/2} \int P(\xi)\, a_n(\xi) \sqrt{g}\, d^d\xi$. In this case $\varsigma(s)$ also posesses an infinite number of zeros at $s = 0, -1, -2, -3, \ldots$. In either case (d even or d odd) $\varsigma(s)$ is analytic at $s = 0$, so the definition of the determinant makes sense and gives a finite value.

3. A Generalized Anomaly Equation for the Effective Action.

Consider a family $\not{D}_\tau$ of first order elliptic Hermitian (self-adjoint) differential operators that depend on a parameter τ. Suppose that

$$\not{D}_\tau = \frac{\sqrt{g_0}}{\sqrt{g_\tau}} \cdot e^{\tau f^\dagger} \cdot \not{D}_0 \cdot e^{\tau f} \tag{3.1}$$

Here $f(x)$ is a (possibly matrix valued) function. The presence of the term involving g_τ is due to the fact that Hermiticity must be defined relative to the appropriate measure * . We shall be mainly interested in the Dirac operator, as is suggested by our choice of notation. Its determinant is defined by:

$$\det(\not{D}) = \sqrt{\det(\not{D}^2)}. \tag{3.2}$$

Since the Dirac operator is self-adjoint its eigenvalues are real. The square root is thus unambiguous up to an overall sign.

Consider the zeta function associated with $\not{D}_\tau^2$:

$$\zeta_\tau(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{tr}'(e^{-t\not{D}_\tau^2}). \tag{3.3}$$

Differentiating with respect to the parameter τ, and using the cyclic property of the trace † yields

$$\begin{aligned}
\frac{d\zeta_\tau(s)}{d\tau} &= -\frac{1}{\Gamma(s)} \int t^s \mathrm{tr}' \left(e^{-t\not{D}_\tau^2} \left[\frac{d\not{D}_\tau}{d\tau} \not{D}_\tau + \not{D}_\tau \frac{d\not{D}_\tau}{d\tau} \right] \right) \\
&= -\frac{2}{\Gamma(s)} \int t^s \mathrm{tr}' \left([-\tfrac{1}{2}(d \ln g_\tau / d\tau) + f(x)^\dagger + f(x)] \cdot \not{D}_\tau^2 \cdot e^{-t\not{D}_\tau^2} \right) \\
&= \frac{2}{\Gamma(s)} \int t^s \frac{d}{dt} \mathrm{tr}' \left([F(x)]\, e^{-t\not{D}_\tau^2} \right).
\end{aligned} \tag{3.4}$$

* For instance, consider the "Laplacian" of appendix III. For δd acting on p–forms in d dimensions, the measure induced factor is $(\sqrt{g_0/g_\tau})^{(1-[2p/d])}$.

† This argument is only formal since $\not{D}$ is not trace class; however a more careful analysis leads to the same result.

Here we define $F(x)$ to be the combination $[-\frac{1}{2}(d\ln g_\tau/d\tau) + f(x)^\dagger + f(x)]$. Integrating by parts, and noting that the boundary terms vanish [for $Re(s) > d/2$, the asymptotic expansion for $e^{-t\not{D}_\tau^2}$ guarantees that the contribution from $t = 0$ vanishes; the contribution from $t = \infty$ vanishes because the potentially dangerous zero–modes do not contribute to the tr$'$] we obtain

$$\begin{aligned}\frac{d\varsigma_\tau(s)}{d\tau} &= -\frac{2s}{\Gamma(s)}\int t^{s-1}\mathrm{tr}'[F(x)\ e^{-t\not{D}_\tau^2}] \\ &= -2s\varsigma_{F,\not{D}_\tau^2}(s)\end{aligned} \tag{3.5}$$

By using the definition of the determinant together with the known analyticity properties of ς we see

$$\begin{aligned}\frac{d\ln\det'\not{D}_\tau}{d\tau} &= -\frac{1}{2}\frac{d}{d\tau}\left(\left.\frac{d\varsigma_{\not{D}_\tau^2}(s)}{ds}\right|_{s=0}\right) \\ &= +\varsigma_{F,\not{D}_\tau^2}(0) \\ &= \int\sqrt{g_\tau}\,d^dx\,F(x)\{(4\pi)^{-d/2}a_{d/2}(x;\not{D}_\tau^2) - K(x;\not{D}_\tau^2)\}\end{aligned} \tag{3.6}$$

Recall that $K(x;\not{D}_\tau^2)$ is the projection operator onto the zero–modes of $\not{D}_\tau^2$. [If f is a matrix we interpret integration to include an implicit trace]. We summarize this calculation as a

<u>Theorem:</u>

If

$$\not{D}_\tau = \frac{\sqrt{g_0}}{\sqrt{g_\tau}}\cdot e^{\tau f^\dagger}\cdot\not{D}_0\cdot e^{\tau f}$$

then

$$\frac{d\ln\det'\not{D}_\tau}{d\tau} = \int\sqrt{g}\,d^dx\,F(x)\{(4\pi)^{-d/2}a_{d/2}(x;\not{D}_\tau^2) - K(x;\not{D}_\tau^2)\} \tag{3.7}$$

where we have defined $F = [-\frac{1}{2}(d\ln g_\tau/d\tau) + f(x)^\dagger + f(x)]$.

Note that this equation for the variation of the one–loop effective action with respect to τ is a generalization of the chiral anomaly equation [10]. There one has $f = f^\dagger = \gamma_5\theta$ and $g_\tau = g_0$, corresponding to an infinetesimal chiral transformation of the Dirac operator keeping the metric fixed. When we consider conformal deformations of the gravitational background, however, the $\sqrt{g_\tau}$ term is vital. The anomaly equation (3.7) is extremely useful since the first few Seeley-de Witt coefficients are known [3, 11]. Our basic strategy will be to use these known coefficients to integrate the anomaly equation explicitly.

Before proceeding further it is useful to deal with the zero–modes. Let $\psi_{i,0}$ be an orthonormal basis for the zero–modes of $\not{D}_0$. Define the objects

$$\psi_{i,\tau} = \exp(-\tau f)\psi_{i,0}; \tag{3.8}$$

these are zero–modes * of $\not{D}_\tau$. Note in particular, that the total number of zero–modes is independent of τ . The $\psi_{i,\tau}$ are in general not orthonormal, so we define

$$\begin{aligned}(\kappa_\tau)_{ij} &= \int \sqrt{g_\tau}\,\psi^*_{i,\tau}\,\psi_{j,\tau} \\ &= \int \sqrt{g_\tau}\,\psi^*_{i,0}\,\exp(-\tau f^\dagger)\,\exp(-\tau f)\,\psi_{j,0}.\end{aligned} \tag{3.9}$$

Since

$$K(x;\not{D}^2_\tau) = \sum \psi_{i,\tau}\,[(\kappa_\tau)^{-1}]^{ij}\,\psi^*_{j,\tau} \tag{3.10}$$

$$\frac{d\kappa_{ij}}{d\tau} = -\int \sqrt{g_\tau}\,\psi^*_{i,\tau}\,[-\tfrac{1}{2}(d\ln g_\tau/d\tau) + f(x)^\dagger + f(x)]\,\psi_{j,\tau}, \tag{3.11}$$

we have

$$\int \sqrt{g_\tau}\,F(x)\,K(x;\not{D}^2_\tau) = -\frac{d}{d\tau}\left(\ln\det[(\kappa_\tau)_{ij}]\right). \tag{3.12}$$

This enables us to rewrite the anomaly equation as a

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* In the interest of generality, we point out that we could have allowed f to be τ dependent, the various factors of $\exp(-\tau f)$ should then be replaced by path-ordered integrals $P[\exp(-\int_0^\tau f)]$.

Corollory:

$$\frac{d}{d\tau}\ln\left[\det{}'\not{D}_\tau/\det(\kappa_\tau)\right] = \int \sqrt{g_\tau}\, d^d x\, F(x)\,(4\pi)^{-d/2}\, a_{d/2}(x;\not{D}_\tau^2) \qquad (3.13)$$

There is yet another form of the anomaly equation which is even simpler but contains slightly less information. Define $\det(\not{D})$ [without the prime!] in the obvious manner: if $\not{D}$ has zero–modes then $\det\not{D} = 0$; if $\not{D}$ has no zero–modes then $\det\not{D} = \det{}'\not{D}$. Then

Corollory:

$$\frac{d}{d\tau}[\det\not{D}_\tau] = (\det\not{D}_\tau)\int \sqrt{g_\tau}\, d^d x \left[F(x)\,(4\pi)^{-d/2}\, a_{d/2}(x;\not{D}_\tau^2)\right] \qquad (3.14)$$

These last two forms of the anomaly equation are considerably more pleasent to deal with since we can avoid a case by case analysis of the zero–modes. If the dimensionality is odd then $a_{(d/2)} = 0$; this reflects the fact that $\zeta_{P,D}(0) = 0$ in odd dimensions. Then the anomaly equation integrates trivially to

$$\det{}'\not{D}_\tau = \det{}'\not{D}_0 \cdot \det(\kappa_\tau); \qquad \det\not{D}_\tau = \det\not{D}_0. \qquad (3.15)$$

Finally we wish to point out that our generalized anomaly theorem holds for second order differential operators (Laplacians) as well as for first-order (Dirac) operators. Merely replace $\not{D}$ by Δ in the statement of the theorem and its corollories. [The proof is slightly different].

4. The Conformal Dependence of the Dirac Operator.

We write the Dirac operator as

$$\not{D} = i\gamma^a e_a{}^\mu(\partial_\mu + \tfrac{1}{2}\omega_{\mu ab}\Sigma^{ab}) \tag{4.1}$$

where $e^a{}_\mu$ are n-beins; $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ and $\omega_{\mu ab}$ is the spin connexion defined by $\nabla_\mu e^a{}_\nu = 0$, that is $\omega_{\mu ab} = -\Gamma_{ab\mu} + e_b{}^\nu \partial_\mu e_{\nu a}$.

Now consider a conformal deformation: $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$, $e_\mu{}^a = e^\sigma \hat{e}_\mu{}^a$, and define $\hat{\partial}_a \equiv \hat{e}_a{}^\mu \partial_\mu$. Then

$$\omega_{\mu ab} = \hat{\omega}_{\mu ab} + \hat{\partial}_a\sigma\, \hat{e}_{b\mu} - \hat{\partial}_b\sigma\, \hat{e}_{a\mu} \tag{4.2}$$

$$\begin{aligned} \not{D} &= e^{-\sigma}(\hat{\not{D}} + \tfrac{1}{2}(d-1)\gamma^a\hat{\partial}_a\sigma) \\ &= \exp(-[(d+1)/2]\sigma) \cdot \hat{\not{D}} \cdot \exp([(d-1)/2]\sigma) \end{aligned} \tag{4.3}$$

We now apply our generalized anomaly equation to the objects

$$\not{D}_\tau = \exp(-[(d+1)/2]\tau\sigma) \cdot \not{D}_0 \cdot \exp([(d-1)/2]\tau\sigma); \tag{4.4}$$

this family of Dirac operators satisfies the hypotheses of our generalized anomaly equation with $(dg/d\tau) = 2g \cdot \sigma \cdot d$, $f = f^\dagger = [(d-1)/2]\sigma$, $F = -\sigma$. We deduce

$$\frac{d}{d\tau} \ln\left[\det{}'\not{D}_\tau / \det(\kappa_\tau)\right] = -(4\pi)^{-d/2} \int \sqrt{g_\tau}\; d^dx\; \sigma(x)\; a_{d/2}(x; \not{D}_\tau^2) \tag{4.5}$$

which may be integrated to obtain the effective action (note: $S_{\text{eff}} = \ln\det{}'\not{D}$; $\det(\kappa_0) = 1$):

$$\begin{aligned} S_{\text{eff}} &= \hat{S}_{\text{eff}} + \ln\det(\kappa) \\ &\quad - (4\pi)^{-d/2} \int \sqrt{\hat{g}}\, d^dx\, \sigma(x) \left\{ \int_0^1 d\tau\, e^{\tau\sigma d} a_{d/2}(x; \not{D}_\tau^2) \right\} \end{aligned} \tag{4.6}$$

Now, any manifold is conformal to a manifold of constant scalar curvature * [12]. Thus, we can without loss of generality choose $\hat{g}$ in such a way that

* This is the well known "Yamabe problem".

$\hat{R} \in \{+1, 0, -1\}$. Our analysis then relates the computation of the Dirac determinant on an arbitrary manifold to the presumably simpler case of constant scalar curvature. In order to apply equation (4.6) we must compute the Seeley-de Witt coefficients $a_{d/2}$. This is particularly simple in two dimensions or if the number of dimensions is odd.

Dimensional analysis is sufficient to show that $a_1(\not{D}^2) = kR \cdot I$; where k is a constant. To calculate k note that (in any number of dimensions) $\not{D}^2 = -(D_\mu D^\mu) + \frac{1}{4}R \cdot I$; then $a_1(\not{D}^2) = a_1(-D^2) - \frac{1}{4}R \cdot I$. Since $(-D^2)$ acting on spinors is in two dimensions locally related by a similarity transformation to Δ acting on scalars, $a_1(-D^2) = a_1(\Delta) \cdot I = \frac{1}{6}R \cdot I$; combining these results yields

$$a_1(\not{D}^2) = -\tfrac{1}{12}R \cdot I. \tag{4.7}$$

This result for a_1 is valid in any number of dimensions [11]. Performing the implicit trace, and using $R = e^{-2\sigma}[\hat{R} + 2\hat{\Delta}\sigma]$ gives

$$\int_0^1 d\tau e^{2\sigma\tau} a_1(\not{D}_\tau^2) = \int_0^1 d\tau e^{2\sigma\tau} \cdot (-\tfrac{1}{6}) \cdot \left(e^{-2\sigma\tau}[\hat{R} + \tau 2\hat{\Delta}\sigma]\right) = -\tfrac{1}{6}(\hat{R} + \hat{\Delta}\sigma) \tag{4.8}$$

and

$$\det{}'\not{D} = \det{}'\hat{\not{D}} \cdot \det(\kappa) \cdot \exp\left(\frac{1}{24\pi}\int \sqrt{\hat{g}}\,\sigma\,[\hat{R} + \hat{\Delta}\sigma]\right). \tag{4.9}$$

The basic structure of this result should be immediately recognisable to string theorists.

We have already observed that for any odd number of dimensions $a_{d/2} = 0$. As a consequence the Dirac determinant is a conformal invariant in any odd number of dimensions.

$$\det{}'\not{D} = \det{}'\hat{\not{D}} \cdot \det(\kappa); \qquad \det\not{D} = \det\hat{\not{D}}. \tag{4.10}$$

Finally we point out that the computation for $d = 4$ is in principle straightforward; "all" one needs to do is to compute $a_2(\not{D}^2)$ and integrate. This computation will be exhibited in a subsequent publication [13].

5. Abelian gauge fields.

We add an Abelian gauge field to the Dirac operator and consider

$$\nabla\!\!\!/ = i\gamma^a e_a{}^\mu(\partial_\mu + \tfrac{1}{2}\omega_{\mu ab}\Sigma^{ab} - iA_\mu) = D\!\!\!\!/ + A\!\!\!/ \,. \tag{5.1}$$

The first thing to notice is that the presence of A_μ does not affect the previous analysis of conformal deformations. Consequently we need only concern ourselves with spaces of constant scalar curvature. To proceed we specialise to two dimensions. We would like to use the Hodge decomposition theorem. Unfortunately, the Hodge decomposition applies only to one–forms defined globally on the manifold. Now, the vector potential corresponding to a magnetic monopole is not a globally defined one–form. In fact, the monopole number (equivalently the first Chern class) is, in two dimensions, an obstruction to the global defintion of the vector potential. We can, however, always decompose $A = \hat{A}^n + A^0$, where $\hat{A}^n$ is a "reference monopole field" and A^0 is globally well defined. We may now apply the Hodge decompostion to A^0, while choosing the "reference monopole field" to be one of constant field strength: $\hat{F} = \omega \int F / \int \omega = n\omega/S$. Here n is the monopole number, ω is the volume two–form, while S is the area of the manifold. We may now write

$$A_\mu = (\hat{A}^n)_\mu + \epsilon_\mu{}^\nu \partial_\nu \chi + h_\mu + \partial_\mu \Theta. \tag{5.2}$$

Here χ,Θ are globally defined scalars while h_μ is a harmonic one–form. The Θ piece of A may be eliminated by a gauge transformation without affecting the eigenvalues or determinant of $\nabla\!\!\!/$. To handle the χ piece we note $[(\gamma_5)^2 = +1]$:

$$\gamma^a \epsilon_a{}^b = i\gamma^b\gamma_5 \tag{5.3}$$

$$\gamma^a\, e_a{}^\mu\, \epsilon_\mu{}^\nu\, \partial_\nu \chi = i\gamma^b\, \gamma_5\, e_b{}^\nu\, \partial_\nu \chi. \tag{5.4}$$

We now apply our generalized anomaly equation to the family

$$\not{\nabla}_\tau = e^{\tau\gamma_5\chi}\,(\not{D} + \hat{\not{A}}^n + \not{h})\;e^{\tau\gamma_5\chi} \tag{5.5}$$

using $g_\tau = g_0$; $f = f^\dagger = \gamma_5\chi$; $F = 2\gamma_5\chi$ to obtain

$$\frac{d}{d\tau}\ln\left[\det{}'\not{\nabla}_\tau/\det(\kappa_\tau)\right] = \frac{1}{2\pi}\int\sqrt{g_\tau}\,d^2x\,\gamma_5\,\chi\,a_1(x;\not{\nabla}_\tau^2). \tag{5.6}$$

In any number of dimensions, $\not{\nabla}^2 = -(\nabla_\mu\nabla^\mu) + \frac{1}{4}R\cdot I + i\Sigma^{\mu\nu}F_{\mu\nu}$; consequently $a_1(\not{\nabla}^2) = a_1(-\nabla^2) - \frac{1}{4}R\cdot I - i\Sigma^{\mu\nu}F_{\mu\nu}$; using the special two dimensional relation $\Sigma_{ab} = i\frac{1}{2}\gamma_5\epsilon_{ab}$ we find

$$a_1(\not{D}^2) = -\tfrac{1}{12}R\cdot I + \tfrac{1}{2}\gamma_5(\epsilon^{\mu\nu}F_{\mu\nu}) \tag{5.7}$$

Performing the implicit spinorial trace and using $F_\tau = \hat{F} + \tau d * d\chi = \hat{F} + \tau * \Delta\chi$; $*F_\tau \equiv \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}(\tau) = *\hat{F} + \tau\Delta\chi$; we obtain

$$\det{}'\not{\nabla} = \det{}'(\not{D} + \hat{\not{A}}^n + \not{h})\cdot\det(\kappa)\cdot\exp\left(\frac{1}{2\pi}\int\chi[\Delta\chi + 2 * \hat{F}]\right). \tag{5.8}$$

This equation is a generalization of the flat space Schwinger model result [14].

Considerable information regarding the determinant $\det{}'(\not{D} + \hat{\not{A}}^n + \not{h})$ can be computed analytically. This determinant depends on the magnetic monopole number n, the harmonic form h, and the spin structure of the manifold. To describe the spin structure of a genus g Riemann surface we first construct a canonical homology basis. That is, we find a set of $2g$ curves $a^i, b^i; 1 \le i \le g$ such that a^i intersects no curve except b^i, which it intersects once, and b^i intersects no curve save a^i. As a (complex) spinor is parallel transported around the curve a^i (resp. b^i) it picks up a phase $e^{2\pi i\theta^i}$ (resp. $e^{2\pi i\phi^i}$). The collection θ^i, ϕ^i is a (complex) spin structure. We shall denote by $\psi(\theta^i, \phi^i)$ a complex spinor consistent with the spin structure θ^i, ϕ^i.

The canonical homology basis allows us to associate phases to the harmonic one–form h. We define $\exp(i\int_{a^i} h) \equiv e^{2\pi i\Theta^i}$ and $\exp(i\int_{b^i} h) \equiv e^{2\pi i\Phi^i}$. These phases are important because if $\psi(\theta^i,\phi^i)$ is an eigenspinor of $\not{D}$, then $\tilde{\psi}(\theta^i + \Theta^i, \phi^i + \Phi^i) \equiv \psi(\theta^i,\phi^i)\exp(i\int h)$ is an eigenspinor of $\not{D} + \not{h}$ with the same eigenvalue. Thus if we know $\det'\not{D}$ for all spin structures we can deduce $\det'(\not{D} + \not{h})$ for all spin structures. The determinant $\det'\not{D}_+$ of the chiral Dirac operator for any spin structure has (for monopole number equal to zero) been calculated by Alvarez-Gaume *et al.* [15]. The determinant $\det'\not{D}(\theta^i,\phi^i)$ is the absolute square of their result, modulo a sign ambiguity discussed previously:

$$\det{}'\not{D}(\theta^i,\phi^i) = |\eta(\Omega)|^2 \left|\vartheta\begin{bmatrix}\theta^i\\ \phi^i\end{bmatrix}(0|\Omega)\right|^2 \tag{5.9}$$

where $\eta(\Omega)$ is a function of the modulii [15] (but is independent of the spin structure), Ω is the period matrix of the manifold, and ϑ is the Riemann theta function of the surface. For genus 1, $\eta(\Omega)$ is related to the Dedekind eta function. Adding a monople field does not change the fibre bundle arguments invoking the Riemann theta function, we find that in our notation

$$\det{}'(\not{D} + \not{A}^n + \not{h}; \theta^i, \phi^i) = |\eta(\Omega, n)|^2 \left|\vartheta\begin{bmatrix}\Theta^i + \theta^i\\ \Phi^i + \phi^i\end{bmatrix}(0|\Omega)\right|^2 \tag{5.10}$$

where η is now a function both of the period matrix (modulii) and the monopole number.

6. Non-Abelian gauge fields.

We consider now a non-Abelian gauge field so that A is a matrix. Our goal is to present a decomposition of the gauge field which will allow us to generalize the Abelian result * . Adopt complex coordinates $z = x + iy$; then the Hermitian gauge field may be written $A = A_\mu dx^\mu = \alpha + \bar{\alpha}$ where $\bar{\alpha} = \alpha^\dagger$. Now consider gauge potentials of the form

$$\alpha = ig^{-1}[(\partial - i\alpha_0)g]; \qquad \bar{\alpha} = ig^\dagger[(\bar{\partial} - i\bar{\alpha}_0)(g^\dagger)^{-1}]. \tag{6.1}$$

Here g is an arbitrary non–singular matrix. Note that g posesses a unique polar decomposition $g = e^{\chi} \cdot U$ where U is a unitary matrix and χ is a Hermitian matrix. For the Abelian case this just reduces to the ordinary Hodge decomposition (5.2) rewritten in complex coordinates, with α_0 playing the role of the monopole field plus harmonic contributions.

For non–Abelian gauge potentials of the above form we proceed as in the Abelian case. As usual, the gauge piece is irrelevant, so we consider a family of Dirac operators with $\alpha_\tau = ie^{-\tau\chi}(\partial - i\alpha_0)e^{\tau\chi}$. Then

$$\begin{aligned} \nabla\!\!\!\!/_\tau &= i\gamma_+(\partial - i\alpha_\tau) + i\gamma_-(\bar{\partial} - i\bar{\alpha}_\tau) \\ &= e^{\tau\gamma_5\chi} \cdot \nabla\!\!\!\!/_0 \cdot e^{\tau\gamma_5\chi} \end{aligned} \tag{6.2}$$

Repeating the Abelian analysis yields

$$\det{}'\nabla\!\!\!\!/ = \det{}'(\nabla\!\!\!\!/_0) \cdot \det(\kappa) \cdot \exp\left(\frac{1}{2\pi} \int_0^1 d\tau \int \sqrt{g}\,\chi\, F^\tau_{\mu\nu}\, \epsilon^{\mu\nu} \right) \tag{6.3}$$

We emphasize that the ansatz (6.1) is sufficiently general to cover a number of interesting cases. We shall see in appendix I that on any manifold diffeomorphic to the plane α is always of type (6.1) with $\alpha_0 = 0$. In appendix II we show

---

* If one attempts to use the Hodge decomposition one rapidly decides that this decomposition is "true but not useful".

that in the field–free case ($F=0$) α is always of type (6.1) with $\chi=0$ and α_0 a holomorphic one–form which may be written $\alpha_0 = c^i\omega_i$. Here ω_i a basis of holomorphic one–forms and c^i a set of commuting matrices. We may simultaneously diagonalise all the c^i in which case $\det{}'\nabla\!\!\!/_0$ becomes the product of operators such as we considered in the Abelian case.

The formula (6.3) is very similar to that derived for the Abelian case. For Abelian fields we had $F_\tau = \hat{F} + \tau(*\Delta\chi)$; so that the integration over τ was trivial. In the non-Abelian case we are not so lucky.

7. Conclusion

We have investigated the two-dimensional Euclidian Dirac operator and have developed formulae for the evaluation of its determinant (equivalently, of the one-loop effective action). Collecting eq. (4.9), (5.8), and (5.10) gives our final result

$$\begin{aligned} det'\not{D} = |\eta(\Omega,n)|^2 \cdot \left|\vartheta\left[{\Theta^i \atop \Phi^i}\right](0|\Omega)\right|^2 \cdot \det(\kappa)\cdot \\ \exp\left(\frac{1}{24\pi}\int\sqrt{\hat{g}}\sigma[\hat{R}+\hat{\Delta}\sigma]\right)\cdot \\ \exp\left(\frac{1}{2\pi}\int\sqrt{\hat{g}}\chi[2*\hat{F}+\hat{\Delta}\chi]\right). \end{aligned} \tag{7.1}$$

This formula explicitly shows how the determinant depends upon the curvature (through the conformal factor σ) and the vector potential (via χ and $\hat{F}$). The dependence upon the Riemann modulii is more subtle. The determinant of the two dimensional chiral Dirac operator is formally the square root of the result just quoted; because of the holomorphic anomaly, however, there is some delicacy in defining the square root precicely [15].

Our eventual goal is to apply some of the techniques of this paper to four dimensional spacetime. Evaluating the gauge field dependence of the effective action appears to be rather difficult though we have some hope of being able to deal with the (anti)self-dual case. On the other hand, the effect of a conformal deformation of a background metric seems computable — see the discussion of section 4.

An important open problem is to generalise our 'anomaly equation' to the (not conformally invariant) case of massive Dirac operators.

Appendix I.

Non-Abelian gauge fields on the plane.

In this appendix we shall demonstrate that in flat space any gauge field admits the decomposition (6.1), even if the field strength is non-vanishing. Introduce complex coordinates $z = x + iy$ and write the gauge field $A = \alpha + \overline{\alpha}$ with $\alpha = \alpha(z, \overline{z})dz$ and $\overline{\alpha} = \alpha^\dagger(z, \overline{z})d\overline{z}$. The function α may be expanded as $\alpha(z, \overline{z}) = \sum_{n,m} \alpha_{n,m} z^n \overline{z}^m$. We associate to α a function of two complex variables $\beta(z, \overline{w}) \equiv \sum_{n,m} \alpha_{n,m} z^n \overline{w}^m$. Note that β is holomorphic in z and antiholomorphic in w and that $\beta(z, \overline{w} = \overline{z}) = \alpha(z, \overline{z})$. Viewed as a function of z (parametrised by $\overline{w}$) the form $\beta \equiv \beta(z, \overline{w})dz$ is field free, unlike α, and

$$g_{p_0}(p, \overline{w}) \equiv P \exp[-i \int_{p_0}^{p} \beta(\overline{w})] \tag{I.1}$$

is a well defined group element, antiholomorphic in w. Moreover,

$$\alpha = i(g^{-1}\partial_z g)\Big|_{\overline{w}=\overline{z}} \tag{I.2}$$

independent of p_0. Since β is a complex field, g is not a unitary matrix. However, g uniquely decomposes to the form $g = e^\chi \cdot U$ with χ Hermitian and U unitary. Inserting this form for g into the above expression for α gives equation (6.1), with $\alpha_0 = 0$. Note that χ vanishes at p_0 while U is the identity there. Now consider a small gauge transformation (which, via a global gauge transformation can be made the identity at a fixed p_0) and a Hermitian matrix χ, in the Lie algebra g of the gauge group which, vanishes at p_0. The object A defined through eq. (I.2) is a gauge field, *viz.*, a well defined g valued function. So one need never suffer the calculation of χ given A, one may define the A in terms of χ. Up to a gauge transformation all gauge fields A are obtained if all Hermetian χ which vanish at p_0 are considered.

When $\alpha_0 = 0$ we may interpret the integral of our generalised anomaly equation, (6.3) in terms of a Wess-Zumino effective action. To find a functional the

τ-derivative of which is $\int \sqrt{g}\chi F$ we observe that the field strength can be written as

$$F = ig^{\dagger}\bar{\partial}(J^{-1}\partial J)g^{\dagger -1} \tag{I.3}$$

where we have introduced the gauge invariant object $J = gg^{\dagger} = \exp(2\tau\chi)$. By using the identities $d = \partial + \bar{\partial}$ and ${}^*d = -i\partial + i\bar{\partial}$ (with the coordinate z determined by the natural complex structure $J_a{}^b = \frac{1}{\sqrt{2}}\epsilon_a{}^b$) one verifies that the natural generalization of the Abelian effective action

$$W_0 = -\frac{1}{16\pi}\int \text{tr}\ (dJ \wedge^* dJ^{-1}) \tag{I.4}$$

has the τ-derivative

$$\frac{dW_0}{d\tau} = \frac{i}{4\pi}\int \text{tr}\ (\chi\bar{\partial}(J^{-1}\partial J)) - \frac{i}{4\pi}\int \text{tr}\ (\chi\partial(J^{-1}\bar{\partial}J)). \tag{I.5}$$

The first term on the right hand is proportional to the variation of the functional we seek. In the Abelian case the two terms are identical and W_0 is (essentially) the effective action. In the non-Abelian case, however, the two terms are different and we must correct for the last one. As pointed out by Polyakov and Wiegmann [16] this can be achieved by adding the so-called Wess-Zumino term [17] to W_0. Let us introduce a 3-manifold $\mathcal{B}$, the boundary of which is $\mathcal{M}$, and extend χ in an arbitrary smooth fashion through $\mathcal{B}$. Denoting exterior differentiation on $\mathcal{B}$ by d_3, the τ derivative of

$$W_{WZ} = -\frac{i}{24\pi}\int_{\mathcal{B}} \text{tr}\ (J^{-1}d_3J)^3 \tag{I.6}$$

becomes

$$\begin{aligned}\frac{dW_{WZ}}{d\tau} &= \frac{i}{4\pi}\int_{\mathcal{B}} \text{tr}(d_3(\chi\, dJ\, dJ^{-1})) \\ &= \frac{i}{4\pi}\int_{\mathcal{M}} \text{tr}(\chi\, \partial(J^{-1}\bar{\partial}J)) + \frac{i}{4\pi}\int_{\mathcal{M}} \text{tr}(\chi\bar{\partial}(J^{-1}\partial J)).\end{aligned} \tag{I.7}$$

Now, adding W_0 and W_{WZ} the undesired term in (I.5) cancels and

$$\frac{d}{d\tau}(W_0 + W_{WZ}) = \frac{1}{2\pi}\int \mathrm{tr}(\chi F) \tag{I.8}$$

as required. From (6.3) and (I.7) we conclude that the effective action for massless fermions on a 2-manifold interacting with a non-Abelian background reads

$$S_{\mathrm{eff}} = S^0_{\mathrm{eff}} + \ln\det(\kappa) + W_0 + W_{WZ}. \tag{I.9}$$

where S^0_{eff} contains only the harmonic part of the gauge potential. By using (4.9) we may furthermore integrate out the conformal part of the gravitational field. In particular, on a conformally flat manifold with vanishing first cohomology-group we have

$$S_{\mathrm{eff}} = \ln\det(\kappa) + \frac{1}{24\pi}\int \sigma\triangle\sigma + W_0 + W_{WZ}. \tag{I.10}$$

Here κ contains both the gravitational and gauge zero–mode contributions.

Finally we observe that if one considers Hermetian gauge fields on the punctured plane, the auxilliary form β remains well-defined but the group element (I.1) does not; homologically distinct paths connecting p_0 with p can give different values for $g_{p_0}(p)$ reflecting possible harmonic contributions to A. We discuss how to treat these harmonic contributions in appendix II.

Appendix II.

Non-Abelian gauge fields for $F = 0$.

Let A be a gauge field with vanishing field strength defined on a two dimensional manifold $\mathcal{M}$. Even though A has vanishing field strength it may not be pure gauge: a genus g manifold admits $2g$ harmonic one–forms which can contribute to A and which cannot be gauged away. Thus we expect A to have the form

$$A = U^{-1}(d + M^i h_i)U \tag{II.1}$$

with h_i a basis of harmonic one–forms and U a unitary matrix. We shall choose to have U represent a 'small' gauge transformation, that is, one deformable to the identity. Even if U is allowed to be 'large', however, A will still generally have harmonic contributions. The main result of this appendix is that the coefficient matrices, M^i, may be chosen to commute.

One manifestation of the harmonic one–forms on $\mathcal{M}$ is that the group element

$$g_{p_0}(p) = P\exp\left[-i\int_{p_0}^{p} A\right] \tag{II.2}$$

(P is the path ordering symbol) is not well defined; homologically distinct paths connecting p_0 with p give different values for $g_{p_0}(p)$ and reflect the harmonic contribution to A. In order to sort out this harmonic contribution we associate a well defined group element $g(\gamma)$ to any closed curve γ by $g(\gamma) \equiv P\exp[-i\int_\gamma A] \equiv \exp[-iM(\gamma)]$. Note that $M(\gamma)$ is not uniquely determined (even though $g(\gamma)$ is), an issue which will concern us (and which we shall resolve) shortly. The path ordering ensures that the matrices $g(\gamma)$ satisfy

$$g(\gamma^i)g(\gamma^j) = g(\gamma^j \circ \gamma^i) \tag{II.3}$$

for any two curves γ^i and γ^j. This relation is useful because any closed curve is homologous to one generated by elements γ^i, of the first homology group $H_1(\mathcal{M})$.

So if we know the matrices $g(\gamma^i)$ we can readily compute $g(\gamma)$ for any closed curve γ. The homology group is Abelian which implies that the matrices $g(\gamma^i)$ commute. The $M(\gamma^i)$ also commute, their ambiguous nature notwithstanding. Let h_i be a basis of real harmonic one–forms dual to the homology basis γ^i: $\int_{\gamma^i} h_j = \delta^i_j$. For any choice of the $M(\gamma^i)$ the object $\tilde{g}_{p_0}(p) = \exp[-i\int_{p_0}^p \sum_i M(\gamma^i)h_i]$ depends on the homology class of the path connecting p_0 to p, but $U_{p_0} \equiv \tilde{g}_{p_0}^{-1} g_{p_0}$ is well defined. We have not needed to include a path ordering symbol in the definition of $\tilde{g}_{p_0}$ because the $M(\gamma^i)$ commute.

From equation (II.2) one sees that

$$A = ig^{-1}dg = iU^{-1}\tilde{g}^{-1}d(\tilde{g}U) = iU^{-1}(d + M(\gamma^i)h_i)U. \tag{II.4}$$

Both the $M(\gamma^i)$ and the U in this decomposition of A are ambiguous. We would like to give a prescription for resolving this ambiguity. To this end, let $\hat{M}(\gamma^i)$ and $M(\gamma^i)$ be two sets of matrices such that $g(\gamma^i) = \exp[-i\hat{M}(\gamma^i)] = \exp[-iM(\gamma^i)]$. These sets lead to matrices $\hat{U}$ or U in the decomposition of A. We shall oblige the $\hat{M}(\gamma^i)$ (resp. $M(\gamma^i)$) to be elements of the Lie algebra $\mathbf{g}$ of the gauge group so that $\hat{U}$ (resp. U) are unitary matrices, *i.e.*, represent gauge transformations. In general, they represent large gauge transformations. The $\hat{M}(\gamma^i)$ and $M(\gamma^i)$ are all mutually commuting. Thus they can all be diagonalised simultaneously. In this diagonal basis

$$M(\gamma^i) = \hat{M}(\gamma^i) - 2\pi n^i{}_j\lambda^j; \qquad U = \exp(-2\pi i n^i{}_j\lambda^j \int_{p_0}^p h_i) \cdot \hat{U}. \tag{II.5}$$

for integers $n^i{}_j$ and a basis λ^j of diagonal matrices in $\mathbf{g}$. Our prescription for resolving the ambiguities in eq. (II.4) is to choose the $n^i{}_j$ so that U is a small gauge transformation. To see that it is always possible to do this, choose any $\hat{M}(\gamma^i)$ and imagine decomposing each member of the one-parameter family of gauge fields $A_s = sA$ $(0 \le s \le 1)$. Choose $\hat{M}_{s=1}(\gamma^i) \equiv \hat{M}(\gamma^i)$ and require the $\hat{M}_s(\gamma^i)$ to vary smoothly with s. For $s = 0$, and in a diagonal basis (the diagonal bases also vary smoothly with s), $\hat{M}_{s=0}(\gamma^i) = 2\pi n^i_{\ j}\lambda^j$. These $n^i_{\ j}$ are the integers we need. That is to say, for $M(\gamma^i) = \hat{M}(\gamma^i) - 2\pi n^i_{\ j}\lambda^j$, $U_{s=0} = 1$.

Thus we have confirmed that, given a base point p_0, any gauge field with vanishing field strength may be uniquely decomposed:

$$A = U^{-1}(d + M^i h_i)U \tag{II.6}$$

where U is a small gauge transformation which is the identity at p_0, and $M^i \equiv M(\gamma^i)$ are a set of mutually commuting matrices in the Lie algebra of the gauge group. Conversely, given a small gauge transformation (which, via a global gauge transformation can be made the identity at a fixed p_0) and a collection of mutually commuting matrices M^i, the object A defined by eq. (II.6)is a gauge field, *viz.*, a well defined **g** valued function. So one need never suffer the calculation of the M^i given A, one may define the A in terms of the M^i. Up to a (small) gauge transformation all gauge fields A are obtained if all sets of mutually commuting matrices of the form $M^i = n^i_{\ j}\lambda^j$ are considered.

Appendix III.

Laplacians on differential forms.

A natural defintion of the kinetic energy for a p-form is

$$
\begin{aligned}
L &= \frac{1}{2p!}\int_{\mathcal{M}} \partial_{[\mu}\omega_{\nu_1\ldots\nu_p]}\partial_{[\mu'}\omega_{\nu_1'\ldots\nu_p']} g^{\mu\mu'} g^{\nu_1\nu_1'}\ldots g^{\nu_p\nu_p'} \\
&= \frac{1}{2p!}\int_{\mathcal{M}} (*d\omega \wedge d\omega) = \frac{1}{2p!}(d\omega, d\omega)
\end{aligned}
\tag{III.1}
$$

after an integration by parts $L = \frac{1}{2p!}(\omega, \delta d\omega)$. We may thus define a Laplacian acting on forms by $\Delta = \delta d$. This is not the usual Hodge-de Rham Laplacian $\Delta_{H-dR} = \delta d + d\delta$. The Laplacian (δd) is more interesting physically in that it implies a gauge invariance under $\omega \mapsto \omega + d\theta$. In particular acting on one–forms (δd) yields the ordinary kinetic energy for Abelian gauge fields. Consider the effect of a conformal deformation $g = e^{2\sigma}\hat{g}$; one easily sees that this influences the Hodge duality operator: $* = e^{[\sigma(d-2p)]}\hat{*}$. Consequently

$$
\Delta = e^{\sigma(2p-d)}\hat{\delta} e^{\sigma(d-2p-2)} d \tag{III.2}
$$

Now choose $p = (d/2) - 1$ [representing zero forms in two dimensions $\mapsto$ scalars in string theory; one forms in four dimensions $\mapsto$ Electromagnetism; two forms in six dimensions $\mapsto$ torsion on Calabi-Yau manifolds]. Then $\Delta = e^{(-2\sigma)}\hat{\Delta}$. Applying our generalized anomaly equation to this Laplacian on $(d/2) - 1$ forms yields:

$$
\frac{d}{d\tau}\ln\left[\det{}'\Delta_\tau/\det(\kappa_\tau)\right] = (4\pi^{-d/2})\int \sqrt{g_\tau}\, d^d x\, \sigma(x) a_{d/2}(x;\Delta_\tau) \tag{III.3}
$$

Thus many of our comments concerning the Dirac operator apply equally well to the operator (δd) acting on $(d/2) - 1$ forms.

Appendix IV.

An Alternative Definition for the Determinant

There are, unfortunately, a number of different definitions for the determinant of a differential operator in common usage. A rather popular variant is the "proper time regularization". See e.g. Alvarez [18] or Schwarz [5]. Motivated by the following identity (valid for finite matrices)

$$\ln \det A - \ln \det B = -\int_0^\infty t^{-1} \left\{ \mathrm{tr}(e^{-tA}) - \mathrm{tr}(e^{-tB}) \right\} dt \qquad \text{(IV.1)}$$

one defines a regulated determinant by

$$\ln \mathrm{Det}'_\epsilon(\not{D}^2) = -\int_\epsilon^\infty t^{-1}[\mathrm{tr}'(e^{-t\not{D}^2})]dt. \qquad \text{(IV.2)}$$

Schwarz [5] , uses the asymptotic expansion of the heat kernel to define a (cut-off independent) determinant from equation (IV.2). For the Dirac operator one may define $\mathrm{Det}'(\not{D}^2)$ by:

$$\ln \mathrm{Det}'(\not{D}^2) \equiv \lim_{\epsilon \to 0} \left\{ \ln \mathrm{Det}'_\epsilon(\not{D}^2) + \sum_{j=0}^{(d/2)-1} \int a_j \cdot \frac{\epsilon^{(j-(d/2))}}{j-(d/2)} - [\int a_{d/2} - \#] \ln \epsilon \right\}. \qquad \text{(IV.3)}$$

The determinant Det defined in this way is <u>not</u> equal to the zeta function definition, there is however a simple relationship (which is not supposed to be obvious)

$$\det'(\not{D}^2) = \mathrm{Det}'(\not{D}^2) \cdot \exp\left[\Gamma'(1)\{[\int (4\pi)^{-d/2} a_{d/2}\sqrt{g}] - \#\} \right]. \qquad \text{(IV.4)}$$

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