FSU-TPI 01/06
DFTUZ–01.12

Nahm Transform and Moduli Spaces of $\mathbb{C}\mathrm{P}^N$-Models on the Torus

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Abstract

There is a Nahm transform for two-dimensional gauge fields which establishes a one-to-one correspondence between the orbit space of $U(N)$ gauge fields with topological charge k defined on a torus and that of $U(k)$ gauge fields with charge N on the dual torus. The main result of this paper is to show that a similar duality transform cannot exist for $\mathbb{C}\mathrm{P}^N$ instantons. This fact establishes a significative difference between 4-D gauge theories and $\mathbb{C}\mathrm{P}^N$ models. The result follows from the global analysis of the moduli space of instantons based on a complete and explicit parametrization of all self-dual solutions on the two-dimensional torus. The boundary of the space of regular instantons is shown to coincide with the space of singular instantons. This identification provides a new approach to analyzing the role of overlapping instantons in the infrared sector of $\mathbb{C}\mathrm{P}^N$ sigma models.

Two-dimensional sigma models have long been used as a testing ground for a variety of ideas in non-perturbative Quantum Field Theory, especially because of some remarkable similarities with non-Abelian gauge theories in 3+1 dimensions [1] (for a review emphasizing this connection, see [2]).

They are scale invariant at the classical level and asymptotically free at the quantum level, some possess topological winding numbers, instantons, a chiral anomaly when coupled to fermions and generate a dynamical mass by non-perturbative effects at zero temperature and a thermal mass $\sim g^2 T$ at finite temperature. In this respect the $O(3)$ nonlinear σ-model with action

$$S = \frac{1}{8g^2} \int d^2x \, (\partial_\mu \vec{n} \cdot \partial_\mu \vec{n}), \qquad \vec{n}^{\,2}(x) = 1, \tag{1.1}$$

has been very extensively studied, specially by its numerous interesting applications to condensed matter: (anti) ferromagnetism, Hall effect, Kondo effect, etc (see Ref. [3] for a review stressing this point of view). The model has also been used to analyze the sphaleron induced fermion-number violation at high temperature [4]. By setting $\vec{n} = \Psi^\dagger \vec{\sigma}\, \Psi$ with a normalized $\Psi \in \mathbb{C}^2$, the action (1.1) can be rewritten in the equivalent form

$$S = \frac{1}{2g^2} \int \mathrm{d}^2x \, \left| D_\mu \Psi \right|^2, \quad \text{with} \quad |\Psi(x)| = 1 \quad \text{and} \quad A_\mu = -i\Psi^\dagger \partial_\mu \Psi. \tag{1.2}$$

It is invariant under gauge transformations $\Psi \to \mathrm{e}^{i\lambda(x)}\Psi$ and hence $\Psi(x)$ may be viewed as an element in $\mathbb{CP}^1$.

There are two natural generalizations of the $O(3) \sim \mathbb{CP}^1$ model: $O(N>3)$ models with action (1.1) but with $\vec{n} \in S^{N-1}$ and $\mathbb{CP}^N$ models with action (1.2), but with $\Psi \in \mathbb{CP}^N$ instead of $\mathbb{CP}^1$ [5] [6]. In contrast to the $O(N)$ models they possess instanton solutions for all N, and a θ term can be added to the action so that their topological properties can be explored [7]. These models are expandable in $1/N$ and have been solved in the large N limit [8]. The role of instantons and related sphalerons [9] is crucial for physical effects at $\theta \neq 0$ [10]. In this paper we shall mainly be interested in the structure of spaces of instantons and hence shall only consider the $\mathbb{CP}^N$ extensions of the $O(3)$-model.

In particular we shall focus on the search of a variant of the Nahm transform for 2-dimensional models. In 4-dimensions this remarkable duality transformation relates different instanton moduli spaces of gauge theories formulated on the four-torus [11]. More explicitly, it transforms a charge k self-dual (instanton) $SU(N)$ gauge potential A on $\mathbb{T}^4$ into a charge N self-dual $U(k)$ potential $\hat{A}$ on the *dual* torus $\hat{\mathbb{T}}^4$ as follows:

$$(\hat{A}_\mu)_{ij}(u) = -i \int_{\mathbb{T}^4} d^4x \; \psi_i^{u\dagger}(x) \frac{\partial}{\partial u^\mu} \psi_j^u(x), \tag{1.3}$$

potential $A_\mu + 2\pi u_\mu \mathbb{1}$, where the constant piece u parametrizes the dual torus. This transformation being involutive means that the moduli space $\mathcal{M}_k^N$ of $SU(N)$ instantons with charge k is equivalent to $\mathcal{M}_N^k$ that of $SU(k)$ instantons with charge N .

Our search for a corresponding Nahm transform for $\mathbb{C}\mathrm{P}^N$ models on the two-torus was motivated by the observation that the complex dimension of the moduli space for charge k instantons in $\mathbb{C}\mathrm{P}^N$ is

$$\dim \mathcal{M}_k^N = k(N+1), \qquad k > 1, \tag{1.4}$$

exhibiting a duality that may be conjectured to hold at the level of moduli spaces, $\mathcal{M}_k^N \approx \mathcal{M}_{N+1}^{k-1}$. This conjecture was further prompted by the fact that there are no charge 1 instantons on the 2-torus for any value of N [12][13], a property shared with gauge theories on the 4-torus [14] [15]. Similarly as for gauge theories this would be a consequence of such a duality, since there is no $\mathbb{C}\mathrm{P}^0$ instanton. If this duality exists the dynamics of the $\mathbb{C}\mathrm{P}^N$ models should simplify in sectors with large k, as it happens for large N.

The aim of the paper is to analyze the existence or not of a generalized Nahm transform for these sigma models. Hence we shall only analyze the instantons on a torus, the only Riemann surface whose dual (Jacobian) is also a torus. This kind of compactification of space time corresponds to the choice of periodic boundary conditions which are appropriate for the study of finite temperature effects [4].

The space time compactification also presents some technical advantages. The action of an instanton does not depend on the parameters of moduli space. This then leads to zero-modes of the fluctuation operator in the instanton background. One expects that for each parameter in moduli space there is one associated zero-mode or that the number of zero-modes is not smaller than the dimension of the moduli space. This expectation is not fulfilled for the sigma models on $\mathbb{R}^2$: if one varies some moduli parameter of the instanton one finds non-normalizable zero-modes [16]. In a compact space this can never happen, thus in our case both methods of counting the dimension of moduli spaces of instantons are equivalent.

Since $\mathbb{C}\mathrm{P}^N$ spaces admit a Kähler structure, 2-dimensional $\mathbb{C}\mathrm{P}^N$-models can be extended to $\mathcal{M}=2$ supersymmetric theories. More general purely bosonic or supersymmetric nonlinear sigma models with Kähler target spaces have been studied on topologically trivial space-times by several authors, see [17]. All these models admit regular instanton solutions, the topological charge of which appears as lower BPS-bound on the action.

In addition to the regular instanton solutions these models possess singular ones. Although they are usually ignored, it has been pointed out recently that these singular

lar, their contribution to the renormalization group flow of supersymmetric theories and correlation functions of topological invariants seems to be crucial.

We shall analyze singular instantons as boundary configurations of moduli spaces of regular instantons. In particular, we shall discuss if they can appear as limit case of strongly overlapping regular instantons as it happens for 4-dimensional Yang-Mills instantons [19].

This paper is organized as follows: In the next section we briefly recall the basic features of classical 2-dimensional $\mathbb{C}\mathrm{P}^N$-models and their instanton solutions. Section 3 contains a detailed analysis of zero-modes of the associated Weyl operator on the 2-dimensional torus. In the following section we study the zero-modes for the shifted gauge potential and explicitly construct the Nahm transform for two-dimensional gauge fields. In particular, we show how this transformation maps $U(1)$ gauge fields with charge k over a 2-dimensional torus into $U(k)$ gauge fields with charge 1 over the dual torus. Section 5 deals with the global structure of $\mathbb{C}\mathrm{P}^N$ moduli spaces of instantons on $\mathbb{T}^2$. A general method for their construction is proposed. It is based on properties of the fiber bundles associated with the $U(1)$ connection, and yields a complete description of the moduli spaces $\mathcal{M}_k^N$. Some simple examples are worked out explicitly. The relevant topological properties of these spaces are investigated in section 6. The main result is that no invertible Nahm transform for regular instantons can exist since the corresponding moduli spaces are topologically distinct. In section 7 we investigate the role of singular instantons as limiting cases of strongly overlapping regular instantons. A summary of our results and conclusions are contained in section 8.

2. Instantons in $\mathbb{C}\mathrm{P}^N$ models

The classical $\mathbb{C}\mathrm{P}^N$ model in 2 Euclidean space-time dimensions is defined by the action

$$S = \frac{N}{2g^2} \int \mathrm{d}^2 x \, \left| D_\mu \Psi \right|^2, \qquad D_\mu \Psi = \left(\partial_\mu - i A_\mu \right) \Psi, \tag{2.1}$$

where $\Psi(x) = (\Psi_a(x)),\ a = 0, \ldots, N$ is a $(N+1)$-component complex field with values in $\mathbb{C}\mathrm{P}^{N+1}$. We consider $\Psi(x)$ to be normalized

$$|\Psi(x)|^2 = \sum_{a=0}^{N} |\Psi_a(x)|^2 = 1 \tag{2.2}$$

and configurations differing by a phase factor are identified,

$$\Psi(x) \sim \mathrm{e}^{i\alpha(x)} \, \Psi(x). \tag{2.3}$$

$U(1)$ gauge field defined in terms of the sigma field by

$$A_\mu(x) := -i\Psi^\dagger \partial_\mu \Psi. \tag{2.4}$$

With this gauge field the symmetry (2.3) can be restated as a usual $U(1)$ gauge invariance. Indeed, the action (2.1) is invariant under the phase transformation (2.3) of $\Psi(x)$ if at the same time the composite field A_μ transforms as a true $U(1)$ connection,

$$A_\mu \to A_\mu + \partial_\mu \alpha(x), \tag{2.5}$$

as it follows from its very definition. Actually, one may view Ψ and A_μ in (2.1) as independent fields. The algebraic equations of motion for the gauge potential are then just (2.4).

A topological charge (instanton number) can be defined

$$k = \frac{1}{2\pi}\int \mathrm{d}^2x\, F_{12} = \frac{1}{2\pi i}\int \mathrm{d}^2x\, \varepsilon_{\mu\nu}\partial_\mu\Psi^\dagger\partial_\nu\Psi = \frac{1}{8\pi i}\int \mathrm{d}^2x\, \varepsilon_{\mu\nu}D_\mu\Psi^\dagger D_\nu\Psi. \tag{2.6}$$

This charge takes integer values for smooth configurations with finite action, and thus the space of configurations splits into disconnected instanton sectors.
Application of the Cauchy-Schwartz inequality to $D_\mu\Psi$, $i\varepsilon_{\mu\nu}D_\nu\Psi$ yields

$$S \geq \frac{N\pi}{g^2}|k|, \tag{2.7}$$

The minimal action is reached by solutions of the first order equations

$$D_\mu\Psi(x) = \mp i\varepsilon_{\mu\nu}D_\nu\Psi(x). \tag{2.8}$$

Antiselfdual solutions correspond to instantons ($-$sign and $k>0$), and selfdual solutions to anti-instantons ($+$sign and $k<0$).

In complex coordinates $z = x^1 + ix^2$ the (anti)selfdual equations (2.8) can be written as follows

$$\begin{aligned} D_{\bar z}\Psi &= (\partial_{\bar z} - iA_{\bar z})\Psi = 0 \qquad \text{instantons} \\ D_z\Psi &= (\partial_z - iA_z)\Psi = 0 \qquad \text{anti}-\text{instantons}, \end{aligned} \tag{2.9}$$

where the complex components of the gauge field are $A_z = \frac{1}{2}(A_1 - iA_2)$, $A_{\bar z} = \frac{1}{2}(A_1 + iA_2)$.

This provides the first characterization of the solutions as holomorphic solutions with respect to the holomorphic bundle structure induced by the gauge field A.

In summary, up to some common normalization factor the components of instanton field solutions are holomorphic sections of a line bundle over space-time. In the plane there and $k=0$. In the torus $\mathbb{T}^2$ the different holomorphic structures are parametrized for fixed k by the points of the dual torus $\hat{\mathbb{T}}^2$ and the space of instantons has a finite dimension [12][13].

The topological charge of such a solution is the sum of multiplicities of the zeros of any non-trivial component of Ψ. From now on we concentrate on instanton solutions ($k>0$). The anti-instanton case is analogous.

3. $\mathbb{C}\mathbf{P}^N$-models on the torus and Fermionic zero-modes

We may view the torus as $\mathbb{R}^2$ modulo a two-dimensional lattice Λ generated by two vectors e_1 and e_2. For simplicity we will restrict to orthonormal vectors e_μ and use dimensionless coordinates. In the sector with instanton number k we shall choose as transition functions U_μ relating the fields at x and $x+e_\mu$,

$$\Psi_a(x+e_\mu)=U_\mu(x)\Psi_a(x) \quad , \quad A(x+e_\mu)=A(x)-iU_\mu^\dagger(x)dU_\mu(x), \tag{3.1}$$

the gauge transformations

$$U_1=e^{i\pi k x^2} \quad \text{and} \quad U_2=e^{-i\pi k x^1}. \tag{3.2}$$

This means that A is defined on a non-trivial line bundle $E_k(\mathbb{T}^2,\mathbb{C})$. In two dimensions any gauge field A induces a holomorphic bundle structure on E_k (in four dimensions this only holds for self-dual gauge fields). Hence there is a local complex gauge transformation h such that

$$A_{\bar z}=ih\,\partial_{\bar z}\,h^{-1} \quad \text{and} \quad D_{\bar z}=\partial_{\bar z}+h\,\partial_{\bar z}\,h^{-1}, \tag{3.3}$$

which trivializes the connection A (see e.g. Ref. [20]).

Next we construct and discuss the zero-modes of the Dirac operator on the 2-dimensional Euclidean torus. By the index theorem the number of right-handed minus the number of left-handed zero-modes of the Dirac equation,

$$\not{D}_A\psi=0$$

depends only on the first Chern class of the gauge field. Since there are only zero-modes of one chirality the total number of such modes in the fundamental representation is k. Since they have definite chirality they are completely determined by one non-trivial component in the Weyl basis, i.e. by one ordinary complex function ψ which we will identify with the spinor field ψ itself.

$$(\partial_{\bar{z}} - iA_{\bar{z}})\,\psi = 0, \qquad k > 0 \tag{3.4}$$

and must satisfy the same boundary condition (3.1) as the components of Ψ. Thus the fermionic zero-modes fulfill the same differential equation and boundary conditions as the components of the sigma field Ψ.

After trivializing the connection as in (3.3) the Weyl equation becomes a holomorphic condition,

$$D_{\bar{z}}\,\psi = 0 \Longleftrightarrow \partial_{\bar{z}}\,\chi = 0, \quad \text{where} \quad \psi = h\chi.$$

However, as pointed out earlier, if $k \neq 0$ the transformation h cannot be globally defined and this shows up in the the change of boundary conditions between ψ and χ,

$$\chi(z+1) = \widetilde{U}_1(z)\chi(z) \quad , \quad \chi(z+i) = \widetilde{U}_2(z)\chi(z),$$

where

$$\widetilde{U}_\mu(z) = h^{-1}(x+e_\mu)\,U_\mu(x)\,h(x)$$

is holomorphic. The holomorphic character of the $\widetilde{U}_\mu$ also reflects the fact that any holomorphic section χ defines the holomorphic structure of the bundle E_k associated to the gauge field A.

Now we consider the particular gauge potential

$$A^I_1 = -\pi k x^2, \qquad A^I_2 = \pi k x^1, \quad \text{or} \quad A^I_{\bar{z}} = \tfrac{i}{2}\pi k z, \tag{3.5}$$

which gives rise to a constant field strength $F_{01} = 2\pi k$ and instanton number k. The complex gauge transformation trivializing A^I reads

$$h = e^{-\pi k z\bar{z}/2} \tag{3.6}$$

and the χ satisfy the holomorphic boundary conditions with transition functions

$$\widetilde{U}_1 = \mathrm{e}^{(1+2z)\pi k/2} \quad \text{and} \quad \widetilde{U}_2 = \mathrm{e}^{(1-2iz)\pi k/2}. \tag{3.7}$$

The zero-modes can be conveniently expressed in terms of Jacobi's theta functions

$$\begin{aligned}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{i\pi\tau(n+a)^2}\mathrm{e}^{2\pi i(n+a)b} \\ &= \eta(i\tau)\,\mathrm{e}^{2\pi iab} q^{\frac{a^2}{2}-\frac{1}{24}} \prod_{n>0}\left(1+q^{n+a-\frac{1}{2}}\mathrm{e}^{2\pi ib}\right)\left(1+q^{n-a-\frac{1}{2}}\mathrm{e}^{-2\pi ib}\right),\end{aligned} \tag{3.8}$$

following shift properties

$$\vartheta\begin{bmatrix} a+m+\frac{n}{\tau} \\ b \end{bmatrix}(\tau) = \mathrm{e}^{2\pi i n(a+b/\tau+n/2\tau)}\,\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) \tag{3.9}$$

and possess first order zeros at the points

$$\tau a + b \in \{m + \tfrac{1}{2} + \tau(n+\tfrac{1}{2}), \qquad m,n \in \mathbb{Z}\}. \tag{3.10}$$

In terms of these θ-functions a basis of linearly independent zero-modes reads

$$\psi_\ell(x) = (2k)^{\frac{1}{4}} h(x)\,\chi_\ell(z), \qquad \chi_\ell(z) = \mathrm{e}^{\pi k z^2/2}\vartheta\begin{bmatrix} z+\frac{\ell}{k} \\ 0 \end{bmatrix}(ik) \tag{3.11}$$

and ψ_ℓ has k zeros at the following points on the torus:

$$x^1 = \langle \tfrac{1}{2} - \tfrac{\ell}{k} \rangle \quad , \quad x^2_p = \langle \tfrac{1}{k}(\tfrac{1}{2}+p) \rangle, \qquad p = 1,2,\dots,k, \tag{3.12}$$

where $\langle a \rangle$ denotes the unique element in the lattice $\{a+\mathbb{Z}\}$ lying in $[0,1)$. The basis $\{\psi_\ell\}$ is orthonormal,

$$(\psi_\ell, \psi_{\ell'}) = \int_{\mathbb{T}^2} \psi^*_\ell \psi_{\ell'} = \delta_{\ell\ell'}.$$

Let $x_{\ell p} \in \mathbb{T}^2$, $p = 1,\dots k$, be the k zeros of ψ_ℓ. Then their sum is independent of ℓ and is given by

$$\sum_{p=1}^k x_{\ell p} = \langle \tfrac{k}{2} \rangle e \quad \text{modulo} \quad \Lambda, \quad \text{where} \quad e = e_1 + e_2. \tag{3.13}$$

That (for fixed k) the sum of the zeros is the same for all ψ_ℓ (modulo the lattice defining the torus), follows from the fact that all holomorphic sections $\chi_l(z)$ satisfy the same boundary conditions. This statement holds true for any choice of a zero-mode basis.

Under a translation by $1/k$ in either of the two directions on the torus the space of zero-modes is left invariant. This is expected on general grounds and is needed for the Nahm transform. More explicitly, let $\boldsymbol{\psi}$ denote the k-dimensional column vector with entries $(\psi_1,\dots,\psi_k)$. Then the transformations read

$$\boldsymbol{\psi}(x + \tfrac{1}{k}e_1) = \mathrm{e}^{i\pi x^2} S_1 \boldsymbol{\psi}(x) \quad \text{and} \quad \boldsymbol{\psi}(x + \tfrac{1}{k}e_2) = \mathrm{e}^{-i\pi x^1} S_2 \boldsymbol{\psi}(x), \tag{3.14}$$

with unitary $k \times k$ matrices

$$S_1 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} \alpha & 0 & & 0 \\ 0 & \alpha^2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \alpha^k \end{pmatrix}, \quad \alpha = \mathrm{e}^{-2\pi i/k}, \tag{3.15}$$

$$S_1^k = S_2^k = \mathbb{1} \quad \text{and} \quad S_1 S_2 = \alpha S_2 S_1. \tag{3.16}$$

These shift identities are consistent with the position of the zeros of ψ_ℓ given in (3.12). Note that $\boldsymbol{\psi}$ may be viewed as a zero mode of the $U(k)$ potential $A^I \mathbb{1}$ on the smaller torus with circumferences $1/k$ and with instanton number 1. The last relation in (3.16) just guarantees that

$$\mathrm{e}^{i\pi x^2} S_1 \quad \text{and} \quad e^{-i\pi x^1} S_2$$

in (3.14) are consistent $U(k)$ transition functions on the smaller torus, that is, they satisfy the cocycle conditions with periods $1/k$.

We could as well have taken an alternative set of orthonormal zero modes,

$$\widetilde{\boldsymbol{\psi}}(x) = S\boldsymbol{\psi}(x), \qquad S^\dagger S = \mathbb{1}. \tag{3.17}$$

For example using the zero-modes

$$\widetilde{\psi}_\ell(x) = (2k)^{\frac{1}{4}}\, \mathrm{e}^{-\pi k z\,(\bar{z}+z)/2} \vartheta \begin{bmatrix} iz + \frac{\ell}{k} \\ 0 \end{bmatrix} (ik) \tag{3.18}$$

instead of the ones in (3.11) amounts to exchanging x^1 and x^2 in the formulae above. With respect to the new basis one again finds the shift identity (3.14) with the replacements

$$S_\mu \longrightarrow S\, S_\mu\, S^{-1}.$$

The algebraic relations (3.16) remain intact and hence are independent of the choice of basis.

4. Nahm transform of gauge fields on 2-dimensional torus $\mathbb{T}^2$

Let A be an arbitrary two-dimensional U(N) gauge field with topological charge

$$c_1(A) = \int_{\mathbb{T}^2} \operatorname{tr} F_{12}(A)$$

defined on a torus $\mathbb{T}^2$. Its Nahm transform is defined in terms of the zero-modes of the Weyl operator for a shifted vector potential

$$A^u_\mu = A^I_\mu + 2\pi u_\mu \mathbb{1} \tag{4.1}$$

which has the same topological charge. By the index theorem the dimension of the space of zero-modes of $\not{D}_{A^u}$ is k. Let $\{\psi^u_j; j = 1, 2, \cdots k\}$ be an orthonormal basis of zero-modes.

dual torus $\mathbb{T}^2$ with topological charge N as follows:

$$(\hat{A}_\mu)_{ij}(u) = -i\int_{\mathbb{T}^2} d^2x\, \psi_i^{u\dagger}(x)\frac{\partial}{\partial u^\mu}\psi_j^u(x), \tag{4.2}$$

Note that this construction does not require any special constraint on the original gauge field A as it does in four dimensions where A must be selfdual. This is because any 2-dimensional gauge field defines a holomorphic structure in the corresponding bundle, whereas in four dimensions this is true only for self-dual gauge fields.

Sigma model fields are associated to Abelian gauge fields. But the Nahm transform does not preserve the Abelian character as we shall see below. This already is the first indication that it might be problematic to extend the Nahm transform to sigma models. To analyze this problem let us now apply the Nahm construction to the Abelian field (3.5).

An orthonormal basis of the zero-modes of the Dirac equation for a shifted vector potential

$$A_\mu^u = A_\mu^I + 2\pi u_\mu \quad \text{or} \quad A_{\bar z}^u = \tfrac{i}{2}\pi k z + \pi w, \qquad w = u^1 + iu^2, \tag{4.3}$$

can be constructed from the solutions with $u = 0$ by shifting the arguments [21]

$$\psi_\ell^u(x) = \mathrm{e}^{i\pi(u,x)}\,\psi_\ell\big(x + \tfrac{1}{k}\varepsilon\, u\big), \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.4}$$

For later purposes it is useful to discuss some properties of these zero-modes:

The k zeros of these modes are related to those of the ψ_ℓ by the shift in (4.4),

$$x_p^1 \in \big\langle \tfrac{1}{2} - \tfrac{1}{k}(\ell + u^2)\big\rangle \quad \text{and} \quad x_p^2 \in \big\langle \tfrac{1}{k}(\tfrac{1}{2} + u^1 + p)\big\rangle, \qquad p = 1,\dots,k. \tag{4.5}$$

Hence two different modes share no common zero unless $u \in \mathbb{Z}^2$.

From (4.4) and (3.14) one sees at once that the vector $\boldsymbol{\psi}^u$ transforms in the same way as $\boldsymbol{\psi}$ when either x^1 or x^2 is translated by $1/k$,

$$\boldsymbol{\psi}^u(x + \tfrac{1}{k}e_1) = \mathrm{e}^{i\pi x^2} S_1 \boldsymbol{\psi}^u(x) \quad \text{and} \quad \boldsymbol{\psi}^u(x + \tfrac{1}{k}e_2) = \mathrm{e}^{-i\pi x^1} S_2 \boldsymbol{\psi}^u(x), \tag{4.6}$$

where the matrices S_μ have been introduced in (3.15) .

The k-dimensional subspace spanned by the zero modes is also left invariant by the following simultaneous rotations of x and u:

$$(x,u) \longrightarrow (\varepsilon^n x, \varepsilon^n u), \quad n \in \{0,1,2,3\}. \tag{4.7}$$

This can be seen by checking that the transformed states satisfy the same differential equation and boundary condition as the original ones. These rotations are represented by of the rotation symmetry on the torus for constant field strength.

In addition, the idempotent transformation

$$(x,u) \longrightarrow (x',u') = (\tfrac{1}{k}\varepsilon u, -k\varepsilon x) \tag{4.8}$$

is projectively represented on the eigenmodes

$$\boldsymbol{\psi}^{u'}(x') = \mathrm{e}^{-2\pi i(x,u)}\,\boldsymbol{\psi}^u(x). \tag{4.9}$$

Under simultaneous translations of x and u the zero modes are invariant, up to a phase

$$\boldsymbol{\psi}^{u+\alpha}\big(x\big) = \mathrm{e}^{i\pi(\alpha,\,x+\frac{1}{k}\varepsilon x)}\boldsymbol{\psi}^u\big(x+\tfrac{1}{k}\varepsilon\alpha\big). \tag{4.10}$$

Later in this paper this shift identity will be rather important.

Finally note, that for $u \neq 0$ the holomorphic gauge transformation (3.6) does not trivialize the gauge field A^u anymore. The modified trivializing transformation reads

$$h^u(x) = \mathrm{e}^{-\frac{\pi}{2}(kz\bar{z}+\frac{1}{k}w\bar{w}-2i\,\bar{z}w)}, \qquad w = u^1 + iu^2.$$

It not only transforms the unitary basis (4.4) into a z-holomorphic basis but also into an w-holomorphic basis,

$$\psi^u_\ell = (2k)^{\frac{1}{4}}\,h^u(x)\,\chi^w_\ell(z), \qquad \chi^w_\ell = \chi_\ell(z-\tfrac{i}{k}w), \tag{4.11}$$

where the χ_ℓ have been introduced in (3.11). This is the essential feature of the Nahm transform. It follows that the Nahm transformed gauge field $\hat{A} = (\hat{A}_{\ell\ell'})$, defined by the Mukai-Nahm construction,

$$(\hat{A}_{\bar{w}})_{\ell\ell'} \stackrel{\mathbf{def}}{=} -i\big(\psi^u_\ell,\,\partial_{\bar{w}}\,\psi^u_{\ell'}\big) = -i\big(\psi^u_\ell,\psi^u_{\ell'}\big)(h^u)^{-1}\partial_{\bar{w}}\,h^u = \frac{i\pi}{2k}w\,\delta^{\mathrm{mod}k}_{\ell\ell'} \tag{4.12}$$

is a reducible $U(k)$ gauge field with constant field strength on the dual torus. The dual torus is given by

$$\hat{\mathbb{T}}^2 = \mathbb{R}^2/\,\hat{\Lambda}, \tag{4.13}$$

where with our choice for the shift in (4.1) the dual lattice $\hat{\Lambda}$ is generated by the two orthonormal vectors $\hat{e}_\mu = e_\mu$. In real notation the potential $\hat{A}$ takes the simple form

$$\hat{A}_1 = \hat{A}_w + \hat{A}_{\bar{w}} = -\frac{\pi}{k}u^2\mathbb{1} \quad , \quad \hat{A}_2 = i(\hat{A}_{\bar{w}} - \hat{A}_w) = \frac{\pi}{k}u^1\mathbb{1}.$$

acter of this $U(k)$ bundle can be seen from the peculiar boundary conditions of the holomorphic structures induced by $\hat{A}$. The corresponding transition functions which relate u and $u+e_\mu$ on the dual torus,

$$\psi^{u+e_1}(x) = \hat{U}_1 \psi^u(x) \quad , \quad \psi^{u+e_2}(x) = \hat{U}_2 \psi^u(x) \tag{4.14}$$

are determined by the shift identity (4.10) and the transformation properties (4.6) as follows,

$$\hat{U}_1 = \mathrm{e}^{2\pi i x^1 + \frac{i}{k}\pi u^2} \hat{S}_1 \quad \text{and} \quad \hat{U}_2 = \mathrm{e}^{2\pi i x^2 - \frac{i}{k}\pi u^1} \hat{S}_2, \tag{4.15}$$

where

$$\hat{S}_1 = S_2^{-1} \quad \text{and} \quad \hat{S}_2 = S_1.$$

Recall that the non-Abelian elements $\hat{S}_\mu$ generate a finite non-Abelian subgroup of $U(k)$

$$\hat{S}_\mu^k = \mathbb{1}, \qquad \hat{S}_1 \hat{S}_2 = \mathrm{e}^{-i\,2\pi/k}\, \hat{S}_2 \hat{S}_1. \tag{4.16}$$

The last relation guarantees that for any fixed x the $\boldsymbol{\psi}^u$ are sections of a $U(k)$-bundle over the dual torus $\hat{\mathbb{T}}^2$ with coordinates u:

$$\hat{U}_2(u+e_1)\, \hat{U}_1(u) = \hat{U}_1(u+e_2)\, \hat{U}_2(u). \tag{4.17}$$

The first Chern class of this bundle follows from the fact that the (non-Abelian) Nahm transformed gauge potential $\hat{A}$ is just k times any of its diagonal elements, hence

$$\int_{\mathbb{T}^2} \operatorname{tr} F(\hat{A}^u) = 2\pi \Longrightarrow c_1(\hat{A}^u) = \hat{k} = 1.$$

It can be shown that the Nahm transform of $\hat{A}$ is A, i.e. the Nahm transformation is involutive. It is a particular case of the more general Mukai transform defined for holomorphic sheaves (which do not necessarily define holomorphic bundle structures)[14][22][1].

[1] see also [23] for a view closer to physical applications

Let us consider an instanton field Ψ on the torus with charge k, that is, a solution of (2.9) subject to the boundary conditions (3.1) . The associated $U(1)$ gauge potential A is a connection defined in a line bundle E_k with first Chern class $c_1(E_k) = k$. The holomorphic structure associated to A in E_k is *globally* equivalent to one of the A^u described in the previous section. This means that there is a global (periodic) complex gauge transformation g: $\mathbb{T}^2 \to \mathbb{C}\backslash\{0\}$ such that

$$A_{\bar{z}} = g\left(A^u_{\bar{z}} + i\,\partial_{\bar{z}}\right)g^{-1} \quad \text{and} \quad D_{\bar{z}} = g\left(\partial_{\bar{z}} - iA^u_{\bar{z}}\right)g^{-1}. \tag{5.1}$$

Therefore, up to U(1) gauge transformations the $N{+}1$ components of Ψ can be expressed in terms of the k independent solutions ψ^u_ℓ of the zero mode equation (4.4) as follows,

$$\Psi = \frac{1}{\sqrt{\boldsymbol{\psi}^{u\dagger}\mathbf{A}^\dagger\mathbf{A}\boldsymbol{\psi}^u}}\,\mathbf{A}\boldsymbol{\psi}^u, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} {a_0}^1 & {a_0}^2 & \cdots & {a_0}^k \\ {a_1}^1 & {a_1}^2 & \cdots & {a_1}^k \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_N^1 & a_N^2 & \cdots & a_N^k \end{pmatrix}. \tag{5.2}$$

Hence, any instanton solution is characterized by a point u in the dual torus $\hat{\mathbb{T}}^2$ and a $(N{+}1)\times k$ matrix $\mathbf{A}$ subject to certain constraints given below. This characterization provides a constructive method to describe the moduli space of instantons with charge k.

The projective nature of the sigma fields Ψ implies that matrices differing by a non-vanishing multiplicative factor must be considered as equivalent,

$$\mathbf{A} \sim \lambda\mathbf{A}, \qquad \lambda \neq 0, \tag{5.3}$$

because they give rise to the same instanton field. Furthermore, in order to satisfy the sigma model condition $\Psi^\dagger(x)\Psi(x){=}1$ the $\mathbf{A}\boldsymbol{\psi}^u$ should never vanish ($\boldsymbol{\psi}^u \in \mathbb{C}^k$ never vanishes since the ψ^u_ℓ have no common zeros) and this imposes a constraint on $\mathbf{A}$. Finally, because of the boundary conditions (4.14) we have the identifications

$$(u, \mathbf{A}) \sim \hat{T}_\mu(u, \mathbf{A}) = \left(u{+}e_\mu\,,\ \mathbf{A}\hat{U}_\mu^{-1}\right), \tag{5.4}$$

since the two pairs give rise to the same $\Psi \in \mathbb{C}\mathrm{P}^N$ and hence must be identified. There is no further identification since a shift $u \to u + \alpha$ with $\alpha \notin \mathbb{Z}^2$ cannot be compensated by a (necessarily) unitary matrix. This would not be compatible with $\boldsymbol{\psi}^u$ being a section of the $U(k)$-bundle over the dual torus with charge 1.

simple example. It is the *basic instanton* of charge k of the $\mathbb{CP}^{k-1}$ model defined by the basis (4.4) of zero-mode sections of E_k,

$$\Psi^u_* = \frac{1}{\sqrt{\boldsymbol{\psi}^{u\dagger}\boldsymbol{\psi}^u}}\,\boldsymbol{\psi}^u. \tag{5.5}$$

In the $(\mathbf{A}, u)$ parametrization this solution corresponds to $\Psi^u_* = (u, \mathbb{1}_k)$. Notice that in this case the constraint is satisfied because sections of the basis (4.4) do not have a common zero [24].

It is not hard to find sufficient conditions on $\mathbf{A}$ for Ψ to be normalizable. Clearly, the denominator in (5.2)

$$\left(\boldsymbol{\psi}^{u\dagger}\,\mathbf{A}^\dagger\mathbf{A}\boldsymbol{\psi}^u\right)^{1/2}$$

is never zero if $\det(\mathbf{A}^\dagger\mathbf{A}) \neq 0$. Since the rank of the $k \times k$ matrix $\mathbf{A}^\dagger\mathbf{A}$ is less or equal that $\min(k, N{+}1)$, this can only be fulfilled for $k \leq N{+}1$. Hence in this case the maximal rank condition is *sufficient*, i.e.

$$\ker\left\{\mathbf{A} : \mathbb{C}^k \to \mathbb{C}^{N+1}\right\} = 0 \qquad \text{if } k \leq N+1. \tag{5.6}$$

However, even in that case this condition is not necessary. The fact that $\Psi(x)$ has to be a non-null vector for any point x requires that the matrix $\mathbf{A}$ be viewed as a projective map $\mathbb{CP}^{k-1} \to \mathbb{CP}^N$ from rays of $\mathbb{CP}^{k-1}$ into rays of $\mathbb{CP}^N$. This just means that the kernel of $\mathbf{A}$ must not lie in the image of $\boldsymbol{\psi}^u(x)$ for any x on the torus, that is

$$\mathrm{range}(\boldsymbol{\psi}^u) \cap \ker(\mathbf{A}) = \emptyset. \tag{5.7}$$

Otherwise $\mathbf{A}\boldsymbol{\psi}^u$ will have zero norm at some point and will not be a true sigma model field. Notice the compatibility of this constraint with the identifications (5.4). This concludes the characterization of instanton solution and provides an explicit procedure for a global description of the moduli space.

Before discussing the subtleties related to (5.7) in the general case we consider some simple examples of moduli spaces $\mathcal{M}^N_k$. First of all is clear that

$$\mathcal{M}^N_0 = \mathbb{CP}^N \qquad \text{and} \qquad \mathcal{M}^N_1 = \emptyset \quad \text{for} \quad N > 0.$$

In the first case because $\Psi^u = 0$ for $u \neq 0$ and Ψ^0 is an arbitrary constant vector in $\mathbb{CP}^N$. The second case follows from the fact that for $k = 1$ there is only one zero mode ψ^u which has exactly one zero on $\mathbb{T}^2$. Then all $N{+}1$ components of Ψ would vanish at this point and hence it could not be normalized.

$\mathcal{M}_2^1$, that is, the charge 2 sector of the $\mathbb{C}\mathrm{P}^1$ model. Since the range of the basic instanton Ψ_*^u completely covers $\mathbb{C}\mathrm{P}^1$ the constraint (5.7) if fulfilled if and only if the matrix $\mathbf{A}$ is regular. In this case the sufficient condition (5.6) is also a necessary one. Since $\mathbf{A}$ maps into $\mathbb{C}\mathrm{P}^1$ we may impose det $\mathbf{A}=1$ and identify $\mathbf{A}$ with $-\mathbf{A}$. It follows that the equivalence classes of matrices are to be regarded as elements of $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 = \mathrm{PSL}(2,\mathbb{C})$.

Because of the identifications (5.4) the moduli $\mathcal{M}_2^1$ is just a non-trivial bundle over the dual torus (with coordinates u) with fiber PSL(2,$\mathbb{C}$). The bundle structure is determined by the coset defined by the lift of the action of the discrete translation group $\mathbb{Z}\times\mathbb{Z}$ on the bundle $\hat{\mathbb{C}}\times$ PSL(2,$\mathbb{C}$), given by [13]:

$$\mathbb{Z}\times\mathbb{Z} = \{(\hat{T}_1^{n_1}, \hat{T}_2^{n_2});\; n_1, n_2 \in \mathbb{Z}\}, \tag{5.8}$$

where $\hat{T}_1$ and $\hat{T}_2$ are the basic generators defined in (5.4). The final result is that

$$\mathcal{M}_2^1 = \frac{\hat{\mathbb{C}} \times \mathrm{PSL}(2,\mathbb{C})}{\mathbb{Z}\ \times\ \mathbb{Z}}. \tag{5.9}$$

In the general case the construction of the moduli space is more involved since the solutions of the constraint (5.7) are not so explicit. But once we have identified the embedding of the space-time torus $\mathbb{T}^2$ into $\mathbb{C}\mathrm{P}^{k-1}$ given by the basic instanton Ψ_*^u, the set of allowed matrices can be parametrized as follows: The basic instanton solution Ψ_*^u defines a map $\mathbb{T}^2 \to \mathbb{C}^k$. Consider all linear subspaces V_n of $\mathbb{C}^k$ of dimension $n<k$ having empty intersection with range(Ψ_*^u). The space of matrices $\mathbf{A}$ which define regular instantons for a fixed u can be identified with the pairs $(V_n, \mathbf{B})$ defined by the subspaces V_n and the non-degenerate linear maps $\mathbf{B}$ mapping the orthogonal complement $V_n^\perp$ of V_n into the target space $\mathbb{C}^{N+1}$.

This means that the moduli space of instantons can be identified with a bundle over the dual torus with fiber isomorphic to the product $\mathcal{V}\times\mathrm{PL}_0(k-n, N+1)$ of the set $\mathcal{V}$ of V_n subspaces and the set $\mathrm{PL}_0(k-n, N+1)$ of non-degenerate projective maps from $\mathbb{C}\mathrm{P}^{k-n-1}$ into $\mathbb{C}\mathrm{P}^N$. The bundle is defined by modding out the trivial bundle $\mathcal{V}\times\mathrm{PL}_0(k-n, N+1)\times\hat{\mathbb{C}}$ by the lift of the discrete translation group $\mathbb{Z}\times\mathbb{Z}$ given by (5.4).

To illustrate how the construction works let us consider a simple non-trivial case in some detail: $\mathcal{M}_3^2$. In this case we have a dense subset $\mathcal{M}_{3(0)}^2$ which is given by the bundle over the dual torus with fiber PSL(3,$\mathbb{C}$)=SL(3,$\mathbb{C}$)/$\mathbb{Z}_3$, the equivalence classes of 3×3 matrices with det $\mathbf{A}=1$. The complex dimension of the sub-manifold, dim $\mathcal{M}_{3(0)}^2 = 9$, equals that of the total space $\mathcal{M}_3^2$. However there is another sub-bundle in $\mathcal{M}_3^2$ with lower dimensional kernel V_1 which does not intersect the image of the map $\psi^u_* : \mathbb{T}^2 \to \mathbb{C}^3$. The complex dimension of this subbundle is six. The total space is the union of the two strata,

$$\mathcal{M}^2_3 = \mathcal{M}^2_{3(0)} \cup \mathcal{M}^2_{3(1)}. \tag{5.10}$$

the second being the boundary of the first one. Notice that the subset of the second stratum $\mathcal{M}^2_{3(1)}$ associated to a fixed kernel can be identified with $\mathcal{M}^1_3$.

6. Global properties of the moduli of instantons

The complex dimensions of the moduli spaces are

$$\dim \mathcal{M}^N_k = (N+1)\, k$$

as follows at once from our matrix representation of the $\mathbb{C}\mathrm{P}^N$-fields. Note that this number is invariant under the interchange of the instanton number k and the number $N+1$ of sigma field components.

In the case $k \geq N+1$ there is a natural stratification of the moduli spaces,

$$\mathcal{M}^N_k = \bigcup_{n=k-N-1}^{k-2} \mathcal{M}^N_{k(n)}\ , \tag{6.1}$$

according to the dimension $n = \dim\ker \mathbf{A}$, but this does not mean that the moduli space is not a smooth manifold. From the matrix parametrization it is obvious that $\mathcal{M}^N_k$ is smooth, although it might seem hidden by the stratification (6.1) introduced in Ref. [25]. Moreover, $\mathcal{M}^N_k$ is a Kähler manifold [17] and the associated Riemannian structure plays an important role in accurate semiclassical expansions of $\mathbb{C}\mathrm{P}^N$ models [13]. The matrix parametrization permits to analyze the global features of these geometric structures, in particular, the incompleteness of the Riemannian metric, but we shall only focus into the analysis of the global topological structure of these moduli spaces. Below we summarize some of the relevant results.

$\mathcal{M}^N_k$ is non-empty and connected for any $k > 1$ and $N > 1$, i.e. $\pi_0(\mathcal{M}^N_k) = 0$. $\mathcal{M}^N_0 = \mathbb{C}\mathrm{P}^N$, and $\mathcal{M}^1_k = \emptyset$ for $k > 1$. The simplest non-trivial moduli space is $\mathcal{M}^1_2$, and because of the identification (5.9) we have

$$\pi_1(\mathcal{M}^1_2) = \mathbb{Z}_2\ \times\ \mathbb{Z}\ \times\ \mathbb{Z}. \tag{6.2}$$

The next case $\mathcal{M}^N_2$ has also a non-trivial bundle structure over the dual torus. Its fiber E_u is the projective set $\mathrm{PL}(2, N+1)$ of equivalence classes $(N+1) \times 2$ matrices $\mathbf{A}$ with defined by excluding rays of the form $(\lambda_1\vec{a}, \lambda_2\,\vec{a}) \in \mathbb{C}^{N+1}\times\mathbb{C}^{N+1}$. This fiber E_u =PL$(2,N)$ is homeomorphic to $\mathbb{C}\mathrm{P}^N \times \mathbb{C}^{N+1}$. The bundle structure is defined by the coset defined by the lift of the action of the translation group $\mathbb{Z}\times\mathbb{Z}$ to the bundle $\hat{\mathbb{C}}\times$PL$(2,N)$ given by (5.8) . Thus,

$$\pi_1(\mathcal{M}_2^N) = \mathbb{Z}_2 \;\times\; \mathbb{Z} \;\times\; \mathbb{Z}. \tag{6.3}$$

A more complex moduli space is $\mathcal{M}_k^1$. In this case the fiber E_u is identified with the (projective) set PL$(k,2)$ of $2\times k$ matrices $\mathbf{A}$ with maximal rank 2 whose kernel has an empty intersection with $\Psi_*^u(x)$ for any x on the torus. In terms of the nodes of the corresponding $\mathbb{C}\mathrm{P}^1$ field, the fiber E_u can be seen as homeomorphic to the set of k pairs $(a_i,b_i)\in\mathbb{T}^2\times\mathbb{T}^2$ (for $i=1,2,\cdots,k$), satisfying the following constraints

$$\sum_{i=1}^k a_i \in \langle u+\tfrac{k}{2}e\rangle, \sum_{i=1}^k b_i \in \langle u+\tfrac{k}{2}e\rangle \quad \text{and} \quad a_i\neq b_j \quad \forall i,j. \tag{6.4}$$

That means that the sum of the zeros are the same (modulo Λ) for both components of the $\mathbb{C}\mathrm{P}^1$ field since they fulfill identical boundary conditions. The space of nodes of any of the two components of Ψ is homeomorphic to $\mathbb{C}\mathrm{P}^{k-1}$ (see appendix). The space $\mathcal{Z}$ of zeros of the two components of Ψ satisfying the constraint of having no common nodes is a bundle over $\mathbb{C}\mathrm{P}^{k-1}$ with fiber $\mathbb{C}\mathrm{P}^{k-1}-\mathcal{C}$, where

$$\mathcal{C}=\big\{\{a_i,b_i\}\in\mathbb{T}^2\times\mathbb{T}^2\big|\, i=1,2,\ldots,k, \quad \text{with} \quad a_i=b_j \quad \text{for some} \quad i,j\big\}.$$

The first homotopy group of $\mathcal{Z}$ is

$$\pi_1(\mathcal{Z})=\mathbb{F}_2\times\mathbb{Z}^{k-3},$$

where $\mathbb{F}_2$ is the free group with 2 generators, i.e. the first homotopy group of a bouquet of two circles $S^1\vee S^1$. This follows from the following characterization of $\mathcal{C}$: The second component Ψ_2 of the sigma field Ψ must be a holomorphic section of the line bundle E_k without common zeros with the first component Ψ_1. Let $a_1,a_2,\ldots,a_k$ be the zeros of Ψ_1. There is only one constraint (6.4) on the position of these zeros. It is always possible to choose a different point b in $\mathbb{T}^2$ such that $b\neq a_i$ for any $i=1,2,\ldots,k$. It is obvious that the points $a_1,\ldots,a_{k-1},b$ will never satisfy the constraint (6.4). Then, the space of vectors $\hat{\mathcal{C}}$ whose rays are in $\mathcal{C}$ is given by all vectors in the subspaces $\Psi(a_1)=0,\Psi(a_2)=0,\ldots,\Psi(a_k)=0$. The space of all holomorphic sections of E_k is parametrized by the coordinates $(\alpha_1,\alpha_2\cdots\alpha_k)$ defined by $\alpha_1=\Psi(a_1),\ldots,\alpha_{k-1}=\Psi(a_{k-1})$, $\alpha_k=\Psi(b)$. In this parametrization $\hat{\mathcal{C}}$ is made out of the first $k-1$ coordinate hyperplanes $\alpha_i=0\,,i=$

$\mathbb{C}\times\mathbb{C}_*^{k-3}\times\mathbb{C}_{**}$, where $\mathbb{C}_{**}$ denotes the complex plane $\mathbb{C}$ without two points $0, 1$. From this construction it is obvious that $\pi_1(\mathbb{CP}^{k-1} - \mathcal{C}) = \mathbb{F}_2 \times \mathbb{Z}^{k-3}$. Then, the first homotopy group of constrained $2 \times k$ matrices $\mathrm{PL}(k,2)_c$ has a non-trivial non-Abelian homotopy group $\pi_1(\mathrm{PL}(k,2)_c) = \mathbb{F}_2 \times \mathbb{Z}^{k-3} \times \mathbb{Z}_2$. This implies that the first homotopy group of the moduli space $\mathcal{M}_k^1$ is

$$\pi_1(\mathcal{M}_k^1) = \mathbb{F}_2 \times \mathbb{Z}^{k-1} \times \mathbb{Z}_2. \tag{6.5}$$

These topological properties of moduli spaces of instantons (6.3) and (6.5) are very different which will be in contradiction with the existence of any kind of Nahm transform for $\mathbb{CP}^N$ sigma models. This turns out to be the major physical consequence of the results of this section.

The fact that the space of unit charge instantons is empty is reminiscent of a similar property of the orbit space of Yang-Mills fields with charge one instantons on the torus $\mathbb{T}^4$ [14][15]. In that case, it appears as a consequence of the existence of Nahm transform and the fact that there are no $U(1)$ instantons. This analogy suggests that perhaps the same property for $\mathbb{CP}^N$ instantons can be derived from a similar duality transform. In addition, the dimensions of $\mathcal{M}_k^N$ and $\mathcal{M}_{N+1}^{k-1}$ are the same.

A first indication that the Nahm transform might not exist for the $\mathbb{CP}^N$ models arises from the fact that the transform $\hat{A}_z$ of the Abelian potential A_z associated with Ψ is non-Abelian and thus cannot be associated to a $\mathbb{CP}^N$ field on the dual torus. One direct way of checking whether such a duality exists is to compare topological properties of $\mathcal{M}_k^N$ and $\mathcal{M}_{N+1}^{k-1}$.

Now, the topological structures of $\mathcal{M}_2^N$ and $\mathcal{M}_{N+1}^1$ given by (6.3) and (6.5) are very different for $N>1$. This already allows us to exclude the existence of an invertible Nahm transform, at least in these cases. The same topological non-equivalence holds for more general moduli spaces, which excludes the existence of a generic duality transformation. The only case where these topological arguments fail to exclude the existence of a (generalized) Nahm transform is the selfdual moduli spaces $\mathcal{M}_{N+1}^N$ because of the trivial identity between both moduli spaces.

7. Compactification of moduli spaces and singular instantons.

The moduli spaces of instantons analyzed in the previous sections have natural compactifications obtained by adding the boundaries consisting of the matrices $\mathbf{A}$ which do not satisfy the constraint (5.7). These configurations correspond to fields which do have common zeros in all its components. Properly speaking these are not $\mathbb{CP}^N$ fields because seen as singular points when one introduces the normalization factor to have a unit norm representation of the field Ψ. These singular points can be interpreted as centers of singular instantons. It is envisable to consider the existence of common zeros as an effective charge reduction induced by the appearance of singular instantons. This observation provides additional information about the structure of the boundary of $\mathcal{M}_k^N$ in $\overline{\mathcal{M}}_k^N$.

The resulting moduli space $\overline{\mathcal{M}}_k^N$ is compact and has a bundle structure over the dual torus with compact fiber $\mathbb{C}\mathrm{P}^{k(N+1)-1}$. This compactified moduli space is stratified according to the number of common zeros of the different components of the field [18]. The generic dense stratum contains all regular instantons. The other strata consist of singular instantons. The degree of singularity is parametrized by the number of common zeros. This is reminiscent of a similar phenomenon occurring in Yang-Mills theory [19].

A single singular instanton can be viewed as a regular one with one topological charge less and with a pointwise singularity at a point x_0. In fact, we can generate singular instantons by adding such singularities to all regular instantons of lower charges, which gives a complete characterization of the subspace of singular instantons. Let Ψ be a (pointwise normalized) $\mathbb{C}P^N$ instanton configuration of charge k. One can obtain a singular instanton $\widetilde{\Psi}$ of charge $k+1$ by simply multiplying each component of Ψ by the phase of a theta function carrying unit charge:

$$\widetilde{\Psi} = \frac{\vartheta_1(z - z_0|i)}{|\vartheta_1(z - z_0|i)|}\Psi. \tag{7.1}$$

Observe that this phase has a singular point at the zero of the theta function, $z = z_0$. The corresponding holomorphic line bundle structure is shifted to $u + z_0$.

The associated gauge potential splits into the previous, charge k piece and a new contribution coming from the singular phase:

$$\widetilde{A}_{\bar{z}} = -i\widetilde{\Psi}^\dagger \partial_{\bar{z}}\widetilde{\Psi} = A_{\bar{z}} + a_{\bar{z}}, \tag{7.2}$$

where

$$a_{\bar{z}} = -i\mathrm{e}^{-i\arg\vartheta_1(z-z_0|i)}\partial_{\bar{z}}\,\mathrm{e}^{i\arg\vartheta_1(z-z_0|i)} = \frac{i}{2}\,\frac{\partial_{\bar{z}}\overline{\vartheta_1(z-z_0|i)}}{\overline{\vartheta_1(z-z_0|i)}}. \tag{7.3}$$

The additional contribution to the topological charge density is singular

$$\frac{1}{4\pi}\varepsilon_{\mu\nu}f_{\mu\nu}(x) = -\frac{2i}{\pi}\partial_z a_{\bar{z}}(x) = \frac{1}{4\pi}\nabla^2\ln\overline{\vartheta_1(z-z_0|i)} = \delta^{(2)}(x - x_0), \tag{7.4}$$

corresponding to a unit point charge at x_0. Thus the total topological density is

$$\frac{1}{4\pi}\varepsilon_{\mu\nu}\widetilde{F}_{\mu\nu}(x) = \frac{1}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu}(x) + \delta^{(2)}(x - x_0). \tag{7.5}$$

nicely illustrates the effect of including singular instantons.

Although the physical interest of singular configurations is not yet understood, field configurations in the vicinity of singular instantons appear in the regular space. To further clarify the structure near singular instantons we shall consider some interesting cases.

A regular instanton in the bulk of $\mathcal{M}_3^1$ can be built by choosing the parameters of the matrix $\mathbf{A}$ in such a way that the two components of the $\mathbb{C}\mathrm{P}^1$ field have well separated nodal points (e.g. the instanton configuration shown in fig.1.)

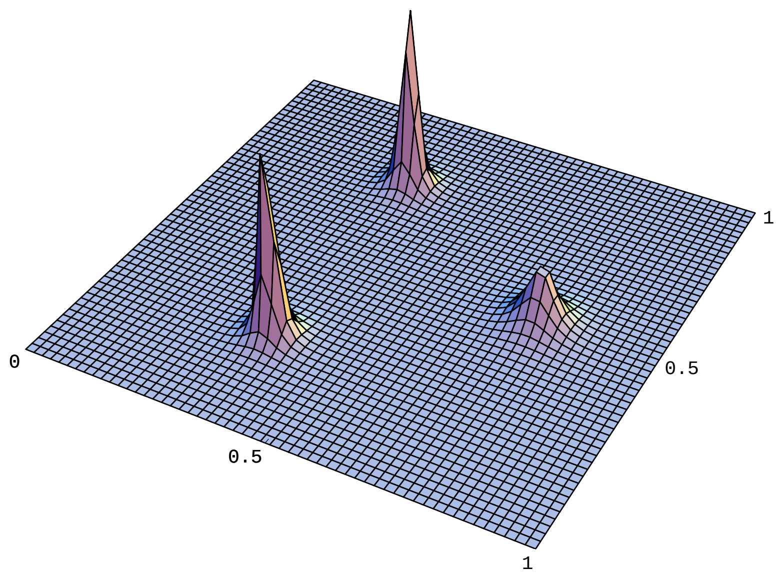

Figure1. Energy density distribution of a $\mathbb{C}\mathrm{P}^1$ regular instanton $u = 0, A = \begin{pmatrix} 2\sqrt{2} & 2 & 0 \\ 0.9 & 0 & 0 \end{pmatrix}$ with charge $k = 3$ in $\mathcal{M}_3^1$

We can approach the boundary of the moduli of regular instantons by choosing parameters of $\mathbf{A}$ in such a way that the nodal points of the two components of Ψ are very close to each other. We are near a singular instanton and find peaks in the energy density located on the nodal points showing the strong localization of energy and topological charge on singular instantons.

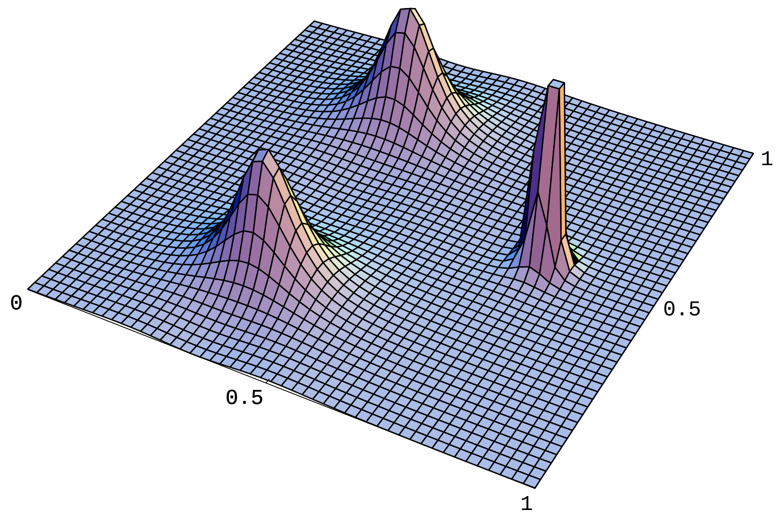

Figure 2. Energy density distribution of a $\mathbb{C}\mathrm{P}^1$ regular instanton $u = 0$, $A = \begin{pmatrix} 2\sqrt{2} & 2 & 0 \\ 0 & 0.1 & -i \end{pmatrix}$ with charge $k = 3$ close to one singular instanton in $\mathcal{M}_3^1$.

In the limit case we obtain a charge 3 instanton with one singular instanton and two regular ones. The picture is quite similar to Fig.2 with two lumps in the topological density, corresponding to two interacting instantons of charge 2 and finite size, and one singular instanton which is not shown in the numerical simulation.

Another way of approaching the boundary of the moduli space is by choosing one of the components of Ψ very small. In this case we approach a completely singular instanton with k singularities and one null component.

Now, the identification of single instantons in a multi-instanton configuration is not always clear. In fact, there are strongly overlapping configurations where it is hard to identify the constituent instantons. In Fig. 3 the configuration seems to contain four lumps whereas its total charge is $k = 2$.

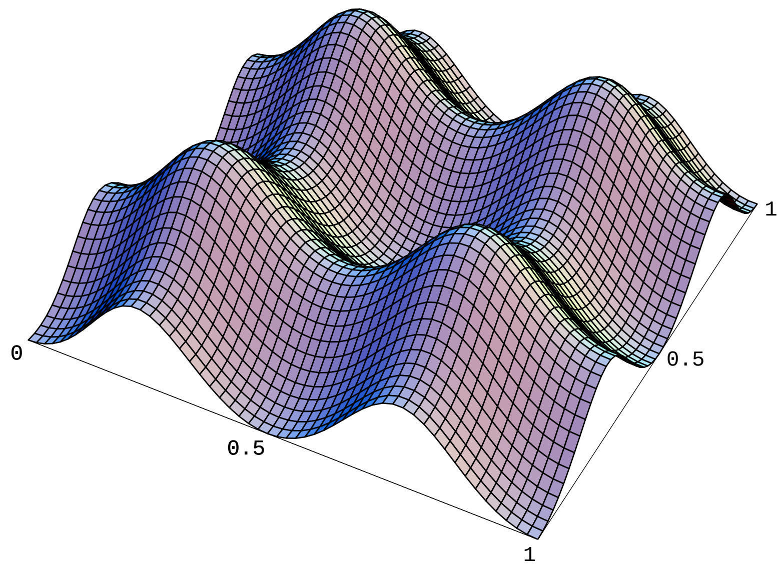

Figure 3. Energy density distribution of a $\mathbb{CP}^1$ regular instanton $u = 0, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with charge $k = 2$ in $\mathcal{M}_2^1$

Moreover, the interaction between instantons can be very involved and we can find densities with the shape of a volcano as in Fig. 4 which seems to describe a ring of instantons whereas its total topological charge is $k = 2$.

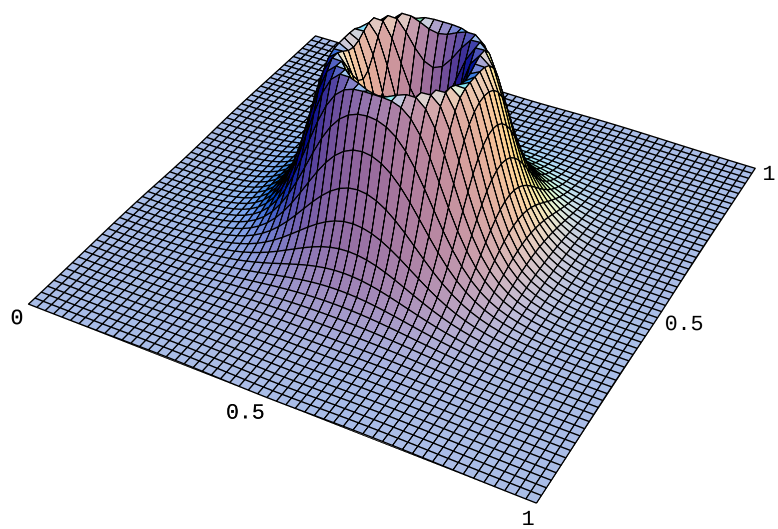

Figure 4. Energy density distribution of a $\mathbb{CP}^1$ strongly interacting regular instanton $u = 0, A = \begin{pmatrix} 0.42 - 0.72i & 0.27 - 0.47i \\ -0.49 - 0.85i & -0.10 - 0.18i \end{pmatrix}$ with charge $k = 2$ in $\mathcal{M}_2^1$

Strongly interacting instantons dominate in dense gas regimes whereas isolated instantons are more relevant for dilute gas phases. In general it is very dificult to identify the number of instatons of a single configuration in dense regimes. This difficulty increases and are made of instantons and anti-instantons.

In any case, singular instantons appear as a limiting case of small size instantons. In particular, they are responsible for the geodesic incompleteness of the moduli space of regular instantons [13]. Such configurations become of physical interest in supersymmetric $\mathbb{C}\mathrm{P}^N$ theories where they are important to localize topological Green functions in the corresponding topological field theory [18].

We close with some more general remarks about the possible role of singular instantons: The semiclassical expansion in a dilute gas of instantons has a different behavior depending on the number of components of the sigma model. For $N > 1$ the expansion is dominated by large instantons and we cannot trust this approximation to describe the deep infrared behavior of the theory [26]. In particular, their relevance for the confinement mechanism is unclear. On the other hand, for the $\mathbb{C}\mathrm{P}^1$ model the expansion is dominated by instantons of small sizes because of the different β function [26]. In the extreme case, singular instantons carry the leading effects and this is self-consistent with the dilute gas approximation. However, an ultraviolet regularization is in any case necessary. In lattice regularization the size of small instantons is bounded below by the lattice spacing and as a consequence the scaling properties of the topological susceptibility are changed. This leads to difficulties when one approaches the continuum limit [26][27]. The above discussion indicates that a continuum approach is feasible. Although similar effects are expected to occur, they might be less severe and lead to the stabilization of the ultraviolet catastrophe seen in the lattice approach.

In some sense the appearance of singular instantons is a dual effect of the existence of reducible instantons. A $\mathbb{C}\mathrm{P}^N$ instanton is reducible when it can be considered as living in a lower dimensional $\mathbb{C}\mathrm{P}^{N-1}$ projective subspace of $\mathbb{C}\mathrm{P}^N$. Reducible instantons belong to the strata of $\mathcal{M}_k^N$ associated with matrices $\mathbf{A}$ with rank lower than N. In the compactified moduli space $\overline{\mathcal{M}}_k^N$ there are two classes of strata, one corresponding to reducible instantons and the other to singular instantons. In one case there is a charge reduction and in the other a dimension reduction. The role of the two kinds of strata are interchanged when we compare the dual cases $\mathcal{M}_k^N$ and $\mathcal{M}_{N+1}^{k-1}$. If we exclude both types of instantons we are left with the modular space of generic regular instantons. The global structure of the space of generic regular instantons in $\mathcal{M}_k^{k-1}$ is much simpler. It is always a bundle with the dual torus as basis and as typical fiber the group PSL(k,$\mathbb{C}$), twisted by the boundary conditions (5.4).

The global structure of the moduli space of instantons in the $\mathbb{CP}^N$ model on a torus has a more explicit description than for Yang-Mills theory. However, this by no means implies that its geometrical and topological properties are simpler. In fact, in the case of gauge theories there exists a Nahm transform establishing a one-to-one correspondence between two a priori very different moduli spaces of instantons, $\mathcal{M}_k^N$ and $\mathcal{M}_{N+1}^{k-1}$. We have shown that such a map cannot exist for the $\mathbb{CP}^N$ sigma models.

We have identified the boundary of the space of regular instantons with the space of singular instantons. This identification of singular instantons as boundary configurations of the space of regular instantons provides a new approach to the analysis of the physical role of overlapping instantons in a dense gas and in topological field theories. The role of instantons in the confinement mechanism seems to be very different in the $\mathbb{CP}^1$ model and higher N models. The dominance of small or large instantons indicates a critical transition or crossover between these two regimes. The behavior of the theory in the presence of a θ term is also very much dependent on the regime and size of leading instanton contributions. In particular, the $\mathbb{CP}^1$ model shows a second order phase transition at $\theta = \pi$ whereas in higher N models it is not yet known whether a similar transition exists or not. It is very plausible that the instantons will play a role in the presence or absence of such a transition. If that case the global structure of $\mathcal{M}_k^N$ analyzed in this paper is expected to have interesting physical effects.

Acknowledgements.

We thank Ugo Bruzzo, Mario Escario, Antonio González-Arroyo and Pierre van Baal for discussions. A. Wipf thanks the University of Zaragoza and the MPI in Munich, where part of the work has been done, for hospitality. M. Aguado is supported by a fellowship of MEC (Programa FPU). M. Aguado and M. Asorey are partially supported by MCyT under grant FPA2000-1252. This work was carried out in the framework of a DAAD-MCyT grant.

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Here we establish the connection between two different geometric characterizations of the space of holomorphic sections of a complex line bundle which have been used in the paper. First, it is clear that this space is linear. On the other hand the holomorphic sections are characterized up to a constant by its zeros (divisors). From the relation between the two approaches it follows that the space of zeros of non-trivial holomorphic sections has a projective space structure. Let us discuss in detail how this projective structure emerges.

Indeed, the space of holomorphic sections of a complex line bundle on the torus with Chern class k is a linear space isomorphic to $\mathbb{C}^k$. Let us consider the basis introduced in (4.11),

$$\chi_\ell^w(z) = \chi_\ell(z - \tfrac{i}{k}w), \qquad \chi_\ell(z) = \mathrm{e}^{\pi k z^2/2} \vartheta \begin{bmatrix} z+\frac{\ell}{k} \\ 0 \end{bmatrix} (ik), \tag{1}$$

for the holomorphic structure defined by $w = u^1 + iu^2$ in $E_k(\mathbb{T}^2,\mathbb{C})$. With this choice for the basis any holomorphic section in E_k is given by the expansion coefficients c_ℓ in

$$\psi(z) = \sum_{\ell=1}^{k} c_\ell\, \chi_\ell^w(z), \tag{2}$$

The nodes of $\chi_\ell^w(z)$ are simple zeros and define the following lattice

$$z_{m,n} = \frac{i}{k} w - \frac{\ell}{k} + \big(m + \frac{1}{2}\big) + \big(n + \frac{1}{2}\big)\frac{i}{k}, \qquad m,\, n \in \mathbb{Z}. \tag{3}$$

Hence, there are k such zeros in the fundamental domain and the section belongs indeed to the bundle of charge k. The boundary conditions

$$\chi_\ell^w(z+1) = \mathrm{e}^{\pi k (z+1/2 - iw/k)} \chi_\ell^w(z) \quad \text{and} \quad \chi_\ell^w(z+i) = \mathrm{e}^{-i\pi k(z+i/2-iw/k)} \chi_\ell^w(z) \tag{4}$$

have been derived in the main body of the paper. They depend on $w \in \hat{\mathbb{T}}^2$ which has been introduced to shift the gauge potential.

To discuss the topology of the $\mathbb{CP}^N$ fields it is advantageous to use an alternative parametrization of the sections for which the nodal structure is explicit (but linearity is not). It is given by the product representation

$$\Theta(z) = \prod_{i=1}^{k} \mathrm{e}^{\frac{\pi}{2}\{(z-a_i)^2 + (1-i)z\}} \theta_1\big(z - a_i | \, i\big), \qquad \vartheta_1(z|i) = \sum_n (-)^n \mathrm{e}^{-\pi(n+z-\frac{1}{2})^2}. \tag{5}$$

Each $\vartheta_1(z - a_i|i)$ has zeros at the points of the lattice $a_i + \Lambda$, i.e. only a simple zero within the fundamental domain. Coalescence of some of these zeros is allowed. Θ fulfills the boundary conditions

$$\Theta(z+1) = \mathrm{e}^{\pi k(z+1) - \pi a + i\pi k/2}\, \Theta(z) \;\; , \;\; \Theta(z+i) = \mathrm{e}^{-i\pi k(z+i) + i\pi a - i\pi k/2}\, \Theta(z), \tag{6}$$

of the same space. This gives rise to the following constraint on the a_i in (5)

$$\sum_{i=1}^{k} a_i = iw + \tfrac{k}{2}\,(1+i). \tag{7}$$

It is easy to see that this sum is identical to that obtained for the nodes in (3).

Since quasiperiodic meromorphic functions on the torus are determined, up to a multiplicative constant, by the boundary conditions and the position and degeneracy of their zeros, one should be able to write *all* sections (5) in the form (1) by mapping the nodal configuration into the set of complex coefficients c_ℓ.

Assume then, for a given configuration $\{a_1, \ldots, a_k\}$ of nondegenerate zeros (the degenerate case will be discussed below) that

$$\Theta(z) = \sum_{\ell=1}^{k} c_\ell\, \chi_\ell^w(z). \tag{8}$$

Then, the k conditions $\Theta(a_i) = 0$ imply the following homogeneous set of equations for the coefficients c_ℓ,

$$\sum_\ell B_{i\ell}\, c_\ell = 0, \quad \text{where} \quad B_{i\ell} = \chi_\ell(a_i). \tag{9}$$

The equivalence of both parametrisations implies that the matrix $B = (B_{i\ell})$ has rank $k-1$ or that its kernel is one-dimensional. Hence the linear system (9) determines the coefficients c_ℓ, up to an overall factor. The overall constant may be fixed by matching the values of the sections in both parametrisations at a non-nodal point. In cases where some zero is degenerate one proceeds in an anologous way. A node a with multiplicity r yields r conditions $\Theta(a) = \Theta'(a) = \ldots = \Theta^{(r)}(a) = 0$, and the corresponding rows in B consist of derivatives of $\chi_\ell^w(z)$ at a.

Hence, any section has the product representation

$$\psi(z) = \lambda \prod_{i=1}^{k} \mathrm{e}^{\frac{\pi}{2}\{(z-a_i)^2+(1-i)z\}}\; \theta_1\big(z - a_i \,|\, i\big) \tag{10}$$

The parameter space for non-trivial holomorphic sections consists of a nonzero complex constant λ and k points on $\mathbb{T}^2$ subject to the constraint

$$\sum a_i = i\omega + \tfrac{k}{2}(1+i) \mod \Lambda. \tag{11}$$

As an example consider $k = 2$ and take $w = 0$. Then the constraint on the two nodes reads

$$a_1 + a_2 = 1 + i. \tag{12}$$

An unambiguous parametrization of the positions of the zeros fulfilling this constraint is achieved by picking the node a_1 in the region $0 < \mathrm{Re}\ a_1 < \frac{1}{2}$, together with the segments $\mathrm{Re}\ a_1 = 0,\ 0 \le \mathrm{Im}\ a_1 \le \frac{1}{2}$ and $\mathrm{Re}\ a_1 = \frac{1}{2},\ 0 \le \mathrm{Im}\ a_1 \le \frac{1}{2}$. The following figure shows the remaining identifications one needs to make:

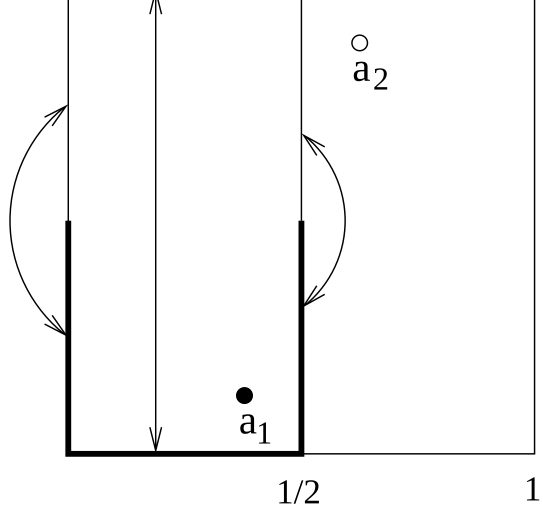

Figure 4. The domain $\mathcal{R}$ for a_1 with the necessary identifications.

This region with identifications is homeomorphic to a two-sphere S^2. The parameter λ in (10) contributes a positive constant $|\lambda|$ and a phase $\arg\lambda$. This phase takes values in the fiber of a principal $U(1)$ bundle over S^2, and to identify it, we need to compute the first Chern class

$$c_1(P) = \frac{1}{4\pi i}\oint_{\partial\mathcal{R}} (\mathrm{d}\,\ln\psi - \mathrm{d}\,\ln\psi^*) = \frac{1}{2\pi i}\oint_{\partial\mathcal{R}} \mathrm{d}\,\ln\psi, \tag{13}$$

where $\mathcal{R}$ is the region defined above. Since

$$\ln\psi = \ln\lambda + \ln\vartheta_1(z-a_1|i) + \ln\vartheta_1(z-a_2|i) + \text{Polynom}\,(z), \tag{14}$$

only the theta function associated with the unique zero $a_1 \in \mathcal{R}$ is relevant for the contour integral in (13). Moreover, from the infinite product expansion for thetas, only a sine factor contributes:

$$c_1(P) = \frac{1}{2\pi i}\oint_{\partial\mathcal{R}} \mathrm{d}\,\ln\vartheta_1(z-a|i) = \frac{1}{2\pi i}\oint_{\partial\mathcal{R}} \mathrm{d}\,\ln\sin[\pi(z-a)] = 1, \tag{15}$$

yielding unit Chern class by the residue theorem. Then the $U(1)$ fibration on S^2 is the Hopf bundle S^3.

The topologically nontrivial space $\mathbb{R}^4 \setminus \{0\}$ is the union of all 3-spheres with radii $|\lambda| \in \mathbb{R}_+$. The null section $\psi(z) = 0$ belonging to a singular instanton completes it to the contractible space $\mathbb{R}^4 \approx \mathbb{C}^2$.

This construction generalizes to arbitrary k, since the space of k points on the torus with fixed sum is topologically equivalent to $\mathbb{C}P^{k-1}$. The phase $\arg\lambda$ defines the sphere S^{2k-1} as a principal $U(1)$ bundle over this projective space. The space of sections $\mathbb{R}^{2k} \approx \mathbb{C}^k$ is constructed as before with the null section and all S^{2k-1} with positive radii. Charge 2 is a particular case since $\mathbb{C}P^1 \approx S^2$.